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Existence of Localized Pulse Solutions to Skew-Gradient Systems

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ABSTRACT

Reaction-diffusion systems have been primary tools for studying pattern formation. A skew-gradient system is well known to encompass a class of activator-inhibitor type reaction-diffusion systems that exhibit localized patterns such as fronts and pulses. In this dissertation, we investigate standing pulse solutions to two extensions of FitzHugh-Nagumo system that possess a skew-gradient structure. Our models exhibit additional nonlinearities that may enable the models to capture more complex behavior of standing pulse solutions. In both extensions, we employ a variational approach that involves a nonlocal term and establish the existence of standing pulse solutions with a sign change. In addition, we explore some qualitative properties of the standing pulse solutions.

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Existence of Localized Pulse Solutions to Skew-Gradient Systems

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Contents

Chapter 1

Introduction

From vegetation patterns in an ecological system to propagating waves in a nerve fiber, fascinating patterns emerge in nature. These self-organizing structures, free of external input, may originate from homogeneous media through some spatial modulation due to diffusion-driven Turing instability. Other patterns can represent phenomena far away from an equilibrium state; both standing and traveling waves are examples of the latter kind. Stationary or moving, these waves consist of one or more localized regions where the change from a trivial background state is substantial. In fact a standing or traveling front connects distinct equilibria, while a pulse returns to the same steady state after undergoing a large amplitude excursion. A pulse resembles a localized sharp spike and results from a delicate balance between gain and loss in the governing reaction kinetics. Competing mechanism, like in activator-inhibitor systems such as FitzHugh-Nagumo and Gierer-Meinhardt equations, are therefore prime examples for pattern formation. Under appropriate circumstances dynamics of these pulses and their mutual interaction can be particle-like, and are referred to as dissipative solitons (Akhmediev and Ankiewicz, 2008; Liehr, 2013). They are the building blocks for more complex structures.

Since Turing proposed the idea of diffusion-driven pattern formation (Turing, 1952), reaction-diffusion systems have been primary tools for studying pattern formation. We restrict ourselves to cases with two species and consider the reaction-diffusion system

$$\begin{cases} \tau_1 u_t = d_1 \Delta u + H_u(u,v), \\ \tau_2 v_t = d_2 \Delta v + (-1)^k H_v(u,v), \end{cases} \tag{1.1}$$

where $\tau_i > 0$ and $d_i > 0$ for $i = 1, 2$, $k \in \{0, 1\}$ and $H : \mathbb{R}^2 \to \mathbb{R}$ is some smooth function. The system in (1.1) is said to have a skew-gradient structure if $k = 1$ (Yanagida, 2002a,b) and a gradient structure if $k = 0$. For a reaction-diffusion system with a gradient structure, there is a Lyapunov functional which eases the analysis of the time dependent problems; it also serves as a natural variational functional for studying the stationary problem. The corresponding analysis of a skew-gradient system is more delicate.

A system with a skew-gradient structure encompasses a class of activator-inhibitor type reaction-diffusion models. A well-studied skew-gradient model that generates standing pulse solutions is

$$\begin{cases} u_t = d u_{xx} + f(u) - v, \\ \tau v_t = v_{xx} + u - \gamma v, \end{cases} \tag{1.2}$$

which is referred to as the FitzHugh-Nagumo equations (the original FitzHugh-Nagumo model does not have the term v_{xx}, see FitzHugh 1961; Nagumo et al. 1962).

For finite domains a variational formulation of the above problem readily yields a global minimizer that corresponds to a steady state solution. However such solution is usually oscillatory and is not a single localized sharp spike when the domain is large. In fact when the domain is unbounded, there is no global minimizer and a more careful treatment is necessary. In Klaasen and Troy (1984), the existence of positive standing pulse solutions to (1.2) was established for large γ and large d by a shooting argument when the parameters allow the presence of multiple constant steady states. Using a special transformation to convert the equations to a quasi-monotone system for large γ and d, Reinecke and Sweers (1999) employed comparison functions and finite domain approximation in $\mathbb{R}^N$ to establish a positive radially symmetric standing pulse solution. In Chen and Choi (2012), a variational approach was applied to find solutions with a sign change when the activator diffusivity is small compared to that of inhibitor, i.e. $d \ll 1$. The solution obtained is a local, rather than global, minimizer. There are also numerous numerical works on this model. Typically they are continuation type methods which require good initial guesses to start the algorithm. Recently a robust steepest descent algorithm for finding the waves numerically without a good initial guess has been proposed in Choi and Connors (2019).

When $f(u)$ is replaced by $f(u)/d$ in (1.2) for small d, this corresponds to studying the equations in a different parameter regime. One can employ other well established methods, for example Γ-convergence or the geometric perturbation method, to study standing pulses and their corresponding stability. Some related models like Ohta-Kawasaki involve a volumetric constraint. See for example Chen, Choi, and Ren (2018); Chen et al. (2018); van Heijster and Sandstede (2014); Wei and Winter (2005); Ren and Wei (2008) and the many references therein.

Over the past two decades, the study of (1.2) has further stretched into various extensions of the model. For example, an extension to a three-component system of (1.2) with an additional inhibitor equation of linear form has been considered in Bode et al. (2002); Doelman et al. (2009); van Heijster et al. (2019) and the references therein. The existence of the corresponding standing and traveling pulse solutions has been investigated both analytically and numerically.

In this dissertation we aim to investigate the existence of standing pulse solutions in the presence of additional nonlinearities in (1.2b). We construct the nonlinearities so that the skew-gradient structure of the model is preserved. In Chapter 2, a skew-gradient model with an extra cubic activator term in (1.2b) is considered. By employing a variational approach, we establish the existence of standing pulse solutions for the new extension. In Chapter 3, we investigate standing pulse solutions to a skew-gradient system in which both activator and inhibitor reaction terms inherit nonlinear structures. The nonlinear structure of the inhibitor equation leads to extra difficulties in applying a variational method as the corresponding functional involves several nonlocal terms. We prove that standing pulse solutions still exist under the effect of nonlinear inhibitor term.

Chapter 2

A skew-gradient system I : linear inhibitor equation

2.1 Introduction

In this chapter, we consider the following skew-gradient system

$$\begin{cases} du'' + f(u) - g'(u)v = 0, \\ v'' - \gamma v + g(u) = 0. \end{cases} \tag{2.1}$$

Here $f(u) = u(u-\beta)(1-u)$ and $g(u) = u + \epsilon u^3$; $0 < \beta < 1/2$ is a fixed constant, and $1 \geq \epsilon > 0$, $d > 0$ and $\gamma > 0$ are constants whose admissible ranges will be determined later. Our goal is to look for weak solutions $(u, v) \in (H^1(\mathbb{R}))^2$ that are even in x and satisfy

$$\lim_{|x| \to \infty} (u(x), v(x)) = (0, 0). \tag{2.2}$$

That is, we are interested in standing pulse solutions of (2.1). We note that when $g(u) = u$, our system is the steady-state FitzHugh-Nagumo equations. It possesses a skew-gradient structure with

$$H(u,v) = \frac{\gamma v^2}{2} + \int_0^u f(\xi)\, d\xi - g(u)v.$$

As we seek solutions which are symmetric about $x = 0$, our analysis is restricted to the interval $[0,\infty)$. The main results of are summarized in the following theorem:

Theorem 2.1. *Let $\beta \in (0, 1/2)$ be given. Then there exists a $\gamma_1 > 0$ so that for any $\gamma \in (0, \gamma_1]$, we have a $d_1 = d_1(\gamma) > 0$ and $\epsilon_1 = \epsilon_1(\gamma) > 0$ such that whenever $\gamma < \gamma_1$, $d < d_1$ and $\epsilon < \epsilon_1$,* (2.1) *has a solution $(u_0, v_0) \in (C^\infty[0,\infty))^2$ that satisfies the zero Neumann boundary condition and decays to 0 exponentially as $x \to \infty$; that is,* (2.1) *possesses a standing pulse solution.*

2.2 Variational formulation

Since (2.1b) is linear in v with constant coefficients, we can find the Green's function for the operator $(\gamma - \frac{d^2}{dx^2})$ associated with the zero Neumann boundary condition at $x = 0$ and decay at large x. Let us write

$$v(x) = \mathcal{L}(g(u))(x) = \int_0^\infty G(x,s) g(u(s))\, ds\,, \tag{2.3}$$

where

$$G(x,s) = \begin{cases} \frac{1}{\sqrt{\gamma}} e^{-\sqrt{\gamma} s} \cosh \sqrt{\gamma} x\,, & \text{if } 0 < x < s, \\ \frac{1}{\sqrt{\gamma}} e^{-\sqrt{\gamma} x} \cosh \sqrt{\gamma} s\,, & \text{if } x > s > 0, \end{cases} \tag{2.4}$$

is symmetric in x and s. Then

$$\begin{aligned}\mathcal{L}(g(u))(x) &= \int_0^x \frac{1}{\sqrt{\gamma}} e^{-\sqrt{\gamma}x} \cosh(\sqrt{\gamma}s)\, g(u(s))\, ds \\ &\quad + \int_x^\infty \frac{1}{\sqrt{\gamma}} e^{-\sqrt{\gamma}s} \cosh(\sqrt{\gamma}x)\, g(u(s))\, ds\,,\end{aligned} \tag{2.5}$$

and $\mathcal{L} : L^2(0,\infty) \to L^2(0,\infty)$ is self-adjoint, i.e., $\int_0^\infty w_1 \mathcal{L} w_2\, dx = \int_0^\infty w_2 \mathcal{L} w_1\, dx$ for any $w_1, w_2 \in L^2(0,\infty)$. Consider a functional $\hat{J} : H^1(0,\infty) \to \mathbb{R}$ defined as

$$\hat{J}(w) = \int_0^\infty \left\{ \frac{d}{2} w'^2 + F(w) + \frac{1}{2} g(w) \mathcal{L} g(w) \right\} dx\,, \tag{2.6}$$

where

$$F(\xi) = -\int_0^\xi f(\eta)\, d\eta = \xi^4/4 - (1+\beta)\xi^3/3 + \beta\xi^2/2.$$

Let $\beta < \beta_1 < 1 < \beta_2$ satisfy $F(\beta_1) = 0$ and $\beta_2 - 1 < \frac{\beta}{2}$. It is easy to check that the Euler-Lagrange equation associated to $\hat{J}$ is

$$du'' + f(u) - g'(u)\mathcal{L}g(u) = 0, \tag{2.7}$$

which is equivalent to our system in (2.1) if we substitute $\mathcal{L}g(u)$ with v. That is, if $u_0 \in H^1(0,\infty)$ is a critical point of $\hat{J}$, then $(u_0, \mathcal{L}g(u_0))$ satisfies (2.1).

The same argument for the case $g(u) = u$ in Chen and Choi (2012) can be employed to show that $\inf_{H^1(0,\infty)} \hat{J} = -\infty$, therefore we turn our attention to a local minimizer of $\hat{J}$. Following Chen and Choi (2012), we define a topological class that is appropriate for the variational formulation of (2.1). Let $M = M(\gamma)$ be a constant such that

$$(1+M) > \beta_2 \quad \text{and} \quad f(\xi) \geq \frac{3g(\beta_2)}{2\gamma}, \quad \forall \xi \leq -M. \tag{2.8}$$

Since f is decreasing on $[-\infty, 0)$, we can always find a large constant M that satisfies the definition in (2.8). We are ready to define the admissible set $\mathcal{A}$ as follows:

$$\begin{aligned}\mathcal{A} \equiv \{w \in H^1(0,\infty) : \beta \le w(0) \le \beta_2;\ &\text{there exist } 0 < x_1 < x_2 \le \infty \text{ such that} \\ &\beta \le w \le \beta_2 \text{ on } [0, x_1],\ 0 \le w \le \beta \text{ on } (x_1, x_2] \text{ and } -(M+1) \le w \le 0 \\ &\text{on } (x_2, \infty)\}. \end{aligned} \tag{2.9}$$

For $w \in \mathcal{A}$, we note that there are $0 < a_1 < b_1 < \infty$ such that $a_1 \le g'(w) \le b_1$. In particular $a_1 = g'(0) = 1$ which is independent of γ. We now impose a restriction on the size of ϵ and assume that $\epsilon \le \epsilon_0 = \frac{1}{6(M+1)^2}$. Observe that $g'(w) \le 1 + 3\epsilon(M+1)^2 \le 3/2$ implies $b_1 = 3/2$ is also independent of γ.

Let $J := \hat{J}\big|_{\mathcal{A}}$. In what follows, let us refer to the terms $\int_0^\infty \frac{d}{2} w'^2\, dx$, $\int_0^\infty F(w)\, dx$ and $\int_0^\infty \frac{1}{2} g(w) \mathcal{L} g(w)\, dx$ as the gradient term, potential term and nonlocal term of J, respectively. We finish this section by showing that $J : \mathcal{A} \to \mathbb{R}$ is well-defined. First let us recall that $H^1(0,\infty) \subseteq L^\infty(0,\infty)$. A proof is included.

Lemma 2.2. *Suppose $u \in H^1(0,\infty)$. Then $\|u\|_{L^\infty(0,\infty)} \le \sqrt{2}\, \|u\|_{H^1(0,\infty)}$ and $u(x) \to 0$ as $x \to 0$.*

Proof. Given any $a \in [0, \infty)$, let $a \le t < x \le a+1$. By integrating both sides of

$$u^2(x) = u^2(t) + \int_t^x D(u^2(s))\, ds$$

with respect to t over the interval $(a, a+1)$, we obtain from the Young's inequality

$$\begin{aligned}\|u\|^2_{L^\infty(a,a+1)} &\leq \|u\|^2_{L^2(a,a+1)} + 2\|u\|_{L^2(a,a+1)}\|Du\|_{L^2(a,a+1)} \\ &\leq 2\|u\|^2_{H^1(a,a+1)}\ .\end{aligned}$$

Taking the supremum over $a \in [0, \infty)$ yields $\|u\|_{L^\infty(0,\infty)} \leq \sqrt{2}\,\|u\|_{H^1(0,\infty)}$. Since $\|u\|_{H^1(a,\infty)} \to 0$ as $a \to \infty$, it is clear that $u \to 0$ as $x \to \infty$. □

Lemma 2.3. *For any $w \in \mathcal{A}$,*

$$|J(w)| < \infty.$$

Proof. For a fixed w, there is a constant C_w, which depends on $\|w\|_{L^\infty(0,\infty)}$, such that if $|\xi| \leq \|w\|_{L^\infty(0,\infty)}$, then $F(\xi) \leq C_w \xi^2$. From (2.1b)

$$\begin{aligned}\int_0^\infty \{(\mathcal{L}g(w))'^2 + \gamma(\mathcal{L}g(w))^2\}\, dx &= \int_0^\infty g(w)\mathcal{L}g(w)\, dx \\ &\leq \|g(w)\|_{L^2}\|\mathcal{L}g(w)\|_{L^2},\end{aligned}$$

which implies that $\|\mathcal{L}g(w)\|_{L^2(0,\infty)} \leq \|g(w)\|_{L^2(0,\infty)}/\gamma$. Since $g(w) \leq b_1|w|$, we have

$$\begin{aligned}|J(w)| &\leq \frac{d}{2}\|w'\|^2_{L^2} + C_w\|w\|^2_{L^2} + \frac{1}{\gamma}\|g(w)\|^2_{L^2} \\ &\leq \frac{d}{2}\|w'\|^2_{L^2} + C_w\|w\|^2_{L^2} + \frac{b_1^2}{\gamma}\|w\|_{L^2} \\ &< \infty.\end{aligned}$$

□

2.3 Existence of a minimizer

To establish the existence of a minimizer of J, we exploit the fact that g is increasing on $[-(M+1), \beta_2]$ with $a_1 \leq g' \leq b_1$.

Lemma 2.4. *Let $w \in \mathcal{A}$. Then $\mathcal{L}g(w) \leq \frac{b_1\beta_2}{\gamma}$ and $|\mathcal{L}g(w)| \leq \frac{b_1(M+1)}{\gamma}$ for all $x \in \mathbb{R}$.*

Proof. Since $g(w) \leq g(\beta_2) \leq b_1\beta_2$ for all $w \leq \beta_2$, we know from (2.5)

$$\begin{aligned}
\mathcal{L}g(w(x)) &= \int_0^x \frac{1}{\sqrt{\gamma}} e^{-\sqrt{\gamma}x} \cosh\left(\sqrt{\gamma}s\right) g(w(s))\, ds \\
&\quad + \int_x^\infty \frac{1}{\sqrt{\gamma}} e^{-\sqrt{\gamma}s} \cosh\left(\sqrt{\gamma}x\right) g(w(s))\, ds \\
&\leq \frac{b_1\beta_2}{\sqrt{\gamma}} e^{-\sqrt{\gamma}x} \int_0^x \cosh\left(\sqrt{\gamma}s\right) ds + \frac{b_1\beta_2}{\sqrt{\gamma}} \cosh\left(\sqrt{\gamma}x\right) \int_x^\infty e^{-\sqrt{\gamma}s}\, ds \\
&\leq \frac{b_1\beta_2}{\gamma}.
\end{aligned}$$

With $\|g(w)\|_{L^\infty} \leq b_1(M+1)$, a similar calculation shows that $|\mathcal{L}g(w)| \leq \frac{b_1(M+1)}{\gamma}$. □

Lemma 2.5. *If $w \in H^1(0,\infty)$, then $\int_0^\infty g(w)\mathcal{L}g(w)\, dx \geq 0$.*

Proof. Since $v = \mathcal{L}g(w)$ is a weak solution of $v'' - \gamma v = -g(w)$, integrating by parts yields

$$\int_0^\infty v'\varphi' + \gamma v\varphi\, dx = \int_0^\infty g(w)\varphi\, dx$$

for all $\varphi \in H^1(0,\infty)$. By choosing $\varphi = \mathcal{L}g(w)$ we observe that

$$\int_0^\infty g(w)\mathcal{L}g(w)\, dx = \int_0^\infty (\mathcal{L}g(w))'^2 + \gamma(\mathcal{L}g(w))^2\, dx \geq 0.$$

□

Lemma 2.6. *There exist $k_1 > 0$ and $k_2 > 0$, both independent of γ, such that for any $\gamma > 0$, whenever $d \le d_0 \equiv k_1\gamma$, there is a $q_0 \in \mathcal{A}$ such that $J(q_0) \le -k_2\sqrt{\gamma} < 0$.*

Proof. Let $0 < a < b$ be constants which will be assigned later. We define a piecewise linear function

$$q_0(x) \equiv \begin{cases} 1\,, & \text{if } 0 \le x \le a \\ \frac{b-x}{b-a}\,, & \text{if } a \le x \le b \\ 0\,, & \text{if } x \ge b\,. \end{cases}$$

A direct calculation shows

$$\int_0^\infty \frac{d}{2}(q_0)'^2\,dx = \frac{d}{2(b-a)}, \tag{2.10}$$

and

$$\int_0^\infty F(q_0)\,dx = -\frac{(1-2\beta)}{12}a + \left(\frac{1}{20} - \frac{(1+\beta)}{12} + \frac{\beta}{6}\right)(b-a). \tag{2.11}$$

For the nonlocal term, we have

$$\int_0^\infty g(q_0)\mathcal{L}g(q_0)\,dx = \int_0^a g(1)\mathcal{L}g(q_0)\,dx + \int_a^b g\left(\frac{b-x}{b-a}\right)\mathcal{L}g(q_0)\,dx. \tag{2.12}$$

For $0 \le x \le a$

$$\begin{aligned}
\mathcal{L}g(q_0)(x) &= \frac{g(1)}{\sqrt{\gamma}} \int_0^x e^{-\sqrt{\gamma}x} \cosh\left(\sqrt{\gamma}s\right) ds + \frac{g(1)}{\sqrt{\gamma}} \int_x^a e^{-\sqrt{\gamma}s} \cosh\left(\sqrt{\gamma}x\right) ds \\
&\quad + \frac{1}{\sqrt{\gamma}} \int_a^b e^{-\sqrt{\gamma}s} \cosh\left(\sqrt{\gamma}x\right) g\left(\frac{b-s}{b-a}\right) ds \\
&\le \frac{g(1)}{\gamma} e^{-\sqrt{\gamma}x} \big(\sinh(\sqrt{\gamma}x) + \cosh(\sqrt{\gamma}x)\big) - \frac{g(1)}{\gamma} e^{-\sqrt{\gamma}a} \cosh(\sqrt{\gamma}x) \\
&\quad + \frac{g(1)}{\sqrt{\gamma}} \cosh\left(\sqrt{\gamma}x\right) \int_a^b e^{-\sqrt{\gamma}s}\, ds \\
&= \frac{g(1)}{\gamma} - \frac{g(1)}{\gamma} e^{-\sqrt{\gamma}b} \cosh\left(\sqrt{\gamma}x\right).
\end{aligned}$$

Therefore we can compute

$$\begin{aligned}
0 \le \int_0^a g(1)\, \mathcal{L}g(q_0)\, dx &= \frac{(g(1))^2}{\gamma} a - \frac{(g(1))^2}{\gamma^{3/2}} e^{-\sqrt{\gamma}b} \sinh\left(\sqrt{\gamma}a\right) \\
&\le \frac{b_1^2}{\gamma} a - \frac{b_1^2}{\gamma^{3/2}} e^{-\sqrt{\gamma}b} \sinh\left(\sqrt{\gamma}a\right).
\end{aligned} \tag{2.13}$$

Similarly for $a \le x \le b$,

$$\begin{aligned}
\mathcal{L}g(q_0)(x) &\le \frac{g(1)}{\sqrt{\gamma}} e^{-\sqrt{\gamma}x} \int_0^x \cosh(\sqrt{\gamma}s)\, ds + \frac{g(1)}{\sqrt{\gamma}} \cosh(\sqrt{\gamma}x) \int_x^b e^{-\sqrt{\gamma}s}\, ds \\
&= \frac{g(1)}{\gamma} - \frac{g(1)}{\gamma} e^{-\sqrt{\gamma}b} \cosh(\sqrt{\gamma}x).
\end{aligned}$$

Then

$$\begin{aligned}
&\int_a^b g\left(\frac{b-x}{b-a}\right) \mathcal{L}g(q_0)\, dx \le \frac{(g(1))^2}{\gamma}(b-a) - \frac{(g(1))^2}{\gamma} e^{-\sqrt{\gamma}b} \int_a^b \cosh(\sqrt{\gamma}x)\, dx \\
&\le \frac{b_1^2}{\gamma} \{(b-a) - \frac{1}{\gamma^{1/2}} e^{-\sqrt{\gamma}b} (\sinh(\sqrt{\gamma}b) - \sinh(\sqrt{\gamma}a))\}.
\end{aligned} \tag{2.14}$$

Combining (2.13) and (2.14) and putting into (2.12), we obtain

$$\begin{aligned}\int_0^\infty g(q_0)\,\mathcal{L}g(q_0)\,dx &\le \frac{{b_1}^2}{\gamma}a - \frac{{b_1}^2}{\gamma^{3/2}}e^{-\sqrt{\gamma}b}\sinh\left(\sqrt{\gamma}a\right)\\ &\quad + \frac{{b_1}^2}{\gamma}(b-a) - \frac{{b_1}^2}{\gamma^{3/2}}e^{-\sqrt{\gamma}b}\left(\sinh(\sqrt{\gamma}b) - \sinh(\sqrt{\gamma}a)\right)\\ &= \frac{{b_1}^2}{\gamma}a + \frac{{b_1}^2}{\gamma}(b-a) - \frac{{b_1}^2}{\gamma^{3/2}}e^{-\sqrt{\gamma}b}\sinh(\sqrt{\gamma}b)\\ &= \frac{{b_1}^2}{\gamma}a + \frac{{b_1}^2}{\gamma}(b-a) - \frac{b_1^2\left(1-e^{-2\sqrt{\gamma}b}\right)}{2\gamma^{3/2}}. \end{aligned} \tag{2.15}$$

As a result of (2.10), (2.11) and (2.15),

$$\begin{aligned} J(q_0) &\le \frac{d}{2(b-a)} - \frac{(1-2\beta)}{12}a + \left(\frac{1}{20} - \frac{(1+\beta)}{12} + \frac{\beta}{6}\right)(b-a)\\ &\quad + \frac{b_1^2}{\gamma}a + \frac{b_1^2}{\gamma}(b-a) - \frac{b_1^2}{2\gamma^{3/2}}\left(1-e^{-2\sqrt{\gamma}b}\right). \end{aligned}$$

If we take $d_0 = (b-a)^2$, then for $d \le d_0$

$$J(q_0) \le (b-a)\left(\frac{7}{15} + \frac{\beta}{12} + \frac{b_1^2}{\gamma}\right) + a\left(\frac{b_1^2}{\gamma} - \frac{1-2\beta}{12}\right) + \frac{b_1^2}{2\gamma^{3/2}}(e^{-2\sqrt{\gamma}b} - 1).$$

It follows from $e^{-x} - 1 \le x^2/2 - x$ that $e^{-2\sqrt{\gamma}b} - 1 \le 2\gamma b^2 - 2\sqrt{\gamma}b$ when $x = 2\sqrt{\gamma}b$. This implies

$$J(q_0) \le \left(\frac{7}{15} + \frac{\beta}{12}\right)(b-a) - \frac{1-2\beta}{12}a + \frac{b_1^2}{\gamma^{1/2}}b^2.$$

Now let $(b-a) = \sqrt{k_1}\gamma^{1/2}$ with $k_1 < \left(\frac{1-2\beta}{24b_1^2}\right)^2$. Then

$$J(q_0) \le \left(\frac{7}{15} + \frac{\beta}{12}\right)\sqrt{k_1}\gamma^{1/2} - \frac{1-2\beta}{12}\left(b - \sqrt{k_1}\gamma^{1/2}\right) + \frac{b_1^2}{\gamma^{1/2}}b^2.$$

Let $b = \frac{(1-2\beta)}{24b_1^2}\gamma^{1/2} > (b-a) = \sqrt{k_1}\gamma^{1/2}$. By choosing k_1 sufficiently small, we have

$$\begin{aligned} J(q_0) &\leq \left(\frac{7}{15}+\frac{\beta}{12}+\frac{1-2\beta}{12}\right)\sqrt{k_1}\gamma^{1/2} - \left(\frac{1-2\beta}{24b_1}\right)^2\gamma^{1/2} \\ &< -k_2\gamma^{1/2} \end{aligned}$$

for some constant $k_2 > 0$. Both k_1 and k_2 do not depend on γ. The proof of Lemma 2.6 is complete. □

Lemma 2.7. *Let* $\gamma_0 := \frac{3a_1^2\beta^2}{2(1-2\beta)}$. *If* $\gamma \leq \gamma_0$ *and* $d \leq d_0 \equiv k_1\gamma$, *then*

(i) $\inf_{w\in\mathcal{A}} J(w) \geq -M_1$ *for some positive constant* $M_1 = M_1(\gamma) > 0$.

(ii) Let $\{w^{(n)}\}_{n=1}^{\infty} \subset \mathcal{A}$ *be any minimizing sequence for* J. *Recall the definition of* x_1 *in* $\mathcal{A}$ *and let* $x_1^{(n)}$ *be a corresponding value for* $w^{(n)}$. *By focusing on the tail of the sequence if necessary, there exists a positive constant* $M_2 = M_2(\gamma)$ *such that* $x_1^{(n)} \leq M_2$ *for all* n.

(iii) There exists a positive constant $M_3 = M_3(d,\gamma)$ *such that* $\|w^{(n)}\|_{H^1} \leq M_3$.

(iv) There exists a positive constant k_3, *which is independent of both* γ *and* d, *such that* $x_1^{(n)} \geq k_3\sqrt{\gamma}$.

Proof. Write $g(w)^+ = \max\{g(w),0\}$ and $g(w)^- = \max\{-g(w),0\}$. Since $\mathcal{L}$ is self-adjoint in the L^2 inner product and maps non-negative functions to non-negative functions, we have

$$\begin{aligned} \int_0^\infty g(w)\mathcal{L}g(w)\,dx &= \int_0^\infty \left(g(w)^+ - g(w)^-\right)\mathcal{L}\left(g(w)^+ - g(w)^-\right)\,dx \\ &\geq \int_0^{x_1} g(w)^+\mathcal{L}g(w)^+\,dx - 2\int_0^{x_2} g(w)^+\mathcal{L}g(w)^-\,dx \\ &\geq g(\beta)\int_0^{x_1}\mathcal{L}g(w)^+\,dx - 2g(\beta_2)\int_0^{x_2}\mathcal{L}g(w)^-\,dx. \end{aligned} \tag{2.16}$$

For $0 \le x \le x_1$,

$$\begin{aligned}\mathcal{L}g(w)^+(x) &= \frac{e^{-\sqrt{\gamma}x}}{\sqrt{\gamma}} \int_0^x g(w(s))^+ \cosh(\sqrt{\gamma}s)\, ds + \frac{\cosh(\sqrt{\gamma}x)}{\sqrt{\gamma}} \int_x^{x_2} g(w(s))^+ e^{-\sqrt{\gamma}s}\, ds \\ &\ge \frac{e^{-\sqrt{\gamma}x}}{\sqrt{\gamma}} \int_0^x g(\beta) \cosh(\sqrt{\gamma}s)\, ds \\ &= \frac{g(\beta)}{\gamma} e^{-\sqrt{\gamma}x} \sinh(\sqrt{\gamma}x)\,, \end{aligned} \tag{2.17}$$

and for $0 \le x \le x_2$,

$$\begin{aligned}\mathcal{L}g(w)^-(x) &= \frac{\cosh(\sqrt{\gamma}x)}{\sqrt{\gamma}} \int_{x_2}^{\infty} g(w(s))^- e^{-\sqrt{\gamma}s}\, ds \\ &\le \frac{\cosh(\sqrt{\gamma}x)}{\sqrt{\gamma}} \int_{x_2}^{\infty} -g(-(M+1)) e^{-\sqrt{\gamma}s}\, ds \\ &= \frac{-g(-(M+1))}{\gamma} e^{-\sqrt{\gamma}x_2} \cosh(\sqrt{\gamma}x). \end{aligned} \tag{2.18}$$

Putting (2.17) and (2.18) into (2.16), we obtain

$$\begin{aligned}\int_0^{\infty} g(w)\mathcal{L}g(w)\, dx \;&\ge\; \frac{(g(\beta))^2}{\gamma} \int_0^{x_1} e^{-\sqrt{\gamma}x} \sinh(\sqrt{\gamma}x)\, dx \\ &\quad + \frac{2g(\beta_2)g(-(M+1))}{\gamma} e^{-\sqrt{\gamma}x_2} \int_0^{x_2} \cosh(\sqrt{\gamma}x)\, dx \\ &\ge\; \frac{(g(\beta))^2}{2\gamma} \left\{ x_1 - \frac{1}{2\sqrt{\gamma}} (1 - e^{-2\sqrt{\gamma}x_1}) \right\} \\ &\quad + \frac{g(\beta_2)g(-(M+1))}{\gamma^{3/2}} \left\{ 1 - e^{-2\sqrt{\gamma}x_2} \right\} \\ &\ge\; \frac{(g(\beta))^2}{2\gamma} x_1 - \frac{(g(\beta))^2}{4\gamma^{3/2}} + \frac{g(\beta_2)g(-(M+1))}{\gamma^{3/2}}. \end{aligned}$$

Together with

$$\int_0^\infty F(w)\,dx \geq \int_0^{x_1} F(w)\,dx \geq F_{\min} x_1 = F(1)x_1 = -\frac{(1-2\beta)}{12}x_1,$$

we have

$$\begin{aligned} J(w) &\geq \int_0^\infty \frac{1}{2}g(w)\mathcal{L}g(w) + F(w)\,dx \\ &\geq \left(\frac{g^2(\beta)}{4\gamma} - \frac{(1-2\beta)}{12}\right)x_1 - \frac{g^2(\beta)}{8\gamma^{3/2}} + \frac{g(\beta_2)g(-(M+1))}{2\gamma^{3/2}} \\ &\geq \left(\frac{a_1^2\beta^2}{4\gamma} - \frac{(1-2\beta)}{12}\right)x_1 - \frac{b_1^2\beta^2}{8\gamma^{3/2}} - \frac{b_1^2\beta_2(M+1)}{2\gamma^{3/2}}. \end{aligned}$$

Let $\gamma_0 \equiv \frac{3a_1^2\beta^2}{2(1-2\beta)}$. For $\gamma \leq \gamma_0$, it is easy to check that $\frac{a_1^2\beta^2}{8\gamma} - \frac{(1-2\beta)}{12} \geq 0$ which implies

$$J(w) \geq -\frac{b_1^2\beta^2}{8\gamma^{3/2}} - \frac{b_1^2\beta_2(M_1+1)}{2\gamma^{3/2}} + \frac{a_1^2\beta^2}{8\gamma}x_1. \tag{2.19}$$

Choosing $M_1 = \frac{b_1^2\beta^2}{8\gamma^{3/2}} + \frac{b_1^2\beta_2(M_1+1)}{2\gamma^{3/2}}$, we establish (i).

For the proofs of (ii)-(iv), in which the nonlocal term is not involved, we refer to Lemma 5 of Chen and Choi (2012). □

With the a priori estimates associated with minimizing sequences of J, we are ready to show the existence of a minimizer.

Lemma 2.8. *Suppose $\gamma \leq \gamma_0$ and $d \leq d_0$. Let $\{w^{(n)}\}_{n=1}^\infty \subset \mathcal{A}$ be a minimizing sequence of J. Then there exists a $u_0 \in \mathcal{A}$ such that* $\liminf J(w^{(n)}) \geq J(u_0)$. *Moreover* *there exists an* $x_1 \in [k_3\sqrt{\gamma}, M_2]$ *(as defined in Lemma 2.7) and* $x_2 \in (x_1, \infty]$ *such that*

$$\begin{cases} \beta \leq u_0(x) \leq \beta_2 \text{ for } x \in [0, x_1], \\ 0 \leq u_0(x) \leq \beta \text{ for } x \in [x_1, x_2], \\ -(M+1) \leq u_0(x) \leq 0 \text{ for } x \in [x_2, \infty) \text{ if } x_2 < \infty. \end{cases} \tag{2.20}$$

Proof. By Lemma 2.7, there exists a minimizing sequence $\{w^{(n)}\}_{n=1}^{\infty} \subset \mathcal{A}$ such that $\lim J(w^{(n)}) = \inf_{w \in \mathcal{A}} J(w)$ and $\|w^{(n)}\|_{H_1} \leq M_3$ for all n. Since $H^1(0, \infty)$ is a reflexive Banach space, we can find a convergent subsequence, still denoted by $\{w^{(n)}\}$, such that $w^{(n)} \rightharpoonup u_0$ weakly in $H^1(0, \infty)$. Moreover since $H^1(0, L)$ is compactly embedded in $L^\infty(0, L)$ for any finite $L > 0$, we can choose a subsequence such that $w^{(n)} \to u_0$ in $L^\infty_{loc}(0, \infty)$ and pointwise a.e. Therefore (2.20) holds as a consequence of Lemma 2.7, and $u_0 \in \mathcal{A}$.

It remains to show that the extracted weakly convergent subsequence $\{w^{(n)}\}$ satisfies $J(u_0) \leq \liminf J(w^{(n)})$. That is, J is weakly lower semicontinuous in $H^1(0, \infty)$. For each term in J, we will verify the followings:

(i) $\|w'^{(n)}\|_{L^2} \geq \|u_0'\|_{L^2}$

(ii) $\liminf \int_0^\infty F(w^{(n)}) \geq \int_0^\infty F(u_0)$

(iii) $\liminf \int_0^\infty g(w^{(n)}) \mathcal{L} g(w^{(n)}) \geq \int_0^\infty g(u_0) \mathcal{L} g(u_0)$

The proofs of (i) and (ii) are shown in Lemma 6 of Chen and Choi (2012). To prove (iii), observe that, since $\{w^{(n)}\}$ is uniformly bounded in $H^1(0, \infty)$, the sequence $\{w^{(n)}\}$ is uniformly bounded in $L^\infty(0, \infty)$. Therefore $|g(w^{(n)})| \leq b_1 |w^{(n)}|$. Hence $g(w^{(n)}) \in H^1(0, \infty)$, and the same is true for $g(u_0)$. Since $g(w^{(n)}) - g(u_0) \in H^1(0, \infty)$, by Lemma 2.5

$$\int_0^\infty \left(g(w^{(n)}) - g(u_0)\right) \mathcal{L} \left(g(w^{(n)}) - g(u_0)\right) dx \geq 0, \quad \forall n.$$

From the self-adjoint property of $\mathcal{L}$, we obtain

$$\int_0^\infty g(w^{(n)})\mathcal{L}g(w^{(n)}) + g(u_0)\mathcal{L}g(u_0)\, dx \geq 2 \int_0^\infty g(w^{(n)})\mathcal{L}g(u_0)\, dx\,. \tag{2.21}$$

By restricting to a subsequence of $\{w^{(n)}\}$ if necessary, we claim that $g(w^{(n)}) \rightharpoonup g(u_0)$ weakly in $H^1(0,\infty)$. Assume its validity for the moment. By taking the lim inf on both sides of (2.21), we conclude

$$\liminf \int_0^\infty g(w^{(n)})\mathcal{L}(g(w^{(n)}))\, dx \geq \int_0^\infty g(u_0)\mathcal{L}g(u_0)\, dx.$$

Finally we need to justify the claim. Since $a_1 \leq g' \leq b_1$, it is clear that

$$\begin{aligned}
&\int_0^\infty \left(|D(g(w^{(n)}))|^2 + |g(w^{(n)})|^2\right)\, dx \\
&\leq b_1^2 \int_0^\infty \left(|Dw^{(n)}|^2 + |w^{(n)}|^2\right)\, dx
\end{aligned}$$

which is uniformly bounded. Going to a subsequence if needed, there exists a $z_0 \in H^1(0,\infty)$ such that $g(w^{(n)}) \rightharpoonup z_0$ weakly in $H^1(0,\infty)$ and pointwise a.e. On the other hand $w^{(n)} \to u_0$ pointwise a.e., which implies the same for $g(w^{(n)}) \to g(u_0)$. Hence we have $z_0 = g(u_0)$ a.e. □

Lemma 2.9. *Let u_0 be changed to $u_{new} \in \mathcal{A}$. Then the change in the nonlocal term* *is*

$$\frac{1}{2}\int_0^\infty \Big(g(u_{new})\mathcal{L}g(u_{new}) - g(u_0)\mathcal{L}g(u_0)\Big)\,dx$$
$$= \frac{1}{2}\int_0^\infty \big(g(u_{new}) - g(u_0)\big)\mathcal{L}\big(g(u_{new}) + g(u_0)\big)\,dx.$$

Proof. Due to the self-adjoint property of $\mathcal{L}$ in the L^2 inner product, the result follows from a direct calculation. □

Lemma 2.10. *Let u_0 be a minimizer obtained in Lemma 2.8. Then* $\min u_0 \geq -M$.

Proof. Suppose for contradiction $\min u_0 < -M$. Consider a truncated function

$$u_{new} = \begin{cases} u_0, & \textit{if } u_0 \geq -M, \\ -M, & \textit{if } u_0 < -M, \end{cases}$$

which lies in $\mathcal{A}$. With $u_{new} - u_0 \equiv p \geq 0$ and $g'(u) = 1 + 3\epsilon u^2$, we obtain

$$g(u_{new}) - g(u_0) \leq \|g'\|_{L^\infty(-(M+1),-M)}(u_{new} - u_0)$$
$$= g'(-M-1)(u_{new} - u_0),$$

which implies

$$
\begin{aligned}
J(u_{new}) &= \int_0^\infty \left\{ \frac{d}{2}(u'_{new})^2 + \frac{1}{2} g(u_{new}) \mathcal{L} g(u_{new}) + F(u_{new}) \right\} dx \\
&= \int_0^\infty \Big\{ \frac{d}{2}(u'_{new})^2 + \frac{1}{2} g(u_0) \mathcal{L} g(u_0) + F(u_{new}) \\
&\quad + \frac{1}{2}(g(u_{new}) - g(u_0)) \mathcal{L}\big(g(u_0) + g(u_{new})\big) \Big\} dx \\
&< \int_0^\infty \left\{ \frac{d}{2} {u'_0}^2 + \frac{1}{2} g(u_0) \mathcal{L} g(u_0) + F(u_0) \right\} dx \\
&\quad + \int_0^\infty \left\{ \frac{1}{2}(g(u_{new}) - g(u_0))\, \mathcal{L}\big(g(u_0) + g(u_{new})\big) + F(u_{new}) - F(u_0) \right\} dx \\
&\le J(u_0) + \int_{\{x:\, u_0(x) < -M\}} \Big\{ \frac{p}{2} g'(-M-1)\, \mathcal{L}\big(g(u_0) + g(u_{new})\big) \\
&\quad + F(u_{new}) - F(u_{new} - p) \Big\} dx \\
&\le J(u_0) + \int_{\{x:\, u_0(x) < -M\}} \left\{ \frac{p}{2} g'(-M-1) \mathcal{L}\big(g(u_0) + g(u_{new})\big) + F'(u_{new}) p \right\} dx\,,
\end{aligned}
$$

where the last inequality is a consequence of the convexity of $F(\xi)$ for $\xi \le 0$. By Lemma 2.4, $\mathcal{L}g(u_0) \le \frac{g(\beta_2)}{\gamma}$ and $\mathcal{L}g(u_{new}) \le \frac{g(\beta_2)}{\gamma}$. By the definition of M in the set $\{x :\, u_0(x) < -M\}$, $F'(u_{new}) = -f(u_{new}) = -f(-M) \le \frac{-3g(\beta_2)}{2\gamma}$. Therefore

$$
J(u_{new}) < J(u_0) + \int_{\{x:\, u_0(x) < -M\}} \left(\frac{g(\beta_2)}{\gamma} g'(-M-1) p - \frac{3g(\beta_2)}{2\gamma}\, p \right) dx.
$$

Together with $|g'(-M-1)| \le b_1 = 3/2$, we conclude that $J(u_{new}) < J(u_0)$. This contradicts the fact that u_0 is a minimizer in $\mathcal{A}$. □

In Lemma 2.10 we have established that the constraint $u_0 \ge -(M+1)$ imposed by $\mathcal{A}$ is in fact inactive. Away from the subset where u_0 equals $0, \beta$ or β_2, there is room for the minimizer u_0 to be perturbed by C_0^∞ functions with small support so that the perturbed function still lies in $\mathcal{A}$. Following the standard regularity argument, we conclude that $u_0 \in C^\infty(0,\infty)$ and satisfies (2.1a) except at those points where $u_0 = 0$, β or β_2. Since $g(u_0) \in H^1(0,\infty)$, we see from (2.1b) that $v_0 = \mathcal{L}g(u_0) \in H^3(0,\infty)$ which implies $v_0 \in C^2[0,\infty)$ and satisfies (2.1b) everywhere.

Lemma 2.11. *Let $x_0 > 0$ and $\ell \in (0, x_0)$. If $u_0(x) \notin \{0, \beta, \beta_2\}$ for $x \in [x_0 - \ell, x_0)$ and $u_0(x_0) \in \{0, \beta, \beta_2\}$, then both $\lim_{x\to x_0^-} u_0'(x)$ and $\lim_{x\to x_0^-} u_0''(x)$ exist. Moreover u_0 can be extended to a $C^\infty[x_0 - \ell, x_0]$ function, satisfying (2.1a) on $[x_0 - \ell, x_0]$. A similar statement holds when considering the neighborhood $(x_0, x_0 + \ell]$.*

Proof. We consider only the case where $u_0(x_0) = \beta$ and $u_0 > \beta$ on $[x_0 - \ell, x_0]$ for the other cases can be proved in the same manner. Note that $u_0 \in C^2[x_0 - \ell, x_0) \cap C[x_0 - \ell, x_0]$ and satisfies (2.1a). With u_0 and $v_0 = \mathcal{L}u_0$ being continuous on $[x_0 - \ell, x_0]$, it is clear from (2.1a) that u_0 is bounded in this neighborhood and $\lim_{x\to x_0^-} u_0''(x)$ exists. By writing

$$u_0'(x) = u_0'(x_0 - \ell) + \int_{x_0-\ell}^{x} u_0''(t)\,dt,$$

we have $\lim_{x\to x_0^-} u_0'(x) = u_0'(x_0 - \ell) + \lim_{x\to x_0^-} \int_{x_0-\ell}^{x} u_0''(t)\,dt$. The boundedness of the integrand guarantees that the limit exists. Hence $u_0 \in C^2[x_0 - \ell, x_0]$ and satisfies (2.1a) on this interval. Using typical regularity bootstrap by differentiating (2.1a), we conclude that $u_0 \in C^\infty[x_0 - \ell, x_0]$. □

2.4 Qualitative properties of the solution

In view of the constraints imposed by $\mathcal{A}$, we need to allow the possibility that there are intervals in which u_0 identically equals to $0, \beta$ or β_2, and u_0 may not satisfy (2.1a) on those intervals as a result. We will eliminate this scenario in the rest of Chapter 2. To that end, we first establish some useful qualitative properties of the solution.

Lemma 2.12. *Suppose x_0 and ℓ are positive numbers such that $u_0(x_0) \in \{0, \beta, \beta_2\}$ and $u_0 \in C^1[x_0 - \ell, x_0] \cap C^1[x_0, x_0 + \ell]$. Then $\lim_{x \to x_0^-} u_0'(x) = \lim_{x \to x_0^+} u_0'(x)$.*

Proof. Suppose for contradiction $u_0(x_0) = 0, \lim_{x \to x_0^-} u_0'(x) = a_1$, and $\lim_{x \to x_0^+} u_0'(x) = a_2$ with $a_1 \neq a_2$. If u_0 were straight lines on either side of x_0, then

$$u_0(x) = \begin{cases} a_1(x - x_0)\,, & \text{if } x_0 - \ell \leq x \leq x_0\,, \\ a_2(x - x_0)\,, & \text{if } x_0 \leq x \leq x_0 + \ell\,. \end{cases}$$

If the points $(x_0 - \ell, u_0(x_0 - \ell))$ and $(x_0 + \ell, u_0(x + \ell))$ are connected by a straight line, the slope is $(a_1 + a_2)/2$. We treat the general case based on this simple observation.

For a general C^1 function u_0, by taking $\ell_1 \leq \ell$ sufficiently small, we can assume

$$\begin{cases} u_0'(x) = a_1 + o(1)\,, & \text{if } x_0 - \ell_1 \leq x \leq x_0\,, \\ u_0'(x) = a_2 + o(1)\,, & \text{if } x_0 \leq x \leq x_0 + \ell_1. \end{cases}$$

The straight line $y = L_1(x)$ joining $(x_0 - \ell_1, u_0(x_0 - \ell_1))$ and $(x_0 + \ell_1, u_0(x_0 + \ell_1))$ has a slope of $(a_1 + a_2)/2 + o(1)$. Consider the function u_{new} obtained from trimming the corner of u_0:

$$u_{new}(x) = \begin{cases} u_0(x)\,, & \text{if } x \leq x_0 - \ell_1\,, \\ L_1(x), & \text{if } x_0 - \ell_1 \leq x \leq x_0 + \ell_1\,, \\ u_0(x)\,, & \text{if } x \geq x_0 + \ell_1\,. \end{cases}$$

Note that $u_{new}(x)$ still lies in $\mathcal{A}$. We now calculate $J(u_{new}) - J(u_0)$. First, the gradient term decreases because

$$\begin{aligned}
&\frac{d}{2}\int_{x_0-\ell_1}^{x_0+\ell_1}\{(u_{new})_x^2-(u_0)_x^2\}\,dx\\
&\quad=\frac{d}{2}\left\{\int_{x_0-\ell_1}^{0}(u_{new})_x^2-(u_0)_x^2\,dx+\int_0^{x_0+\ell_1}(u_{new})_x^2-(u_0)_x^2\,dx\right\}\\
&\quad=\frac{d\ell_1}{2}\left\{2\left(\frac{(a_1+a_2)}{2}+o(1)\right)^2-(a_1+o(1))^2-(a_2+o(1))^2\right\}\\
&\quad=\frac{d\ell_1}{2}\left\{\frac{(a_1+a_2)^2}{2}-a_1^2-a_2^2+o(1)\right\}\\
&\quad=-\frac{d\ell_1}{4}\left\{(a_1-a_2)^2+o(1)\right\}<0.
\end{aligned}$$

We employ the mean value theorem for the potential term to write

$$\int_{x_0-\ell_1}^{x_0+\ell_1}\{F(u_{new})-F(u_0)\}\,dx=-\int_{x_0-\ell_1}^{x_0+\ell_1}f(\tilde{u})(u_{new}-u_0)\,dx$$

for some $\tilde{u}$ between u_0 and u_{new}. With $\max_{-M\le\xi\le\beta_2}|f(\xi)|$ being bounded and $u_{new}-u_0=O(\ell_1)$,

$$\left|\int_{x_0-\ell_1}^{x_0+\ell_1}\{F(u_{new})-F(u_0)\}\,dx\right|\le\ell_1O(\ell_1)$$

which is negligible to the change in the gradient term of J. For the nonlocal term of J, we have from Lemma 2.9 and the mean value theorem on g that

$$\begin{aligned}
&\left|\frac{1}{2}\int_{x_0-\ell_1}^{x_0+\ell_1}g(u_{new})\mathcal{L}g(u_{new})-g(u_0)\mathcal{L}g(u_0)\,dx\right|\\
&\quad=\left|\frac{1}{2}\int_{x_0-\ell_1}^{x_0+\ell_1}\left(g(u_{new})-g(u_0)\right)\left(\mathcal{L}g(u_{new})+\mathcal{L}g(u_0)\right)\,dx\right|\\
&\quad=\left|\frac{1}{2}\int_{x_0-\ell_1}^{x_0+\ell_1}g'(\tilde{u})(u_{new}-u_0)\left(\mathcal{L}g(u_{new})+\mathcal{L}g(u_0)\right)\,dx\right|
\end{aligned}$$

for some $\tilde{u}$ between u_{new} and u_0. Since $|g'|, |\mathcal{L}g(u_{new})|, |\mathcal{L}g(u_0)|$ are all bounded and $|u_{new} - u_0| = O(\ell_1)$, we get

$$\left| \frac{1}{2} \int_{x_0-\ell_1}^{x_0+\ell_1} g(u_{new})\mathcal{L}g(u_{new}) - g(u_0)\mathcal{L}g(u_0)\, dx \right| \leq \ell_1 O(\ell_1),$$

which is also negligible compared to the change in the gradient term. Hence $J(u_{new}) < J(u_0)$ with $u_{new} \in \mathcal{A}$. This contradicts u_0 being a minimizer in $\mathcal{A}$. □

Remark 2.13. In what follows, Lemma 2.12 is referred to as a corner lemma, which does not require u_0 satisfy (2.1a) on either $[x_0 - \ell, x_0]$ or $[x_0, x_0 + \ell]$. Consider, for example, the case where $u_0(x_0) = \beta_2$ and $u_0 \in C^1[x_0, x_0+\ell]$ for some $\ell > 0$. By taking ℓ smaller if necessary, on the left side of a neighborhood of x_0 there are eventually three possibilities for the behavior of u_0:

(P1) $u_0 < \beta_2$ on $[x_0 - \ell, x_0)$;

(P2) $u_0 = \beta_2$ on $[x_0 - \ell, x_0]$;

(P3) There exist $a_1 < b_1 \leq a_2 < b_2 \leq a_3 < b_3 \cdots$ in the interval $[x_0 - \ell, x_0]$ such that

$$\begin{cases} u_0 \text{ satisfies (2.1a) on each interval } (a_n, b_n), \quad n = 1, 2, \cdots, \\ u_0 = \beta_2 \text{ on } [x_0 - \ell, x_0] \setminus \cup_{n=1}^{\infty}(a_n, b_n), \end{cases}$$

with both $a_n \to x_0^-$ and $b_n \to x_0^-$.

The case of $u_0(x_0) = 0$ and $u_0(x_0) = \beta$ can be treated similarly with corresponding cases referred to as (Q1)-(Q3) and (R1)-(R3), respectively. In case of (P1), $u_0 \in C^\infty[x_0 - \ell, x_0]$ follows from Lemma 2.11. It is clear that $u_0 \in C^\infty[x_0 - \ell, x_0]$ in (P2). The next lemma shows that $u_0 \in C^1[x_0 - \ell, x_0]$ in cases (P3), (Q3) and (R3). Consequently $u_0 \in C^1[0, \infty)$.

Lemma 2.14. *If $d \le d_0$, then $u_0 \in C^1[0,\infty)$. Let $v_0 = \mathcal{L}g(u_0)$, then (u_0, v_0) satisfies the following properties:*

(a) If x_0 is a limit point stated in (P3), it is necessary that $u_0(x_0) = \beta_2, u_0'(x_0) = v_0'(x_0) = 0$ and $v_0(x_0) = \frac{f(\beta_2)}{g'(\beta_2)} < 0$.

(b) In case x_0 is a limit point stated in (Q3), then $u_0(x_0) = 0$, $u_0'(x_0) = v_0'(x_0) = 0$ and $v_0(x_0) = \frac{f(0)}{g'(0)} = 0$.

(c) If x_0 is a limit point stated in (R3), then $u_0(x_0) = \beta$, $u_0'(x_0) = v_0'(x_0) = 0$ and $v_0(x_0) = \frac{f(\beta)}{g'(\beta)} = 0$.

Proof. In all cases of (P3), (Q3) and (R3), $u_0'(a_i) = u_0'(b_i) = 0$ and $u_0 \in C^1[x_0 - \ell, x_0)$ by Lemma 2.11. It remains to study u_0' at $x = x_0$. On the interval $[a_n, b_n]$, u_0 satisfies (2.1a)

$$du_0'' = -f(u_0) + g'(u_0)v_0\,,$$

and there is a $s_n \in (a_n, b_n)$ such that $u_0'(s_n) = 0$. Since $\|-f(u_0)+g'(u_0)v_0\|_{L^\infty(a_n,b_n)} \le C_1$ for some constant C_1 not depending on x_0 or n, a simple integration yields

$$d\,|u_0'(x)| \le C_1 \left| \int_{s_n}^{x} dt \right|,$$

so that

$$|u_0'(x)| \le \frac{C_1(b_n - a_n)}{d}.$$

As $n \to \infty$, it follows from $|b_n - a_n| \to 0$ that $\|u_0'\|_{L^\infty(a_n,b_n)} \to 0$. Then $u_0 \in C^1[a_1, x_0]$ if we set $u_0'(x_0^-) = 0$. Since the same argument shows $u_0 \in C^1[x_0, x_0 + \delta]$, invoking the corner lemma yields $u_0 \in C^1[x_0 - \ell, x_0 + \ell]$ for some $\ell > 0$. This completes the proof of $u_0 \in C^1[0,\infty)$.

Next we prove (a) where (P3) prevails on the left-hand side of x_0. Since $u_0 \le \beta_2$

everywhere, u_0 has a minimum at s_n, and by (2.1a)

$$f(u_0(s_n)) - g'(u_0(s_n))v_0(s_n) = -du_0''(s_n) \leq 0. \tag{2.22}$$

As $s_n \to x_0^-$, it follows that

$$f(u_0(x_0)) - g'(u_0(x_0))v_0(x_0) \leq 0 \tag{2.23}$$

and $u_0'(x_0) = 0$.

From our assumption, $u_0 \in C^1(a_n, b_n)$, and it follows from Lemma 2.11 that $u_0 \in C^2[a_n, b_n]$. In view of $u_0(b_n) = \beta_2$ and $u_0'(b_n) = 0$, we know $u_0''(b_n) \leq 0$ and consequently

$$f(u_0(b_n)) - g'(u_0(b_n))v_0(b_n) = -du_0''(b_n) \geq 0. \tag{2.24}$$

Passing to a limit as $n \to \infty$ gives

$$f(u_0(x_0)) - g'(u_0(x_0))v_0(x_0) \geq 0. \tag{2.25}$$

This together with (2.23) yields

$$f(u_0(x_0)) - g'(u_0(x_0))v_0(x_0) = 0. \tag{2.26}$$

In other words, $v_0(x_0) = \frac{f(\beta_2)}{g'(\beta_2)} < 0$.

We claim $v_0'(x_0) = 0$. Suppose for contradiction $v_0'(x_0) < 0$. This together with $u_0'(x_0) = 0$ gives $(f(u_0) - g'(u_0)v_0)'|_{x=x_0} = -g'(u_0(x_0))v_0'(x_0) > 0$. Since $f(u_0) - g'(u_0)v_0 = 0$ at $x = x_0$, it follows that $f(u_0(x)) - g'(u_0(x))v_0(x) < 0$ on an interval

$[x_0 - \delta, x_0)$ for some $\delta > 0$. But this is incompatible with (2.24) as there exists $N > 0$ such that $b_n \in [x_0 - \delta, x_0)$ for all $n \geq N$. Similarly $v_0'(x_0) > 0$ would contradict (2.22). Now the claim is justified, which completes the proof of (a).

The proofs of (b) and (c), when $u_0(x_0) = 0$ and $u_0(x_0) = \beta$ are slightly different, since u_0 can cross 0 and β in (a_1, x_0); nevertheless due to the fact that $u_0 \in \mathcal{A}$ is allowed to cross 0 and β only once, by choosing a_1 sufficiently close to x_0, u_0 does not change sign in $[a_1, x_0]$. Then the rest of the proofs for (b) and (c) are similar to the proof of of (a). We omit the details. □

In the remainder of the Section 2.4, we will use the fact that $u_0 \in C^1[0, \infty)$ to establish the positivity of v_0.

Lemma 2.15. *If u_0 is a minimizer obtained in Lemma 6 and $v_0 = \mathcal{L}g(u_0)$, then $v_0(0) > 0$.*

Proof. If $u_0 \geq 0$ for all $x \in [0, \infty)$, $v_0(0) > 0$ trivially follows from the positivity of the Green's function. Therefore assume u_0 changes sign at $x = x_2$. For contradiction, suppose that $v_0(0) \leq 0$. We claim that if $v_0(0) \leq 0$, then

(i) $v_0(x) < v_0(0)$ and $v_0'(x) < 0$ for all $x \in (0, x_2]$, and

(ii) $v_0(x) < 0$ on $[x_2, \infty)$.

Suppose the claim holds. Then, $v_0(x) < 0$ for all $x \in [0, \infty)$. Let x_0 be a point where u_0 attains its global minimum. Then $u''(x_0) \geq 0$. Since u_0 is assumed to have changed sign at $x = x_2$, we know $u_0(x_0) < 0$ and $u_0(x) < 0$ on $(x_0 - \epsilon, x_0 + \epsilon)$ for some $\epsilon > 0$. With $f(u(x_0)) > 0$,

$$g'(u_0(x_0))v_0(x_0) = du''(x_0) + f(u_0(x_0)) > 0$$

as u_0 satisfies (2.1a) at x_0. Hence $v_0(x_0) > 0$. However, this contradicts our claim that $v_0 < 0$ on $[0, \infty)$. This finishes the proof of the lemma, and it remains to verify our claim. To prove (i), first consider when $v_0(0) < 0$. On $[0, x_2]$, we have from (2.1b)

$$v_0'' - \gamma v_0 = -g(u_0) \leq 0,$$

so v_0 cannot attain an interior non-positive minimum on $(0, x_2)$. If $x = 0$ is the minimum, the Hopf lemma gives $v_0'(0) > 0$, which is a contradiction to the Neumann boundary condition. Therefore, $v_0(x) < v_0(0)$ and $v_0'(x) < 0$ for $x \in (0, x_2]$. Next, if $v_0(0) = 0$, it follows from (2.1b)

$$v_0''(0) = -g(u_0(0)) < 0.$$

Since $u_0 \in \mathcal{A}$, $u_0(0) > 0$. With $v_0'(0) = 0$, we have $v_0(x) < 0$ on a small neighborhood of 0, say $(0, \epsilon)$, which is a result from simple Taylor expansion. By using the maximum principle again, we have $v_0'(x) < 0$ on $[\epsilon, x_2]$ for an arbitrary ϵ, i.e., on $(0, x_2]$.

From (i), we know $v_0(x_2) < 0$. By our assumption, u_0 changes sign at x_2, so $u_0(x) \leq 0$ on $[x_2, \infty)$. In (2.1b),

$$v_0''(x) - \gamma v_0(x) = -g(u_0) \geq 0$$

on $[x_2, \infty]$, so v_0 cannot have a non-negative maximum by the maximum principle. Together with $v_0 \to 0$ as $x \to 0$, it follows that $v_0 < 0$ on $[x_2, \infty)$. This finishes the proof of pur claim, and therefore we conclude $v_0(0) > 0$. □

To continue the proof of positivity of $\mathcal{L}g(u_0)$, we extract information for (u_0, v_0) from the linearization of the system (2.1) at $(0,0)$.

Let $d_1 \equiv \min\{d_0, \frac{\beta^2}{4((g'(0))^2+\beta\gamma)}\}$ and assume $d \le d_1$ in what follows. We can express the system (2.1) as

$$\begin{cases} u_0'' - \frac{\beta}{d}u_0 - \frac{g'(0)}{d}v_0 = \frac{-u_0^2(1+\beta-u_0)}{d} + \frac{g'(u_0)}{d}v_0 - \frac{g'(0)}{d}v_0\,, \\ v_0'' + g'(0)u_0 - \gamma v_0 = -g(u_0) + g'(0)u_0\,, \end{cases}$$

or equivalently,

$$\begin{pmatrix} u_0 \\ v_0 \end{pmatrix}'' - A \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} = - \begin{pmatrix} \frac{u_0^2(1+\beta-u_0)}{d} - \frac{g'(u_0)}{d}v_0 + \frac{g'(0)}{d}v_0 \\ g(u_0) - g'(0)u_0 \end{pmatrix} \tag{2.27}$$

where

$$A = \begin{pmatrix} \frac{\beta}{d} & \frac{g'(0)}{d} \\ -g'(0) & \gamma \end{pmatrix}.$$

We begin with simple calculation on the eigenvalues, and left and right eigenvectors of A.

(a) An eigenvalue λ of A satisfies

$$\lambda(\beta - d\lambda) = (g'(0))^2 + \gamma\beta - d\gamma\lambda, \tag{2.28}$$

which is an intersection point of the parabola $z = \lambda(\beta - d\lambda)$ and the straight line $z = (g'(0))^2 + \gamma\beta - d\gamma\lambda$ in the (λ, z) plane. The parabola has zeros at $\lambda = 0$ and $\lambda = \beta/d$ and its maximum height is $\beta^2/4d$ at $\lambda = \beta/2d$, while the straight line passes through the points $(0, (g'(0))^2 + \gamma\beta)$ and $(\beta/d, (g'(0))^2)$. Given $d \leq d_1$, we have $\beta^2/4d \geq (g'(0))^2 + \gamma\beta$. Thus the parabola and the straight line intersect at λ_1 and λ_2, which are two real eigenvalues of A. Moreover $\lambda_1 + \lambda_2 = trace(A) = \gamma + \beta/d$. Hence

$$0 < \lambda_1 < \frac{\beta}{2d} < \frac{\beta}{2d} + \frac{\gamma}{2} = \frac{1}{2}\left(\gamma + \frac{\beta}{d}\right) < \lambda_2 < \frac{\beta}{d} \tag{2.29}$$

In addition, by (2.29)

$$\gamma < \frac{\beta}{d}. \tag{2.30}$$

(b) Let $\mathbf{a}$ and $\mathbf{b}$ be the right eigenvectors of A associated with λ_1 and λ_2, respectively. Observe that the second row of matrix $A - \lambda_2 I$ is $(-g'(0), \gamma - \lambda_2)$. Since (2.29) and (2.30) imply that

$$\gamma - \lambda_2 < \gamma - \frac{1}{2}(\gamma + \frac{\beta}{d}) < 0,$$

and $(A - \lambda_2 I)\mathbf{b} = 0$, we conclude that the two components of $\mathbf{b}$ must have opposite signs. From (2.29), we know $\beta/2d > \lambda_1$. Since the first row of matrix $A - \lambda_1 I$ is $(\beta/d - \lambda_1, g'(0)/d)$, the components of $\mathbf{a}$ must have opposite signs as well.

(c) Let $\mathbf{l}_1^{\mathrm{T}}$ and $\mathbf{l}_2^{\mathrm{T}}$ be the left eigenvectors of A associated with λ_1 and λ_2, respectively. Since $\lambda_1 \neq \lambda_2$, it is known that $\mathbf{l}_1 \cdot \mathbf{b} = \mathbf{l}_2 \cdot \mathbf{a} = 0$. From (b) we see that $(-g'(0), \gamma - \lambda_2)$ is perpendicular to $\mathbf{b}$. This allows us to take $\mathbf{l}_1 = (g'(0), \lambda_2 - \gamma)^{\mathrm{T}} = (g'(0), \beta/d - \lambda_1)^{\mathrm{T}} = (g'(0), \alpha_2)^{\mathrm{T}}$ where $\alpha_2 \equiv \beta/d - \lambda_1$, while $\mathbf{l}_1 \perp \mathbf{b}$ allows us to pick $\mathbf{b} = (-\alpha_2, g'(0))^{\mathrm{T}}$. Similarly due to the fact that $\mathbf{l}_2$ is parallel to the first row of $A - \lambda_1 I$, $(\alpha_2, g'(0)/d)^{\mathrm{T}}$, we may choose $\mathbf{l}_2 = (g'(0), \frac{(g'(0))^2}{d\alpha_2})^{\mathrm{T}} = (g'(0), \alpha_1)^{\mathrm{T}}$ by setting $\alpha_1 \equiv \frac{(g'(0))^2}{d\alpha_2} = \frac{(g'(0))^2}{\beta - d\lambda_1}$

and $\mathbf{a} = (-g'(0), \frac{(g'(0))^2}{\alpha_1})^{\mathrm{T}} = (-g'(0), d\alpha_2)^{\mathrm{T}}$. In summary, we have

$$\mathbf{a} = \begin{pmatrix} -g'(0) \\ d\alpha_2 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} -\alpha_2 \\ g'(0) \end{pmatrix}, \quad \mathbf{l}_1 = \begin{pmatrix} g'(0) \\ \alpha_2 \end{pmatrix}, \quad \mathbf{l}_2 = \begin{pmatrix} g'(0) \\ \alpha_1 \end{pmatrix}.$$

With $trace(A) = \lambda_1 + \lambda_2 = \gamma + \beta/d$ and $det(A) = \lambda_1\lambda_2 = ((g'(0))^2 + \gamma\beta)/d$,

$$\begin{aligned} \alpha_1 + \lambda_2 &= \frac{(g'(0))^2}{\beta - d\lambda_1} + \lambda_2 \\ &= \frac{(g'(0))^2 + \lambda_2\beta - d\lambda_1\lambda_2}{\beta - d\lambda_1} \\ &= \frac{\beta(\lambda_2 - \gamma)}{\beta - d\big(\frac{\beta}{d} + \gamma - \lambda_2\big)} \\ &= \frac{\beta}{d}. \end{aligned}$$

We also have $\alpha_2 + \lambda_1 = (\beta/d - \lambda_1) + \lambda_1 = \beta/d$. From (a), the graph of the parabola $z = \lambda(\beta - d\lambda)$ is symmetric with respect to the vertical line $\lambda = \beta/2d$, and the line $z = 1 + \gamma\beta - d\gamma\lambda$ is a decreasing function of λ. Since $\lambda_1 + \alpha_2 = \lambda_2 + \alpha_1 = \beta/d$, it is clear that

$$0 < \alpha_1 < \lambda_1 < \frac{\beta}{2d} < \alpha_2 < \lambda_2 < \frac{\beta}{d}. \tag{2.31}$$

(d) Knowing that $d \le d_1 < \beta^2/4(g'(0))^2$, we get $d\alpha_2^2 - (g'(0))^2 > d\big(\frac{\beta}{2d}\big)^2 - (g'(0))^2 = \frac{\beta^2}{4d} - (g'(0))^2 > 0$. Then by direct calculation,

$$\mathbf{l}_1 \cdot \mathbf{a} = d\alpha_2^2 - (g'(0))^2 > 0, \tag{2.32}$$

and

$$\mathbf{l}_2 \cdot \mathbf{b} = -g'(0)(\alpha_2 - \alpha_1) < 0. \tag{2.33}$$

(e) Suppose there is an $a_0 \in \mathbb{R}$ such that $u_0 \not\equiv 0$ on any subinterval of $[a_0, \infty)$. Since $(u_0, v_0) \to (0,0)$ as $x \to \infty$, we can ensure that $|u_0| < \beta < M$ for sufficiently large x. Thus for some $a_1 \geq a_0$, (u_0, v_0) is a smooth function and satisfies (2.1) on $[a_1, \infty)$. The dominant behavior of (u_0, v_0) can be studied by linearizing (2.27) about $(u, v) = (0, 0)$:

$$\begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix}'' - A \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} = \mathbf{0} \quad \text{on } [a_1, \infty). \tag{2.34}$$

Denote by s_2, s_3, the roots of $s^2 - \lambda_1 = 0$ and s_1, s_4, those of $s^2 - \lambda_2 = 0$. By plotting the curve $z = s^2$ and the horizontal linse $z = \lambda_1$ and $z = \lambda_2$, it is readily seen that

$$s_1 < s_2 < 0 < s_3 < s_4,$$

since $\lambda_2 > \lambda_1 > 0$. Moreover $\begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} \in span\{e^{s_1 x}\mathbf{b},\, e^{s_2 x}\mathbf{a},\, e^{s_3 x}\mathbf{a},\, e^{s_4 x}\mathbf{b}\}$. For a solution of (2.34) decaying to $(0,0)$ as $x \to \infty$,

$$\begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} = C_1 e^{-\sqrt{\lambda_1} x}\mathbf{a} + C_2 e^{-\sqrt{\lambda_2} x}\mathbf{b} \tag{2.35}$$

with C_1 and C_2 being constants.

Recall that $g(u) = u + \epsilon u^3$ with $\epsilon \leq \epsilon_0 = \frac{1}{6(M+1)^2}$. We now impose an additional restriction on the size of ϵ.

Lemma 2.16. *Let* $\epsilon_1 := \min\left\{\epsilon_0, \frac{\beta}{(M+1)(9/\gamma+1/2)}\right\}$ *and define* $\psi_1 = g'(0)u_0 + \alpha_2 v_0$, $\psi_2 = g'(0)u_0 + \alpha_1 v_0$*. If* $\epsilon \le \epsilon_1$*, then* $\psi_i > 0$ *everywhere for* $i = 1, 2$*.*

Proof. **Step 1** By Lemma 2.14, $u_0 \in C^1[0,\infty)$ irrespective if there are intervals on which u_0 equals $0, \beta$, or β_2. Consequently, $v_0 \in C^2[0,\infty)$. Define $\psi_2 \equiv g'(0)u_0 + \alpha_1 v_0 = \boldsymbol{l_2} \cdot \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}$, then $\psi_2 \in C^1[0,\infty)$. Away from those intervals on which u_0 is identically equal to 0, β, or β_2, we know from Lemma 2.11 that $u_0 \in C^\infty$ and (u_0, v_0) satisfies (2.27). Multiplying (2.27) through by $\boldsymbol{l}_2^{\mathrm{T}}$ yields

$$\psi_2'' - \lambda_2\psi_2 = -\Big(\frac{g'(0)}{d}u_0^2(1+\beta-u_0) + \frac{g'(0)}{d}\big(g'(0) - g'(u_0)\big)v_0 + \alpha_1\big(g(u_0) - g'(0)u_0\big)\Big).$$

With $g(u) = u + \epsilon u^3$ and $\alpha_1 < \frac{\beta}{2d}$ from (2.31),

$$\begin{aligned} \psi_2'' - \lambda_2\psi_2 &= -\left(\frac{1}{d}u_0^2(1+\beta-u_0) - \frac{3}{d}u_0^2 v_0\epsilon + \alpha_1 u_0^3\epsilon\right) \\ &\le -\frac{1}{d}u_0^2\left((1+\beta-u_0) - 3v_0\epsilon - \frac{\beta}{2}|u_0|\epsilon\right). \end{aligned}$$

With $b_1 = 3/2$, it follows from $|u_0| \le \beta_2$ and $|v_0| \le \frac{b_1(M+1)}{\gamma} \le \frac{3(M+1)}{2\gamma}$ that

$$\begin{aligned} (1+\beta-u_0) - 3v_0\epsilon - \frac{\beta}{2}|u_0|\epsilon &\ge (1+\beta-\beta_2) - \left(\frac{9(M+1)}{2\gamma} + \frac{\beta(M+1)}{2}\right)\epsilon \\ &\ge \frac{\beta}{2} - \left(\frac{9}{2\gamma} + \frac{1}{4}\right)(M+1)\epsilon \\ &\ge 0, \end{aligned}$$

and we obtain the following differential inequality on ψ_2:

$$\psi_2'' - \lambda_2\psi_2 \le 0. \tag{2.36}$$

Step 2 Since $u_0 \to 0$ and $v_0 \to 0$ as $x \to \infty$, it follows that $\psi_2 \to 0$ as $x \to \infty$. Suppose that $\psi_2 < 0$ somewhere. Define $b = \sup\{x : \psi_2(x) < 0\}$, where the possibility that $b = \infty$ is not excluded. Since $\psi_2(b) = 0$ (or $\psi_2 \to 0$, if $b = \infty$), there exists $b_1 \in (0, \infty)$ such that $\psi_0(b_1) \equiv -t_0 < 0$ and $\psi_2'(b_1) \equiv t_1 > 0$. We claim $\psi_2'(x) \geq t_1$ for $x \in [a_1, b_1]$, where a_1 is a point in $(0, b_1)$. This claim will be verified under three cases:

Case (A): Suppose that $u_0 \neq 0$, $u_0 \neq \beta$ and $u_0 \neq \beta_2$ on (a_1, b_1) for some $a_1 \in (0, b_1)$. Then ψ_2 satisfies (2.36) on $[a_1, b_1]$ so that ψ_2 cannot have an interior non-positive minimum in (a_1, b_1). Furthermore, as a consequence of the Hopf lemma, $\psi_2' > 0$ on $[a_1, b_1)$. With such an information put back into (2.36) gives, for all $x \in [a_1, b_1]$,

$$\psi_2''(x) \leq \lambda_2 \psi_2(x) \leq \lambda_2 \psi_2(b_1) = -\lambda_2 t_0,$$

and

$$\psi_2'(b_1) - \psi_2'(x) = \int_x^{b_1} \psi_2''(t)\, dt \leq \int_x^{b_1} -\lambda_2 t_0 \, dt \leq -\lambda_2 t_0 (b_1 - x).$$

With $\psi_2'(b_1) = t_1$, we therefore have $\psi_2'(x) \geq t_1 + \lambda_2 t_0 (b_1 - x) \geq t_1$ on $[a_1, b_1]$.

Case (B): Suppose $u_0 = \beta_2$ on $[a_1, b_1]$ for some $a_1 \in (0, b_1)$. Then $u_0(b_1) = \beta_2$, and $u_0'(b_1) = 0$ by the corner lemma. From $\psi_2(b_1) = g'(0)u_0(b_1) + \alpha_1 v_0(b_1) = -t_0$ and $\psi_2'(b_1) = g'(0)u_0'(b_1) + \alpha_1 v_0'(b_1) = t_1$, we obtain $v_0(b_1) = -(t_0 + g'(0)\beta_2)/\alpha_1$ and $v_0'(b_1) = t_1/\alpha_1 > 0$. In view of (2.1b)

$$v_0'' - \gamma v_0 = -g(\beta_2) < 0 \text{ on } [a_1, b_1],$$

the maximum principle dictates that v_0 cannot attain an interior non-positive minimum on (a_1, b_1). Since $v_0'(b_1) > 0$, the Hopf lemma says $v_0(x) < v_0(b_1) = -(t_0 + g'(0)\beta_2)/\alpha_1$ and $v_0'(x) > 0$ for $x \in [a_1, b_1)$. Then

$$\begin{aligned} v_0''(x) &= \gamma v_0(x) - g(\beta_2) \\ &< \frac{-\gamma(t_0 + g'(0)\beta_2)}{\alpha_1} - g(\beta_2) \\ &= -\{g(\beta_2) + \frac{\gamma}{\alpha_1}(t_0 + g'(0)\beta_2)\}, \end{aligned}$$

and by integrating over (x, b_1),

$$v_0'(x) > \frac{t_1}{\alpha_1} + \big(g(\beta_2) + \frac{\gamma}{\alpha_1}(t_0 + g'(0)\beta_2)\big)(b_1 - x)$$

on $[a_1, b_1)$. Combining with $u_0(x) = \beta_2$ and $u_0'(x) = 0$ yields $\psi_2'(x) = g'(0)u_0'(x) + \alpha_1 v_0'(x) \geq t_1$ and $\psi_2(x) = g'(0)\beta_2 + \alpha_1 v_0(x) \leq -t_0$ on $[a_1, b_1]$.

Case (C): If $u_0 = 0$ or $u_0 = \beta$ on $[a_1, b_1]$ for some $0 \leq a_1 < b_1$, replacing β_2 by 0 or β in the above calculation will do.

Recalling case (P3) or (Q3) as indicated in Remark 2.13, we claim neither can occur, i.e., there is no accumulation point in $[0, \infty)$. For if it does happen as $x \downarrow x_0^+$ with one of cases (A)–(C) occurs alternatively in adjacent subintervals of (x_0, ∞) or possibly a combination of such distributions, then $\psi_2 \in C^1$ and $\psi_2'(x_0) \geq t_1$. However by Lemma 2.14, we get $u_0'(x_0) = v_0'(x_0) = 0$, so is $\psi_2'(x_0) = 0$. This contradiction completes the proof of the claim that no accumulation point exists. The same is true for $x \uparrow x_0^-$, as a limit point from the left.

As a result of our claim, we may assume that the interval $[0, b_1]$ consists of a finite combination of cases (A), (B), and (C). Hence $\psi_2'(x_0) \geq t_1$ for all $x \in [0, b_1]$ and $\psi_2(0) < 0$. However, we know $u_0(0) > 0$ due to Lemma 2.8 and $v_0(0) > 0$ from Lemma 2.15, which gives $\psi_2(0) > 0$. This contradiction allows us to conclude that $\psi_2 \geq 0$ on $[0, \infty)$. Note that we cannot exclude (P3), (Q3) or (R3) yet.

Step 3 Next we show that $\psi_2 > 0$ on $[0, \infty)$ by arguing indirectly. Suppose $\psi_2 = 0$ at some $\xi > 0$, then $\psi_2'(\xi) = 0$. Without loss of generality, we may assume that $\psi_2 > 0$ on $(\xi, \xi + \delta]$ for some $\delta > 0$ (if $\psi_2 = 0$ on $(\xi, \xi + \delta]$, then we take the interval $(\xi + \delta, \xi + \delta + \delta_0]$ where $\psi_2 > 0$ for some $\delta_0 > 0$).

Case (A1): Suppose that $u_0(\xi) \notin \{0, \beta, \beta_2\}$. Applying the Hopf lemma to (2.36) on $[\xi, \xi+\delta]$ for δ sufficiently small yields $\psi_2'(\xi) > 0$, and thus case (A1) is not possible.

Case (B1): Suppose that $u_0(\xi) = \beta_2$. With u_0 having a maximum at ξ, $u_0'(\xi) = 0$. Then from $\psi_2(\xi) = 0$ and $\psi_2'(\xi) = 0$, we get $v_0(\xi) = -g'(0)\beta_2/\alpha_1 < 0$ and $v_0'(\xi) = 0$. At $x = \xi$, we also have from (2.1b) that

$$v_0''(\xi) = \gamma v_0(\xi) - g(\beta_2) < 0.$$

Hence if $x > \xi$ and is sufficiently close to ξ, then $v_0(x) < v_0(\xi) = -g'(0)\beta_2/\alpha_1$ by a simple Taylor expansion, and consequently

$$\psi_2(x) = g'(0)u_0(x) + \alpha_1 v_0(x) < g'(0)\beta_2 + \alpha_1 v_0(\xi) = 0.$$

This is absurd, so case (B1) is eliminated.

Case (C1): Suppose $u_0(\xi) = 0$ or $u_0(\xi) = \beta$. We prove it for when $u_0(\xi) = 0$ and the argument for the case $u_0(\xi) = \beta$ is similar. First consider when $u_0'(\xi) \neq 0$. Since $u_0(x) \neq 0$ on $(\xi, \xi+\delta]$ for some $\delta > 0$, by taking δ smaller if necessary, $u_0 \in C^2[\xi, \xi+\delta]$ and satisfies (2.1a). Consequently $\psi_2 \in C^2[\xi, \xi+\delta]$, and by the assumption, ψ_2 attains minimum at $x = \xi$ on this interval. Applying the Hopf lemma to (2.36) yields $\psi_2'(\xi) > 0$, which is a contradiction.

Next we consider when $u_0'(\xi) = 0$. Since $\psi_2(\xi) = 0$ and $\psi_2'(\xi) = 0$, we infer that $v_0(\xi) = v_0'(\xi) = 0$. If $u_0 \neq 0$ on $(\xi, \xi+\delta]$, the same proof as above will do. It remains to study the situation when there exist $b_1 > a_1 \geq b_2 > a_2 \geq b_3 > a_3 \ldots$ in the interval $[\xi, \xi+\delta]$ such that

$$\begin{cases} u_0 \neq 0 & \text{on intervals } (a_i, b_i) \quad i = 1, 2, \ldots, \\ u_0 = 0 & \text{on } [x_0 - \ell, x_0] \setminus \cup_{i=1}^{\infty} (a_i, b_i), \end{cases}$$

with both $a_i \to \xi^+$ and $b_i \to \xi^+$ as $i \to \infty$. Since u_0 changes the sign only once if it does, there is $j > 1$ such that u_0 does not change sign on $[\xi, b_j]$. Suppose $u_0 \geq 0$ on the interval. Let us consider the solution of (2.1b) under the initial condition $v(\xi) = v'(\xi) = 0$. Applying a comparison theorem for the initial value problem we see that $v_0 \leq 0$ on $[\xi, b_j]$ if we take $v = 0$ as a comparison function. Then $du_0'' - (1-u_0)(\beta - u_0)u_0 = g'(u_0)v_0 \leq 0$ on (a_j, b_j). Invoking the Hopf lemma yields $u_0'(a_j) > 0$ which is a contrary to the corner lemma.

The case of $u_0 \leq 0$ on $[\xi, b_j]$ can be treated similarly. Hence it is clear that (C1) cannot occur.

In conclusion, $\psi_2 > 0$ on $[0, \infty)$. A similar argument yields $\psi_1 > 0$ on $[0, \infty)$. $\square$

Lemma 2.17. *If $d \leq d_1$, then $v_0 > 0$ everywhere, and cases (P3), (Q3) and (R3) cannot occur. Moreover, if for some x_2, $u_0 \geq 0$ on $[0, x_2]$ and $u_0 \leq 0$ on $[x_2, \infty)$, then $v_0' < 0$ on $[x_2, \infty)$ and v_0 decreases to 0 as $x \to \infty$. Furthermore, once u_0 turns negative, then it keeps negative for all larger value of x.*

Proof. If $u_0 \geq 0$ on $[0, \infty)$, the positivity of v_0 follows from the positivity of the Green's function. Hence, assume u_0 changes sign at $x = x_2$. Due to positivity of ψ_2 and Lemma 2.15, v_0 can only be non-positive somehwere on $(0, x_2)$. However this cannot happen because $v_0'' - \gamma v_0 = -g(u_0) \leq 0$ on $[0, x_2]$. Therefore, $v_0 > 0$ everywhere and due to Lemma 2.14, cases (P3), (Q3) and (R3) cannot occur.

Next we show $v_0' < 0$ on $[x_2, \infty)$. Applying the maximum principle to

$$v_0'' - \gamma v_0 = -g(u_0) \geq 0,$$

v_0 cannot have an interior non-negative maximum on (x_2, ∞). With $v_0(x_2) > 0$ and $v_0(x) \to 0$ as $x \to \infty$, we have from the Hopf lemma that $v_0'(x) < 0$ on $[x_2, \infty)$.

Lastly, suppose for contradiction u_0 touches 0 again after it turns negative at $x = x_2$. With $v_0 > 0$ on $[0, \infty)$, we have from (2.1a)

$$du_0'' + f(u_0) = g'(u_0)v_0 \geq 0$$

on $[x_2, \infty)$. Set $h(x) \equiv -(u_0(x) - \beta)(1 - u_0(x))$, then $f(u_0) = -h(x)u_0$ and $h(x)$ is positive on $[x_2, \infty)$. By the maximum principle, u_0 cannot attain an interior non-negative maximum on (x_2, ∞). Moreover the Hopfs lemma dictates that the slope is non-zero at any maximum point where $u_0 = 0$. This is a contradiction, and therefore u_0 must keep negative for all $x > x_2$. □

2.5 On the constraints imposed by the admissible set

With the help of qualitative properties of the solution $(u_0, \mathcal{L}g(u_0))$, we are ready to eliminate the possibility of intervals on which $u_0 = 0, \beta$ or β_2. In this section, u_0 always stands for the minimizer of J and $v_0 = \mathcal{L}g(u_0)$.

Lemma 2.18. *Suppose $\gamma \leq \gamma_0$, $d \leq d_1$ and $\epsilon \leq \epsilon_1$. Then $\beta_1 < \max u_0 < 1$.*

Proof. Assume $a_0 \equiv \max u_0 > 1$. Take a small δ such that $a_0 - \delta > 1$. Define a truncated function

$$u_{new} = \begin{cases} u_0\,, & \text{if } a_0 - \delta \geq u_0 \geq -M\,, \\ a_0 - \delta\,, & \text{if } a_0 \geq u_0 > a_0 - \delta\,, \end{cases} \tag{2.37}$$

which lies in $\mathcal{A}$. Let us consider $J(u_{new}) - J(u_0)$. It is clear that

$$\frac{d}{2}\int_0^\infty (u'_{new})^2\,dx < \frac{d}{2}\int_0^\infty (u'_0)^2\,dx,$$

and, since $F(\xi)$ is increasing on $a_0 - \delta < \xi \leq a_0$,

$$\int_0^\infty F(u_{new})\,dx < \int_0^\infty F(u_0)\,dx.$$

Recall that $\mathcal{L}g(u_0) > 0$ by Lemma 2.17. The continuity of $\mathcal{L}$ together with that of g guarantees that, by making δ smaller if necessary, $\mathcal{L}g(u_{new}) > 0$ on the interval $[0, x_1]$. With $u_{new} < u_0$, we have $g(u_{new}) - g(u_0) = (u_{new} - u_0) + \epsilon(u_{new} - u_0)(u_{new}^2 +$

$u_{new}u_0 + u_0^2) \leq 0$ and its support lies inside $[0, x_1]$. Applying Lemma 2.9,

$$\begin{aligned}
&\frac{1}{2}\int_0^\infty g(u_{new})\mathcal{L}g(u_{new})\,dx - \frac{1}{2}\int_0^\infty g(u_0)\mathcal{L}g(u_0)\,dx \\
&\quad = \frac{1}{2}\int_0^\infty \big(g(u_{new}) - g(u_0)\big)\mathcal{L}\big(g(u_{new}) + g(u_0)\big)\,dx \\
&\quad \leq 0.
\end{aligned}$$

Hence $J(u_{new}) < J(u_0)$, which contradicts the fact that u_0 is a minimizer in $\mathcal{A}$. Therefore $\max u_0 \leq 1$.

We claim that in fact $u_0 < 1$. For contradiction, suppose $u_0 = 1$ at some $x = x_0$. Set $h(x) \equiv u_0(x)(u_0(x) - \beta)$ and take $a < x_0 < b$ so that on $[a, b]$, $\beta < u_0(x) \leq 1$ and $u_0 \not\equiv 1$. Thus, $h(x) > 0$ and u_0 satisfies (2.1a) on the interval $[a, b]$. By rewriting (2.1a),

$$d(u_0 - 1)'' - h(x)(u_0 - 1) = g'(u_0)v_0 > 0$$

which implies that $u_0 - 1$ cannot have an interior non-negative maximum on $[a, b]$. Furthermore the Hopf lemma dictates that at any maximum point where $u_0 = 1$, the slope has to be non-zero. This is a contradiction.

We are left to show $u_0 > \beta_1$. By Lemma 2.5, the nonlocal term of $J(u_0)$, $\int_0^\infty g(u_0)\mathcal{L}g(u_0)\,dx$, is non-negative. For $J(u_0) < 0$, we need $\int_0^\infty F(u_0)\,dx < 0$. Thus $\max u_0 > \beta_1$ must hold. □

Lemma 2.19. *Suppose $\gamma \leq \gamma_0$, $d \leq d_1$ and $\epsilon \leq \epsilon_1$. Let x_1 and x_2 be the values defined in Lemma 2.8. Furthermore, take the unique x_1 so that $u_0 < \beta$ on some small neighborhood $(x_1, x_1 + \delta]$, and the smallest x_2 (if finite) such that $u_0 > 0$ on some neighborhood of $[x_2 - \delta, x_2)$. Then $u_0' < 0$ on the interval $[x_1, x_2)$ (including possibly $x_2 = \infty$). Moreover, $u_0'(x_2) < 0$ if u_0 changes sign at x_2. In fact there is no* *(finite or infinite) interval* $[a_1, a_2]$ *with* $a_1 \geq x_2$ *such that* $u_0 \geq 0$ *on* $[0, a_1]$ *and* $u_0 = 0$ *on* $[a_1, a_2]$.

Proof. Suppose for contradiction there exist $x_1 < y_1 < y_2 < x_2$ such that $0 < u_0(y_1) < u_0(y_2)$. Since $u_0(x_2) = 0$, there is a local maximum of u_0 between y_1 and x_2. Whether the maximum of this hump touches β or not, we can use the trimming technique in Lemma 2.18 to derive a contradiction due to all three terms of J decrease strictly. Therefore u_0 is monotone non-increasing on $[x_1, x_2]$. Since $0 < u_0(x) < \beta$ on (x_1, x_2), $u_0(x_1) = \beta$ and $u_0(x_2) = 0$, we can apply Lemma 2.11 to conclude $u_0 \in C^\infty[x_1, x_2]$. Thus $u_0' \leq 0$ on $[x_1, x_2]$.

Next we show $u_0' < 0$ on $[x_1, x_2)$. As $du_0'' = g'(u_0)v_0 - f(u_0) > 0$ on $[x_1, x_2]$, u_0 cannot attain an interior maximum there. The Hopf lemma dictates that $u_0'(x) < 0$ on $[x_1, x_2)$.

Assume now u_0 touches 0 at some finite x_2. We claim that u_0 can never turn positive again. Suppose for contradiction, this were the case. Since $u_0 \to 0$ as $x \to \infty$, there would be a positive hump of u_0 beyond x_2. Then the same argument as above would yield a contradiction. Therefore either

(a) u_0 becomes negative on a small neighborhood $(x_2, x_2 + \delta_1]$, or

(b) $u_0 = 0$ on $[x_2, x_2 + \delta]$ and $u_0 < 0$ on a small neighborhood of $(x_2 + \delta, x_2 + \delta + \delta_1]$,

if u_0 eventually changes sign.

We are going to eliminate case (b). Assume (b) holds. Take $h(x) = -(u_0(x) - \beta)(1 - u_0(x))$, which is positive on $[x_2 + \delta, \infty)$. Since $u_0(x) < 0$ on $(x_2 + \delta, x_2 + \delta + \delta_1]$, the corner lemma dictates that

$$u_0'(x_2+\delta) = \lim_{x\to(x_2+\delta)^-} u_0'(x) = 0\,.$$

This is a contradiction since $du_0''(x) - h(x)u_0 = g'(u_0)v_0 > 0$ on $[x_2+\delta, x_2+\delta+\delta_1]$ requires $u_0'(x_2+\delta) < 0$ by the Hopf lemma.

Hence only case (a) holds. The Hopf lemma requires $u_0'(x_2) < 0$. □

Lemma 2.20. *Suppose $\gamma \le \gamma_0$ and $d \le d_1$. No matter whether u_0 changes sign or not, there cannot be an interval $[a,b]$ on which $u_0 = 0$.*

Proof. Assume $u_0 = 0$ on an interval $[a,b]$. Take a small δ, and define

$$u_{new} = \begin{cases} -\ell \sin\left(\frac{\pi(x-a)}{b-a}\right), & \text{if } x \in [a,b], \\ u_0\,, & \text{otherwise.} \end{cases} \tag{2.38}$$

If u_0 stays non-negative, $u_0 = 0$ on $[a,\infty)$ by Lemma 2.19 which states that once u_0 touches 0, it cannot go back up to positive. If u_0 changes sign, $u_0 \le 0$ on (b,∞) since $u_0 \in \mathcal{A}$. Hence, whether or not u_0 changes sign, $u_{new} \in \mathcal{A}$. Using Lemma 2.9 to calculate the change in the nonlocal term gives

$$J(u_{new}) - J(u_0) = \int_a^b \left\{ \frac{d}{2}(u_{new}')^2 + F(u_{new}) + \frac{1}{2} g(u_{new})(v_0 + \mathcal{L}g(u_{new})) \right\} dx.$$

Notice that both gradient and integral terms are in the order of ℓ^2. For the nonlocal term, we can write

$$\mathcal{L}g(u_{new}) = v_0 + \mathcal{L}(g(u_{new}) - g(u_0)) = v_0 + \mathcal{L}(g'(\tilde{u})(u_{new} - u_0)).$$

Since $|u_{new} - u_0| \leq \ell$, as in the calculation in Lemma 2.4, we get $\mathcal{L}g(u_{new}) = v_0 + O(\ell)$. Then with $g(u_{new}) = -\ell \sin\left(\frac{\pi(x-a)}{b-a}\right) - \epsilon\ell^3 \sin^3\left(\frac{\pi(x-a)}{b-a}\right)$,

$$\begin{aligned}
J(u_{new}) - J(u_0) &= \int_a^b \left\{ O(\ell^2) + \frac{1}{2} g(u_{new})(2v_0 + O(\ell)) \right\} dx \\
&\leq \int_a^b \left\{ O(\ell^2) - \ell \sin\left(\frac{\pi(x-a)}{b-a}\right) v_0 \right\} dx \\
&\leq O(\ell^2)(b-a) - \ell \min_{x\in[a,b]} v_0(x) \int_a^b \sin\left(\frac{\pi(x-a)}{b-a}\right) dx \\
&= -\min_{x\in[a,b]} v_0(x) \frac{2(b-a)}{\pi} \ell + O(\ell^2)(b-a) \\
&< 0
\end{aligned}$$

by choosing ℓ sufficiently small. This contradicts the fact that u_0 is a minimizer. □

Lemma 2.21. *Suppose $\gamma \leq \gamma_0$ and $\epsilon \leq \epsilon_1$. Then the minimizer u_0 changes sign.*

Proof. If u_0 stays non-negative, then $u_0 > 0$ by Lemma 2.19 and 2.20. If u_0 turns negative, it does so at one point x_2 by Lemma 2.19. Moreover it will never get back to 0 again by Lemma 2.17. Hence $u_0 \neq 0$ on $[y_1, \infty)$ for some large y_1. It can be seen from the linearization of (u_0, v_0) about $(0,0)$ in (2.35) that, for $C_1 \neq 0$,

$$\begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} \sim C_1 e^{-\sqrt{\lambda_1}x} \mathbf{a}.$$

Since we have shown that the components of the eigenvector $\mathbf{a}$ have opposite sign, u_0 and v_0 must take up different signs at large x. Given $v_0 > 0$, we must have $u_0 < 0$ at large x. The case when $C_1 = 0$ and $C_2 \neq 0$ is similar. Hence u_0 has to change sign. □

The next corollary is an immediate consequence of Lemma 2.21.

Corollary 2.22. *Suppose $\gamma \leq \gamma_0$, $d \leq d_1$ and $\epsilon \leq \epsilon_1$. Let $x_1 = \inf\{y : u_0(x) < \beta$ if $x \in (y, \infty)\}$. Then the minimizer $u_0 \in C^\infty[x_1, \infty)$. In fact u_0 changes sign and satisfies (2.1a) on this interval. Moreover u_0 crosses 0 at only one point, say x_2.*

After eliminating the possibility of $u_0 = 0$ on an interval, we turn our attention to verify that the constraint β is in active as well.

Lemma 2.23. *Even if there are intervals on which $u_0 = \beta, (u_0, v_0)$ satisfies*

$$\frac{1}{2}{v_0'}^2 - \frac{\gamma}{2}v_0^2 + g(u_0)v_0 - \frac{d}{2}{u_0'}^2 + F(u_0) = 0 \quad \textit{on } [0, \infty). \tag{2.39}$$

Proof. From linearization about $(u, v) = (0, 0)$ in (2.35), it can be seen that both u_0 and v_0 die down exponentially as $x \to \infty$. With u_0'' and v_0'' being bounded for large x, we conclude from the standard interpolation theorem that u_0' and v_0' also die down exponentially.

On the interval $[x_1, \infty)$, where $u_0 \in C^\infty$, we multiply (2.1a) by $-u_0'$ and (2.1b) by v_0' and sum the resulting equations to get

$$-du_0'u_0'' - u_0'f(u_0) + u_0'g'(u_0)v_0 + v_0'v_0'' + v_0'g(u_0) - \gamma v_0 v_0' = 0.$$

By integrating, we obtain

$$\frac{1}{2}{v_0'}^2 - \frac{\gamma}{2}v_0^2 + g(u_0)v_0 - \frac{d}{2}{u_0'}^2 + F(u_0) = \text{constant}. \tag{2.40}$$

If we take the limit as $x \to \infty$, it is clear that the integration constant has to be 0. Hence equation (2.39) holds on $[x_1, \infty)$ and continues to be valid to the left of x_1 until

$u_0 = \beta$ on an interval $[a, b]$ with $b \leq x_1$ when (2.1a) fails to hold. Note that $u_0'(b^+) = u_0'(b^-) = 0$ by the corner lemma. Now on $[a, b], u_0 = \beta$ gives $v_0'' - \gamma v_0 + g(\beta) = 0$ in (2.1b). Therefore

$$\frac{d}{dx}\left(\frac{1}{2}{v_0'}^2 - \frac{\gamma}{2}v_0^2 + g(\beta)v_0\right) = 0.$$

With $u_0 = \beta$, $u_0' = 0$ on $[a, b]$, it is clear that left-hand side of (2.39) does not change for $x \in [a, b]$. Since (2.1a) holds at $x = b$ by Lemma 2.11, we conclude that the left-hand side of (2.39) is identically 0 on $[a, b]$. Therefore, (2.39) holds on $[a, b]$ where $u_0 = \beta$.

Once $x < a$ with $u_0 > \beta$, then (2.1a) and (2.1b) are satisfied so that (2.40) holds. Here the integration constant has to be zero again by evaluation at $x = a$. The proof of lemma is complete. □

Lemma 2.24. *(a)* $0 \leq \max_{\beta \leq \xi \leq 1} f(\xi) \leq \frac{1}{4}$;
(b) $\min_{\beta \leq \xi \leq 1} \frac{F(\xi)}{\xi} \geq -\frac{1}{9}$.

Proof. The proof can be found in Lemma 18 of Chen and Choi (2012). □

Lemma 2.25. *The minimizer u_0 has the property $u_0'(0) = 0$.*

Proof. We know that $\beta \leq u_0(0) < 1$. If $u_0(0) > \beta$, u_0 satisfies (2.1a) in a neighborhood of the origin. As a minimizer, the zero Neumann boundary condition follows from the standard argument in calculus of variations. In case $u_0(0) = \beta$, we divide the proof into two cases:

(i) If $u_0 = \beta$ on some interval $[0, \delta]$ for some $\delta > 0$, then $u_0'(0) = 0$ immediately follows.

(ii) Suppose $u_0(0) = \beta$ and $u_0 > \beta$ on some small interval $(0, \delta]$. By Lemma 2.11,

$u_0 \in C^2[0,\delta]$ and satisfies (2.1a) on $[0,\delta]$. Assume $u_0'(0) \equiv a > 0$. We define

$$u_{new} = \begin{cases} u_0, & \text{if } x \geq \delta, \\ u_0(\delta), & \text{if } x < \delta, \end{cases}$$

where $u_0' > 0$ on $[0,\delta]$ by taking δ sufficiently small. Observe that $u_{new} \in \mathcal{A}$ and u_{new} has a zero slope in the neighborhood of the origin. When δ is sufficiently small, the change in the gradient term of J is

$$\begin{aligned} \frac{d}{2}\int_0^\delta (u_{new}')^2 - (u_0')^2\,dx &= \frac{d}{2}\int_0^\delta -(a+o(1))^2\,dx \\ &= -\frac{d}{2}\int_0^\delta (a^2+o(1))\,dx \\ &= -\frac{da^2}{2}\delta(1+o(1)). \end{aligned}$$

Since $F(u_0(\delta)) < F(u_0(x))$ for $x \in [0,\delta)$, the integral of F decreases as well. For the nolocal term, we use Lemma 2.9 and the fact $\|\mathcal{L}g(u)\|_{L^\infty(0,\infty)} \leq \frac{b_1(M+1)}{\gamma}$ to obtain

$$\begin{aligned} &\left|\frac{1}{2}\int_0^\delta \left(g(u_{new}(x)) - g(u_0(x))\right)\left(\mathcal{L}g(u_0(x)) + \mathcal{L}g(u_{new}(x))\right)\,dx\right| \\ &\leq \frac{b_1(M+1)}{\gamma}\int_0^\delta |g(u_0(\delta)) - g(u_0(x))|\,dx \\ &\leq \frac{b_1(M+1)}{\gamma}\int_0^\delta g'(u_0(\tilde{x}))u_0'(\tilde{x})(\delta - x)\,dx \quad (\tilde{x} \in [x,\delta]) \\ &\leq \frac{b_1(M+1)}{\gamma}\int_0^\delta b_1(a+o(1))\delta\,dx \\ &= \frac{ab_1^2(M+1)}{\gamma}\delta^2(1+o(1)). \end{aligned}$$

By making δ sufficiently small, it turns out $J(u_{new}) < J(u_0)$, contradicting that u_0 is a minimizer in $\mathcal{A}$. Therefore, $u_0'(0) = 0$ in case (ii).

We remark that $x = 0$ cannot be a limit point stated in (Q3) for we have eliminated such a scenario in Lemma 2.17. Hence the proof is complete. □

In what follows, define $\gamma_1 = \min\{\gamma_0, 4\beta a_1^2\}$. Observe that $\gamma_1 = \gamma_0$ for β close to 1/2, while $\gamma_1 = 4\beta a_1^2$ for β close to 0.

Lemma 2.26. *Suppose $\gamma \leq \gamma_1$ and $d \leq d_1$. Let $c = u_0(0)$. Then*

$$v_0(0) = \frac{g(c) - g(c)\sqrt{1 + 2\gamma F(c)/g^2(c)}}{\gamma}. \tag{2.41}$$

Proof. Recall that $v_0'(0) = 0$ always holds and $u_0'(0) = 0$ by Lemma 2.25. Evaluating (2.39) at $x = 0$, we obtain a quadratic equation in $v_0(0)$:

$$\frac{\gamma}{2} v_0^2(0) - g(c)v_0(0) - F(c) = 0. \tag{2.42}$$

The quadratic equation can be solved to yield

$$v_0(0) = \frac{g(c) \pm \sqrt{g^2(c) + 2\gamma F(c)}}{\gamma}.$$

Since $v_0(0)$ is real, it follows that $g^2(c) + 2\gamma F(c) \geq 0$. This inequality in general may not be valid for all the value c in $[\beta, 1]$. However if $\gamma \leq \gamma_1$, we can verify $g^2(c) + 2\gamma F(c) \geq 0$ for all $c \in [\beta, 1]$. Since F is decreasing on $[\beta, 1]$ while g is decreasing on the interval, we have

$$\begin{aligned}
&\inf_{\beta\in(0,\frac{1}{2})}\left[\min_{\beta\le c\le 1}\{1+\frac{2\gamma F(c)}{g^2(c)}\}\right] = \inf_{\beta\in(0,\frac{1}{2})}\left[\min_{\beta_1\le c\le 1}\{1+\frac{2\gamma F(c)}{g^2(c)}\}\right]\\
&\ge \inf_{\beta\in(0,\frac{1}{2})}\left[\min_{\beta_1\le c\le 1}\left\{1+\frac{2\gamma F(c)}{a_1^2c^2}\right\}\right] \ (\because F(c)<0 \text{ for } \beta_1<c\le 1 \text{ and } g(c)\ge a_1c)\\
&= \inf_{\beta\in(0,\frac{1}{2})}\left[1+\frac{2\gamma}{a_1^2}\min_{\beta_1\le c\le 1}\frac{F(c)}{c^2}\right]\\
&\ge \inf_{\beta\in(0,\frac{1}{2})}\left[1+\frac{2\gamma}{a_1^2}\min_{\beta_1\le c\le 1}\left(\frac{c^2}{4}-\frac{1+\beta}{3}c+\frac{\beta}{2}\right)\right]\\
&= \inf_{\beta\in(0,\frac{1}{2})}\left[1+\frac{2\gamma}{a_1^2}\left(\frac{\beta(5-2\beta)-2}{18}\right)\right] \ (\because \text{min. attained at the c. point in } (\beta_1,1))\\
&\ge \inf_{\beta\in(0,\frac{1}{2})}\left[1-\frac{8\beta}{9}\right] \ (\because \gamma\le 4\beta a_1^2 \text{ and by Lemma 2.24}(b))\\
&> 0.
\end{aligned}$$

Proving the lemma amounts to saying that only the smaller root is admissible. We argue indirectly. Assume that

$$v_0(0)=\frac{g(c)+\sqrt{g^2(c)+2\gamma F(c)}}{\gamma}.$$

Since $\gamma\le\gamma_1$,

$$v_0(0)\ge\frac{g(c)}{\gamma}\ge\frac{g(\beta)}{\gamma}\ge\frac{g(\beta)}{4\beta a_1^2}\ge\frac{1}{4a_1}. \tag{2.43}$$

At $x=0$, we need to consider the following two cases as the limit point stated in (Q3) does not exist:

Case (a) Suppose there is an $\ell_0>0$ such that $u_0>\beta$ on $(0,\ell_0)$. Then (u_0,v_0) satisfies (2.1a) in a small neighborhood of $[0,\ell_0]$ containing the origin, and continues to do so until u_0 touches β at some $x \geq \ell_0$. Hence at $x = 0$,

$$v_0''(0) = \gamma v_0(0) - g(u_0(0)) > g(c) - g(u_0(0)) = 0\,,$$

and

$$du_0''(0) = g'(u_0)v_0 - f(u_0) > a_1 \cdot \frac{1}{4a_1} - f(u_0) \geq 0$$

from (2.43) and Lemma 2.24(a) where we showed $\max_{\beta \leq \xi \leq 1} f(\xi) \leq \frac{1}{4}$. Then $u_0''(x) > 0$ and $v_0''(x) > 0$ in a neighborhood $[0, \delta]$. Since $u_0'(0) = v_0'(0) = 0$, it follows from the mean value theorem that $u_0' > 0$ and $v_0' > 0$ on $(0, \delta]$. Clearly u_0' and v_0' will become 0 at some points in (δ, ∞). We claim that v_0 does that first. Suppose not, say $u_0'(a) = 0$ while $v_0' > 0$ on $[0, a)$. Since we have on $[0, a]$

$$v_0(x) > v_0(0) > \frac{1}{4a_1},$$

it follows that

$$du_0'' = g'(u_0)v_0 - f(u_0) > \frac{1}{4} - f(u_0) \geq 0$$

on [0, a]. This implies $u_0'(a) > 0$, which leads to a contradiction.

Therefore we consider $v_0'(a) = 0$ with $v_0' > 0$, $u_0' > 0$, $u_0'' > 0$ on $(0, a)$. Again, we have $v_0(x) > v_0(0) > \frac{1}{4a_1}$ on $[0, a]$. Together with Lemma 2.24,

$$\begin{aligned}
\frac{g(u_0)v_0}{2} + F(u_0) &\geq \frac{a_1 u_0 v_0}{2} + F(u_0) \\
&= u_0 \left(\frac{a_1 v_0}{2} + \frac{F(u_0)}{u_0} \right) \\
&> u_0 \left(\frac{1}{8} - \frac{1}{9} \right) \\
&> 0
\end{aligned}$$

on $[0, a)$ since $u_0(0) > 0$ and $u' > 0$ on $[0, a)$. It follows that the integral $J_1 \equiv \int_0^a \left\{ \frac{du_0'^2}{2} + \frac{u_0 v_0}{2} + F(u_0) \right\} dx > 0$. Let us cut out the interval $[0, a]$; that is, define

$$u_{new}(x) \equiv u_0(x+a), \quad v_{new}(x) \equiv v_0(x+a) \quad \text{for } x \geq 0.$$

Since v_0 satisfies (2.1a) and $v_0'(a) = 0$, we see that $v''_{new} - \gamma v_{new} + g(u_{new}) = 0$ holds on $[0, \infty)$ and $v'_{new}(0) = 0$. Hence $v_{new} = \mathcal{L}g(u_{new})$. Moreover $u_{new} \in \mathcal{A}$. Then

$$\begin{aligned} J(u_0) &= \int_0^a \left\{ \frac{du_0'^2}{2} + \frac{g(u_0)v_0}{2} + F(u_0) \right\} dx + \int_a^\infty \left\{ \frac{du_0'^2}{2} + \frac{g(u_0)v_0}{2} + F(u_0) \right\} dx \\ &= J_1 + \int_0^\infty \left\{ \frac{du_{new}'^2}{2} + \frac{g(u_{new})v_{new}}{2} + F(u_{new}) \right\} dx \\ &= J_1 + J(u_{new}) \\ &> J(u_{new}), \end{aligned}$$

which is contrary to u_0 being a minimizer. Therefore the choice of larger root of $v_0(0)$ in (2.42) is not permitted in case (a).

Case (b) Consider when there exists an $\ell_0 > 0$ such that $u_0 = \beta$ on $[0, \ell_0]$. Without loss of generality, we may assume $u_0 > \beta$ in a neighborhood $(\ell_0, \ell_0 + \ell_1]$. Then u_0 satisfies (2.1a) in $[\ell_0, \ell_0 + \ell_1]$. Recall that v_0 satisfies (2.1b) on $[0, \infty)$ with $v_0'(0) = 0$. Similar to the proof of (a), we see that

$$v_0''(0) = \gamma v_0(0) - g(u_0(x)) > g(c) - g(\beta) \geq 0$$

and therefore, $v_0'' > 0$ and $v_0' > 0$ on $(0, \ell_0]$. With $u_0(\ell_0) = \beta$, $v_0(\ell_0) > v_0(0)$, $v_0'(\ell_0) > 0$ and (u_0, v_0) satisfying (2.1) on $[\ell_0, \ell_0 + \ell_1]$, we can replicate the proof in case (a) to conclude that the choice of the larger root $v_0(0)$ in (2.42) is also not permitted.

Hence the proofs for both cases are complete which finish the proof of the Lemma. $\square$

Lemma 2.27. *Suppose* $\gamma \leq \gamma_1$ *and* $d \leq d_1$. *Then* $u_0(0) > \beta_1$. *Moreover* $v_0 < g(\beta)/\gamma$, $v_0' \leq 0$ *and* $v_0'' < 0$ *on the interval* $[0, x_1]$. *In fact*

$$v_0 < \frac{1}{6g(\beta_1)}(1-2\beta) \quad \textit{on} \;\; [0, x_1], \tag{2.44}$$

which provides an upper bound not depending on γ.

Proof. **Step 1** Recall that $c = u_0(0)$ and $\beta \leq c < 1$. By Lemma 2.26,

$$v_0(0) = \frac{g(c) - \sqrt{g^2(c) + 2\gamma F(c)}}{\gamma}. \tag{2.45}$$

From the definition of β_1, $F(\beta_1) = 0$. It follows that when $c = \beta_1$, $v_0(0) = 0$. In addition, for $\beta_1 < c < 1$, we have $F(c) < 0$ so that $v_0(0) > 0$. Otherwise if $c < \beta_1$, then $F(c) > 0$ so that $v_0(0) < 0$, which is not true. Hence $u_0(0) > \beta_1$.

Note from the graph of F that

$$\max_{\beta_1 \leq c \leq 1} |F(c)| = |F(1)| = \frac{1-2\beta}{12}.$$

Together with $\gamma \leq \gamma_0 \equiv \frac{3a_1^2\beta^2}{2(1-2\beta)}$, $g(\xi) \geq a_1\xi$ and $g(\beta) < g(\beta_1) < g(c)$, we get

$$\begin{aligned}
\gamma v_0(0) &= g(c) - \sqrt{g^2(c) + 2\gamma F(c)} \\
&\leq g(c) - g(c)\sqrt{1 - \frac{\gamma}{6g^2(c)}(1-2\beta)} \\
&\leq g(c) - g(c)\left(1 - \frac{\gamma}{6g^2(c)}(1-2\beta)\right) \quad (\because \frac{\gamma}{6g^2(c)}(1-2\beta) \leq \frac{\beta^2}{4c^2} < 1) \\
&= \frac{\gamma}{6g(c)}(1-2\beta) \\
&\leq \frac{a_1^2\beta^2}{4g(c)} \\
&\leq \frac{g^2(\beta)}{g(\beta_1)} \\
&< g(\beta).
\end{aligned}$$

Therefore $v_0(0) < g(\beta)/\gamma$ follows immediately.

Step 2 By continuity there exists a maximal interval $[0, \delta)$ such that $v_0(x) < g(\beta)/\gamma$. Suppose $\delta < x_1$. This implies

$$\begin{aligned}
0 &= v_0'' - \gamma v_0 + g(u_0) \\
&> v_0'' - \gamma\frac{g(\beta)}{\gamma} + g(u_0) \\
&\geq v_0''
\end{aligned}$$

on $[0, \delta)$ since $g(u_0) \geq g(\beta)$ on the interval $[0, x_1]$. This together with $v_0'(0) = 0$, we know $v_0' < 0$ on $[0, \delta]$ from the mean value theorem and $v_0 < \beta/\gamma$ at $x = \delta$. Then $v_0(x) < \frac{\beta}{\gamma}$ holds on an interval larger than $[0, \delta)$. This contradiction allows us to conclude that $v_0 < g(\beta)/\gamma$ and $v_0'' < 0$ on $[0, x_1]$. It is also clear that $v_0' < 0$ on $(0, x_1]$.

Step 3 For $x \in [0, x_1]$, since $v_0(x) \leq v_0(0)$, in the view of calculations in Step 1,

$$\gamma v_0(x) \leq \frac{\gamma}{6g(\beta_1)}(1 - 2\beta)$$

which completes the proof. □

Lemma 2.28. *Suppose $\gamma \leq \gamma_1$ and $d \leq d_1$. Then u_0 cannot equal to β on an interval $[a, b] \subset [0, x_1]$. In fact there is no point at which $u_0 = \beta$ and $u_0' = 0$.*

Proof. Assume for contradiction there is an interval $[a, b] \subset [0, x_1]$ such that $u_0 = \beta$. By the corner lemma, $u_0'(b) = 0$. Evaluating (2.39) at $x = b$, we obtain

$$\frac{1}{2}(v_0'(b))^2 = v_0(b)\left\{\frac{\gamma}{2}v_0(b) - g(\beta)\right\} - F(\beta). \tag{2.46}$$

From (2.44), it is clear that

$$\begin{aligned} \frac{\gamma}{2}v_0(b) - g(\beta) &\leq \frac{\gamma_0}{2}\frac{1}{6g(\beta_1)}(1 - 2\beta) - g(\beta) \\ &= \frac{a_1^2\beta^2}{8g(\beta_1)} - g(\beta) \\ &\leq g(\beta)(\frac{g(\beta)}{8g(\beta_1)} - 1) \\ &< 0. \end{aligned}$$

With $F(\beta) > 0$, the right-hand side of (2.46) is negative, which is incompatible with the left-hand side. Hence no such interval $[a, b]$ exists. It is also clear from the proof that there is no point at which $u_0 = \beta$ and $u_0' = 0$. □

Corollary 2. Suppose $\gamma \leq \gamma_1$, $d \leq d_1$ and $\epsilon \leq \epsilon_1$. There is a unique x_1 at which

$u_0 = \beta$. Moreover $u_0'(x_1) < 0$.

Proof. This is a consequence of Lemma 2.28 and of Lemma 2.19, which states that u_0 cannot cross β again once it goes below β. $\square$

We therefore conclude that (u_0, v_0) is a standing pulse solution to (2.1) with

$$\lim_{|x|\to\infty} (u(x)), v(x))) = (0, 0).$$

Chapter 3

A skew-gradient system II : nonlinear inhibitor equation

3.1 Introduction

We study the existence of standing pulse solutions for a system of reaction-diffusion equations of the form

$$\begin{cases} u_t = du_{xx} + f(u) - v, \\ \tau v_t = v_{xx} - \gamma v - v^3 + u, \end{cases} \tag{3.1}$$

where $f(u) = u(1-u)(u-\beta)$ and d, τ, γ and β are positive constants. Specifically, we study the steady-state of (3.1), namely the system

$$\begin{cases} du_{xx} + f(u) - v = 0, \\ v_{xx} - \gamma v - v^3 + u = 0, \end{cases} \tag{3.2}$$

on $(-\infty, \infty)$ for small γ and d. Observe that the system (3.2) has a skew-gradient structure with

$$H(u, v) = \frac{1}{2}\gamma v^2 + \frac{1}{4}v^4 - uv - F(u),$$

where $F(u) = -\int_0^u f(x)\,dx = u^4/4 - (1+\beta)u^3/3 + \beta u^2/2$. To the best of our knowledge, this work is the first attempt to show the existence of standing pulse solutions on $(-\infty, \infty)$ in the skew-gradient system (1.2) that accounts for the nonlinear dependence of inhibitor reaction term. The use of the explicit form of the Green's function in the case of linear inhibitor equation needs to be substantially modified. The additional nonlinearity may enable the model to capture more complex behavior of standing pulse solutions. Other kinds of nonlinearity associated with more general skew-gradient systems can be studied later on as the techniques we develop in this work may apply to a broader class of skew-gradient systems. We will look for solutions (u, v) that are even in x and

$$\lim_{|x|\to\infty} (u, v) = (0, 0).$$

Due to the symmetry, we restrict our attention to $[0, \infty)$. The anchor at the origin prevents the solution from translation, which is important in analyzing the equations. Our main result is summarized in the following theorem:

Theorem 3.1. *Let $\beta \in (1/3, 1/2)$ be given. There exists a $\gamma_1 > 0$ so that for any $\gamma \in (0, \gamma_1]$, we have a $d_1 = d_1(\gamma) > 0$ such that whenever $\gamma < \gamma_1$ and $d < d_1$, then* (3.2) *has a solution which is denoted by (u_0, v_0) with $u_0, v_0 \in C^\infty(0, \infty)$ and exponentially decay to 0 as $x \to \infty$; that is,* (3.2) *possesses a standing pulse solution.*

We have an explicit estimate for γ_1 in Lemma 3.45. To get a sense of the constraint

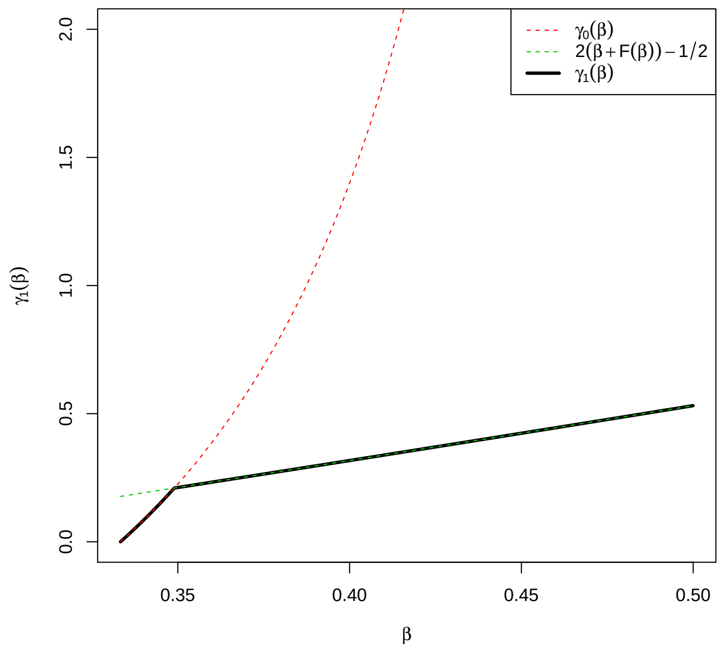

FIGURE 3.1: A plot of γ_1 versus β when $\beta \in (1/3, 1/2)$. Here $\gamma_0 = 3\beta^2/(1-2\beta) - 1$ and $\gamma_1 = \min\{\gamma_0, 2(\beta + F(\beta)) - 1/2\}$. For $\gamma < \gamma_1$, there is a standing pulse solution when d is sufficiently small.

on γ in the above theorem, a plot of γ_1 for $\beta \in (1/3, 1/2)$ is presented in Figure 3.1. For $\gamma < \gamma_1$ a standing pulse solution exists if $d \leq d_1(\gamma)$. To simplify notation, we suppress the dependence of β when we refer to γ_1 or d_1; for instance, we write $d_1(\gamma)$ rather than $d_1(\beta, \gamma)$. Some qualitative properties of the above standing pulse solution is established in the next theorem.

Theorem 3.2. *Suppose that (u_0, v_0) is a standing pulse solution obtained by Theorem 3.1. Then*

(i) There is a pair of unique points $0 < x_1 < x_2 < \infty$ such that $u_0(x_1) = \beta$ and $u_0(x_2) = 0$, respectively. Moreover $u_0'(x_1) < 0$ and $u_0'(x_2) < 0$.

(ii) $u_0 > \beta$ on $[0, x_1)$, $0 < u_0 < \beta$ on (x_1, x_2) and $u_0 < 0$ on (x_2, ∞).

(iii) $u_0' < 0$ on $[x_1, x_2]$.

(iv) u_0 possesses one global negative minimum on (x_2, ∞); this is also the unique local minimum point of u_0 on $[x_1, \infty)$.

(v) $v_0 > 0$ *on* $[0, \infty)$.

3.2 The nonlinear inhibitor equation

When a variational method is employed to find a standing pulse solution of (1.2), one introduces a linear operator $\mathcal{L}$ associated with the inhibitor equation so that $v = \mathcal{L}u$. This section serves as a counterpart when we are confronted with a nonlinear inhibitor equation. For any given $u \in H^1(0, \infty)$, we show that there exists a nonlinear operator $\mathcal{N} : H^1(0, \infty) \to H^3(0, \infty)$ such that $v = \mathcal{N}u$ satisfies (3.2b). It is also necessary to examine the (Fréchet) differentiability of this operator $\mathcal{N}$ in our new variational formulation. While properties for the linear operator $\mathcal{L}$ is more or less obvious, the same cannot be said about $\mathcal{N}$. We begin with some basic estimates.

Lemma 3.3. *Suppose* $u \in H^1(0, \infty)$ *and* $u_1, u_2 \in H^1(0, \infty)$. *Then*

(i) $\|u\|_{L^\infty(0,\infty)} \leq \sqrt{2}\,\|u\|_{H^1(0,\infty)}$ *and* $u(x) \to 0$ *as* $x \to 0$.

(ii) $\|u_1 u_2\|_{H^1(0,\infty)} \leq \sqrt{6}\,\|u_1\|_{H^1(0,\infty)} \|u_2\|_{H^1(0,\infty)}$.

Proof. The proof of (i) can be found in Lemma 2.2. Statement (ii) follows from

$$
\begin{aligned}
\|u_1 u_2\|_{H^1} &= \left(\|u_1 u_2\|_{L^2}^2 + \|u_2\, Du_1 + u_1\, Du_2\|_{L^2}^2\right)^{\frac{1}{2}} \\
&\leq \left(2\|u_1\|_{H^1}^2 \|u_2\|_{H^1}^2 + 2\|Du_1\|_{L^2}^2 \|u_2\|_{L^2}^2 + 2\|u_1\|_{L^2}^2 \|Du_2\|_{L^2}^2\right)^{\frac{1}{2}} \\
&\leq \left(6\|u_1\|_{H^1}^2 \|u_2\|_{H^1}^2\right)^{\frac{1}{2}}.
\end{aligned}
$$

□

Lemma 3.4. *Assume $\gamma > 0$, $f \in H^1(0,\infty)$ and $p \in H^1(0,\infty)$ with $p \geq 0$. If $v \in H^1(0,\infty)$ satisfies*

$$\int_0^\infty \{v_x\varphi_x + (\gamma+p)v\varphi\}\, dx = \int_0^\infty f\varphi\, dx\,, \quad \forall\varphi \in H^1(0,\infty) \tag{3.3}$$

then $v \in H^3(0,\infty)$ and $\|v\|_{H^3(0,\infty)} \leq C_0\|f\|_{H^1(0,\infty)}$ for some positive constant $C_0 = C_0(\gamma, \|p\|_{H^1})$.

Proof. As $p \in L^\infty(0,\infty)$ by Lemma 3.3, we have $pv \in L^2(0,\infty)$. By choosing $\varphi = v$ in (3.3), it is immediate from regularity estimate that $v \in H^2(0,\infty)$ and satisfies $v_{xx} = -f + (\gamma+p)v$ a.e. Moreover, we see that $\|v\|_{H^1(0,\infty)} \leq \max\{1, 1/\gamma\}\|f\|_{L^2(0,\infty)}$. Finally we observe

$$\begin{aligned}\|v_{xx}\|_{H^1} &\leq \gamma\|v\|_{H^1} + \sqrt{6}\,\|p\|_{H^1}\|v\|_{H^1} + \|f\|_{H^1}\\ &\leq \left(\gamma + \sqrt{6}\,\|p\|_{H^1}\right)\max\{1, 1/\gamma\}\|f\|_{H^1} + \|f\|_{H^1}\,.\end{aligned}$$

Therefore, $\|v\|_{H^3} \leq C_0\|f\|_{H^1}$ for some positive constant $C_0 = C_0(\gamma, \|p\|_{H^1})$. □

Lemma 3.5. *Given $u \in H^1(0,\infty)$, define a functional $\mathcal{K} : H^1(0,\infty) \to \mathbb{R}$ such that whenever $z \in H^1(0,\infty)$*

$$\mathcal{K}(z) \equiv \int_0^\infty \left\{\frac{z_x^2}{2} + \frac{\gamma z^2}{2} + \frac{z^4}{4} - uz\right\} dx.$$

Then the followings hold:

(i) $\mathcal{K}$ is well defined.

(ii) $\mathcal{K}$ is Fréchet differentiable with

$$\mathcal{K}'(z)\varphi = \int_0^\infty \left\{ z_x\varphi_x + \gamma z\varphi + z^3\varphi - u\varphi \right\} dx, \quad \forall \varphi \in H^1(0,\infty).$$

(iii) $\mathcal{K}$ has a minimizer $v \in H^1(0,\infty)$ which is a weak solution of (3.2b), i.e.

$$\int_0^\infty \left\{ v_x\varphi_x + \gamma v\varphi + v^3\varphi - u\varphi \right\} dx = 0, \quad \forall \varphi \in H^1(0,\infty).$$

Moreover, $v \in H^3(0,\infty)$ and satisfies $v_{xx} - \gamma v - v^3 + u = 0$ a.e.

(iv) The weak solution v is unique.

Proof. For any $z \in H^1(0,\infty)$, it follows from Lemma 3.3 that $\|z\|_{L^\infty(0,\infty)} \leq \sqrt{2}\,\|z\|_{H^1(0,\infty)}$. Therefore

$$|\mathcal{K}(z)| \leq \frac{1}{2}\max\{1,\gamma\}\|z\|_{H^1}^2 + \frac{1}{4}\|z\|_{L^\infty}^2\|z\|_{L^2}^2 + \|u\|_{L^2}\,\|z\|_{L^2} < \infty.$$

This completes the proof of (i). Statement (ii) is standard. As a consequence of convexity and coercivity of $\mathcal{K}$, a minimizer $v \in H^1(0,\infty)$ exists and $\mathcal{K}'(v)\varphi = 0$ for all $\varphi \in H^1(0,\infty)$. Observe that with $p := v^2 \in H^1(0,\infty)$, it follows from Lemma 3.4 that $v \in H^3$. These prove (iii). Next, suppose v_1 and v_2 are weak solutions of (3.2b) with $v_1 \neq v_2$. Then

$$\int_0^\infty \{(v_1 - v_2)_x\varphi_x + \gamma(v_1 - v_2)\varphi + (v_1^3 - v_2^3)\varphi\}\, dx = 0, \quad \forall \varphi \in H^1(0,\infty).$$

By choosing $\varphi = v_1 - v_2$, we have

$$\int_0^\infty \{(v_1 - v_2)_x^2 + \gamma(v_1 - v_2)^2 + (v_1^2 + v_1v_2 + v_2^2)(v_1 - v_2)^2\}\, dx = 0.$$

From $(v_1^2 + v_1 v_2 + v_2^2) \geq 0$, it is clear that $v_1 - v_2 = 0$ as desired in (iv). □

Lemma 3.6. *If v is a critical point of $\mathcal{K}$ defined in Lemma 3.5, then $v_x(0) = 0$.*

Proof. A critical point $v \in H^2(0,\infty)$ of $\mathcal{K}$ satisfies

$$\begin{aligned} 0 = \mathcal{K}'(v)\varphi &= \int_0^\infty \left\{ v_x \varphi_x + \gamma v \varphi + v^3 \varphi - u\varphi \right\} \, dx \\ &= \int_0^\infty \{ -v_{xx} + \gamma v + v^3 - u \} \varphi \, dx \; - \; v_x(0)\varphi(0) \end{aligned}$$

for all compactly supported $\varphi \in C^\infty[0,\infty)$. Since we know from Lemma 3.5 that v satisfies (3.2b) a.e., we have $v_x(0)\varphi(0) = 0$ for any arbitrary $\varphi(0)$. Therefore, $v_x(0) = 0$. □

Remark 3.7. The property in Lemma 3.6 is well known and often referred to as a natural boundary condition.

Suppose $u \in H^1(0,\infty)$ and let $v \in H^3(0,\infty)$ be the unique minimizer of $\mathcal{K}$ in Lemma 3.5. Then we write $v := \mathcal{N}u$ so that $\mathcal{N} : H^1(0,\infty) \to H^3(0,\infty)$ and $v_x(0) = 0$. We remark that $u \in C^{1/2}[0,\infty)$ and $v \in C^{2+1/2}[0,\infty)$ by the Sobolev embedding and therefore (u_0, v_0) satisfies (3.2b) in a classical sense. Finding a symmetric solution to the system (3.2) becomes equivalent to studying the integral-differential equation

$$du_{xx} + f(u) - \mathcal{N}u = 0$$

with boundary condition $u_x(0) = 0$. Before closing this section, we present some properties of the nonlinear operator $\mathcal{N}$.

Lemma 3.8. *For any* $w \in H^1$,

$$\|\mathcal{N}w\|_{H^1(0,\infty)} \leq \max\{1, 1/\gamma\}\|w\|_{L^2(0,\infty)}. \tag{3.4}$$

Proof. Multiplying (3.2b) through by $\mathcal{N}w$ and integrating by parts,

$$\int_0^\infty \{(\mathcal{N}w)'^2 + \gamma(\mathcal{N}w)^2 + (\mathcal{N}w)^4\}\,dx = \int_0^\infty w\mathcal{N}w\,dx$$

and the result follows. □

The next lemma shows that $\mathcal{N}$ is Frechét differentiable. Its derivative will be denoted by $\mathcal{N}'$.

Lemma 3.9. *The nonlinear map* $\mathcal{N}$ *is Frechét differentiable. To be precise, given any* $w \in H^1(0,\infty)$ *and* $v = \mathcal{N}w$*, we have* $\mathcal{N}'(w) : H^1(0,\infty) \to H^3(0,\infty)$ *such that for any given* $\hat{w} \in H^1(0,\infty)$

$$\hat{v} = \mathcal{N}'(w)\hat{w}$$

is the unique solution in $H^3(0,\infty)$ *of*

$$\hat{v}'' - \gamma\hat{v} - 3v^2\hat{v} = -\hat{w} \tag{3.5}$$

with $v'(0) = 0$.

Proof. Fix $w \in H^1(0,\infty)$ and set $v = \mathcal{N}w$. Given $\hat{w} \in H^1(0,\infty)$, let $A : H^1(0,\infty) \to H^3(0,\infty)$ be a map such that $\hat{v} = A\hat{w}$ is the unique H^3 solution of (3.5). The existence of A is guaranteed by using a similar variational argument as in Lemma 3.5, resulting a $\hat{v}$ satisfying $\hat{v}'(0) = 0$. We claim that $A = \mathcal{N}'(w)$. It is clear that A is linear.

With $\hat{v}'' = \gamma\hat{v} + 3v^2\hat{v} - \hat{w}$, it follows from Lemma 3.4 that there exists a constant $C_0 = C_0(\gamma, \|3v^2\|_{H^1})$ such that $\|\hat{v}\|_{H^3} \leq C_0\|\hat{w}\|_{H^1}$. Hence A is a bounded operator. To finish our proof, it suffices to check that

$$\|\mathcal{N}(w + \hat{w}) - \mathcal{N}w - A\hat{w}\|_{H^3} = o(\|\hat{w}\|_{H^1}) \tag{3.6}$$

for any $\hat{w} \in H^1$ with norm at most 1. Let $\tilde{v} = \mathcal{N}(w + \hat{w}) - \mathcal{N}w$. Since both $(w + \hat{w}, v + \tilde{v})$ and (w, v) satisfy (3.2b), we have

$$\begin{cases} (v + \tilde{v})'' - \gamma(v + \tilde{v}) - (v + \tilde{v})^3 = -(w + \hat{w}), \\ v'' - \gamma v - v^3 = -w. \end{cases}$$

Subtracting from one another yields

$$\tilde{v}'' - \gamma\tilde{v} - 3v^2\tilde{v} - 3v\tilde{v}^2 - \tilde{v}^3 = -\hat{w}, \tag{3.7}$$

and we subtract (3.5) from (3.7) to get

$$(\tilde{v} - \hat{v})'' - (\gamma + 3v^2)(\tilde{v} - \hat{v}) = \tilde{v}^3 + 3v\tilde{v}^2.$$

By applying Lemma 3.4 and the estimate from Lemma 3.3, there exists a positive constant $C_0 = C_0(\gamma, \|3v^2\|_{H^1})$ such that

$$\|\tilde{v} - \hat{v}\|_{H^3} \leq \sqrt{6}\, C_0\|\tilde{v} + 3v\|_{H^1}\|\tilde{v}\|^2_{H^1} .$$

Since $\gamma\tilde{v} + 3v^2\tilde{v} + 3v\tilde{v}^2 + \tilde{v}^3 = (\gamma + (\tilde{v} + 3v/2)^2 + 3v^2/4)\,\tilde{v}$, the weak formulation of

(3.7) implies $\|\tilde{v}\|_{H^1} \leq \max\{1, 1/\gamma\}\|\hat{w}\|_{H^1}$. Together with $\|v\|_{H^1} \leq \max\{1, 1/\gamma\}\|w\|_{H^1}$ from Lemma 3.8, we finally have

$$\begin{aligned}\|\tilde{v} - \hat{v}\|_{H^3} &\leq \sqrt{6}\, C_0 \left(\|\tilde{v}\|_{H^1} + \|3v\|_{H^1}\right) \|\tilde{v}\|_{H^1}^2 \\ &\leq \sqrt{6}\, C_0 \max\{1, 1/\gamma^3\} \left(\|\hat{w}\|_{H^1} + 3\|w\|_{H^1}\right) \|\hat{w}\|_{H^1}^2 \\ &\leq C_1 \|\hat{w}\|_{H^1}^2\end{aligned}$$

for some $C_1 = C_1(\gamma, \|w\|_{H^1})$, which implies (3.6) as desired. □

Lemma 3.10. *Suppose $w_1, w_2 \in H^1(0,\infty)$ are distinct with $w_1 \geq w_2$, then $\mathcal{N}w_1 > \mathcal{N}w_2$.*

Proof. Let $w_1, w_2 \in H^1(0,\infty)$ with $w_1 \geq w_2$. Then $v_1 = \mathcal{N}w_1$ and $v_2 = \mathcal{N}w_2$ satisfy $v_1'' - \gamma v_1 - v_1^3 + w_1 = 0$ and $v_2'' - \gamma v_2 - v_2^3 + w_2 = 0$, respectively. By subtracting the two equations, we obtain

$$(v_1 - v_2)'' - \gamma(v_1 - v_2) - (v_1^2 + v_1 v_2 + v_2^2)(v_1 - v_2) = -(w_1 - w_2) \leq 0. \tag{3.8}$$

Let $z = v_1 - v_2 \in H^1(0,\infty)$. Then $z'(0) = 0$ and $z \to 0$ as $x \to \infty$. Since $v_1^2 + v_1 v_2 + v_2^2 \geq 0$, the maximum principle is applicable to (3.8) and z cannot attain an interior non-positive minimum unless $z \equiv 0$. The last possibility is excluded as w_1 and w_2 are distinct.

Suppose $z(0) \leq 0$, then $z'(0) > 0$ as a result of the Hopf lemma. This is a contradiction and hence $z(0) > 0$. Coupled with the absence of a non-positive interior minimum point, we see that $z > 0$ everywhere and the proof of the lemma is complete.

□

Lemma 3.11. *Suppose $w_1, w_2 \in L^2(0,\infty)$, then*

$$\|\mathcal{N}w_2 - \mathcal{N}w_1\|_{H^1(0,\infty)} \le \max\{1, 1/\gamma\}\|w_2 - w_1\|_{L^2(0,\infty)}.$$

Proof. The same proof as in Lemma 3.10 leads us to (3.8) (but without ≤ 0 at the end). Now multiply by $v_1 - v_2$ and integrate over the interval $(0,\infty)$. □

3.3 Variational formulation

In this section, we introduce a variational formulation that corresponds to the system (3.2) or, equivalently, to

$$du'' + f(u) - \mathcal{N}u = 0. \tag{3.9}$$

Consider the functional $\hat{J} : H^1(0,\infty) \to \mathbb{R}$ defined by

$$\hat{J}(w) = \int_0^\infty \left\{\frac{d}{2}w'^2 + \frac{1}{2}w\mathcal{N}w + F(w) + \frac{1}{4}(\mathcal{N}w)^4\right\} dx,$$

where

$$F(\xi) = -\int_0^\xi f(\eta)\, d\eta = \frac{\xi^4}{4} - \frac{(1+\beta)\xi^3}{3} + \frac{\beta\xi^2}{2}.$$

Let $0 < \beta_1 < 1 < \beta_2$ such that $F(\beta_1) = F(\beta_2) = 0$. We will first verify that (3.9) is the Euler-Lagrange equation associated with $\hat{J}$.

Lemma 3.12. *The functional $\hat{J}$ is well defined for all $w \in H^1(0,\infty)$.*

Proof. Let $w \in H^1(0,\infty)$. Then $\|w\|_{L^\infty(0,\infty)} \le \sqrt{2}\|w\|_{H^1(0,\infty)}$ and $\|\mathcal{N}w\|_{L^\infty(0,\infty)} \le \sqrt{2}\,\|\mathcal{N}w\|_{H^1(0,\infty)}$ by Lemma 3.3. For a fixed w, there is a positive constant C_w, which depends on $\|w\|_{L^\infty(0,\infty)}$, such that $|F(\xi)| \le C_w\xi^2$ for $|\xi| \le \|w\|_{L^\infty(0,\infty)}$. Together with

Lemma 3.8, we obtain

$$\begin{aligned} |\hat{J}(w)| &\leq \frac{d}{2}\|w\|_{H^1}^2 + \frac{1}{2}\|w\|_{L^2}\|\mathcal{N}w\|_{L^2} + C_w\|w\|_{L^2}^2 + \frac{1}{4}\|\mathcal{N}w\|_{L^\infty}^2\|\mathcal{N}w\|_{L^2}^2 \\ &< \frac{d}{2}\|w\|_{H^1}^2 + \frac{1}{2}\max\{1, 1/\gamma\}\|w\|_{L^2}^2 + C_w\|w\|_{H^1}^2 + \frac{1}{2}\|\mathcal{N}w\|_{H^1}^4 \\ &< \infty. \end{aligned}$$

□

Lemma 3.13. *Let* $v = \mathcal{N}w$. *Then*

$$\int_0^\infty \left\{\frac{1}{4}v^4 + \frac{1}{2}wv\right\} dx = \int_0^\infty \left\{ -\frac{1}{2}v'^2 - \frac{\gamma}{2}v^2 - \frac{1}{4}v^4 + wv\right\} dx.$$

Proof. For (w, v) satisfies (3.2b) weakly,

$$\int_0^\infty \frac{1}{2}\left(-v'\varphi' - \gamma v\varphi - v^3\varphi + w\varphi\right) dx = 0, \quad \forall \varphi \in H^1(0,\infty).$$

We choose $\varphi = v$ and add $\int_0^\infty \left(\frac{1}{4}v^4 + \frac{1}{2}wv\right) dx$ on both sides to get the result. □

Lemma 3.14. *If* $u_0 \in H^1(0,\infty)$ *is a critical point of* $\hat{J}$, *then* $(u_0, \mathcal{N}u_0)$ *is a weak solution of* (3.2).

Proof. Given any $w \in H^1(0,\infty)$, define $v = \mathcal{N}w$. By Lemma 3.13 we can write

$$\hat{J}(w) = \int_0^\infty \left\{\frac{d}{2}w'^2 - \frac{1}{2}v'^2 - \frac{\gamma}{2}v^2 - \frac{1}{4}v^4 + wv + F(w)\right\} dx.$$

With $\hat{v} = \mathcal{N}'(w)\hat{w}$, the Fréchet derivative of $\hat{J}$ is

$$\hat{J}'(w)\hat{w} = \int_0^\infty \left\{dw'\hat{w}' - v'\hat{v}' - \gamma v\hat{v} - v^3\hat{v} + w\hat{v} + v\hat{w} - f(w)\hat{w}\right\} dx. \tag{3.10}$$

Since $\hat{J}'(u_0)\hat{w} = 0$ and $v_0 = \mathcal{N}u_0$ satisfies (3.2b), the equation (3.10) becomes

$$\int_0^\infty \{du_0'\hat{w}' - f(u_0)\hat{w} + v_0\hat{w}\}\,dx = 0,$$

which implies that (u_0, v_0) satisfies (3.2a) weakly. □

Remark 3.15. The critical point u_0 of $\hat{J}$ satisfies the natural boundary condition $u_0'(0) = 0$.

To find a standing pulse solution of (3.2), we now consider a minimizing problem for $\hat{J}$. Define a class of admissible functions $\mathcal{A}$ as

$$\begin{aligned}\mathcal{A} \equiv \{w \in\, & H^1(0,\infty) : \beta \le w(0) \le 1;\ \text{there exist } 0 \le x_1 < x_2 \le \infty \text{ such that} \\ & \beta \le w \le 1 \text{ on } [0, x_1]\,,\ 0 \le w \le \beta \text{ on } (x_1, x_2]\,,\ \text{and } -(M+1) \le w \le 0 \\ & \text{on } (x_2, \infty)\},\end{aligned} \tag{3.11}$$

where $M = M(\gamma)$ is a constant such that $f(\xi) \ge 1 + 1/\gamma$ for all $\xi \le -M$. We note that the initial condition $\beta \le w(0) \le 1$ is vacuous if $x_1 = 0$. Without any constraint we expect there is no global minimizer of $\hat{J}$, a fact demonstrated in the work of Chen and Choi (2012). We therefore restrict our attention to $J \equiv \hat{J}|_{\mathcal{A}}$ for a minimizer. In what follows, let us refer to the terms $\int_0^\infty \frac{d}{2}w'^2\,dx$, $\int_0^\infty F(w)\,dx$, and $\int_0^\infty \left(\frac{1}{2}w\mathcal{N}w + \frac{1}{4}(\mathcal{N}w)^4\right)\,dx$ as the gradient term, potential term, and nonlocal term of J, respectively.

The presence of the nonlocal term imposes a difficulty in showing the existence of a minimizer. To attain a minimizer in the next section, we discuss some estimates of the nonlocal term that will be useful.

Lemma 3.16. *Let $w \in \mathcal{A}$. Then $-(M+1) \le \mathcal{N}w \le 1$.*

Proof. Set $v = \mathcal{N}w$ and $\bar{v} = 1$. Since $w \le 1$,

$$\bar{v}'' - \gamma\bar{v} - \bar{v}^3 = -\gamma - 1 \le -w.$$

By subtracting $v'' - \gamma v - v^3 = -w$ from above,

$$(\bar{v} - v)'' - \gamma(\bar{v} - v) - (\bar{v}^2 + \bar{v}v + v^2)(\bar{v} - v) \le 0.$$

Let $z = \bar{v} - v$. The same maximum principle argument stated after (3.8) enables us to conclude that $z \ge 0$ everywhere, i.e. $v \le 1$. Similarly for the lower bound, set $\underline{v} = -(M+1)$ and observe that, since $w \ge -(M+1)$,

$$(v - \underline{v})'' - \gamma(v - \underline{v}) - (v^2 + \underline{v}v + \underline{v}^2)(v - \underline{v}) = -(w + \gamma(M+1) + (M+1)^3) \le 0.$$

The argument as before leads to $v - \underline{v} \ge 0$. □

Next, we use a comparison to obtain an estimate of $\mathcal{N}$. Consider the following linear equations

$$\begin{cases} V'' - \gamma V \ + w = 0, \\ V_0'' - (\gamma+1)V_0 + w = 0, \end{cases} \tag{3.12}$$

with zero Neumann boundary conditions at $x = 0$ for a fixed $w \in L^2(0,\infty)$. By solving (3.12a), we write

$$V(x) = \mathcal{L}w(x) = \int_0^\infty G(x,s)w(s)\,ds, \tag{3.13}$$

where $\mathcal{L} : L^2(0,\infty) \to L^2(0,\infty)$ is a linear operator with the Green's function

$$G(x,s) = \begin{cases} \frac{1}{\sqrt{\gamma}} e^{-\sqrt{\gamma}s} \cosh \sqrt{\gamma}x \,, & \text{if } x < s, \\ \frac{1}{\sqrt{\gamma}} e^{-\sqrt{\gamma}x} \cosh \sqrt{\gamma}s \,, & \text{if } x > s. \end{cases}$$

It can be verified that $\int_0^\infty w_1 \mathcal{L} w_2 \, dx = \int_0^\infty w_2 \mathcal{L} w_1 \, dx$ for any $w_1, w_2 \in L^2(0,\infty)$, i.e. $\mathcal{L}$ is self-adjoint with respect to the L^2 inner product. Moreover, a direct calculation shows

$$\begin{aligned} \mathcal{L}w(x) &= \int_0^x \frac{1}{\sqrt{\gamma}} e^{-\sqrt{\gamma}x} \cosh(\sqrt{\gamma}s)\, w(s)\, ds + \int_x^\infty \frac{1}{\sqrt{\gamma}} e^{-\sqrt{\gamma}s} \cosh(\sqrt{\gamma}x)\, w(s)\, ds \\ &\leq \frac{1}{\sqrt{\gamma}} e^{-\sqrt{\gamma}x} \int_0^x \cosh(\sqrt{\gamma}s)\, ds + \frac{\cosh(\sqrt{\gamma}x)}{\sqrt{\gamma}} \int_x^\infty e^{-\sqrt{\gamma}s}\, ds \\ &= \frac{1}{\gamma}. \end{aligned} \tag{3.14}$$

Similarly, for (3.13b), we can set $\mathcal{L}_0 = \left((\gamma+1) - \frac{d^2}{dx^2} \right)^{-1}$ and write

$$V_0(x) = \mathcal{L}_0 w(x) = \int_0^\infty G_0(x,s) w(s)\, dx, \tag{3.15}$$

where

$$G_0(x,s) = \begin{cases} \frac{1}{\sqrt{\gamma+1}} e^{-\sqrt{\gamma+1}\,s} \cosh\left(\sqrt{\gamma+1}\,x\right), & \text{if } x < s, \\ \frac{1}{\sqrt{\gamma+1}} e^{-\sqrt{\gamma+1}\,x} \cosh\left(\sqrt{\gamma+1}\,s\right), & \text{if } x > s. \end{cases}$$

Lemma 3.17. *For a non-negative, non-trivial function $w \in \mathcal{A}$,*

$$0 < \mathcal{L}_0 w \leq \mathcal{N} w \leq \mathcal{L} w.$$

Proof. Let $V_0 = \mathcal{L}_0 w$ and $V = \mathcal{L} w$. The positivity of the Green's function G_0 implies that $V_0 > 0$. Since $v = \mathcal{N}w$ satisfies (3.2b), we have $v'' - (\gamma + v^2)v \leq 0$ so that $v > 0$ by the maximum principle. In addition

$$v'' - (\gamma + 1)v + w = v^3 - v. \tag{3.16}$$

By subtracting (3.12b) from (3.16) and using $0 \leq v \leq 1$, we obtain

$$(v - V_0)'' - (\gamma + 1)(v - V_0) = v^3 - v \leq 0.$$

We now conclude $V_0 \leq v$ using the maximum principle. The proof for $v \leq V$ is similar. □

Lemma 3.18. *If $w \in H^1$, then*

$$\int_0^\infty w\mathcal{N}w\,dx \geq 0,$$

and for any $w_1, w_2 \in H^1$,

$$\int_0^\infty (w_1 - w_2)(\mathcal{N}w_1 - \mathcal{N}w_2)\,dx \geq 0.$$

Proof. Let $v = \mathcal{N}w$, then (w, v) satisfies (3.2b). Multiplying (3.2b) by v and integrating by parts gives $\int_0^\infty w\mathcal{N}w\,dx = \int_0^\infty (v'^2 + \gamma v^2 + v^4)\,dx \geq 0$. Next let $v_1 = \mathcal{N}w_1$ and $v_2 = \mathcal{N}w_2$. Subtracting the equations (3.2b) for v_1 and v_2 from one another, we get $(v_1 - v_2)'' - \gamma(v_1 - v_2) - p(x)(v_1 - v_2) = -(w_1 - w_2)$ where $p = v_1^2 + v_1 v_2 + v_2^2 \geq 0$. The same integration by parts argument yields the next inequality. □

Lemma 3.19. *Let $w = f - g$ with $f \equiv \max\{w, 0\} \geq 0$ and $g \geq 0$ being its positive* *and negative parts, respectively. Then*

$$\int_0^\infty w\mathcal{N}w\,dx \geq \int_0^\infty (f-g)(\mathcal{N}f - \mathcal{N}g)\,dx - 4\int_0^\infty \mathcal{L}f\,\mathcal{L}g\,dx,$$

where $\mathcal{L}$ *is the linear operator defined in* (3.13)*.*

Proof. Let $v_f = \mathcal{N}f$, $v_g = \mathcal{N}g$ and $v_{f-g} = \mathcal{N}(f-g)$. Notice from (3.2b) that $\mathcal{N}u = \mathcal{L}u - \mathcal{L}((\mathcal{N}u)^3)$ for any u. Hence

$$\begin{aligned}\int_0^\infty w\mathcal{N}w\,dx &= \int_0^\infty (f-g)\mathcal{N}(f-g)\,dx \\ &= \int_0^\infty (f-g)(\mathcal{L}f - \mathcal{L}g - \mathcal{L}v_{f-g}^3)\,dx \\ &= \int_0^\infty (f-g)(v_f + \mathcal{L}v_f^3 - v_g - \mathcal{L}v_g^3 - \mathcal{L}v_{f-g}^3)\,dx.\end{aligned}$$

Since $\mathcal{L}$ is self-adjoint with respect to the L^2 inner product,

$$\int_0^\infty w\mathcal{N}w\,dx = \int_0^\infty (f-g)(v_f - v_g)\,dx - \int_0^\infty (v_g^3 - v_f^3 + v_{f-g}^3)\mathcal{L}(f-g)\,dx. \quad (3.17)$$

It remains to show that $\int_0^\infty (v_g^3 - v_f^3 + v_{f-g}^3)\mathcal{L}(f-g)\,dx \leq 4\int_0^\infty \mathcal{L}f\,\mathcal{L}g\,dx$. Recall from Lemma 3.17 that $\mathcal{L}f$ and $\mathcal{L}g$ are non-negative. Since $\mathcal{N}$ is non-decreasing by Lemma 3.10, we have $-v_g \leq v_{f-g} \leq v_f$. Then $|v_{f-g}| \leq \max\{v_f, v_g\}$, which implies that $v_{f-g}^3 \leq v_f^3 + v_g^3$. Observe that

$$\begin{aligned}\int_0^\infty (v_g^3 - v_f^3 + v_{f-g}^3)\mathcal{L}(f-g)\,dx &= \int_0^\infty \{(v_g^3 - v_f^3)(\mathcal{L}f - \mathcal{L}g) + v_{f-g}^3(\mathcal{L}f - \mathcal{L}g)\}\,dx \\ &\leq \int_0^\infty \{(v_g^3 - v_f^3)(\mathcal{L}f - \mathcal{L}g) + (v_g^3 + v_f^3)(\mathcal{L}f + \mathcal{L}g)\}\,dx \\ &= 2\int_0^\infty (v_g^3\mathcal{L}f + v_f^3\mathcal{L}g)\,dx.\end{aligned}$$

Together with $0 \le v_f \le 1$, $0 \le v_g \le 1$, $v_g \le \mathcal{L}g$ and $v_f \le \mathcal{L}f$,

$$\int_0^\infty (v_g^3 - v_f^3 + v_{f-g}^3)\mathcal{L}(f-g)\,dx \le 4\int_0^\infty \mathcal{L}f\,\mathcal{L}g\,dx.$$

□

Lemma 3.20. *Suppose there exists a sequence* $\{u^{(n)}\}_{n=1}^\infty$ *such that* $u^{(n)} \rightharpoonup u_0$ *weakly in* $H^1(0,\infty)$ *with* $\|u^{(n)}\|_{H^1(0,\infty)} < \infty$. *Then* $\int_0^\infty u_0 \mathcal{N}u^{(n)}\,dx \to \int_0^\infty u_0\mathcal{N}u_0\,dx$.

Proof. Let $\epsilon > 0$ be given. Since $u_0 \in H^1(0,\infty)$, there exists a large $a > 0$ such that

$$\int_a^\infty u_0^2\,dx \le \epsilon. \tag{3.18}$$

By compactness we can find a subsequence of $\{u^{(n)}\}_{n=1}^\infty$, still denoted by $\{u^{(n)}\}$, such that $u^{(n)} \to u_0$ in $L^2(0,a)$. In conjugation with (3.18) and Lemma 3.11, for any arbitrary $\epsilon > 0$ there is a $N_0 > 0$ such that whenever $n \ge N_0$,

$$\begin{aligned}\int_0^\infty |u_0\mathcal{N}u^{(n)} - u_0\mathcal{N}u_0|\,dx &= \int_0^a |u_0||\mathcal{N}u^{(n)} - \mathcal{N}u_0|\,dx + \int_a^\infty |u_0||\mathcal{N}u^{(n)} - \mathcal{N}u_0|\,dx \\ &\le \epsilon\,\|u_0\|_{L^2(0,a)} + \epsilon\,\|\mathcal{N}u^{(n)} - \mathcal{N}u_0\|_{L^2(a,\infty)}\,.\end{aligned}$$

As $\|\mathcal{N}u^{(n)}\|_{L^2(0,\infty)}$ is bounded because of Lemma 3.8, our result follows. □

3.4 Existence of a minimizer

To extract a minimizer from a minimizing sequence $\{w^{(n)}\}_{n=1}^\infty$ of J, we need some a priori estimates on the sequence.

Lemma 3.21. *There exists a positive constant d_0, which may depend on γ, such that if $d \le d_0$, there are a $q_0 \in \mathcal{A}$ and a positive constant $M_0 = M_0(\gamma)$, which is independent of d, such that $J(q_0) \le -M_0$.*

Proof. Let $0 < a < b$ be constants whose values will be assigned later. We first impose a constraint $b - a \le 1$. Define a piecewise linear function

$$q_0(x) \equiv \begin{cases} 1\,, & \text{if } 0 \le x \le a, \\ \frac{b-x}{b-a}\,, & \text{if } a \le x \le b, \\ 0\,, & \text{if } x \ge b\,. \end{cases}$$

Let $v = \mathcal{N}q_0$ and $V = \mathcal{L}q_0$, where $\mathcal{L}$ is the linear operator defined in (3.13). Since (q_0, v) satisfies (3.2b), we obtain $\int_0^\infty v^4\,dx \le \int_0^\infty q_0 v\,dx$ from the weak formulation of (3.2b). Then,

$$J(q_0) = \int_0^\infty \left\{ \frac{d}{2}{q_0'}^2 + \frac{1}{2}q_0 v + F(q_0) + \frac{1}{4}v^4 \right\} dx \le \int_0^\infty \left\{ \frac{d}{2}{q_0'}^2 + F(q_0) + \frac{3}{4}q_0 v \right\} dx. \tag{3.19}$$

A direct computation yields

$$\int_0^\infty \frac{d}{2}{q_0'}^2\,dx = \frac{d}{2(b-a)}$$

and, with $F(\xi) = \xi^4/4 - (1+\beta)\xi^3/3 + \beta\xi^2/2$,

$$\int_0^\infty F(q_0)\,dx = -\frac{(1-2\beta)}{12}a + (b-a)\left\{ \frac{1}{20} - \frac{(1+\beta)}{12} + \frac{\beta}{6} \right\}.$$

For the nonlocal term, it follows from Lemma 3.3 and Lemma 3.8 that

$$\int_0^\infty q_0 v\,dx \le \sqrt{2}\|v\|_{H^1}\|q_0\|_{L^1} \le \sqrt{2}\max\{1, 1/\gamma\}\|q_0\|_{L^2}\|q_0\|_{L^1}\ .$$

Then by computing the norms of q_0 directly, we obtain

$$\begin{aligned}\int_0^\infty \frac{3}{4}q_0 v\,dx &\le \frac{3\sqrt{2}}{4}\max\{1, 1/\gamma\}\left(a+\frac{1}{3}(b-a)\right)^{\frac{1}{2}}\left(a+\frac{1}{2}(b-a)\right)\\ &\le \frac{3\sqrt{2}}{4}\left(1+\frac{1}{\gamma}\right)\left(a+\frac{1}{2}(b-a)\right)^{\frac{3}{2}}.\end{aligned}$$

Take $d_0 = (b-a)^2$, and let $d \le d_0$. Plugging the gradient, potential and nonlocal terms into (3.19),

$$J(q_0) \le (b-a)\left\{\frac{11}{20} - \frac{(1+\beta)}{12} + \frac{\beta}{6}\right\} - \frac{1-2\beta}{12}a + \frac{3\sqrt{2}}{4}\left(1+\frac{1}{\gamma}\right)\left(a+\frac{1}{2}(b-a)\right)^{3/2}.$$

Let $C_0 \equiv \frac{11}{20} - \frac{(1+\beta)}{12} + \frac{\beta}{6}$ and note that $C_0 \ge 7/15$, the lower limit being attained when $\beta = 0$. Assume $a \le 24C_0/(1-2\beta)$ and $(b-a) \le \frac{(1-2\beta)}{24C_0}a \le 1$, then

$$\begin{aligned}J(q_0) &\le -\frac{1-2\beta}{24}a + \frac{3\sqrt{2}}{4}\left(1+\frac{1}{\gamma}\right)\left(1+\frac{1-2\beta}{48C_0}\right)^{3/2} a^{3/2}\\ &\le -\frac{1-2\beta}{48}a\end{aligned}$$

by choosing $a = \frac{2}{9}\left(\frac{1-2\beta}{24}\right)^2\left(1+\frac{1}{\gamma}\right)^{-2}\left(1+\frac{1-2\beta}{48C_0}\right)^{-3} \le 24\,C_0/(1-2\beta)$. We therefore obtain

$$J(q_0) \le -\frac{1}{9}\left(\frac{1-2\beta}{24}\right)^3\left(1+\frac{1}{\gamma}\right)^{-2}\left(1+\frac{1-2\beta}{48C_0}\right)^{-3} := -M_0.$$

Recall that C_0 is independent of γ. As $\gamma \to 0$ we see that a can go to 0, which in turn forces $d_0 = (b-a)^2 \leq \left(\frac{1-2\beta}{24C_0}\right)^2 a^2 \to 0$. Hence there exist a $d_0 = (b-a)^2$, which may depend on γ, and a positive constant $M_0 := M_0(\gamma)$ such that if $d \leq d_0$, we have $J(q_0) \leq -M_0$. □

In what follows, let $\gamma_0 \equiv \frac{3\beta^2}{1-2\beta} - 1$. We remark that $\gamma_0 > 0$ for $\beta \in (\frac{1}{3}, \frac{1}{2})$.

Lemma 3.22. *If $\gamma \leq \gamma_0$ and $d \leq d_0$, then*

(i) $\inf_{w \in \mathcal{A}} J(w) \geq -M_1$ *for some positive constant $M_1 = M_1(\gamma)$.*

(ii) Recall the definition of x_1 in (3.11)*. For any minimizing sequence $\{w^{(n)}\}$ of J, let $x_1^{(n)}$ be a corresponding value for $w^{(n)}$. By focusing on the tail of the sequence if necessary, $0 < m_2 \leq x_1^{(n)} \leq M_2 < \infty$ for all n, where $M_2 = M_2(\gamma)$ and $m_2 = m_2(\gamma)$ are positive constants which are independent of n.*

(iii) A minimizing sequence $\{w^{(n)}\}$ is uniformly bounded for all n in $H^1(0,\infty)$ norm.

Proof. Let $w = f - g$ where $f \geq 0$ and $g \geq 0$ are the positive and negative parts of w, respectively, as in Lemma 3.19, thus we can write

$$\int_0^\infty w\mathcal{N}w\, dx \geq I - 4\, II, \tag{3.20}$$

where $I \equiv \int_0^\infty (f-g)(\mathcal{N}f - \mathcal{N}g)\, dx$ and $II \equiv \int_0^\infty \mathcal{L}f\, \mathcal{L}g\, dx$. Since $\mathcal{L}_0 w \leq \mathcal{N}w \leq \mathcal{L}w$ for any $w \geq 0$ by Lemma 3.17 and $\int_0^\infty g\mathcal{N}g\, dx \geq 0$ by Lemma 3.18, together with the self-adjointness of $\mathcal{L}$, we get

$$\begin{aligned} I &\geq \int_0^\infty \{f\mathcal{N}f - g\mathcal{N}f - f\mathcal{N}g\}\, dx \\ &\geq \int_0^\infty \{f\mathcal{L}_0 f - 2f\mathcal{L}g\}\, dx \\ &\geq \beta \int_0^{x_1} \mathcal{L}_0 f\, dx - 2\int_0^{x_2} \mathcal{L}g\, dx. \end{aligned} \tag{3.21}$$

For $0 \le x \le x_1$, we use the definition of $\mathcal{L}_0$ in (3.15) to compute

$$\begin{aligned}
\mathcal{L}_0 f(x) &= \frac{e^{-\sqrt{\gamma+1}\,x}}{\sqrt{\gamma+1}} \int_0^x f(s)\cosh(\sqrt{\gamma+1}\,s)\,ds + \frac{\cosh(\sqrt{\gamma+1}x)}{\sqrt{\gamma+1}} \int_x^{x_2} f(s)e^{-\sqrt{\gamma+1}s}\,ds \\
&\ge \frac{\beta e^{-\sqrt{\gamma+1}\,x}}{\sqrt{\gamma+1}} \int_0^x \cosh(\sqrt{\gamma+1}\,s)\,ds \\
&= \frac{\beta}{\gamma+1} e^{-\sqrt{\gamma+1}x}\sinh(\sqrt{\gamma+1}\,x). \qquad (3.22)
\end{aligned}$$

Similarly for $0 \le x \le x_2$, we obtain from the definition of $\mathcal{L}$ in (3.13) that

$$\begin{aligned}
\mathcal{L}g(x) &= \frac{\cosh(\sqrt{\gamma}x)}{\sqrt{\gamma}} \int_{x_2}^{\infty} g(s)e^{-\sqrt{\gamma}s}\,ds \\
&\le \frac{(M+1)\cosh(\sqrt{\gamma}x)}{\sqrt{\gamma}} \int_{x_2}^{\infty} e^{-\sqrt{\gamma}s}\,ds \\
&= \frac{(M+1)}{\gamma} e^{-\sqrt{\gamma}x_2}\cosh(\sqrt{\gamma}x). \qquad (3.23)
\end{aligned}$$

Finally by plugging in (3.22) and (3.23) into (3.21),

$$\begin{aligned}
I &\ge \frac{\beta^2}{\gamma+1} \int_0^{x_1} e^{-\sqrt{\gamma+1}\,x}\sinh(\sqrt{\gamma+1}x)\,dx - \frac{2(M+1)}{\gamma} e^{-\sqrt{\gamma}x_2} \int_0^{x_2} \cosh(\sqrt{\gamma}x)\,dx \\
&= \frac{\beta^2}{2(\gamma+1)} \int_0^{x_1} (1 - e^{-2\sqrt{\gamma+1}x})\,dx - \frac{2(M+1)}{\gamma^{3/2}} e^{-\sqrt{\gamma}x_2}\sinh(\sqrt{\gamma}x_2) \\
&= \frac{\beta^2}{2(\gamma+1)} \left(x_1 - \frac{1}{2\sqrt{\gamma+1}}(1 - e^{-2\sqrt{\gamma+1}x_1}) \right) - \frac{(M+1)}{\gamma^{3/2}} \left(1 - e^{-2\sqrt{\gamma}x_2}\right) \\
&\ge \frac{\beta^2}{2(\gamma+1)} x_1 - \frac{\beta^2}{4(\gamma+1)^{3/2}} - \frac{(M+1)}{\gamma^{3/2}}\ . \qquad (3.24)
\end{aligned}$$

Next let us find an upper bound of II. Recall from (3.14) that $\mathcal{L}f \le \frac{1}{\gamma}$; a similar calculation shows that $\mathcal{L}g \le \frac{M+1}{\gamma}$. Then

$$II \le \frac{1}{\gamma} \int_0^{x_2} \mathcal{L}g\,dx + \frac{(M+1)}{\gamma} \int_{x_2}^{\infty} \mathcal{L}f\,dx\ . \qquad (3.25)$$

For $0 \le x \le x_2 \le \infty$, it follows from (3.23) that

$$\int_0^{x_2} \mathcal{L}g\,dx \le \frac{(M+1)}{\gamma} e^{-\sqrt{\gamma}x_2} \int_0^{x_2} \cosh(\sqrt{\gamma}x)\,dx$$
$$\le \frac{(M+1)}{2\gamma^{3/2}}\ . \tag{3.26}$$

At the same time when $x_2 \le x < \infty$, we use the definition of $\mathcal{L}$ in (3.13) to obtain

$$\mathcal{L}f(x) = \frac{e^{-\sqrt{\gamma}x}}{\sqrt{\gamma}} \int_0^{x_2} f(s)\cosh(\sqrt{\gamma}s)\,ds$$
$$\le \frac{e^{-\sqrt{\gamma}x}}{\gamma}\sinh(\sqrt{\gamma}x_2),$$

which implies that

$$\int_{x_2}^{\infty} \mathcal{L}f(x)\,dx \le \frac{1}{\gamma}\sinh(\sqrt{\gamma}x_2) \int_{x_2}^{\infty} e^{-\sqrt{\gamma}x}\,dx$$
$$\le \frac{1}{2\gamma^{3/2}}\ . \tag{3.27}$$

Substituting (3.26) and (3.27) into (3.25),

$$II \le \frac{(M+1)}{\gamma^{5/2}}\ . \tag{3.28}$$

Now using the bounds in (3.24) and (3.28) to estimate (3.25), we get

$$\int_0^{\infty} w\mathcal{N}w\,dx \ge I - 4\,II \ge \frac{\beta^2}{2(\gamma+1)}x_1 - \frac{\beta^2}{4(\gamma+1)^{3/2}} - \frac{(M+1)}{\gamma^{3/2}} - \frac{4(M+1)}{\gamma^{5/2}}\ .$$

Since $F \geq 0$ when $x \geq x_1$, with $\int_0^\infty F(x)\,dx \geq F_{\min} x_1 := F(1)\,x_1 = -\frac{(1-2\beta)}{12} x_1$,

$$\begin{aligned} J(w) &= \int_0^\infty \left\{ \frac{d}{2} w'^2 + \frac{1}{2} w \mathcal{N} w + F(w) + \frac{1}{4} (\mathcal{N} w)^4 \right\} dx \\ &\geq \int_0^\infty \left\{ F(w) + \frac{1}{2} w \mathcal{N} w \right\} dx \\ &\geq \left(-\frac{(1-2\beta)}{12} + \frac{\beta^2}{4(\gamma+1)} \right) x_1 - \frac{\beta^2}{8(\gamma+1)^{3/2}} - \frac{(M+1)}{2\gamma^{3/2}} - \frac{2(M+1)}{\gamma^{5/2}} . \end{aligned} \tag{3.29}$$

Observe that $-\frac{(1-2\beta)}{12} + \frac{\beta^2}{4(\gamma+1)} > 0$ for $\gamma < \gamma_0 = \frac{3\beta^2}{(1-2\beta)} - 1$. Choosing $M_1 = \frac{\beta^2}{8(\gamma+1)^{3/2}} + \frac{(M+1)}{2\gamma^{3/2}} + \frac{2(M+1)}{\gamma^{5/2}}$, we establish (i).

By Lemma 3.21 we can assume that a minimizing sequence $w^{(n)}$ satisfies $J(w^{(n)}) \leq -M_0 < 0$ by focusing on the tail of the sequence if needed. We can include the gradient term on the right hand side of (3.29); doing so, we have

$$\frac{d}{2} \|w_x^{(n)}\|_{L^2}^2 + \left(-\frac{(1-2\beta)}{12} + \frac{\beta^2}{4(\gamma+1)} \right) x_1^{(n)} \leq M_1, \tag{3.30}$$

which implies that there is a positive constant $M_2 = M_2(\gamma) := M_1 / \left(-\frac{(1-2\beta)}{12} + \frac{\beta^2}{4(\gamma+1)} \right)$, independent of n, such that $x_1^{(n)} \leq M_2$. Moreover, since the nonlocal term is non-negative and $F(\xi) \geq 0$ for $\xi \leq \beta_1$,

$$\begin{aligned} -\frac{1}{2} M_0 \geq \ J(w^{(n)}) &\geq \int_0^\infty F(w^{(n)})\,dx \\ &\geq \int_{\{x:\, w^{(n)}(x) \geq \beta_1\}} F(w^{(n)})\,dx \\ &\geq -|F_{\min}|\, |\{x :\ w^{(n)}(x) \geq \beta_1\}|, \end{aligned}$$

which implies that $x_1^{(n)} \geq \{x :\ w^{(n)}(x) \geq \beta_1\} \geq 6M_0/(1-2\beta) := m_2 > 0$; hence there is always a non-trivial positive part of $w^{(n)}$. The proof of (ii) is complete. To show (iii), first observe from (3.30) that $\|w_x^{(n)}\|_{L^2}$ is bounded for all n. Next, it follows from (ii) that

$$\int_{\{x:\, w^{(n)}(x)\geq\beta\}} (w^{(n)})^2\, dx \leq \int_{\{x: w^{(n)}(x)\geq\beta\}} 1\, dx = x_1^{(n)} \leq M_2.$$

On $\{x : w^{(n)}(x) < \beta\}$, for there exists a $C_1 > 0$ independent of d and γ such that $F(\xi) \geq C_1\xi^2$,

$$\begin{aligned}
\int_{\{x:\, w^{(n)}(x)<\beta\}} (w^{(n)})^2\, dx &\leq \frac{1}{C_1}\int_{\{x: w^{(n)}(x)<\beta\}} F(w^{(n)})\, dx \\
&= \frac{1}{C_1}\left\{\int_0^\infty F(w^{(n)})\, dx - \int_{\{x: w^{(n)}(x)\geq\beta\}} F(w^{(n)})\, dx\right\} \\
&\leq \frac{1}{C_1}\left\{J(w^{(n)}) - \int_{\{x: w^{(n)}(x)\geq\beta_1\}} F(w^{(n)})\, dx\right\} \\
&\leq \frac{1}{C_1}\{|F_{\min}|\, M_2\}.
\end{aligned}$$

Therefore, $\|w^{(n)}\|_{L^2}$ is uniformly bounded for all n. This completes the proof that $\|w^{(n)}\|_{H^1}$ is uniformly bounded. □

We now extract a minimizer $u_0 \in \mathcal{A}$ from the minimizing sequence. Due to the constraints imposed on the admissible set $\mathcal{A}$, u_0 may not satisfy (3.2a) on the intervals where it is identically equal to one of the constraints. To eliminate this possibility in the later sections, a truncation technique is used routinely in which we truncate u_0 to obtain a new function $u_{new} \in \mathcal{A}$ with $J(u_{new}) < J(u_0)$. With the help of the truncation technique, the constraint $-(M+1)$ will be released in this section.

Lemma 3.23. *Suppose $\gamma \leq \gamma_0$ and $d \leq d_0$. Let $\{w^{(n)}\}_{n=1}^\infty \subset \mathcal{A}$ be a minimizing sequence of J. Then there exists a $u_0 \in \mathcal{A}$ such that* $\liminf J(w^{(n)}) \geq J(u_0)$. *Moreover* *there exist* $0 < x_1 < x_2 \leq \infty$ *such that*

$$\begin{cases} \beta \leq u_0(x) \leq 1 \text{ for } x \in [0, x_1], \\ 0 \leq u_0(x) \leq \beta \text{ for } x \in [x_1, x_2], \\ -(M+1) \leq u_0(x) \leq 0 \text{ for } x \in [x_2, \infty) \text{ if } x_2 < \infty \end{cases} \tag{3.31}$$

with $m_2 \leq x_1 \leq M_2$.

Proof. By Lemma 3.22 there is a minimizing sequence $\{w^{(n)}\}_{n=1}^{\infty}$ such that $\lim J(w^{(n)}) = \inf_{w \in \mathcal{A}} J(w)$ with $\|w^{(n)}\|_{H_1(0,\infty)}$ uniformly bounded in n; this sequence is therefore compact in the weak topology. By choosing a subsequence, still denoted by $\{w^{(n)}\}$, there exists a $u_0 \in H^1(0,\infty)$ such that $w^{(n)} \rightharpoonup u_0$ weakly in $H^1(0,\infty)$ and strongly in $L^{\infty}_{loc}(0,\infty)$. As a consequence of Lemma 3.22, (3.31) holds with $m_2 \leq x_1 \leq M_2$ and $u_0 \in \mathcal{A}$.

Next we show that the weakly convergent subsequence satisfies $\liminf J(w^{(n)}) \geq J(u_0)$. The weak convergence in H^1 implies that

$$\liminf \int_0^{\infty} (w^{(n)\prime})^2 \, dx \geq \int_0^{\infty} (u_0')^2 \, dx. \tag{3.32}$$

Since $w^{(n)} \to u_0$ on $[0, x_1]$ and $\|F(w^{(n)})\|_{L^{\infty}(0,\infty)} < \infty$ for $w^{(n)} \in \mathcal{A}$,

$$\lim \int_0^{x_1} F(w^{(n)}) \, dx = \int_0^{x_1} F(u_0) \, dx.$$

Moreover since $w^{(n)} \leq \beta$ on $[x_1, \infty)$, Fatou's lemma implies that

$$\liminf \int_{x_1}^{\infty} F(w^{(n)}) \, dx \geq \int_{x_1}^{\infty} F(u_0) \, dx.$$

Therefore we conclude that

$$\liminf \int_0^\infty F(w^{(n)})\,dx \geq \int_0^\infty F(u_0)\,dx. \tag{3.33}$$

It remains to treat the nonlocal term. With $w_1 = w^{(n)}$ and $w_2 = u_0$, it follows from Lemma 3.18 that

$$\int_0^\infty (w^{(n)}\mathcal{N}w^{(n)} + u_0\mathcal{N}u_0)\,dx \geq \int_0^\infty (u_0\mathcal{N}w^{(n)} + w^{(n)}\mathcal{N}u_0)\,dx.$$

Since $\int_0^\infty u_0\mathcal{N}w^{(n)}\,dx \to \int_0^\infty u_0\mathcal{N}u_0\,dx$ by Lemma 3.20 and $\int_0^\infty w^{(n)}\mathcal{N}u_0\,dx$ goes to the same limit because $w^{(n)} \rightharpoonup u_0$ weakly in L^2,

$$\liminf \int_0^\infty w^{(n)}\mathcal{N}w^{(n)}\,dx + \int_0^\infty u_0\mathcal{N}u_0\,dx \geq 2\int_0^\infty u_0\mathcal{N}u_0\,dx.$$

Then it is clear that

$$\liminf \int_0^\infty \left(w^{(n)}\mathcal{N}w^{(n)} + (\mathcal{N}w^{(n)})^4\right)\,dx \geq \int_0^\infty \left(u_0\mathcal{N}u_0 + (\mathcal{N}u_0)^4\right)\,dx. \tag{3.34}$$

Combining (3.32), (3.33) and (3.34), $\liminf J(w^{(n)}) \geq J(u_0)$ follows immediately. Therefore u_0 is a minimizer of J satisfying $J(u_0) = \inf_{\mathcal{A}} J$. □

Lemma 3.24. *Let u_0 be changed to $u_{new} \in A$. Then the change in the nonlocal term is*

$$\int_0^\infty \left\{\left(\frac{1}{2}u_{new}\mathcal{N}u_{new} + \frac{1}{4}(\mathcal{N}u_{new})^4\right) - \left(\frac{1}{2}u_0\mathcal{N}u_0 + \frac{1}{4}(\mathcal{N}u_0)^4\right)\right\}\,dx$$
$$= \frac{1}{2}\int_0^\infty (u_{new} - u_0)(\mathcal{N}u_{new} + \mathcal{N}u_0)\,dx + \frac{1}{4}\int_0^\infty (\mathcal{N}u_{new} + \mathcal{N}u_0)(\mathcal{N}u_{new} - \mathcal{N}u_0)^3\,dx.$$

Moreover

$$\left|\frac{1}{4}\int_0^\infty (\mathcal{N}u_{new} + \mathcal{N}u_0)(\mathcal{N}u_{new} - \mathcal{N}u_0)^3\,dx\right| \le \frac{(M+1)^2}{4}\max\{1,\frac{1}{\gamma^2}\}\int_0^\infty (u_{new} - u_0)^2\,dx.$$

Proof. Set $v_0 = \mathcal{N}u_0$ and $v_{new} = \mathcal{N}u_{new}$. Then

$$\int_0^\infty \left(v''_{new} - \gamma v_{new} - v^3_{new}\right) v_0\,dx = \int_0^\infty -u_{new}v_0\,dx, \tag{3.35}$$

$$\int_0^\infty \left(v''_0 - \gamma v_0 - v^3_0\right) v_{new}\,dx = \int_0^\infty -u_0 v_{new}\,dx. \tag{3.36}$$

After integrating by parts each equation, we subtract one from the other to get

$$\int_0^\infty (u_0 v_{new} - u_{new}v_0)\,dx = \int_0^\infty (v_0^3 v_{new} - v^3_{new}v_0)\,dx. \tag{3.37}$$

Observe that

$$\begin{aligned}&\int_0^\infty \left(\frac{1}{2}u_{new}v_{new} + \frac{1}{4}v^4_{new} - \frac{1}{2}u_0v_0 - \frac{1}{4}v_0^4\right)\,dx\\ &= \int_0^\infty \left\{\frac{1}{2}(u_{new} - u_0)(v_{new} + v_0) + \frac{1}{2}(u_0v_{new} - u_{new}v_0) + \frac{1}{4}(v^4_{new} - v^4_0)\right\}\,dx.\end{aligned} \tag{3.38}$$

For the last two term in the integral, it follows from (3.37) that

$$\begin{aligned}&\int_0^\infty \left\{\frac{1}{2}(u_0v_{new} - u_{new}v_0) + \frac{1}{4}(v^4_{new} - v^4_0)\right\}\,dx\\ &= \int_0^\infty \left\{\frac{1}{2}v_0v_{new}(v_0^2 - v^2_{new}) + \frac{1}{4}(v^2_{new} + v_0^2)(v^2_{new} - v_0^2)\right\}\,dx\\ &= \int_0^\infty \frac{1}{4}(v_{new} + v_0)(v_{new} - v_0)^3\,dx\end{aligned}$$

and therefore, our first inequality holds. To show the next inequality, note that $\|v_{new} - v_0\|_{L^2} \leq \max\{1, 1/\gamma\}\ \|u_{new} - u_0\|_{L_2}$ from Lemma 3.11. Together with $\|v_{new}\|_{L^\infty} \leq M+1$ and $\|v_0\|_{L^\infty} \leq M+1$ from Lemma 3.16, we obtain

$$\begin{aligned}\left|\frac{1}{4}\int_0^\infty (v_{new}+v_0)(v_{new}-v_0)^3\,dx\right| &\leq \frac{1}{4}\int_0^\infty \max\{v_{new}^2, v_0^2\}\,(v_{new}-v_0)^2\,dx \\ &\leq \frac{(M+1)^2}{4}\max\{1,\frac{1}{\gamma^2}\}\int_0^\infty (u_{new}-u_0)^2\,dx.\end{aligned}$$

□

Lemma 3.25. *Let $d \leq d_0$ and u_0 be a minimizer obtained in Lemma 3.23. Then $\min u_0 \geq -M$.*

Proof. Suppose $\min u_0 < -M$. Take a small positive $\delta < 1$ so that $-M \geq \min u_0 + \delta$. Consider a truncated function

$$u_{new} = \begin{cases} u_0, & \text{if } u_0 \geq \min u_0 + \delta, \\ \min u_0 + \delta, & \text{if } u_0 < \min u_0 + \delta, \end{cases}$$

so that $u_{new} - u_0 \equiv p(x) \leq \delta$. Then $u_{new} \in \mathcal{A}$. We now define a positive constant $M_\delta := -(\min u_0 + \delta)$ to simplify the notation. Since the energy associated with the gradient term decreases by the change,

$$\begin{aligned} J(u_{new}) - J(u_0) \ &< \ \int_{\{x: u_0 \leq -M_\delta\}} \{F(u_{new}) - F(u_0)\}\,dx \\ &\quad + \int_0^\infty \left\{\frac{1}{2}\left(u_{new}\mathcal{N}u_{new} - u_0\mathcal{N}u_0\right) + \frac{1}{4}\left((\mathcal{N}u_{new})^4 - (\mathcal{N}u_0)^4\right)\right\} dx \end{aligned}$$

and applying Lemma 3.24 gives

$$\begin{aligned} J(u_{new}) - J(u_0) &< \int_{\{x: u_0 \le -M_\delta\}} \{F(u_{new}) - F(u_0) + \frac{p}{2} (\mathcal{N} u_{new} + \mathcal{N} u_0)\} \, dx \\ &\quad + \frac{1}{4} \int_0^\infty (\mathcal{N} u_{new} + \mathcal{N} u_0) (\mathcal{N} u_{new} - \mathcal{N} u_0)^3 \, dx \\ &\le \int_{\{x: u_0 \le -M_\delta\}} \{F(u_{new}) - F(u_0)\} + p + \frac{(M+1)^2}{4} \max\{1, \frac{1}{\gamma^2}\} \, p^2\} \, dx. \end{aligned}$$

By choosing δ smaller if necessary, we can ensure that $\frac{(M+1)^2}{4} \max\{1, 1/\gamma^2\} \, p \le 1/2\gamma$. The convexity of $F(\xi)$ for $\xi \le 0$ then implies that

$$J(u_{new}) - J(u_0) < \int_{\{x: u_0 \le -M_\delta\}} p \left\{ F'(u_{new}) + 1 + \frac{1}{2\gamma} \right\} \, dx.$$

As $M_\delta \ge M$, it is immediate from the definition of M that $F'(u_{new}) = -f(u_{new}) = -f(-M_\delta) \le -1 - \frac{1}{\gamma}$ on the set $\{x : u_0 \le -M_\delta\}$. Therefore $J(u_{new}) - J(u_0) < 0$. This contradicts the fact that u_0 is a minimizer in $\mathcal{A}$. □

By Lemma 3.25, the minimizer u_0 is greater than $-(M+1)$. Away from where u_0 equals $0, \beta$ or 1, we can perturb u_0 by C_0^∞ functions with small support to ensure that the perturbed function still lies inside $\mathcal{A}$. Setting $v_0 = \mathcal{N} u_0$, we can conclude after regularity bootstrap that $v_0 \in C^3[0, \infty)$. Moreover $u_0 \in C^2$ and satisfies (3.2a) except where u_0 equals $0, \beta$ or 1.

3.5 Qualitative properties of the solution

To establish that $(u_0, \mathcal{N} u_0)$ is a standing pulse solution of (3.2), we need to eliminate the possibility of an interval where u_0 equals $0, \beta$ or 1. This requires a better understanding of the qualitative properties of u_0. In this section, we investigate the derivatives of the minimizer u_0 of J. From now on, u_0 always stands for the minimizer of J and $v_0 = \mathcal{N} u_0$.

Lemma 3.26. *Let $x_0 > 0$ and $\ell \in (0, x_0)$. If $u_0(x) \notin \{0, \beta, 1\}$ for $x \in [x_0 - \ell, x_0)$ and $u_0(x_0) \in \{0, \beta, 1\}$, then both $\lim_{x \to x_0^-} u_0'(x)$ and $\lim_{x \to x_0^-} u_0''(x)$ exist. Moreover u_0 can be extended to a $C^\infty[x_0 - \ell, x_0]$ function, satisfying (3.2a) on $[x_0 - \ell, x_0]$. A similar statement holds on the interval $(x_0, x_0 + \ell]$.*

Proof. We only consider $u_0(x_0) = \beta$ and $u_0(x) \neq \beta$ on $[x_0 - \ell, x_0)$; the proof for the other cases are not different. Observe that $u_0 \in C^2[x_0 - \ell, x_0) \cap C[x_0 - \ell, x_0]$ and (u_0, v_0) satisfies (3.2a) on $[x_0 - \ell, x_0)$. It is clear from (3.2a) that $|u_0''|$ is bounded on $[x_0 - \ell, x_0)$, and $\lim_{x \to x_0^-} u_0''(x) = \frac{1}{d} \lim_{x \to x_0^-} (v_0(x) - f(u_0(x))$ exists. In view of

$$\lim_{x \to x_0^-} u_0'(x) = u_0'(x_0 - \ell) + \lim_{x \to x_0^-} \int_{x_0 - \ell}^{x} u_0''(t)\, dt,$$

the boundedness of the integrand guarantees that the limit exists. Hence $u_0 \in C^2[x_0 - \ell, x_0]$ and satisfies (3.2a) on $[x_0 - \ell, x_0]$. Using typical regularity bootstrap by differentiating (3.2a), we conclude that $u_0 \in C^\infty[x_0 - \ell, x_0]$. □

Following a similar idea in Chen and Choi (2012), the next lemma excludes the possibility of a sharp corner in the profile of u_0.

Lemma 3.27. *Suppose x_0 and ℓ are positive numbers such that $u_0(x_0) \in \{0, \beta, 1\}$ and $u_0 \in C^1[x_0 - \ell, x_0] \cap C^1[x_0, x_0 + \ell]$. Then $\lim_{x \to x_0^-} u_0'(x) = \lim_{x \to x_0^+} u_0'(x)$.*

Proof. We first prove the case $u_0(x_0) = 0$. Suppose $\lim_{x \to x_0^-} u_0' = a_1$ and $\lim_{x \to x_0^+} u_0' = a_2$ with $a_1 \neq a_2$. By taking a sufficiently small $\ell_1 \leq \ell$, we may assume $u_0' = a_1 + o(1)$

on $[x_0-\ell_1, x_0]$ and $u_0' = a_2 + o(1)$ on $[x_0, x_0+\ell_1]$. Let $y = L_1(x)$ be a straight line joining $(x_0-\ell_1, u_0(x_0-\ell_1))$ and $(x_0+\ell_1, u_0(x_0+\ell_1))$, whose slope is then given by $(a_1+a_2)/2 + o(1)$. We obtain u_{new} by trimming the corner of u_0 as follows:

$$u_{new} = \begin{cases} u_0(x), & \text{if } x \le x_0 - \ell_1, \\ L_1(x), & \text{if } x_0 - \ell_1 \le x \le x_0 + \ell_1, \\ u_0(x), & \text{if } x \ge x_0 + \ell_1. \end{cases}$$

As this is a small perturbation from u_0, $u_{new} \in \mathcal{A}$. We will show that $J(u_{new}) < J(u_0)$. The change in the gradient term decreases, because

$$\begin{aligned} \frac{d}{2}\int_{x_0-\ell_1}^{x_0+\ell_1} (u_{new}'^2 - u_0'^2)\,dx &= \frac{d}{2}\left\{\int_{x_0-\ell_1}^{0} (u_{new}'^2 - u_0'^2)\,dx + \int_{0}^{x_0+\ell_1} (u_{new}'^2 - u_0'^2)\,dx\right\} \\ &= \frac{d\ell_1}{2}\left\{2\left(\frac{(a_1+a_2)}{2} + o(1)\right)^2 - (a_1+o(1))^2 - (a_2+o(1))^2\right\} \\ &= \frac{d\ell_1}{2}\left\{\frac{(a_1+a_2)^2}{2} - a_1^2 - a_2^2 + o(1)\right\} \\ &= -\frac{d\ell_1}{4}\left\{(a_1-a_2)^2 + o(1)\right\} \\ &< 0. \end{aligned}$$

By the mean value theorem,

$$\int_{x_0-\ell_1}^{x_0+\ell_1} \{F(u_{new}) - F(u_0)\}\,dx = -\int_{x_0-\ell_1}^{x_0+\ell_1} f(\tilde{u})(u_{new} - u_0)\,dx$$

for some $\tilde{u}$ between u_0 and u_{new}. With $\max_{-M\le\xi\le 1} |f(\xi)|$ being bounded and $|u_{new} - u_0| = O(\ell_1)$,

$$\left|\int_{x_0-\ell_1}^{x_0+\ell_1} \{F(u_{new}) - F(u_0)\}\,dx\right| \le \ell_1 O(\ell_1).$$

The change in the nonlocal term can be calculated by applying Lemma 3.24. Since both $\|\mathcal{N}u_0\|_{L^\infty}$ and $\|\mathcal{N}u_{new}\|_{L^\infty}$ are bounded,

$$\begin{aligned}
&\left|\int_0^\infty \left\{\frac{1}{2}\left(u_{new}\mathcal{N}u_{new} - u_0\mathcal{N}u_0\right) + \frac{1}{4}\left((\mathcal{N}u_{new})^4 - (\mathcal{N}u_0)^4\right)\right\} dx\right| \\
\leq& \left|\frac{1}{2}\int_{x_0-\ell_1}^{x_0+\ell_1} (u_{new} - u_0)\left(\mathcal{N}u_{new} + \mathcal{N}u_0\right) dx\right| + \frac{(M+1)^2}{4}\max\{1, \frac{1}{\gamma^2}\} \int_{x_0-\ell_1}^{x_0+\ell_1} (u_{new} - u_0)^2\, dx \\
\leq& \,\ell_1 O(\ell_1).
\end{aligned}$$

Observe that the changes in the potential term and in the nonlocal term are both negligible compared to that in the gradient term. Then $J(u_{new}) < J(u_0)$ contradicts the fact that u_0 is a minimizer in $\mathcal{A}$. The same argument can be used to treat the other cases. $\square$

Remark 3.28. In what follows, Lemma 3.27 is referred to as a corner lemma, which does not require u_0 satisfy (3.2a) on either $[x_0 - \ell, x_0]$ or $[x_0, x_0 + \ell]$.

Let us consider the case $u_0(x_0) = 1$ and $u_0 \in C^1[x_0, x_0 + \ell]$ for some $\ell > 0$. By taking ℓ sufficiently small, there are three possibilities for the behavior of u_0 on the left side of a neighborhood of x_0:

(P1) $u_0 < 1$ on $[x_0 - \ell, x_0)$;

(P2) $u_0 = 1$ on $[x_0 - \ell, x_0]$;

(P3) There exist $a_1 < b_1 \leq a_2 < b_2 \leq a_3 < b_3 \ldots$ in the interval $[x_0 - \ell, x_0]$ such that

$$\begin{cases} u_0 \text{ satisfies (3.2a) on intervals } (a_n, b_n), \quad n = 1, 2, \ldots, \\ u_0 = 1 \quad \text{ on } [x_0 - \ell, x_0] \setminus \cup_{n=1}^\infty (a_n, b_n), \end{cases}$$

with both $a_n \to {x_0}^-$ and $b_n \to {x_0}^-$.

The case of $u_0(x_0) = 0$ can be studied similarly with corresponding cases referred to as (Q1), (Q2), and (Q3), respectively. We denote the cases for $u(x_0) = \beta$ by (R1), (R2), and (R3). Except in the case (P3), (Q3), or (R3), $u_0 \in C^\infty[x_0 - \ell, x_0 + \ell]$ follows from Lemma 3.26 and the corner lemma. Moreover for (P3), (Q3), or (R3), the next lemma states that $\lim_{x \to x_0} u_0'(x)$ exists. As a consequence, we conclude that $u_0 \in C^1[0, \infty)$.

Lemma 3.29. *Assume that $d \le d_0$. If x_0 is a limit point stated in (P3), (Q3), or (R3), then $u_0'(x_0) = 0$ and $v_0(x_0) = v_0'(x_0) = 0$.*

Proof. On the interval $[a_n, b_n] \subseteq [x_0 - \ell, x_0)$, where u_0 satisfies (3.2a), there is a $s_n \in (a_n, b_n)$ such that $u_0'(s_n) = 0$. Since $\| - f(u_0) + v_0\|_{L^\infty(a_n, b_n)} \le C_1$ for some constant C_1 not depending on x_0 or n, integrating (3.2a) yields $d|u_0'(x)| \le C_1 \left| \int_{s_n}^x dt \right|$, which implies $|u_0'(x)| \le C_1(b_n - a_n)/d$. For $|b_n - a_n| \to 0$ as $n \to \infty$, it follows that $\|u_0'\|_{L^\infty(a_n, b_n)} \to 0$. Then $u_0 \in C^1[x_0 - \ell, x_0]$ if we set $u_0'(x_0^-) = 0$. Suppose Cases (P1), (P2), (Q1), (Q2), (R1) or (R2) occurs on the interval $[x_0, x_0 + \ell]$, we see that $u_0 \in C^1[x_0, x_0 + \ell]$ so that $u_0'(x_0) = 0$ is immediate from the corner lemma. On the other hand if u_0 satisfies an analogous situation (P3), (Q3) or (R3) on the interval $[x_0, x_0 + \ell]$, the same argument as for $[x_0 - \ell, x_0]$ shows $u_0 \in C^1[x_0, x_0 + \ell]$ with $u_0'(x_0^+) = 0$. Hence in all scenario irrespective of what cases we have on the right of x_0, we have $u_0'(x_0) = 0$.

Next let us prove that $v_0(x_0) = 0$ and $v_0'(x_0) = 0$. Consider (P3) first. Since $u_0 \le 1$ everywhere, by (3.2b)

$$f(u_0(s_n)) - v_0(s_n) = -du_0''(s_n) \le 0. \tag{3.39}$$

As $s_n \to x_0^-$, we see that $f(u_0(x_0)) - v_0(x_0) \le 0$. On the other hand, since $u_0 \in$

$C^2[a_n, b_n]$ by Lemma 3.26 and $u_0(b_n) = 1$ with $u_0'(b_n) = 0$,

$$f(u_0(b_n)) - v_0(b_n) = -du_0''(b_n) \geq 0. \tag{3.40}$$

In this case, taking $b_n \to x_0^-$ gives $f(u_0(x_0)) - v_0(x_0) \geq 0$. Therefore $f(u_0(x_0)) - v_0(x_0) = 0$ from (3.39) and (3.40) which implies that $v_0(x_0) = f(1) = 0$. Suppose now that $v_0'(x_0) < 0$. This together with $u_0'(x_0) = 0$ gives $(f(u_0)-v_0)'\big|_{x=x_0} = -v_0'(x_0) > 0$. Since $f(u_0(x_0)) - v_0(x_0) = 0$, it follows that $f(u_0(x)) - v_0(x) < 0$ on $[x_0 - \delta, x_0)$ for some $\delta > 0$. This is incompatible with (3.40). Similarly $v_0'(x_0) > 0$ would contradict (3.39). Therefore $v_0'(x_0) = 0$.

If $u_0(x_0) = 0$ or $u_0(x_0) = \beta$, the proof of $v_0(x_0) = v_0'(x_0) = 0$ is slightly different since u_0 can cross 0 or β in (a_1, x_0); nevertheless due to the fact that u_0 can cross 0 or β only once, by choosing a_1 sufficiently close to x_0, u_0 does not change sign on $[a_1, x_0]$ in the case of (Q3), and either $u_0 \geq \beta$ or $u_0 \leq \beta$ on $[a_1, x_0]$ in the case of (R3). Then the rest of the proof is similar as above. We omit the details. □

Another essential qualitative property of the minimizer u_0 is the positivity of v_0. When the sign of v_0 is known, the energy change in the nonlocal term associated with the modification of u_0 becomes easier to quantify. As a result, Lemma 3.24 turns out to be more useful when we apply the truncation technique. We begin with two lemmas which show that v_0 is partially positive. Then, we follow the idea in Chen and Choi (2015) to study the linearization of (3.2) which provides information crucial for showing $v_0 > 0$ everywhere.

Lemma 3.30. *If $u_0 \geq 0$ on $[0, \infty)$ is non-trivial, then $v_0 > 0$ everywhere.*

Proof. If $u_0 \geq 0$, then $v_0 \geq \mathcal{L}u_0 > 0$ follows from Lemma 3.17. □

Lemma 3.31. *No matter whether u_0 changes sign or not, $v_0(0) > 0$.*

Proof. If u_0 stays non-negative, the assertion follows immediately from Lemma 3.30. Therefore assume u_0 changes sign at $x = x_2$. Suppose $v_0(0) \leq 0$. We claim that $v_0(x) < 0$ for all $x \in (0, \infty)$. Let us prove our claim on $(0, x_2]$ first. Its proof is divided into two cases:

Case 1: Assume $v_0(0) < 0$. Since $v_0'' - (\gamma + v_0^2)v_0 = -u_0 \leq 0$ on $[0, x_2]$, v_0 cannot have an interior negative minimum by the maximum principle. Moreover, with the boundary condition $v_0'(0) = 0$, the Hopf lemma implies that the minimum occurs at $x = x_2$ and $v_0' < 0$ on $(0, x_2]$. Hence $v_0(x) < v_0(0) < 0$ on $(0, x_2]$.

Case 2: Assume $v_0(0) = 0$. Then $v_0''(0) = -u_0(0) < 0$, and the boundary condition $v_0'(0) = 0$ implies that $v_0'(x) < 0$ in a neighborhood of 0. This leads to the same conclusion as in Case 1.

On the interval (x_2, ∞), since $v_0'' - (\gamma + v_0^2)v_0 = -u_0 \geq 0$, v_0 cannot attain a non-negative interior maximum. From the fact $v_0(x_2) < 0$ and $v_0 \to 0$ as $x \to \infty$, it follows that $v_0 < 0$ on $[x_2, \infty)$. This finishes the proof of our claim. Now let x_0 be a point where u_0 attains its global minimum. Then $u_0''(x_0) \geq 0$. Since u_0 is negative in a neighborhood of x_0, we have $f(u_0(x_0)) > 0$, which implies $v_0(x_0) = du_0''(x_0) + f(u_0(x_0)) > 0$. This contradicts our claim that $v_0 < 0$ on $(0, \infty)$. $\square$

Let $d_1 \equiv \min\{d_0, \frac{\beta^2}{4(1+\beta\gamma)}\}$. It what follows, it is assumed that $d \leq d_1$. Observe that the system (3.2) can be expressed as

$$\begin{pmatrix} u_0 \\ v_0 \end{pmatrix}'' - A \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} = \begin{pmatrix} -\frac{u_0^2}{d}(1 + \beta - u_0) \\ v_0^3 \end{pmatrix}, \tag{3.41}$$

where

$$A = \begin{pmatrix} \frac{\beta}{d} & \frac{1}{d} \\ -1 & \gamma \end{pmatrix}.$$

We now document the eigenvalues, and corresponding left and right eigenvectors of A. Details can be found in Chen and Choi (2015).

(a) Eigenvalues λ_1, λ_2 of A are real and positive. Moreover they satisfy

$$0 < \lambda_1 < \frac{\beta}{2d} < \frac{1}{2}\left(\gamma + \frac{\beta}{d}\right) < \lambda_2 < \frac{\beta}{d}. \tag{3.42}$$

(b) For the eigenvalue λ_1, it has a right eigenvector $\mathbf{a} = (-1, d\alpha_2)^T$ and a left eigenvector $\mathbf{l}_1 = (1, \alpha_2)^T$, where $\alpha_2 := \beta/d - \lambda_1 > 0$. For the eigenvalue λ_2, it has a right eigenvector $\mathbf{b} = (-\alpha_2, 1)^T$ and a left eigenvector $\mathbf{l}_2 = (1, \alpha_1)^T$, where $\alpha_1 := 1/d\alpha_2 > 0$.

It can be checked that

$$0 < \alpha_1 < \lambda_1 < \frac{\beta}{2d} < \alpha_2 < \lambda_2 < \frac{\beta}{d},$$

and

$$\mathbf{l}_1 \cdot \mathbf{a} > 0, \qquad \mathbf{l}_2 \cdot \mathbf{b} < 0. \tag{3.43}$$

(c) The asymptotic behavior of (u_0, v_0) at large x can be studied by linearizing (3.41) about $(u, v) = (0, 0)$:

$$\begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix}'' - A \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} = \mathbf{0}. \tag{3.44}$$

For (3.44) all the solutions decaying to $(0, 0)$ as $x \to \infty$ are of the form

$$\begin{pmatrix} u_0 \\ v_0 \end{pmatrix} \sim \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} = C_1 e^{-\sqrt{\lambda_1} x} \mathbf{a} + C_2 e^{-\sqrt{\lambda_2} x} \mathbf{b}. \tag{3.45}$$

While the linearization of (3.41) is the same whether the additional nonlinearity v_0^3 on its right hand side is present or not, this nonlinearity has to be taken into account when studying the solution on the entire interval $[0, \infty)$.

Lemma 3.32. *Let $\psi_1 = u_0 + \alpha_2 v_0$ and $\psi_2 = u_0 + \alpha_1 v_0$. Then for $i = 1, 2$, $\psi_i \geq 0$ everywhere.*

Proof. We give a proof for $i = 2$. A similar argument works when $i = 1$.

Step 1: Define $\psi_2 = u_0 + \alpha_1 v_0 = \mathbf{l}_2 \cdot \binom{u_0}{v_0}$. Then $\psi_2 \in C^1[0, \infty)$. Away from the intervals where u_0 is identically $0, \beta$, or 1, $u_0 \in C^\infty$ and (u_0, v_0) satisfies (3.41). Premultiplying (3.41) by $\mathbf{l}_2^T$ yields

$$\psi_2'' - \lambda_2 \psi_2 = -\frac{u_0^2(1 + \beta - u_0)}{d} + \alpha_1 v_0^3.$$

Let us subtract $\frac{1}{\alpha_1^2}\psi_2^3$ from both sides to get

$$\begin{aligned} \psi_2'' - \lambda_2 \psi_2 - \frac{1}{\alpha_1^2}\psi_2^3 &= -\frac{u_0^2(1 + \beta - u_0)}{d} + \alpha_1 v_0^3 - \frac{1}{\alpha_1^2}\psi_2^3 \\ &= -\frac{u_0^2(1 + \beta - u_0)}{d} + \alpha_1 v_0^3 - \frac{1}{\alpha_1^2}(u_0^3 + 3\alpha_1 u_0 v_0 \psi_2 + \alpha_1^3 v_0^3) \\ &= -\frac{u_0^2}{d}(1 + \beta - u_0) - \frac{u_0^3}{\alpha_1^2} - \frac{3 u_0 v_0}{\alpha_1}\psi_2, \end{aligned}$$

which is equivalent to

$$\psi_2'' - (\lambda_2 + \frac{1}{\alpha_1^2}\psi_2^2 - \frac{3u_0v_0}{\alpha_1})\psi_2 = -\frac{u_0^2}{d}(1 + \beta - (1 - \frac{d}{\alpha_1^2})u_0). \tag{3.46}$$

Since $d \le \frac{\beta^2}{4(1+\beta\gamma)} < \frac{\beta^2}{4}$ and $\frac{1}{\alpha_1} = \beta - d\lambda_1 < \beta$, we have $\frac{d}{\alpha_1^2} < \frac{\beta^4}{4} < 1$. Hence, $0 < 1 - \frac{d}{\alpha_1^2} < 1$. Together with $u_0 \le 1$, it is clear that the sign of the right hand side of (3.46) is non-positive. With $h(x) = \lambda_2 + \frac{1}{\alpha_1^2}\psi_2^2 - \frac{3u_0v_0}{\alpha_1}$,

$$\psi_2'' - h(x)\psi_2 \le 0. \tag{3.47}$$

We remark that

$$\begin{aligned} h(x) &= \lambda_2 + \frac{1}{\alpha_1^2}(\psi_2^2 - 3\alpha_1 u_0 v_0) \\ &= \lambda_2 + \frac{1}{\alpha_1^2}(u_0^2 + \alpha_1^2 v_0^2 - \alpha_1 u_0 v_0) \\ &> 0. \end{aligned}$$

Step 2: Suppose for contradiction $\psi_2 < 0$ somewhere. Define $b \equiv \sup\{x : \psi_2(x) < 0\}$, where $b = \infty$ is allowed. Since $\psi_2 \to 0$ as $x \to \infty$, it follows that $\psi_2(b) = 0$ or $\psi_2 \to 0$ if $b = \infty$. In either case, there exists a $b_1 \in (0, \infty)$ such that $\psi_2(b_1) := -t_0 < 0$ and $\psi_2'(b_1) := t_1 > 0$. We claim $\psi_2'(x) > t_1$ on $(0, b_1]$. Let a_1 be a point in $(0, b_1)$. First we verify that $\psi_2(x) < -t_0$ and $\psi_2'(x) > t_1$ on $[a_1, b_1]$ under three possibilities:

Case (A): Suppose that $u_0 \ne 0$, $u_0 \ne \beta$, and $u_0 \ne 1$ on $[a_1, b_1]$. Since ψ_2 satisfies (3.47) on $[a_1, b_1]$, it cannot attain a non-positive minimum on (a_1, b_1) by the maximum principle. Moreover, with $\psi_2'(b_1) > 0$, it follows from the Hopf lemma that $\psi_2' > 0$ on $[a_1, b_1]$. Therefore $\psi_2(x) < \psi_2(b_1) = -t_0$ on $[a_1, b_1)$. Putting this information back into (3.47), $\psi_2''(x) \le h(x)\psi_2(x) < -h(x)t_0$ for all $x \in [a_1, b_1)$; consequently, $\psi_2'(x) \ge t_0 \int_x^{b_1} h(\xi)\, d\xi + t_1 > t_1$.

Case (B): Suppose $u_0 = 1$ on $[a_1, b_1]$. Since $u_0(b_1) = 1$ and $u_0'(b_1) = 0$ by the corner lemma, it follows that $v_0(b_1) = -(t_0+1)/\alpha_1$ and $v_0'(b_1) = t_1/\alpha_1$. In view of

$$v_0'' - (\gamma + v_0^2)v_0 = -1 < 0 \quad \text{on } [a_1, b_1], \tag{3.48}$$

the maximum principle together with the Hopf lemma implies that $v_0(x) < v_0(b_1) = -(t_0+1)/\alpha_1$ on $[a_1, b_1)$. Then $v_0'' < -\{1 + (\gamma + v_0^2)(t_0+1)/\alpha_1\}$ and consequently $v_0'(x) > t_1/\alpha_1 + \int_x^{b_1} \{1 + (\gamma + v_0^2(\xi))(t_0+1)/\alpha_1\}\, d\xi$ for all $x \in [a_1, b_1)$. Combining with $u_0' = 0$ on $[a_1, b_1]$ yields $\psi_2'(x) > t_1 + \int_x^{b_1} \{\alpha_1 + (\gamma + v_0^2(\xi))(t_0+1)\}\, d\xi > t_1$ and therefore $\psi_2(x) < -t_0$.

Case (C): If $u_0 = \beta$ or $u_0 = 0$ on $[a_1, b_1]$, replacing 1 by β and 0, respectively, the calculation in Case (B) will do.

To finish our claim that $\psi_2'(x) > t_1$ for all $x \in (0, b_1]$, it suffices to show that $(0, b_1]$ is a finite combination of Cases (A)-(C). Suppose there is an accumulation point x_0 such that $x \downarrow x_0^+$ with one of Cases (A)-(C) occurs alternatively in adjacent subintervals of (x_0, b_1) or possibly a combination of such distributions. From what we have shown $\psi_2' \ge t_1$ on (x_0, b_1), so $\psi_2'(x_0) \ge t_1 > 0$ follows from $\psi_2 \in C^1[0, \infty)$. However, $u_0'(x_0) = v_0'(x_0) = 0$ by Lemma 3.29, which implies that $\psi_2'(x_0) = 0$. This is a contradiction. The same is true when x_0 is a limit point from the left. Hence there is no accumulation point, and therefore $\psi_2' \ge t_1 > 0$ on $(0, b_1)$. On the other hand, with $v_0(0) > 0$ from Lemma 3.31 and $\psi_2(0) < -t_0$, we see that $u_0(0) < 0$. This is a contradiction since it follows from $x_1 > 0$ proved in Lemma 3.22 that $u(0) > 0$. An alternative proof that does not require $u(0) > 0$ to be known is given in the following:

the continuity of ψ_2' implies that $\psi_2'(0) \geq t_1 > 0$. Since $u_0(0) \notin \{0, \beta, 1\}$, u_0 is smooth and satisfies (3.2) in a neighborhood $[0, \delta_1)$ of $x = 0$. By choosing a test function φ with support on $[0, \delta_1)$, a duplication of the proof of Lemma 3.6 shows that the natural boundary condition $u_0'(0) = 0$ is satisfied. Coupled with $v_0'(0) = 0$, it follows that $\psi_2'(0) = 0$, which is absurd. This completes the proof of $\psi_2 \geq 0$. □

Lemma 3.33. *If $d \leq d_1$, then $v_0 > 0$ everywhere. Moreover, if $u_0 \geq 0$ on $[0, x_2]$ and $u_0 \leq 0$ on $[x_2, \infty)$ for some x_2, then $v_0' < 0$ on $[x_2, \infty)$ and $v_0 \downarrow 0$ as $x \to \infty$. Once u_0 turns negative, then $u_0 < 0$ for all $x \in (x_2, \infty)$.*

Remark 3.34. The fact that u_0 changes sign at some $x_2 < \infty$ will be shown later in Lemma 3.42. The qualitative properties of (u_0, v_0) stated in Lemma 3.33 will therefore always hold.

Proof. It suffices to consider the case when u_0 changes the sign at $x = x_2$, otherwise Lemma 3.30 implies the positivity of v_0. Let us first consider the interval $[0, x_2]$, where

$$v_0'' - (\gamma + v_0^2)v_0 = -u_0 \leq 0.$$

By the maximum principle, v_0 cannot attain an interior non-positive minimum. Since $v_0(0) > 0$ and $v_0(x_2) = \psi(x_2)/\alpha_1 \geq 0$, it follows $v_0 > 0$ on $[0, x_2)$. We claim that $v_0(x_2) > 0$. For if not, the Hopf lemma implies $v_0'(x_2) < 0$, and thus $v_0 < 0$ on $(x_2, x_2 + \epsilon)$ for some $\epsilon > 0$. However from a different perspective,

$$v_0 = \frac{1}{\alpha_1}(\psi_2 - u_0) \geq 0 \quad \text{on } [x_2, \infty) \tag{3.49}$$

by using Lemma 3.32. Therefore $v_0 > 0$ on $[0, x_2]$.

Next consider the interval $[x_2, \infty)$. The maximum principle applied to

$$v_0'' - (\gamma + v_0^2)v_0 = -u_0 \geq 0 \tag{3.50}$$

implies v_0 cannot have an interior non-negative maximum. If v_0 touches 0, it cannot go back up since v_0 then attains a positive maximum as it decays to 0. Thus v_0 has to satisfy one of the following cases:

(A) v_0 decreases to 0 on $[x_2, \infty)$ with $v_0' < 0$ by the Hopf lemma, or

(B) $v_0(z_0) = 0$ for some $z_0 > x_2$, where z_0 is the first point at which v_0 touches 0.

To eliminate (B), we apply the Hopf lemma to (3.50) on $[z_0, \infty)$ and conclude that $v_0'(z_0) < 0$. This gives a rise to a contradiction since $v_0 \geq 0$ on $[x_2, \infty)$ as seen in (3.49).

The last statement is a consequence of the maximum principle applied to $du_0'' - h(u_0)u_0 \geq 0$ on (x_2, ∞) with $h(u) = (1-u)(\beta - u) \geq 0$ on the interval. □

The next corollary is an immediate consequence of the positivity of v_0 and Lemma 3.29.

Corollary 3.35. *The cases (P3), (Q3) and (R3) cannot occur.*

From Lemma 3.23 we have $x_1 > 0$. The above Corollary then implies either Case (P1) or (P2) happens near $x = 0$. In the former case, solution u_0 will be smooth near $x = 0$ so that the natural boundary condition $u_0'(0) = 0$ holds. For the latter case when $u_0 = 1$ in a neighborhood of $x = 0$, it is clear that $u_0'(0) = 0$. Thus we can conclude the followings.

Corollary 3.36. *The minimizer u_0 satisfies $u_0'(0) = 0$.*

With the new information $v_0 > 0$ everywhere, we in fact exclude the possibility that $u_0 = 1$ on any interval.

Lemma 3.37. *If* $d \leq d_1$ *then* $\beta_1 < \max u_0 < 1$.

Proof. To show $\max u_0 < 1$, suppose there exists an $x_0 \in [0, x_1)$ such that $u_0(x_0) = 1$. Without loss of generality we can assume $u_0 < 1$ on $(x_0, x_0 + \delta_1]$ for some small $\delta_1 > 0$. Consequently u_0 satisfies (3.2a) on $[x_0, x_0 + \delta_1]$. By making δ_1 smaller if needed, we set $h(x) := u_0(x)(u_0(x) - \beta) > 0$ on $[x_0, x_0 + \delta_1]$. This gives

$$d(u_0 - 1)'' - h(x)(u_0 - 1) = v_0 \geq 0.$$

By the maximum principle, $u_0 - 1$ cannot attain an interior non-negative maximum on $(x_0, x_0 + \delta_1)$ and moreover, the Hopf lemma dictates that $u_0'(x_0) < 0$. Thus $x_0 \neq 0$ by Corollary 3.36. With $x_0 > 0$, we see that $u_0 > 1$ in some interval $[x_0 - \delta_2, x_0)$ for some small $\delta_2 > 0$. This is a contradiction.

Lastly, we need $\int_0^\infty F(u_0)\, dx < 0$ for $J(u_0) < 0$ since all other terms are positive. It must hold that $\max u_0 > \beta_1$. □

3.6 On the constraints imposed by the admissible set

At the moment we have not eliminated the possibilities of Cases (Q2) and (R2), i.e. there may exist intervals on which $u_0 = \beta$ or $u_0 = 0$. As a consequence x_1 and x_2 as defined in the admissible set $\mathcal{A}$ may not be unique. Let $\{x_2\}$ denote the set of points that represent any x_2. With the established qualitative properties of (u_0, v_0), we are ready to show that there are no intervals on which u_0 is identical to 0; to be more precise the set $\{x_2\}$ has only 1 point at which u_0 changes sign. The truncation argument will serve as the key tool for the proofs in this section.

Lemma 3.38. *Suppose we trim u_0 on a compact support such that $u_{new} \in \mathcal{A}$ with $u_{new} \leq u_0$. If $\|u_{new} - u_0\|_{L^\infty(0,\infty)} = O(\epsilon)$ for some $\epsilon > 0$, then for ϵ sufficiently small, the nonlocal energy decreases as well. That is,*

$$\int_0^\infty \left\{ \left(\frac{1}{2} u_{new} \mathcal{N} u_{new} + \frac{1}{4} (\mathcal{N} u_{new})^4 \right) - \left(\frac{1}{2} u_0 \mathcal{N} u_0 + \frac{1}{4} (\mathcal{N} u_0)^4 \right) \right\} dx < 0 \, .$$

Proof. With $u_{new} - u_0 \leq 0$, Lemma 3.10 gives $\mathcal{N} u_{new} \leq \mathcal{N} u_0 = v_0$. Let $I = \{x : u_{new} - u_0 \neq 0\}$ have a compact support. If ϵ is sufficiently small, by continuity $\mathcal{N} u_{new}$ will remain positive on I. Then it follows from Lemma 3.24 and Lemma 3.16 that

$$\begin{aligned}
&\int_0^\infty \left\{ \left(\frac{1}{2} u_{new} \mathcal{N} u_{new} + \frac{1}{4} (\mathcal{N} u_{new})^4 \right) - \left(\frac{1}{2} u_0 \mathcal{N} u_0 + \frac{1}{4} (\mathcal{N} u_0)^4 \right) \right\} dx \\
&= \frac{1}{2} \int_I (u_{new} - u_0)(\mathcal{N} u_{new} + \mathcal{N} u_0)\, dx + \frac{1}{4} \int_0^\infty (\mathcal{N} u_{new} + \mathcal{N} u_0)(\mathcal{N} u_{new} - \mathcal{N} u_0)^3\, dx \\
&\leq \frac{1}{2} \min_{x \in I} v_0(x) \int_I (u_{new} - u_0)\, dx + \frac{(M_1 + 1)}{2} \int_0^\infty |\mathcal{N} u_{new} - \mathcal{N} u_0|^3\, dx \\
&= -\frac{1}{2} \min_{x \in I} v_0(x) \|u_{new} - u_0\|_{L^1(0,\infty)} + \frac{(M_1 + 1)}{2} \|\mathcal{N} u_{new} - \mathcal{N} u_0\|^3_{L^3(0,\infty)} \\
&\leq -\frac{1}{2} \min_{x \in I} v_0(x) \|u_{new} - u_0\|_{L^1(0,\infty)} + \frac{C_0(M_1 + 1)}{2} \|\mathcal{N} u_{new} - \mathcal{N} u_0\|^3_{H^1(0,\infty)}
\end{aligned}$$

for some positive constant C_0, where the last inequality follows from the Sobolev embedding $H^1(0,\infty) \hookrightarrow L^3(0,\infty)$. Finally by applying Lemma 3.11, we obtain

$$\begin{aligned}
&\int_0^\infty \left\{ \left(\frac{1}{2} u_{new} \mathcal{N} u_{new} + \frac{1}{4} (\mathcal{N} u_{new})^4 \right) - \left(\frac{1}{2} u_0 \mathcal{N} u_0 + \frac{1}{4} (\mathcal{N} u_0)^4 \right) \right\} dx \\
&\leq -\frac{1}{2} \min_{x \in I} v_0(x) \|u_{new} - u_0\|_{L^1(0,\infty)} + \frac{C_0(M_1+1)}{2} \max\{1, 1/\gamma^3\} \|u_{new} - u_0\|^3_{L^2(0,\infty)} \\
&\leq -\frac{1}{2} \min_{x \in I} v_0(x) \|u_{new} - u_0\|_{L^1(0,\infty)} + \sqrt{2}\, C_0 (M_1+1)^{5/2} \max\{1, 1/\gamma^3\} \|u_{new} - u_0\|^{3/2}_{L^1(0,\infty)} \\
&< 0 \qquad (3.51)
\end{aligned}$$

for sufficiently small ϵ, as $\|u_{new} - u_0\|_{L^1(0,\infty)}$ can be made arbitrarily small. □

When we refer to d_1 in the following lemmas, we understand that it depends on γ, i.e. $d_1 = d_1(\gamma)$.

Lemma 3.39. *Suppose $\gamma \leq \gamma_0$ and $d \leq d_1$. Take the largest x_1 so that $u_0 < \beta$ on some small neighborhood $(x_1, x_1 + \delta]$ and, if $\{x_2\}$ is non-empty, take the smallest x_2 such that $u_0 > 0$ on some neighborhood $[x_2 - \delta, x_2)$. Then $u_0' < 0$ on the interval $[x_1, x_2)$; the same is true on $[x_1, \infty)$ if $\{x_2\}$ is empty. Moreover if u_0 changes sign at x_2, then $u_0'(x_2) < 0$ and $u_0 < 0$ on (x_2, ∞).*

Remark 3.40. We will establish in Lemma 3.41 and Lemma 3.42 that u_0 changes sign at a unique $x_2 < \infty$; therefore u_0 will satisfy all the qualitative properties stated in Lemma 3.39.

Proof. Suppose $\{x_2\}$ is nonempty and there exist $x_1 < y_1 < y_2 < x_2$ such that $0 < u_0(y_1) < u_0(y_2)$. Since $u_0(x_2) = 0$, a local maximum of u_0 is attained between y_1 and x_2, thereby creating a hump. The top of the hump can even go up all the way and form an interval on which $u_0 = \beta$. Take a small positive ϵ and let

$$u_{new}(x) = \begin{cases} u_0(x) - \epsilon, & \text{if } x \geq y_1 \text{ and } u_0(x) \geq \max_{[y_1,x_2]} u_0 - \epsilon, \\ u_0(x), & \text{otherwise.} \end{cases}$$

In other words we trim a small height ϵ from the top of the hump and obtain a $u_{new} \in \mathcal{A}$. Upon trimming, it is clear that the gradient energy decreases. As $F(\xi)$ is strictly monotone increasing for $\xi \in [0, \beta]$, the potential energy also decreases. Finally since $u_{new} - u_0$ has a compact support, the nonlocal energy decreases as well by Lemma 3.38. These lead to $J(u_{new}) < J(u_0)$, contradicting u_0 being a minimizer. It forces us to conclude that u_0 is non-increasing on $[x_1, x_2]$. By Lemma 3.26, $u_0 \in C^\infty[x_1, x_2]$ and satisfies (3.2a) on the interval. Since $du_0'' = v_0 - f(u_0) > 0$ on $[x_1, x_2]$, the Hopf lemma implies that $u_0' < 0$ on $[x_1, x_2)$. If $\{x_2\}$ is empty, the same argument still works if we take $x_1 < y_1 < y_2 < \infty$. This leads to $u_0' < 0$ on $[x_1, \infty)$ in this case.

For finite x_2, only one of the followings will happen:

(a) u_0 becomes negative on $(x_2, x_2 + \delta]$ for some finite $\delta > 0$,

(b) $u_0 = 0$ on $[x_2, x_2 + \delta]$ and $u_0 < 0$ on $(x_2 + \delta, x_2 + \delta + \delta_1]$ for some positive δ and δ_1, or

(c) $u_0 = 0$ on $[x_2, \infty)$.

Assume u_0 changes sign at x_2. Then we need to consider only cases (a) and (b). Suppose case (b) occurs. Let $h(x) = -(u_0 - \beta)(1 - u_0)$, which is positive on $[x_2 + \delta, \infty)$. Since $du_0'' - h(x)u_0 = v_0 > 0$ on the interval, we apply the Hopf lemma to conclude that $u_0'(x_2 + \delta) < 0$. This contradicts the result from the corner lemma that $u_0'(x_2 + \delta) = 0$. Therefore only case (a) holds and $u_0'(x_2) < 0$ follows from the Hopf lemma on $[x_2, x_2 + \delta]$. The same Hopf lemma argument will also prevent u_0

from touching zero again on (x_2, ∞), which leads us to conclude that $u_0 < 0$ on (x_2, ∞). $\square$

Next we eliminate case (c) in the above proof.

Lemma 3.41. *Suppose $\gamma \leq \gamma_0$ and $d \leq d_1$. Whether u_0 changes sign or not, there cannot be an interval $[a, b]$ where $u_0 = 0$.*

Proof. In view of Lemma 3.39, it suffices to eliminate case (c) in its proof. Let $u_0 = 0$ on a finite interval $[a, b] \subset [x_2, \infty)$. Take $\epsilon > 0$ small and define

$$u_{new}(x) = \begin{cases} -\epsilon \sin\left(\frac{\pi(x-a)}{b-a}\right), & \text{if } x \in [a, b], \\ u_0, & \text{otherwise.} \end{cases}$$

It is clear that $u_{new} \in \mathcal{A}$. Let us set $v_0 = \mathcal{N} u_0$ and $v_{new} = \mathcal{N} u_{new}$. With $\|u_{new} - u_0\|_{L^1(0,\infty)} = \frac{2(b-a)}{\pi}\epsilon$, it follows from the calculation in (3.51) that the change in nonlocal energy is given by

$$\begin{aligned} &\int_0^\infty \left\{ \left(\frac{1}{2} u_{new} v_{new} + \frac{1}{4} v_{new}^4 \right) - \left(\frac{1}{2} u_0 v_0 + \frac{1}{4} {v_0}^4 \right) \right\} dx \\ &\leq -\min_{x \in I} v_0(x) \frac{(b-a)}{\pi} \epsilon + \sqrt{2}\, C_0 (M_1 + 1)^{5/2} \max\{1, 1/\gamma^3\} \left(\frac{2(b-a)}{\pi} \epsilon \right)^{3/2}. \end{aligned}$$

Since $0 \leq F(\xi) \leq (1 + \beta)\xi^2/2$ for small ξ, together with $u_0' = 0$ and $F(u_0) = 0$ on $[a, b]$, we obtain

$$\begin{aligned} J(u_{new}) - J(u_0) &= \int_a^b \left\{ \frac{d}{2} u_{new}'^2 + F(u_{new}) \right\} dx \\ &\quad + \int_0^\infty \left\{ \left(\frac{1}{2} u_{new} \mathcal{N} u_{new} + \frac{1}{4} (\mathcal{N} u_{new})^4 \right) - \left(\frac{1}{2} u_0 \mathcal{N} u_0 + \frac{1}{4} (\mathcal{N} u_0)^4 \right) \right\} dx \\ &= O(\epsilon^2) - C\epsilon + O(\epsilon^{3/2}) \end{aligned}$$

for some positive constant C. Then $J(u_{new}) < J(u_0)$ if we choose ϵ sufficiently small, but this contradicts the fact that u_0 is a minimizer. □

Lemma 3.42. *If $\gamma \leq \gamma_0$ and $d \leq d_1$, then*

(i) u_0 has a slow decay at $+\infty$;

(ii) u_0 changes sign.

Proof. By Lemmas 3.39 and 3.41, there exists some large $y_1 > 0$ such that u_0 vanishes to 0 and $u_0 \neq 0$ on $[y_1, \infty)$. Therefore, we can study the behavior of (u_0, v_0) near $+\infty$ from the linearization in (3.45). If u_0 has a fast decay at $+\infty$, then $\begin{pmatrix} u_0 \\ v_0 \end{pmatrix} \sim C_2 e^{-\sqrt{\lambda_2}x}\mathbf{b}$ with $C_2 \neq 0$. Therefore $\psi_2 = \mathbf{l_2} \cdot \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} \sim C_2 e^{-\sqrt{\lambda_2}x}\mathbf{l_2} \cdot \mathbf{b}$. Since $\psi_2 \geq 0$ by Lemma 3.32 and $\mathbf{l}_2 \cdot \mathbf{b} < 0$ by (3.43), it follows that $C_2 < 0$. Recall that $\mathbf{b} = (-\alpha_2, 1)^T$, where $\alpha_2 = \beta/d - \lambda_1 > 0$. Then $u_0 > 0$ and $v_0 < 0$ at large x, which contradicts the positivity of v_0. The proof of (a) is now complete.

With known slow decay, $\begin{pmatrix} u_0 \\ v_0 \end{pmatrix} \sim C_1 e^{-\sqrt{\lambda_1}x}\mathbf{a}$ with $C_1 \neq 0$. Taking inner product with $\mathbf{l_1}$ yields $\psi_1 \sim C_1 e^{-\sqrt{\lambda_1}x}\mathbf{l_1} \cdot \mathbf{a}$. It follows again from (3.43) and the positivity of v_0 that $C_1 > 0$. With $\mathbf{a} = (-1, d\alpha_2)^T$, it is clear that u_0 is negative at large x. Therefore, u_0 must change sign at some finite x_2. □

The above lemma eliminates case (c) in the proof of Lemma 3.39. As a consequence we have the following corollary.

Corollary 3.43. *Suppose $\gamma \leq \gamma_0$ and $d \leq d_1$. Let $x_1 = \inf\{y : u_0(x) < \beta$ if $x \in (y, \infty)\}$. Then the minimizer $u_0 \in C^\infty[x_1, \infty)$. In fact u_0 changes sign and* *satisfies (3.2a) on this interval. Moreover the set* $\{x_2\}$ *contains a single point, i.e.* u_0 *crosses* 0 *at only one point.*

In this section, we establish that (u_0, v_0) is the standing pulse solution of (3.2) by ruling out the possibility that u_0 equals to β on any interval. We exploit the fact that (3.2) is a Hamiltonian system with

$$\frac{1}{2}v_0'^2 - \frac{\gamma}{2}v_0^2 - \frac{1}{4}v_0^4 + u_0v_0 - \frac{d}{2}u_0'^2 + F(u_0) = 0$$

and that this identity is still valid on $(-\infty, \infty)$ even when $u_0 = \beta$ on an interval where (3.2) fails. Note that in the event of such an interval exists, u_0 may not be C^2 at the boundary points of the interval.

Lemma 3.44. *Even if there are intervals where* $u_0 = \beta$*,* (u_0, v_0) *satisfies*

$$\frac{1}{2}v_0'^2 - \frac{\gamma}{2}v_0^2 - \frac{1}{4}v_0^4 + u_0v_0 - \frac{d}{2}u_0'^2 + F(u_0) = 0. \tag{3.52}$$

Proof. It can be seen from the linearization of (u_0, v_0) that both u_0 and v_0 die down exponentially as $x \to \infty$. Since u_0'' and v_0'' are bounded, the standard interpolation theorem implies that u_0' and v_0' also die down exponentially.

On the interval $[x_1, \infty)$, we multiply (3.2a) by $-u_0'$ and (3.2b) by v_0', sum the resulting equations, and then integrate to obtain

$$\frac{1}{2}v_0'^2 - \frac{\gamma}{2}v_0^2 - \frac{1}{4}v_0^4 + u_0v_0 - \frac{d}{2}u_0'^2 + F(u_0) = \text{constant}. \tag{3.53}$$

By taking the limit as $x \to \infty$, the integration constant is clearly zero as in (3.52). This continues to be valid from the right until $u_0 = \beta$ on an interval $[a, b]$ with $b \leq x_1$

when (3.2a) fails to hold. Note that $u_0'(b^+) = u_0'(b^-) = 0$ by the corner lemma. On $[a, b]$ where $u_0 = \beta$, it follows from (3.2b) that $v_0'' - \gamma v_0 - v_0^3 + \beta = 0$. Therefore

$$\frac{d}{dx}\left(\frac{1}{2}v_0'^2 - \frac{\gamma}{2}v_0^2 - \frac{1}{4}v_0^4 + \beta v_0\right) = 0,$$

which implies that the left-hand side of (3.53) does not change on $[a, b]$. Since (3.52) holds at $x = b$, this constraint continues to be valid on $[a, b]$. Once $x < a$ with $u_0 > \beta$, both (3.2a) and (3.2b) are satisfied so that (3.53) holds. Then evaluating (3.53) at $x = a$ implies that the integration constant is zero. Hence, (3.52) holds everywhere. □

Lemma 3.45. *Let $\beta \in (1/3, 1/2)$ (so that $\gamma_0 > 0$). Suppose $\gamma < \gamma_1 \equiv \min\{\gamma_0, 2(\beta + F(\beta)) - 1/2\}$ and $d \leq d_1$. Then there cannot be an interval $[a, b] \subset [0, x_1]$ where $u_0 = \beta$. In fact there is no point at which $u_0 = \beta$ and $u_0' = 0$, and the set $\{x_1\}$ has only a unique point.*

Proof. Assume there is an interval $[a, b] \subset [0, x_1]$ such that $u_0 = \beta$. By the corner lemma, $u_0'(b) = 0$. Since $v_0 \leq 1$ as shown in Lemma 3.16 and $F(\beta) = (2\beta^3 - \beta^4)/12 > 0$, on evaluating (3.52) at $x = b$ we obtain

$$\begin{aligned}\frac{1}{2}(v_0'(b))^2 &= v_0(b)\left\{\frac{\gamma}{2}v_0(b) + \frac{1}{4}v_0^3(b) - \beta\right\} - F(\beta)\\ &\leq v_0(b)\left\{\frac{\gamma}{2}v_0(b) + \frac{1}{4}v_0^3(b) - \beta - F(\beta)\right\}\\ &< 0\end{aligned}$$

when $\frac{\gamma}{2}v_0(b) + \frac{1}{4}v_0^3(b) - \beta - F(\beta) \leq \frac{\gamma}{2} + \frac{1}{4} - \beta - F(\beta) < 0$. But $\frac{1}{2}(v_0'(b))^2 < 0$ is absurd, and thus no such interval $[a, b]$ exists. It is clear from the proof that there is no point at which $u_0 = \beta$ and $u_0' = 0$. As a consequence, the point x_1 is unique. □

At this point, we have completely removed the possibility that u_0 equals to one of the constraints imposed on $\mathcal{A}$. By the regularity estimates u_0 and v_0 are $C^\infty[0,\infty)$ functions. Extending them to be even functions on $(-\infty,\infty)$, we conclude that (u_0, v_0) is a standing pulse solution to (3.2) satisfying $\lim_{|x|\to\infty}(u_0, v_0) = (0,0)$. This finishes the proof of Theorem 3.1. We note that a plot of γ_1 in Lemma 3.45 versus β has been represented in Figure 3.1. A better estimate can make γ_1 larger.

Various qualitative properties of u_0 have already been investigated in the previous lemmas. To finish the proof of Theorem 3.2, we show the following qualitative property of u_0.

Lemma 3.46. *Suppose $\gamma < \gamma_1$ and $d \le d_1$. Then u_0 has a unique negative local minimum point on $[0,\infty)$ which is also the global minimum point.*

Proof. Recall from Lemma 3.33 that $v_0' < 0$ on $[x_2,\infty)$. Then $du_0''' + f'(u_0)u_0' = v_0' < 0$ on $[x_2,\infty)$. Since $u_0 \le 0$ on $[x_2,\infty)$, we have $f'(u_0) \le 0$ on that interval.

Next from Lemma 3.42 we know that $u_0 \sim -C_1 e^{-\sqrt{\lambda_1}x}$ at $+\infty$ which implies that $u_0' > 0$ at some large x. Since u_0' cannot attain a non-positive minimum on $[x_2,\infty)$ by the maximum principle, u_0' has to increase from a negative value at x_2 to 0 at some $y_0 \in (x_2,\infty)$. Moreover $u_0'(y_0) > 0$ by using the Hopf lemma on $[y_0,\infty)$, and once u_0' turns positive it cannot become negative again. Correspondingly u_0 decreases from 0 at x_2 to a negative local minimum at y_0, and then increases to 0 as $x \to \infty$. Hence u_0 has a unique negative local minimum at $x = y_0$ on the interval $[x_2,\infty)$, which then is the global minimum of u_0 on the entire interval $[0,\infty)$. □

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