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Gaussian Limits and Polynomials on High Dimensional Spheres

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Gaussian Limits and Polynomials on High Dimensional Spheres

Amy Peterson, Ph.D.

University of Connecticut, 2019

ABSTRACT

We show in detail that the limit of spherical surface integrals taken over slices of a high dimensional sphere is a Gaussian integral on an affine plane of finite codimension in infinite dimensional space. We then utilize these ideas to show that a natural class of orthogonal polynomials on high dimensional spheres limit to Hermite polynomials.

Gaussian Limits and Polynomials on High Dimensional Spheres

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M.S. Mathematics Auburn University

B.S. Mathematics Auburn University

A Dissertation

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Doctor of Philosophy

at the

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APPROVAL PAGE

Doctor of Philosophy Dissertation

Gaussian Limits and Polynomials on High Dimensional Spheres

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Chapter 1

Introduction

This thesis has two sections to it. First, in chapters 2 through 6, we show the Gaussian limit of certain high dimensional spherical integrals. Second, in chapter 7, we prove results about polynomials on spheres that stem from the ideas in chapters 2 through 5 where we are looking at a polynomial basis for the space of L^2 functions on the sphere and seeing that in a suitable sense they converge to Hermite polynomials.

In chapter 2, we begin by introducing the following spherical integrals and the main theorem proving their limit is an infinite dimensional Gaussian integral. Let A be a closed affine plane of finite codimension m in l^2 . We take \mathbb{R}^N to be a subspace of l^2 by identifying it with the subspace $\mathbb{R}^N \times \{0\}$. Now let $A_N = A \cap \mathbb{R}^N$ and $S^{N-1}(\sqrt{N}) \subset \mathbb{R}^N$ be the sphere of radius \sqrt{N} , centered at the origin. Then $S_{A_N} = A_N \cap S^{N-1}(\sqrt{N})$ is the circle formed by slicing the sphere $S^{N-1}(\sqrt{N})$ with the plane A_N . (Refer to Figure 1.0.1 for an example.) Let σ be the standard surface area measure on a sphere and $\bar{\sigma}$ be the standard surface area measure normalized to have unit total mass. Now let $\pi_{(k)} : l^2 \rightarrow \mathbb{R}^k$ be the coordinate projection $z \mapsto z_{(k)} = (z_1, \dots, z_k)$.

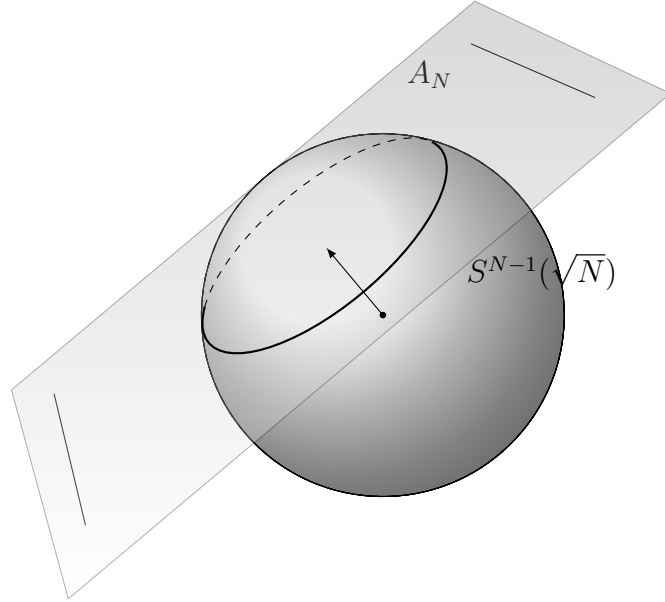


FIGURE 1.0.1: The plane A_N slicing the sphere $S^{N-1}(\sqrt{N})$

The main result proved in chapters 2 through 5 is the following theorem:

Theorem 1.0.1. *Let A be a finite-codimension affine subspace in l^2 . Let k be a positive integer and suppose that the image of A under the coordinate projection $\pi_{(k)}$ is all of \mathbb{R}^k . Let ϕ be a bounded Borel function on \mathbb{R}^k . Then*

$$\lim_{N \rightarrow \infty} \int_{S_{A_N}} \phi(x_1, \dots, x_k) d\bar{\sigma}(x_1, \dots, x_N) = \int_{\mathbb{R}^k} \phi(z_{(k)}) d\mu(z), \quad (1.0.1)$$

where $\bar{\sigma}$ is the normalized standard surface area measure on S_{A_N} and μ is the probability measure on \mathbb{R}^k specified by the characteristic function

$$\int_{\mathbb{R}^k} \exp(i\langle t, x \rangle) d\mu(x) = \exp\left(i\langle t, z^0 \rangle - \frac{1}{2} \|P_0 t\|^2\right) \quad \text{for all } t \in \mathbb{R}^k, \quad (1.0.2)$$

where z^0 is the point on A closest to the origin and P_0 is the orthogonal projection in l^2 onto the subspace $A - z^0$.

To prove Theorem 2.1.1 we construct, in chapter 2, an important disintegration formula for spherical integrals. This is followed in chapter 3 with an explanation of many of the geometric elements involved in the proof of Theorem 2.1.1 and certain related lemmas including an important theorem for limits of projection operators. Then in chapter 4 we proceed to complete the proof of Theorem 2.1.1 utilizing the results from chapter 2 and chapter 3. In chapter 6 we will discuss and prove a version of this result in terms of a more general Banach and Hilbert space.

In chapter 6 we continue to investigate the relationship between spherical integration and Gaussian integration. First we note the relationship between spherical integration over $S^{N-1}(\sqrt{N})$ with respect to the spherical surface area measure and Gaussian integration over \mathbb{R}^N with respect to standard Gaussian measure on \mathbb{R}^N for polynomials. This leads to the following theorem relating the respective inner products:

Proposition 1.0.2. *Let f and g be homogeneous Borel functions on \mathbb{R}^N , of degrees d_f and d_g , respectively, and square-integrable with respect to standard Gaussian measure μ . Then*

$$\langle f, g \rangle_{L^2(\mathbb{R}^N, \mu)} = a_{d,N} \langle f, g \rangle_{L^2(S^{N-1}(\sqrt{N}), \bar{\sigma})} \quad (1.0.3)$$

where $d = (d_f + d_g)/2$. If $d_f + d_g$ is odd then both sides in (6.2.8) are 0, and where

$$a_{d,N} = \prod_{j=1}^d \left(1 + 2 \left(\frac{j-1}{N} \right) \right). \quad (1.0.4)$$

Following from this relationship between inner products if we take polynomial functions p and q defined on k variables for some $k < N$ then we can take the

limit as $N \rightarrow \infty$ of each side of (1.0.3). This will result in the limit of the inner product on $L^2(S^{N-1}(\sqrt{N}), \bar{\sigma})$ being the inner product on $L^2(\mathbb{R}^\infty, \mu)$ for polynomials of k variables.

For \mathcal{P}_k^d , the space of polynomials of k variables and total degree d , there is a natural basis coming from the monomials that have degree d . Now we can remove the projection onto lower degree polynomial spaces of monomials of degree d to create an orthogonal basis of polynomials of $\mathcal{P}_k^{\leq d}$ in the sense of Gram-Schmidt orthogonalization.

Using these spherical inner products and Gaussian inner product we can construct related orthogonal projections from the space of polynomials of k variables and total degree $\leq d$, which we denote as $\mathcal{P}_k^{\leq d}$. Let $\Pi^{\leq d}$ be the orthogonal projection, for all $d \geq 1$ and $k \geq 1$, from $\mathcal{P}_k^{\leq d}$ to $\mathcal{P}_k^{\leq d-1}$ associated to the Gaussian integral inner product in \mathbb{R}^k . Let

$$\dot{\Pi}_{k,N}^{\leq d} : \mathcal{P}_k^{\leq d} \rightarrow \mathcal{P}_k^{\leq d-1} \quad (1.0.5)$$

be the orthogonal projection using the inner-product $\langle \cdot, \cdot \rangle_N = \langle \cdot, \cdot \rangle_{L^2(S^{N-1}(\sqrt{N}), \bar{\sigma})}$. Now for monomials on the sphere we have

Proposition 1.0.3. *For $\dot{\Pi}_{k,N}^{\leq d}$ defined above, and $j_1 + \dots + j_k = d$,*

$$\lim_{N \rightarrow \infty} (I - \dot{\Pi}_{k,N}^{\leq d})(X_1^{j_1} \dots X_k^{j_k}) = H_{j_1}(X_1) \dots H_{j_k}(X_k). \quad (1.0.6)$$

1.0.1 Related Literature

In this section we briefly mention several works that have influenced the areas of study related to this dissertation. First the relationship between spherical surface

area measures and Gaussian measure has many origins the most notable being in the study of gas molecules in physics by Maxwell [12] and Boltzmann [2] and in its mathematical form by Wiener [17], Levy [11], and Hida [7].

The study of Gaussian measures on infinite dimensional spaces has a rich history and we reference here two important texts on the topic by Kuo [10] and Bogachev [1]. We also note that Hertle [5], [6] studied the Radon transform with respect to surface measures on spheres in infinite dimensions but using a different framework than the one we use here.

Next we note that many of the results in this paper have been published in [14] this paper is the third in a sequence of papers. The first [9] develops the Gaussian Radon Transform for Banach spaces, the second [15] shows the Gaussian Radon Transform results as a limit of high dimensional spherical integrals on a hyperplane, and the third [14], one of the topics of this dissertation, shows that the results of [15] hold for an affine subspace of any finite codimension.

In 1866, Mehler [13] showed the limit of spherical polynomials was Hermite polynomials and Umemura and Kono [16] in 1965, following work done by Hida and Nomoto [8], show spherical polynomials limit to Hermite polynomials using the limiting behavior of the spherical Laplacian. We also discuss in this dissertation the limit of monomials on the sphere giving Hermite polynomials but using only projections and a limit of inner products.

Chapter 2

Gaussian Limit of High Dimensional Spherical Means

2.1 Introduction

To begin our study of the Gaussian limit of high dimensional spherical integrals we define some important notation and state one of the main results of this dissertation that will be proven throughout chapters 2 through 5. Let \mathbb{R}^∞ be the space of all real sequences $(x_n)_{n \geq 1}$ and l^2 be the subspace of \mathbb{R}^∞ consisting of all sequences $(x_n)_{n \geq 1}$ whose standard l^2 norm, $(\sum_{n \geq 1} x_n^2)^{1/2}$, is finite. Let A be a closed affine space of finite codimension m in l^2 . We take \mathbb{R}^N to be a subspace of l^2 by identifying it with the subspace $\mathbb{R}^N \times \{0\}$. Now let $A_N = A \cap \mathbb{R}^N$ and $S^{N-1}(\sqrt{N}) \subset \mathbb{R}^N$ be the sphere of radius \sqrt{N} , centered at the origin. Then

$$S_{A_N} = A_N \cap S^{N-1}(\sqrt{N})$$

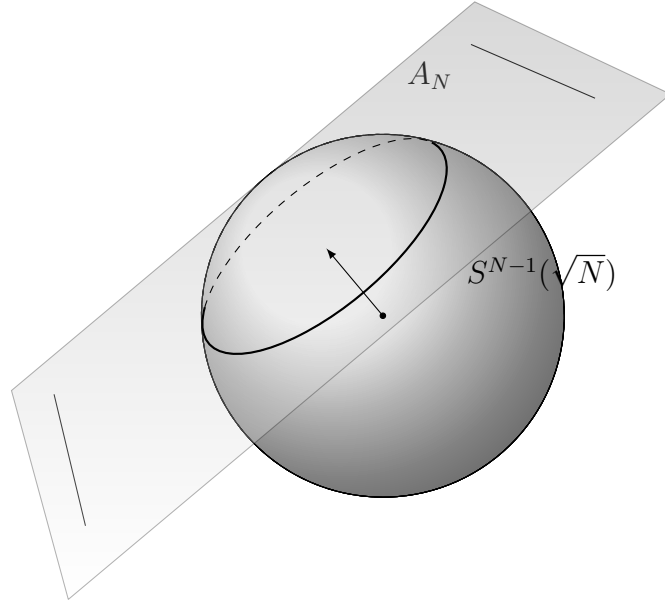


FIGURE 2.1.1: The plane A_N slicing the sphere $S^{N-1}(\sqrt{N})$

is the circle formed by slicing the sphere $S^{N-1}(\sqrt{N})$ with the plane A_N . (Refer to Figure 2.1.1 for an example.) Let σ be the standard surface area measure on a sphere. There are many ways to define the spherical surface measure but we take the very simple approach of defining it using the ambient space \mathbb{R}^N and its Lebesgue measure.

For any set $E \in \mathcal{B}(S^n(a))$ then we define

$$\sigma_n(E) = \frac{n+1}{a} \lambda_{n+1}(C_E)$$

where $S^n(a)$ is the sphere in \mathbb{R}^{n+1} centered at the origin of radius a and λ_{n+1} is the Lebesgue measure on \mathbb{R}^{n+1} . And C_E is the cone formed with base E and point at the origin:

$$C_E = \bigcup_{t \in [0,1]} tE.$$

We will further take $\bar{\sigma}$ to be the normalization of σ so that it has unit total mass by dividing it by the total surface area of the sphere.

Now let $\pi_{(k)} : l^2 \rightarrow \mathbb{R}^k$ be the coordinate projection $z \mapsto z_{(k)} = (z_1, \dots, z_k)$.

The main result is the following theorem:

Theorem 2.1.1. *Let A be a finite-codimension affine subspace in l^2 . Let k be a positive integer and suppose that the image of A under the coordinate projection $\pi_{(k)}$ is all of \mathbb{R}^k . Let ϕ be a bounded Borel function on \mathbb{R}^k . Then*

$$\lim_{N \rightarrow \infty} \int_{S_{A_N}} \phi(x_1, \dots, x_k) d\bar{\sigma}(x_1, \dots, x_N) = \int_{\mathbb{R}^k} \phi(z_{(k)}) d\mu(z), \quad (2.1.1)$$

where $\bar{\sigma}$ is the normalized standard surface area measure on S_{A_N} and μ is the probability measure on \mathbb{R}^k specified by the characteristic function

$$\int_{\mathbb{R}^k} \exp(i\langle t, x \rangle) d\mu(x) = \exp\left(i\langle t, z^0 \rangle - \frac{1}{2} \|P_0 t\|^2\right) \quad \text{for all } t \in \mathbb{R}^k, \quad (2.1.2)$$

where z^0 is the point on A closest to the origin and P_0 is the orthogonal projection in l^2 onto the subspace $A - z^0$.

It is important to assume that the image of $\pi_{(k)}(A)$ is all of \mathbb{R}^k because the left side of (2.1.1) ϕ is only evaluated on the image of $\pi_{(k)}(A)$.

To prove Theorem 2.1.1 we begin by constructing, in the next chapter, an important disintegration formula for spherical integrals.

2.2 Spherical Disintegration

To prove our main result we will need a spherical disintegration formula that we will build up through a series of theorems in this section. These proofs appear in [14] and [15]. We state without proof the following scaling property:

$$\int_{S^d(r)} f d\sigma = (r/a)^d \int_{S^d(a)} f((r/a)z) d\sigma(z), \quad (2.2.1)$$

whenever either side exists, where $S^d(t)$ denotes the sphere of radius t and center 0 in \mathbb{R}^{d+1} . We will also use the following polar disintegration formula:

$$\int_{\mathbb{R}^{d+1}} f dx = \int_{r \in (0, \infty)} \left[\int_{S^d(r)} f d\sigma \right] dr. \quad (2.2.2)$$

Proofs of both formulas can be found in [15].

Let V be a finite-dimensional real inner-product space and $S_V(a)$ be the sphere in V of radius $a > 0$, centered at the origin. Let W be a proper subspace of V .

Theorem 2.2.1. *Let f be a non-negative or bounded Borel function on the sphere $S_V(a)$ and $P : V \rightarrow V$ be the orthogonal projection onto the subspace W . Then*

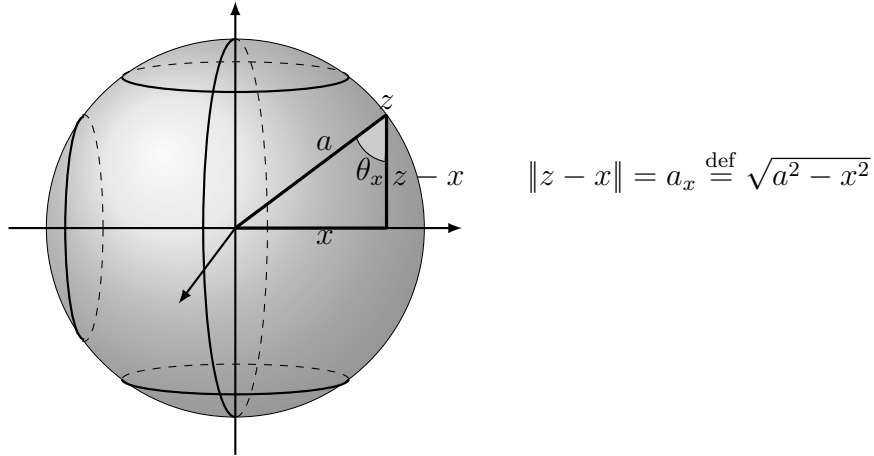
$$\int_{S_V(a)} f d\sigma = \int_{B_W(a)} \left[\int_{S_V(a) \cap P^{-1}(x)} f d\sigma \right] \frac{a}{a_x} dx, \quad (2.2.3)$$

where

$$a_x = \sqrt{a^2 - \|x\|^2}, \quad (2.2.4)$$

and $B_W(a)$ is the open ball of radius a , and centered at 0, in W .

Thinking geometrically, the slice $S_V(a) \cap P^{-1}(x)$ is a circle whose radius is a_x . To

FIGURE 2.2.1: Illustration for a_x and θ_x

see this, let $z \in S_V(a) \cap P^{-1}(x)$ then

$$z = Pz + z - Pz = x + (I - P)z$$

since $(I - P)z \in \ker P$ it is orthogonal to $\text{Im}(P)$ and therefore to x . This gives us

$$\|z - x\|^2 = \|z\|^2 \|x\|^2 = a_x^2$$

and shows that $z \in S_V(a) \cap P^{-1}(x)$ lies at fixed distance a_x from x . Further

$$\frac{a}{a_x} = \frac{\|z\|}{\|z - x\|} = \frac{1}{\cos \theta_x}$$

where θ_x is the angle between the vector from x to z and the vector z itself. This is illustrated in Figure 2.2.1. We proceed to the proof of Theorem 2.2.1.

Proof. We assume that $f \geq 0$; all other cases follow by taking real and imaginary parts if f is complex-valued, and positive and negative parts for real-valued f . By

choosing an orthonormal basis e_1, \dots, e_k in W , and extending to an orthonormal basis $e_1, \dots, e_k, e_{k+1}, \dots, e_{d+1}$ of V , we will assume that $V = \mathbb{R}^{d+1}$ and $W = \mathbb{R}^k \oplus \{0\}$. Thus the formula we have to establish is

$$\int_{S^d(a)} f \, d\sigma = \int_{x \in B_k(a)} \left[\int_{y \in S^{d-k}(a_x)} f(x, y) \, d\sigma(y) \right] \frac{a}{a_x} \, dx, \quad (2.2.5)$$

where $S^d(a)$ is the sphere of radius a , centered at 0, in \mathbb{R}^{d+1} , and $B_k(a)$ is the ball of radius a , center 0, in \mathbb{R}^k .

Let F be the function on \mathbb{R}^{d+1} given by

$$F(z) = f\left(\frac{a}{\|z\|}z\right), \quad (2.2.6)$$

with $F(0)$ defined arbitrarily. Thus F is constant along radial rays and equal to f on the sphere $S^d(a)$.

Let ψ be any non-negative Borel function on $[0, \infty)$. We work out the integral

$$\int_{\mathbb{R}^{d+1}} F(z)\psi(\|z\|^2) \, dz$$

in two ways.

Using the polar disintegration formula (2.2.2) and scaling (2.2.1) we have

$$\begin{aligned} \int_{\mathbb{R}^{d+1}} F(z)\psi(\|z\|^2) \, dz &= \int_0^\infty \left[\int_{S^d} F(rw) r^d \, d\sigma(w) \right] \psi(r^2) \, dr \\ &= \left(\int_{S^d(a)} f(w) \, d\sigma(w) \right) \int_0^\infty \psi(r^2)(r/a)^d \, dr. \end{aligned} \quad (2.2.7)$$

This expresses the spherical integral on the right in terms of the volume integral on the left.

Next we will split \mathbb{R}^{d+1} into \mathbb{R}^k and \mathbb{R}^{d+1-k} and disintegrate the left side in (2.2.7) by repeated use of Fubini's theorem:

$$\begin{aligned}
& \int_{\mathbb{R}^{d+1}} F(z)\psi(\|z\|^2) dz \\
&= \int_{\mathbb{R}^k} \left[\int_{\mathbb{R}^{d+1-k}} F(x, y)\psi(\|x\|^2 + \|y\|^2) dy \right] dx \\
&= \int_{\mathbb{R}^k} \left[\int_{R \in (0, \infty)} \left\{ \int_{w \in S^{d-k}} F(x, Rw)\psi(\|x\|^2 + R^2) d\sigma(w) \right\} R^{d-k} dR \right] dx \\
&= \int_{\mathbb{R}^k \times (0, \infty) \times S^{d-k}} F(x, Rw)\psi(\|x\|^2 + R^2) d\sigma(w) R^{d-k} dR dx.
\end{aligned} \tag{2.2.8}$$

Here we have used the assumption that W is a *proper* subspace of V , which in the present notation means that $k < d + 1$. Now, for *fixed* $x \in \mathbb{R}^k$, we change variables from R to $r \geq \|x\|$ given by

$$r^2 = R^2 + \|x\|^2. \tag{2.2.9}$$

Then

$$r dr = R dR. \tag{2.2.10}$$

Hence, using (2.2.8), we have

$$\begin{aligned}
& \int_{\mathbb{R}^{d+1}} F(z)\psi(\|z\|^2) dz \\
&= \int_{x \in \mathbb{R}^k, R \in (0, \infty), w \in S^{d-k}, r \geq \|x\|} F(x, Rw)\psi(\|x\|^2 + R^2) d\sigma(w) R^{d-k} dR dx \\
&= \int_{r \in (0, \infty), x \in B_k(r), w \in S^{d-k}} F(x, r_x w)\psi(r^2) d\sigma(w) r_x^{d-k-1} r dr dx
\end{aligned} \tag{2.2.11}$$

where we have now written r_x for R :

$$r_x = \sqrt{r^2 - \|x\|^2}. \quad (2.2.12)$$

Recalling the choice of the function F , we have:

$$F(z) = f\left(\frac{a}{\|z\|}z\right) = f\left(\frac{a}{r}x, \frac{aR}{r}w\right) \quad \text{if } z = (x, Rw) \text{ with } w \in S^{d-k}.$$

Thus

$$\begin{aligned} & \int_{\mathbb{R}^{d+1}} F(z)\psi(\|z\|^2) dz \\ &= \int_{r \in (0, \infty), x \in B_k(r)} \left[\int_{w \in S^{d-k}} f(ax/r, ar_x w/r) d\sigma(w) \right] r_x^{d-k-1} dx \psi(r^2) r dr. \end{aligned} \quad (2.2.13)$$

Keeping in mind that f is evaluated only at points on the sphere $S^d(a)$, we change coordinates to make clearer use of this. For fixed r and x , we change from variable w to

$$w' = \frac{ar_x}{r}w = a_{x'}w, \quad \text{where } x' = \frac{a}{r}x, \quad (2.2.14)$$

which changes the spherical integral on the right side of (2.2.13) to

$$a_{x'}^{k-d} \int_{w' \in S^{d-k}(a_{x'})} f(x', w') d\sigma(w').$$

Thus:

$$\begin{aligned}
& \int_{\mathbb{R}^{d+1}} F(z) \psi(\|z\|^2) dz \\
&= \int_{r \in (0, \infty), x \in B_k(r)} \left[\int_{S^{d-k}(a_{x'})} f(x', w') a_{x'}^{k-d} d\sigma(w') \right] r_x^{d-k-1} dx \psi(r^2) r dr \\
&= \int_{r \in (0, \infty), x \in B_k(r)} \left[\int_{S^{d-k}(a_{x'})} f(x', w') d\sigma(w') \right] a_{x'}^{k-d} r_x^{d-k-1} dx \psi(r^2) r dr.
\end{aligned}$$

Note that since $x \in B_k(r)$ we have $x' \in B_k(a)$, and, by (2.2.14),

$$dx = (r/a)^k dx'$$

and

$$r_x = \frac{r}{a} a_{x'}.$$

Thus:

$$\begin{aligned}
& \int_{\mathbb{R}^{d+1}} F(z) \psi(\|z\|^2) dz \\
&= \int_{r \in (0, \infty), x' \in B_k(a)} \left[\int_{S^{d-k}(a_{x'})} f(x', w') d\sigma(w') \right] \frac{a}{a_{x'}} dx' \psi(r^2) (r/a)^d dr.
\end{aligned} \tag{2.2.15}$$

Choosing ψ for which

$$\int_0^\infty \psi(r^2) (r/a)^d dr = 1,$$

and comparing (2.2.15) with the earlier expression (2.2.7) we obtain:

$$\int_{S^d(a)} f d\sigma = \int_{x' \in B_k(a)} \left[\int_{S^{d-k}(a_{x'})} f(x', w') d\sigma(w') \right] \frac{a}{a_{x'}} dx'. \tag{2.2.16}$$

□

2.3 Spherical Disintegration for non-orthogonal projections

We generalize the previous theorem by looking at a projection that is not orthogonal.

Theorem 2.3.1. *Let V be a finite-dimensional real inner-product space and let*

$$L : V \rightarrow X$$

be a linear surjection onto a real inner-product space X , where $0 < \dim X < \dim V$.

Then L restricts to an isomorphism

$$L_0 : (\ker L)^\perp \rightarrow X, \tag{2.3.1}$$

and

$$\begin{aligned} & \int_{S_V(a)} f \, d\sigma \\ &= \int_{x \in X, \|L_0^{-1}x\| < a} \left\{ \int_{S_V(a) \cap L^{-1}(x)} f \, d\sigma \right\} \frac{a}{\sqrt{a^2 - \|L_0^{-1}x\|^2}} \frac{dx}{\sqrt{|\det LL^*|}}, \end{aligned} \tag{2.3.2}$$

for any non-negative or bounded Borel function f , defined on the sphere $S_V(a)$.

Note that for an isomorphism $T : V \rightarrow W$ between finite-dimensional real inner product spaces V and W we denote by $|\det(T)|$ the absolute value of the determinant of T with respect to the orthonormal basis of V and W and

$$|\det(T)| = \sqrt{\det(TT^*)}.$$

Also we note that if

$$P_{\perp} : V \rightarrow V \tag{2.3.3}$$

is the orthogonal projection onto $(\ker L)^{\perp}$ then

$$L_0 P_{\perp} = L. \tag{2.3.4}$$

Proof. We use the standard formula for transformation of integrals

$$\int_{X'} \phi(x') dx' = \int_X \phi(Jx) |\det J| dx \tag{2.3.5}$$

where $J : X \rightarrow X'$ is an isomorphism of a finite-dimensional inner-product space X onto an inner-product space X' . This is valid for any Borel function ϕ on X' for which either side of (2.3.5) exists. We apply this with $J = L_0^{-1} : X \rightarrow (\ker L)^{\perp}$ to obtain

$$\int_{(\ker L)^{\perp}} \phi(x') dx' = \int_X \phi(L_0^{-1}x) \frac{dx}{|\det L_0|}. \tag{2.3.6}$$

The Jacobian term $|\det L_0|$ is computed as the absolute value of the determinant of any matrix of L_0 relative to orthonormal bases in $(\ker L)^{\perp}$ and X ; in terms of L it is given by:

$$|\det L_0| = \sqrt{|\det LL^*|}. \tag{2.3.7}$$

Let us note that if $z \in L^{-1}(x)$ then $Lz = x$ and so, with $z_0 = L_0^{-1}x \in (\ker L)^{\perp}$, we have

$$L(z - z_0) = 0, \tag{2.3.8}$$

and so

$$z \in z_0 + \ker L. \quad (2.3.9)$$

Thus any point in $L^{-1}(x)$ is $L_0^{-1}(x)$ plus a vector orthogonal to z_0 and so

$$\text{the element of smallest norm in } L^{-1}(x) \text{ is } z_0 = L_0^{-1}x. \quad (2.3.10)$$

For ϕ we use the function on $(\ker L)^\perp$ given by

$$\phi(x') = \frac{a}{a_{x'}} \int_{S_V(a) \cap P_\perp^{-1}(x')} f d\sigma, \quad (2.3.11)$$

where $P_\perp : V \rightarrow V$ is the orthogonal projection onto the subspace $(\ker L)^\perp$, as in (2.3.3), and the right side in (2.3.11) is taken to be 0 when $\|x'\| \geq a$. If f is continuous then ϕ is continuous on the open ball of radius a , and 0 outside this ball. Then by standard limiting arguments ϕ is Borel when f is the indicator function of a compact set, and hence ϕ is Borel for any non-negative or bounded Borel function f .

Then

$$\phi(L_0^{-1}x) = \frac{a}{a_{L_0^{-1}x}} \int_{S_V(a) \cap (L_0 P_\perp)^{-1}(x)} f d\sigma = \frac{a}{a_{L_0^{-1}x}} \int_{S_V(a) \cap L^{-1}(x)} f d\sigma, \quad (2.3.12)$$

on using the relation (2.3.4). Here, and often, we take the integral over the empty set to be 0; thus:

$$\phi(L_0^{-1}x) \text{ is 0 if } L^{-1}(x) \cap S_V(a) \text{ is empty.}$$

By (2.3.10), this means

$$\phi(L_0^{-1}x) = 0 \quad \text{if } \|L_0^{-1}x\| > a. \quad (2.3.13)$$

We assume for now that $f \geq 0$; then $\phi \geq 0$. Applying (2.3.6) and (2.3.12), we have

$$\begin{aligned} & \int_{(\ker L)^\perp} \left[\int_{S_V(a) \cap P^{-1}(x')} f d\sigma \right] \frac{a}{a_{x'}} dx' \\ &= \int_X \left[\int_{S_V(a) \cap L^{-1}(x)} f d\sigma \right] \frac{a}{a_{L_0^{-1}x}} \frac{dx}{|\det L_0|}. \end{aligned} \quad (2.3.14)$$

The integrand on the left is 0 outside the ball of radius a in $(\ker L)^\perp$ and, by (2.3.13), the integrand on the right is 0 unless $\|L_0^{-1}x\| < a$. By Theorem 2.2.1 the left side is equal to $\int_{S_V(a)} f d\sigma$. This proves the identity (2.3.2) for $f \geq 0$.

For general complex-valued bounded f the result follows by considering real and imaginary parts and then positive and negative parts. Since f is bounded, all the integrals over $S_V(a)$ involved are finite.

We are mainly interested in the case where $\dim V$ is large compared to $\dim X$, and, in particular, $m = \dim V - \dim X$ is ≥ 2 . Then in the definition (2.3.11) of $\phi(x')$ the integral of f is over a sphere, of dimension $m - 1 \geq 1$, of radius $a_{x'}$, and so, for bounded f , the integral is bounded by a constant times $a_{x'}^{m-1}$. Therefore, $\phi(x')$ itself is bounded by a constant times a non-negative power of $a_{x'}$. Thus, ϕ is bounded if f is bounded. \square

Let Z , W , and X be finite-dimensional real inner-product spaces, and $\mathcal{L} : Z \rightarrow X$ and $Q : Z \rightarrow W$ be linear surjections. We consider the sphere $S_Z(a)$, centered at 0 and of radius $a > 0$, in Z . The sphere is sliced by an affine subspace $Q^{-1}(w^0)$, where w^0 is some point in W . We denote by z^0 the point on $Q^{-1}(w^0)$ closest to the origin, and

$$x^0 = \mathcal{L}(z^0) \in X$$

the ‘projection’ of z^0 on X by \mathcal{L} .

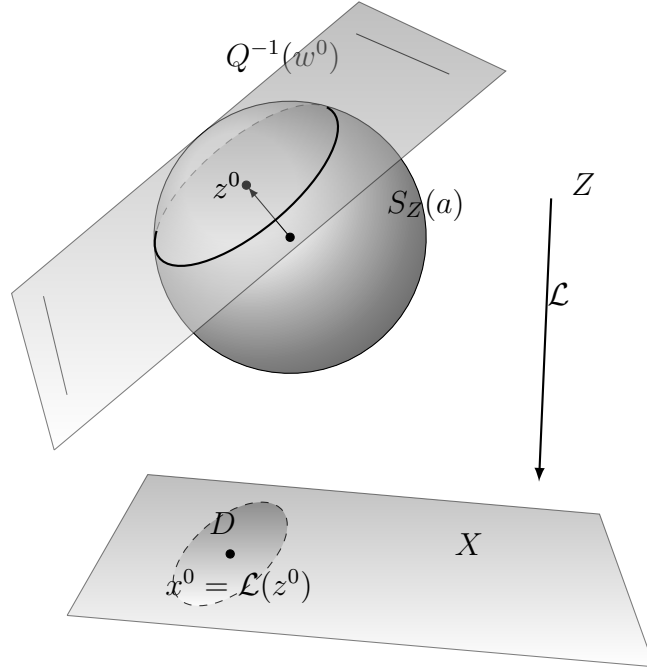


FIGURE 2.3.1: The affine subspace $Q^{-1}(w^0) \subset Z$ slicing the sphere $S_Z(a)$, the ‘projection’ $\mathcal{L} : Z \rightarrow X$, the points z^0 , closest in $Q^{-1}(w^0)$ to the center, and $x^0 = \mathcal{L}(z^0)$, and the ellipsoid D which is the projection on X of the slice of the ball by $Q^{-1}(w^0)$.

Let L_0 be the restriction of \mathcal{L} to the subspace of $\ker Q$ that is the orthogonal complement of $\ker(\mathcal{L}|_{\ker Q})$:

$$L_0 : \ker Q \ominus \ker \mathcal{L} \rightarrow \mathcal{L}(\ker Q) : z \mapsto \mathcal{L}z,$$

where on the left we have the orthogonal complement of $\ker(\mathcal{L}|_{\ker Q})$ within $\ker Q$. As before in (2.3.7), the determinant $|\det L_0|$ is the absolute value of the determinant of the matrix of L_0 relative to orthonormal bases in its domain and range; we take $|\det L_0|$ to be 1 in the degenerate case where L_0 is 0.

Theorem 2.3.2. (Figure 2.3.1) *Let f a bounded, or non-negative, Borel function defined on the ‘circular slice’ $S_Z(a) \cap Q^{-1}(w^0)$ for some $w^0 \in W$. Let z^0 be the point*

on $Q^{-1}(w^0)$ closest to 0, $x^0 = \mathcal{L}z^0 \in X$. Let L_0 be the restriction of \mathcal{L} to the subspace of $\ker Q$ that is the orthogonal complement of $\ker(\mathcal{L}|_{\ker Q})$. Then

$$\begin{aligned} & \int_{S_Z(a) \cap Q^{-1}(w^0)} f \, d\sigma \\ &= \int_{x \in D} \left\{ \int_{S_Z(a) \cap Q^{-1}(w^0) \cap \mathcal{L}^{-1}(x)} f \, d\sigma \right\} \frac{a_{z^0}}{\sqrt{a_{z^0}^2 - \|L_0^{-1}(x - x^0)\|^2}} \frac{dx}{|\det L_0|}, \end{aligned} \quad (2.3.15)$$

where D consists of all $x \in x^0 + \mathcal{L}(\ker Q) \subset X$ for which the term under the square-root is positive:

$$D = x^0 + \{y \in \mathcal{L}(\ker Q) : \|L_0^{-1}(y)\| < a_{z^0}\}. \quad (2.3.16)$$

On the left in (2.3.15) is the integral of f over the ‘circular’ slice of the sphere $S_Z(a)$ by the affine subspace $Q^{-1}(w^0)$. On the right is the disintegration of this with respect to the values of \mathcal{L} . In this disintegration each fiber $S_Z(a) \cap Q^{-1}(w^0) \cap \mathcal{L}^{-1}(x)$ is a sphere of radius $\sqrt{a_{z^0}^2 - \|L_0^{-1}(x - x^0)\|^2}$. The set D is an ‘ellipsoid.’ (In the degenerate case where \mathcal{L} is actually zero on $\ker Q$ the integral over dx drops out and we have a trivial equality in (2.3.15).)

Figure 2.3.1 illustrates some of the objects involved here. In the picture, $\ker Q$ is a two-dimensional subspace (through the origin, parallel to $Q^{-1}(w^0)$). Since \mathcal{L} maps $\ker Q$ onto the two-dimensional space X , its kernel is, in this picture, just $\{0\}$.

Proof. We will apply the disintegration result Theorem 2.3.1, taking for V the subspace $\ker Q \subset Z$, and L the restriction of \mathcal{L} to V :

$$L = \mathcal{L}|_{\ker Q} : V \rightarrow \mathcal{L}(\ker Q). \quad (2.3.17)$$

Then L_0 is, as in Theorem 2.3.1, the restriction:

$$L_0 : (\ker L)^\perp \rightarrow \text{Im}(L) = \mathcal{L}(\ker Q), \quad (2.3.18)$$

where $(\ker L)^\perp$ is the subspace of V consisting of all vectors in V orthogonal to $\ker L$.

In more detail,

$$(\ker L)^\perp = \{z \in \ker Q : z \in (\ker \mathcal{L} \cap \ker Q)^\perp\}. \quad (2.3.19)$$

The center of the ‘circle’ $S_Z(a) \cap Q^{-1}(w^0)$ is the point on $Q^{-1}(w^0)$ closest to 0. Let us check that this point is given by

$$z^0 = Q^*(QQ^*)^{-1}(w^0); \quad (2.3.20)$$

here we note that since Q is surjective, QQ^* is invertible because any vector in its kernel would also be in $\ker Q^* = [\text{Im}(Q)]^\perp$. Clearly,

$$Qz^0 = w^0,$$

and if $v \in \ker Q$ then

$$\langle v, z^0 \rangle = \langle Qv, (QQ^*)^{-1}(w^0) \rangle = 0. \quad (2.3.21)$$

This implies that the point $z^0 + v \in Q^{-1}(w^0)$ has norm

$$\|v + z^0\|^2 = \|v\|^2 + \|z^0\|^2 \geq \|z^0\|^2,$$

thus showing that z^0 is the unique point on $Q^{-1}(w^0)$ closest to the origin.

To disintegrate

$$\int_{S_Z(a) \cap Q^{-1}(w^0)} f d\sigma$$

we write this as an integral over the sphere of radius a_{z^0} in $V = \ker Q$:

$$\int_{S_Z(a) \cap Q^{-1}(w^0)} f d\sigma = \int_{S_V(a_{z^0})} f(z + z^0) d\sigma(z), \quad (2.3.22)$$

which we see by observing that

$$\begin{aligned} S_V(a_{z^0}) + z^0 &= \{v + z^0 : v \in \ker Q, \|v\|^2 = a^2 - \|z^0\|^2\} \\ &= \{v + z^0 : Q(v + z^0) = w^0, \|v + z^0\|^2 = a^2\} \\ &= S_Z(a) \cap Q^{-1}(w_0), \end{aligned} \quad (2.3.23)$$

where in the second equality we used the orthogonality (2.3.21).

Applying the disintegration formula (2.3.2) for \mathcal{L} in (2.3.22) we obtain:

$$\begin{aligned} &\int_{S_Z(a) \cap Q^{-1}(w^0)} f d\sigma \\ &= \int_{y \in D_0} \left\{ \int_{S_V(a_{z^0}) \cap L^{-1}(y)} f(z + z_0) d\sigma(z) \right\} \frac{a_{z^0}}{\sqrt{a_{z^0}^2 - \|L_0^{-1}y\|^2}} \frac{dy}{|\det L_0|}, \end{aligned} \quad (2.3.24)$$

where D_0 is the set of all $y \in \text{Im}(L_0) = \mathcal{L}(\ker Q)$ for which $\|L_0^{-1}y\| < a$. Changing variables by translation with $y = x - x^0$, we then have

$$\int_{S_Z(a) \cap Q^{-1}(w^0)} f d\sigma = \int_{x \in D} I(x) \frac{a_{z^0}}{\sqrt{a_{z^0}^2 - \|L_0^{-1}(x - x^0)\|^2}} \frac{dx}{|\det L_0|}, \quad (2.3.25)$$

where $D = D_0 + x^0$ and

$$\begin{aligned} I(x) &= \int_{S_V(a_{z^0}) \cap L^{-1}(x-x^0)} f(z+z_0) d\sigma(z) \\ &= \int_{[S_V(a_{z^0}) \cap L^{-1}(x-x^0)] + z^0} f(z) d\sigma(z). \end{aligned} \quad (2.3.26)$$

Now

$$[S_V(a_{z^0}) \cap L^{-1}(x-x^0)] + z^0 = [S_V(a_{z^0}) + z^0] \cap \mathcal{L}^{-1}(x), \quad (2.3.27)$$

because a point p lies in the right hand side if and only if $p = p^0 + z^0$, where $p^0 \in S_V(a_{z^0})$ and $\mathcal{L}(p^0) = x - \mathcal{L}(z^0) = x - x^0$. Then, using (2.3.23), we have

$$[S_V(a_{z^0}) \cap L^{-1}(x-x^0)] + z^0 = S_Z(a) \cap Q^{-1}(w_0) \cap \mathcal{L}^{-1}(x). \quad (2.3.28)$$

Hence

$$I(x) = \int_{S_Z(a) \cap Q^{-1}(w_0) \cap \mathcal{L}^{-1}(x)} f(z) d\sigma(z). \quad (2.3.29)$$

Using this value of I in (2.3.25) gives us the desired disintegration formula (2.3.15). □

2.3.1 Disintegration of slices expressed in coordinates

Now let us work out some details of the disintegration of slices formula (2.3.15). We apply Theorem 2.3.2 with $Z = \mathbb{R}^{d+1}$, $X = \mathbb{R}^k$, where $0 < k < d$, and with \mathcal{L} being the projection $\mathcal{L} = P_{(k)} : \mathbb{R}^{d+1} \rightarrow \mathbb{R}^k$.

Suppose u_1, \dots, u_m form an orthonormal basis of $(\ker Q)^\perp$. Then

$$Qu_1, \dots, Qu_m \text{ form a basis of } W = \text{Im}(Q).$$

Thus

$$Qz = \langle z, u_1 \rangle Qu_1 + \dots + \langle z, u_m \rangle Qu_m \quad \text{for all } z \in (\ker Q)^\perp. \quad (2.3.30)$$

As before, let

$$V = \ker Q,$$

and L the restriction of the projection \mathcal{L} to V :

$$L : V \rightarrow X = \mathbb{R}^k : z \mapsto z_{(k)} \stackrel{\text{def}}{=} (z_1, \dots, z_k). \quad (2.3.31)$$

Let us note that

$$\ker \mathcal{L} = \{0\} \oplus \mathbb{R}^{d+1-k},$$

and, of more interest,

$$\begin{aligned} \ker L &= \{(0, y) \in \mathbb{R}^{d+1} : \langle y, (u_1)_{(k)'} \rangle = 0, \dots, \langle y, (u_m)_{(k)'} \rangle = 0\} \\ &= \{0\} \oplus [\text{span of } (u_1)_{(k)'}, \dots, (u_m)_{(k)'}]^\perp \end{aligned} \quad (2.3.32)$$

where $(u_i)_{(k)'} = ((u_i)_{k+1}, \dots, (u_i)_{d+1}) \in \mathbb{R}^{d+1-k}$. The space $(\ker L)^\perp$ consists of all $z \in \ker Q$ that are orthogonal to the subspace $\ker L$. Thus a vector $z = (x, y) \in \mathbb{R}^{d+1}$ lies in $(\ker L)^\perp$ if and only if $z \in \ker Q$ and the component x is unrestricted but the

component y is orthogonal to $[\text{span of } (u_1)_{(k)', \dots, (u_m)_{(k)'}]}^\perp$:

$$(\ker L)^\perp = \{z = (x, y) \in \mathbb{R}^k \oplus [\text{span of } (u_1)_{(k)', \dots, (u_m)_{(k)'}]} : z \in \ker Q\}. \quad (2.3.33)$$

Thus $(\ker L)^\perp$ consists of all elements of \mathbb{R}^{d+1} of the form

$$(x, 0) + (0, c_1(u_1)_{(k)'} + \dots + c_m(u_m)_{(k)'})$$

that are orthogonal to u_1, \dots, u_m :

$$\langle (u_a)_{(k)}, x \rangle + \sum_{b=1}^m \langle (u_a)_{(k)'}, (u_b)_{(k)'} \rangle c_b = 0 \quad (2.3.34)$$

for $a \in \{1, \dots, m\}$. These m equations yield a solution for (c_1, \dots, c_m) :

$$c = U^{-1}\vec{x} \quad (2.3.35)$$

where $c = (c_1, \dots, c_m)$, the linear mapping

$$U : \mathbb{R}^m \rightarrow \mathbb{R}^m$$

has matrix

$$[\langle (u_a)_{(k)'}, (u_b)_{(k)'} \rangle],$$

and

$$\vec{x} = (\langle (u_1)_{(k)}, x \rangle, \dots, \langle (u_m)_{(k)}, x \rangle) \in \mathbb{R}^m.$$

The mapping L restricted to $(\ker L)^\perp$ is given by

$$\begin{aligned} L_0 : (\ker L)^\perp &\rightarrow X = \mathbb{R}^k \\ (x, c_1(u_1)_{(k)'} + \dots + c_m(u_m)_{(k)'}) &\mapsto x. \end{aligned} \quad (2.3.36)$$

The inverse of this mapping is given by

$$L_0^{-1}x = (x, c_1(u_1)_{(k)'} + \dots + c_m(u_m)_{(k)'}), \quad (2.3.37)$$

where (c_1, \dots, c_m) is given by (2.3.35).

Next we work out the adjoint L_0^* . For any $z \in (\ker L)^\perp$, which is the subspace of $\ker Q$ orthogonal to $\ker Q \cap \ker \mathcal{L}$, we have

$$\begin{aligned} \langle L_0^*x, z \rangle &= \langle x, L_0z \rangle \\ &= \langle x, z_{(k)} \rangle \\ &= \langle (x, 0), z \rangle \\ &= \langle (x, 0), P_{\ker Q}z \rangle \\ &= \langle P_{\ker Q}(x, 0), z \rangle \\ &= \langle (I - P_{(\ker Q)^\perp})(x, 0), z \rangle \\ &= \left\langle (x, 0) - \sum_{a=1}^m \langle x, (u_a)_{(k)} \rangle u_a, z \right\rangle. \end{aligned} \quad (2.3.38)$$

The element

$$P_{\ker Q}(x, 0) = (x, 0) - \sum_{a=1}^m \langle x, (u_a)_{(k)} \rangle u_a \quad (2.3.39)$$

lies in $\ker Q$ and is also in the subspace

$$\mathbb{R}^k \oplus [\text{span of } (u_1)_{(k)'}, \dots, (u_m)_{(k)'}]. \quad (2.3.40)$$

Thus it is in $(\ker L)^\perp$. Hence

$$L_0^* x = P_{\ker Q}(x, 0) = (x, 0) - \sum_{a=1}^m \langle x, (u_a)_{(k)} \rangle u_a. \quad (2.3.41)$$

From this and the fact that L_0 is just the projection onto the first k coordinates, we have

$$L_0 L_0^* x = x - \sum_{a=1}^m \langle x, (u_a)_{(k)} \rangle (u_a)_{(k)}. \quad (2.3.42)$$

For future reference let us rewrite this in different notation:

$$\begin{aligned} L_0 L_0^* x &= x - \sum_{a=1}^m \langle P_{(k)}^* x, u_a \rangle P_{(k)} u_a = \left(I - \sum_{a=1}^m P_{(k)} P_{u_a} P_{(k)}^* \right) x \\ &= (I - P_{(k)} P_{(\ker Q)^\perp} P_{(k)}^*) x \\ &= P_{(k)} P_{\ker Q} P_{(k)}^* x \end{aligned} \quad (2.3.43)$$

where

$$P_{(k)} : \mathbb{R}^{d+1} \rightarrow \mathbb{R}^k : z \mapsto z_{(k)}$$

is the projection onto the first k coordinates.

Now let

$$P_{k,a} : \mathbb{R}^k \rightarrow \mathbb{R}^k : x \mapsto \langle x, \widehat{(u_a)_{(k)}} \rangle \widehat{(u_a)_{(k)}} \quad (2.3.44)$$

be the orthogonal projection onto the ray spanned by $(u_a)_{(k)}$, assumed to be nonzero.

Then

$$L_0 L_0^* = I - \sum_{a=1}^m \|(u_a)_{(k)}\|^2 P_{k,a}. \quad (2.3.45)$$

Now recall the disintegration formula (2.3.15):

$$\begin{aligned} & \int_{S^d(a) \cap Q^{-1}(w^0)} f d\sigma \\ &= \int_{x \in D} \left\{ \int_{S^d(a) \cap Q^{-1}(w^0) \cap \mathcal{L}^{-1}(x)} f d\sigma \right\} \frac{a_{z^0}}{\sqrt{a_{z^0}^2 - \|L_0^{-1}(x - x^0)\|^2}} \frac{dx}{|\det L_0|} \end{aligned} \quad (2.3.46)$$

where

$$x^0 = \mathcal{L}z^0, \quad (2.3.47)$$

and

$$D = x^0 + \{y \in \mathcal{L}(\ker Q) : \|L_0^{-1}(y)\| \leq a_{z^0}\}. \quad (2.3.48)$$

We have now both a way to compute L_0^{-1} , given in (2.3.37), and an expression for the determinant factor:

$$|\det L_0| = \sqrt{\det \left(I - \sum_{a=1}^m \|(u_a)_{(k)}\|^2 P_{k,a} \right)}. \quad (2.3.49)$$

Chapter 3

Projections on Subspaces

Before we implement our spherical disintegration formula to evaluate the limit discussed in Theorem 2.1.1 we first want to look at some of the geometric intuition and structure.

3.1 Geometry and “large enough” N

We are interested in an affine space A of finite codimension m to l^2 . Let $Q : l^2 \rightarrow \mathbb{R}^m$ be a continuous linear surjection and $w^0 \in \mathbb{R}^m$ such that A is a level set of Q :

$$A = Q^{-1}(w^0). \tag{3.1.1}$$

Letting w_1, \dots, w_m be an orthonormal basis of $(\ker Q)^\perp$ we can also write A as

$$A = \{v \in l^2 : \langle v, w_1 \rangle = p_1, \langle v, w_2 \rangle = p_2, \langle v, w_3 \rangle = p_3, \dots, \langle v, w_m \rangle = p_m\} \tag{3.1.2}$$

where $p_1, \dots, p_m \in \mathbb{R}$. Now we want to take the part of A in \mathbb{R}^N , $A_N \cap \mathbb{R}^N$, by thinking of $\mathbb{R}^N \in l^2$ by $\mathbb{R}^N = \mathbb{R}^N \oplus \{0\}$. Using the definition 3.1.2 we can write any vector $x \in A_N$ as

$$\langle x, (w_1)_{(N)} \rangle = p_1, \langle x, (w_2)_{(N)} \rangle = p_2, \dots, \langle x, (w_m)_{(N)} \rangle = p_m$$

where $(w)_{(N)} = (w_1, w_2, \dots, w_N)$. For N large enough the vectors $(w_1)_{(N)}, \dots, (w_m)_{(N)}$ are linearly independent as shown below. The lemma is presented in more broad terms for application to more general scenarios later on.

Lemma 3.1.1. *Suppose u_1, \dots, u_m are linearly independent in a Hilbert space H and $Z_1 \subset Z_2 \subset \dots$ are closed subspaces of H whose union is dense in H . Let P_{Z_N} be the orthogonal projection onto Z_N . Then, for N large enough, $P_{Z_N}u_1, \dots, P_{Z_N}u_m$ are linearly independent vectors in Z_N .*

Proof. Let C be the infimum of $\|\sum_{a=1}^m r_a u_a\|$ with $r = (r_1, \dots, r_m)$ running over the unit “diamond” in \mathbb{R}^m , which consists of all r for which $|r_1| + \dots + |r_m| = 1$.

$$C = \inf \left\{ \left\| \sum_{a=1}^m r_a u_a \right\| \mid r \in \mathbb{R}^m, |r_1| + \dots + |r_m| = 1 \right\}$$

Since this is the infimum over a compact set in \mathbb{R}^m it must contain its minimum, so the infimum is attained at some r^* . Since this $r^* > 0$ then $C > 0$ by the linear independence of u_a .

Now let u'_1, \dots, u'_m be such that

$$\max_a \|u'_a - u_a\| < C/2. \tag{3.1.3}$$

If $\sum_{a=1}^m \lambda_a u'_a = 0$ then

$$\begin{aligned} \left\| \sum_a \lambda_a u_a \right\| &= \left\| \sum_a \lambda_a (u_a - u'_a) \right\| \\ &\leq \frac{C}{2} \sum_a |\lambda_a|. \end{aligned} \tag{3.1.4}$$

On the other hand we have $\|\sum_a \lambda_a u_a\| \geq C \sum_a |\lambda_a|$, and so $\sum_a |\lambda_a|$ must be 0, which means that each λ_a is 0.

Thus u'_a are linearly independent provided they satisfy (3.1.3). \square

Now if p_N^0 is the point on A_N closest to the origin we can write

$$A_N = p_N^0 + [(w_1)_{(N)}, (w_2)_{(N)}, \dots, (w_m)_{(N)}]^\perp$$

where $[(w_1)_{(N)}, (w_2)_{(N)}, \dots, (w_m)_{(N)}]$ is the span of the vectors $(w_1)_{(N)}, (w_2)_{(N)}, \dots, (w_m)_{(N)}$.

Let J_N be the inclusion map:

$$J_N : \mathbb{R}^N \oplus \{0\} \rightarrow l^2.$$

Recall the definition (3.1.1) of A as a level set of Q we will now let

$$Q_N = QJ_N : \mathbb{R}^N \oplus \{0\} \rightarrow \mathbb{R}^m$$

We will show below that for large enough N , Q_N will be a surjection and so we can also write A_N as

$$A_N = Q_N^{-1}(w^0).$$

And the point p_N^0 , the point on A_N closest to the origin, will be given by

$$p_N^0 = z_N^0 = Q_N^*(Q_N Q_N^*)^{-1}(w_0) \quad (3.1.5)$$

as seen in (2.3.20).

Proposition 3.1.2. *Let H be a Hilbert space and $Z_1 \subset Z_2 \subset \dots$ a sequence of closed subspaces such that $\cup_{N \geq 1} Z_N$ is dense in H . Suppose*

$$R : H \rightarrow Y$$

is a continuous linear surjection onto a finite-dimensional vector space Y . Then $R|_{Z_N}$ is surjective onto Y for large N .

Proof. Since $R(Z_1) \subset R(Z_2) \subset \dots$ is an increasing sequences of subspaces of the finite-dimensional space Y , the subspaces stabilize, in the sense that $R(Z_{N_0}) = R(Z_N)$ for all $N \geq N_0$, where N_0 is such that

$$\dim R(Z_{N_0}) = \max_{N \geq 1} \dim R(Z_N).$$

Let $y^* \in Y$ be orthogonal to $R(Z_{N_0})$; here we have equipped Y with an arbitrary inner product. Let $J_N : Z_N \rightarrow H$ be the inclusion map. Then $J_N^* : H \rightarrow Z_N$ is the orthogonal projection onto the subspace Z_N . Since, for all $N \geq N_0$, the vector $y^* \in Y$ is orthogonal to $R(Z_N) = \text{Im}(R J_N)$ then $y^* \in \ker(R J_N)^* = [\text{Im}(R J_N)]^\perp$. Thus,

$$J_N^* R^* y^* = 0.$$

This means

$$P_{Z_N}(R^*y^*) = 0,$$

since $J_N^*z = P_{Z_N}z$ for all $z \in H$ and all $N \geq N_0$. Then, using Lemma 3.2.1,

$$R^*y^* = \lim_{N \rightarrow \infty} P_{Z_N}(R^*y^*) = 0,$$

and so $y^* \in \ker R^* = \text{Im}(R)^\perp = 0$. Thus $R(Z_{N_0}) = Y$, and so $R|_{Z_N}$ is surjective for all $N \geq N_0$. \square

Now recall $S_{A_N} = A_N \cap S^{N-1}(\sqrt{N})$ where $S^{N-1}(\sqrt{N})$ is the sphere in \mathbb{R}^N centered at the origin and of radius \sqrt{N} . Since by viewing $\mathbb{R}^N = \mathbb{R}^N \oplus \{0\}$ we have nested the spaces we also have $A_m \subset A_n$ for $m < n$. This means since the center of the circle S_{A_N} is given by p_N^0 we have:

$$\|p_m^0\| \leq \|p_n^0\|$$

and so we will also require that N be large enough such that the radius of the circle S_{A_N} :

$$\text{radius}(S_{A_N}) = \sqrt{N - \|p_N^0\|^2}$$

is a positive number and therefore S_{A_N} is not an empty sphere.

3.2 Limit of Projections and finite codimension

As mentioned before we think of $\mathbb{R}^N = \mathbb{R}^N \oplus \{0\}$ as a subspace of l^2 . Thus this creates a nested sequences of subspaces $\mathbb{R}^N \subset \mathbb{R}^{N+1} \subset \dots$ whose union is dense in l^2 . Letting $Z_N = \mathbb{R}^N$ and denoting the orthogonal projection onto Z_N by P_{Z_N} we now

show that the limit of such projection operators is just the identity.

Lemma 3.2.1. *Let $Z_1 \subset Z_2 \subset \dots$ be a sequence of closed subspaces of a Hilbert space. Then the following are equivalent:*

(i) $\cup_{N \geq 1} Z_N$ is dense in H ;

(ii) $\lim_{N \rightarrow \infty} P_{Z_N} z = z$ for all $z \in H$, where P_{Z_N} is the orthogonal projection onto Z_N .

Proof. Suppose (i) holds. Let $z \in H$. Then there is a sequence of points $w_n \in \cup_{N \geq 1} Z_N$ converging to z . For each k there is an integer N_k such that $w_k \in Z_{N_k}$. Then, by definition of the orthogonal projection, $P_{Z_{N_k}} z$ is the point on Z_{N_k} closest to z , so we have

$$\left\| z - P_{Z_{N_k}} z \right\| \leq \|z - w_k\| \rightarrow 0 \quad \text{as } k \rightarrow \infty.$$

So for any $\epsilon > 0$ there is an integer k such that

$$\left\| z - P_{Z_{N_k}} z \right\| < \epsilon.$$

For $N > N_k$ the subspace Z_{N_k} is contained in Z_N and so $P_{Z_N} z$ being the point on Z_N closest to z , we have

$$\|z - P_{Z_N} z\| < \epsilon$$

for all $N > N_k$. Thus (ii) holds.

The implication (ii) \implies (i) holds since all the points $P_{Z_N} z$ lie in the union $\cup_{N \geq 1} Z_N$. □

We now show that a limit of projections onto the spaces $\ker Q_N$ is the projection

onto $\ker Q$ and show why the condition of finite codimension for our affine space A is required. Again we prove the following theorems in more general settings.

Theorem 3.2.2. *Suppose H is a Hilbert space, and $Z_1 \subset Z_2 \subset \dots$ is a sequence of finite dimensional subspaces whose union is dense in H . Let A_0 be a closed subspace of H of finite codimension. Then*

$$\lim_{N \rightarrow \infty} P_{A_0 \cap Z_N} z = P_{A_0} z. \quad (3.2.1)$$

for all $z \in H$. In particular, if $A_0 = \ker Q$, for some continuous linear mapping $Q : H \rightarrow W$ onto a finite-dimensional space, then

$$\lim_{N \rightarrow \infty} P_{\ker Q_N} z = P_{\ker Q} z \quad \text{for all } v \in H, \quad (3.2.2)$$

where $Q_N = Q|_{Z_N}$.

Let us note that (3.2.1) implies that any element $z \in A_0$ is the limit of a sequence of elements $P_{A_0 \cap Z_N} z \in A_0 \cap Z_N$; thus

$$\cup_{N \geq 1} (A_0 \cap Z_N) \text{ is dense in } A_0. \quad (3.2.3)$$

This conclusion requires that A_0 be of finite codimension; otherwise, we could just choose A_0 to be the line through any non-zero vector outside $\cup_{N \geq 1} Z_N$ and obtain a contradiction.

Proof. Let u_1, \dots, u_m be an orthonormal basis of A_0^\perp . Then for any $v \in Z_N$ we have

$$\langle P_{Z_N} u_a, v \rangle = \langle u_a, P_{Z_N} v \rangle = \langle u_a, v \rangle.$$

Thus if $v \in Z_N$ is orthogonal to $[P_{Z_N}u_1, \dots, P_{Z_N}u_m]$ then $v \in [u_1, \dots, u_m]^\perp$ and so $v \in A_0$. Conversely, if $v \in Z_N$ is also in A_0 then v is orthogonal to each u_a and hence to each $P_{Z_N}u_a$. Thus the orthogonal complement of $[P_{Z_N}u_1, \dots, P_{Z_N}u_m]$ in Z_N is $A_0 \cap Z_N$:

$$Z_N \ominus P_{Z_N}(A_0^\perp) = Z_N \cap A_0. \quad (3.2.4)$$

Consequently,

$$Z_N \ominus A_0 = P_{Z_N}(A_0^\perp). \quad (3.2.5)$$

By Lemma 3.2.3 (proven below) the orthogonal projection in H onto $P_{Z_N}(A_0^\perp)$ converges pointwise, as $N \rightarrow \infty$, to the orthogonal projection onto A_0^\perp . Thus, using (3.2.5),

$$\lim_{N \rightarrow \infty} P_{Z_N \ominus A_0} z = P_{A_0^\perp} z \quad \text{for all } z \in H. \quad (3.2.6)$$

Since H is the sum of the mutually orthogonal subspaces $Z_N \ominus A_0$, $Z_N \cap A_0$, and Z_N^\perp , we have

$$z = P_{Z_N \ominus A_0} z + P_{Z_N \cap A_0} z + P_{Z_N^\perp} z,$$

and so

$$P_{Z_N \cap A_0} z = z - P_{Z_N^\perp} z - P_{Z_N \ominus A_0} z = P_{Z_N} z - P_{Z_N \ominus A_0} z, \quad (3.2.7)$$

for all $z \in H$. Then

$$\begin{aligned} \lim_{N \rightarrow \infty} P_{A_0 \cap Z_N} z &= \lim_{N \rightarrow \infty} (P_{Z_N} z - P_{Z_N \ominus A_0} z) \\ &= z - P_{A_0^\perp} z \\ &= P_{A_0} z \end{aligned} \quad (3.2.8)$$

□

We prove the lemma used in the preceding proof.

Lemma 3.2.3. *Let H be a Hilbert space and K a finite-dimensional subspace of H . Suppose that R_1, R_2, \dots are orthogonal projections in H such that*

$$R_N z \rightarrow z \quad \text{for all } z \in H, \text{ as } N \rightarrow \infty.$$

Let S_N be the orthogonal projection in H onto $R_N(K)$. Then S_N converges pointwise to the orthogonal projection onto K :

$$\lim_{N \rightarrow \infty} S_N z = Sz \quad \text{for all } z \in H,$$

where S is the orthogonal projection onto K .

In this result the hypothesis that K is finite-dimensional is needed. For, consider $H = l^2$, R_N the orthogonal projection given by $R(x_1, x_2, \dots) = (x_1, \dots, x_N, 0, 0, \dots)$, and $K = v^\perp$, where v is the vector $(1, 1/2, 1/3, \dots)$. Then $R_N(K) = \mathbb{R}^N \times \{(0, 0, \dots)\}$, and $S_N = R_N$. The pointwise limit of S_N is I , which is not the same as the orthogonal projection onto K .

Proof. Let u_1, \dots, u_m be an orthonormal basis of $K = S(H)$ (if $S = 0$ the result is obvious). By Lemma 3.1.1 (below) we may assume that N is large enough that the vectors $R_N u_1, \dots, R_N u_m$ are linearly independent, and thus form a basis of $R_N(K)$. Since S_N is the orthogonal projection onto this subspace, for any $z \in H$ the vector $S_N z$ can be expressed in terms of $R_N u_1, \dots, R_N u_m$ as follows:

$$S_N z = \sum_{i=1}^m c_i R_N u_i. \tag{3.2.9}$$

The coefficients c_i depend on N . Since $S_N z$ is the orthogonal projection of z on the span of $\{R_N u_1, \dots, R_N u_m\}$, the inner product of $S_N z$ with each $R_N u_j$ is the same as the inner product of z with $R_N u_j$:

$$\langle z, R_N u_j \rangle = \langle S_N z, R_N u_j \rangle = \sum_{i=1}^m \langle R_N u_j, R_N u_i \rangle c_i. \quad (3.2.10)$$

Thus the vector $c(N)$ of coefficients c_j is

$$c(N) \stackrel{\text{def}}{=} \begin{bmatrix} c_1 \\ \vdots \\ c_m \end{bmatrix} = [\langle R_N u_a, R_N u_b \rangle]^{-1} \begin{bmatrix} \langle z, R_N u_1 \rangle \\ \vdots \\ \langle z, R_N u_m \rangle \end{bmatrix}. \quad (3.2.11)$$

Letting $N \rightarrow \infty$ we obtain (using continuity of matrix inversion):

$$\lim_{N \rightarrow \infty} c(N) = [\langle u_a, u_b \rangle]^{-1} \begin{bmatrix} \langle z, u_1 \rangle \\ \vdots \\ \langle z, u_m \rangle \end{bmatrix} = \begin{bmatrix} \langle z, u_1 \rangle \\ \vdots \\ \langle z, u_m \rangle \end{bmatrix}. \quad (3.2.12)$$

Going back to (3.2.9) we conclude that

$$\lim_{N \rightarrow \infty} S_N z = \sum_{i=1}^m \langle z, u_i \rangle u_i, \quad (3.2.13)$$

and this is just the orthogonal projection of z onto the subspace $S(H)$ spanned by u_1, \dots, u_m . \square

We will often use the following notation:

$$X \ominus Y = X \cap (X \cap Y)^\perp, \quad (3.2.14)$$

which is the orthogonal complement of $X \cap Y$ within X , where X and Y are subspaces of any given inner-product space.

3.2.1 Subspaces and Projections

Before we return to our proof of the main result we note some important lemmas regarding the projections that will be used.

Lemma 3.2.4. *Let A and B be closed subspaces of H . Then*

$$P_A P_{A \cap B + A^\perp} P_A = P_{A \cap B}. \quad (3.2.15)$$

Moreover,

$$P_A^{-1}(B) = (A \cap B) + A^\perp. \quad (3.2.16)$$

Proof. Decomposing H as the orthogonal sum of subspaces $A \cap B$, $A \ominus B$, and A^\perp , we have

$$\begin{aligned} P_A P_{A \cap B + A^\perp} P_A &= (P_{A \cap B} + P_{A \ominus B})(P_{A \cap B} + P_{A^\perp})(P_{A \cap B} + P_{A \ominus B}) \\ &= P_{A \cap B} P_{A \cap B} P_{A \cap B} + 0 \\ &= P_{A \cap B}, \end{aligned} \quad (3.2.17)$$

where in the second line we have 0 because the subspaces are mutually orthogonal.

Next, if a point $x \in H$ lies in $P_A^{-1}(B)$ then $P_A x$ lies in B and hence in $A \cap B$, and so since $x - P_A x \in A^\perp$, we have $x \in A \cap B + A^\perp$. Conversely, if $x \in A \cap B + A^\perp$ then $x = x_0 + x_A^\perp$, with $x_A^\perp \in A^\perp$ and $x_0 \in A \cap B$, which implies $P_A x = P_A x_0 = x_0$, which is in B . □

We apply this observation to establish formula (3.2.19) below.

Proposition 3.2.5. *Let $Q : H \rightarrow W$ be a continuous linear surjection from a Hilbert space H onto a Hilbert space W . Then $QQ^* : W \rightarrow W$ is an isomorphism and the orthogonal projection in H onto the subspace $\ker Q$ is given by*

$$P_{\ker Q} = I - Q^*(QQ^*)^{-1}Q. \quad (3.2.18)$$

Moreover, for any orthogonal projection $P : H \rightarrow H$, the operator $PP_{\ker(QP)}P$ is the orthogonal projection in H onto the subspace $\text{Im}(P) \cap (\ker Q)$:

$$PP_{\ker(QP)}P = P_{\text{Im}(P) \cap (\ker Q)}. \quad (3.2.19)$$

Proof. The mapping $QQ^* : W \rightarrow W$ is continuous linear and its kernel is $\ker Q^* = [\text{Im}(Q)]^\perp$, which is $\{0\}$ because Q is surjective. Moreover, if w is orthogonal to the image of QQ^* then $\langle w, QQ^*w \rangle = 0$ and so $w \in \ker Q^* = \{0\}$. Hence, QQ^* is invertible and, by the open mapping theorem, $(QQ^*)^{-1}$ is also continuous. Next,

$$Q^*(QQ^*)^{-1}Q$$

is a continuous linear mapping $H \rightarrow H$ which is self-adjoint and whose square equals itself. Thus, it is an orthogonal projection. Clearly, the image is contained in the image of Q^* , which is $[\ker Q]^\perp$. In fact, it is the orthogonal projection onto this subspace because if y is any point on $[\ker Q]^\perp = \text{Im}(Q^*)$, we can write y as Q^*z and

compute

$$\begin{aligned}
\langle y, Q^*(QQ^*)^{-1}Qv \rangle &= \langle Q^*z, Q^*(QQ^*)^{-1}Qv \rangle \\
&= \langle z, QQ^*(QQ^*)^{-1}Qv \rangle \\
&= \langle z, Qv \rangle \\
&= \langle y, v \rangle,
\end{aligned} \tag{3.2.20}$$

which shows that the point $Q^*(QQ^*)^{-1}Qv$ in $\text{Im}(Q^*)$ has the same inner-product with any vector $y \in \text{Im}(Q^*)$ as does v . Thus

$$P_{(\ker Q)^\perp} = Q^*(QQ^*)^{-1}Q, \tag{3.2.21}$$

which implies (3.2.18).

Let

$$R = PP_{\ker(QP)}P.$$

Now

$$\ker(QP) = \{x \in H : Q(Px)\} = P^{-1}(\ker Q) = \text{Im}(P) \cap (\ker Q) \cap +\text{Im}(P)^\perp,$$

where we used (3.2.16) in the last equality, along with $(\ker P)^\perp = \text{Im}(P)$. Then, using (3.2.15), we see that R equals $P_{\text{Im}(P) \cap (\ker Q)}$. \square

The following is a basic observation about how subspaces in a vector space may be situated relative to each other.

Lemma 3.2.6. *Let $\mathcal{L} : H \rightarrow X$ and $Q : H \rightarrow W$ be surjective linear maps between vector spaces. Then the following are equivalent:*

- (i) \mathcal{L} maps $\ker Q$ surjectively onto X ;
- (ii) Q maps $\ker \mathcal{L}$ surjectively onto W ;
- (iii) $\ker \mathcal{L} + \ker Q = H$.

More generally, \mathcal{L} maps a subspace $V \subset H$ onto X if and only if $V + \ker \mathcal{L} = H$.

Proof. Let V be a subspace of H . Then

$$\mathcal{L}^{-1}(\mathcal{L}(V)) = V + \ker \mathcal{L}.$$

If $V + \ker \mathcal{L} = H$ then $X = \mathcal{L}(H) = \mathcal{L}(V)$. Conversely, if $\mathcal{L}(V) = X$ then $V + \ker \mathcal{L} = \mathcal{L}^{-1}(\mathcal{L}(V)) = \mathcal{L}^{-1}(X) = H$.

Thus (iii) is equivalent to (i) and also (with \mathcal{L} and Q interchanged) to (ii). \square

Let H be a Hilbert space, K and M closed subspaces of H such that $K^\perp \subset M$. If we split $v \in M$ as $P_K v + P_{K^\perp} v$, then in this the second vector is in M and hence so is the first vector; thus $P_K v \in M$ if $v \in M$. This means that the point $P_K v$ on K closest to v is in fact in $K \cap M$. Thus

$$P_K v = P_{K \cap M} v \quad \text{if } v \in M. \quad (3.2.22)$$

We conclude with another observation on how subspaces are situated relative to each other.

Proposition 3.2.7. *Let R be an orthogonal projection in a Hilbert space H and K a closed subspace of H . Then*

$$\text{Im}(R) \cap K^\perp = \text{Im}(R) \cap [R(K)]^\perp. \quad (3.2.23)$$

Moreover, the orthogonal complement of $\text{Im}(R) \cap K$ within $\text{Im}(R)$ is the image under R of the orthogonal complement of $\text{Im}(R) \cap K$:

$$\text{Im}(R) \ominus (\text{Im}(R) \cap K) = R([\text{Im}(R) \cap K]^\perp). \quad (3.2.24)$$

Take $R = P_{Z_N}$, the orthogonal projection onto the closed subspace Z_N , and $K = \ker Q$, where Q is any continuous linear mapping on H , we have

$$Z_N \cap (\ker Q)^\perp = Z_N \cap [P_{Z_N}(\ker Q)]^\perp \quad (3.2.25)$$

and

$$P_{Z_N}([\ker Q_N]^\perp) = Z_N \ominus (\ker Q_N), \quad (3.2.26)$$

where $Q_N = Q|_{Z_N}$.

Proof. If $v \in \text{Im}(R) \cap K^\perp$ then $Rv = v$ and for any $z \in K$ we have

$$\langle v, R(z) \rangle = \langle Rv, z \rangle = \langle v, z \rangle = 0,$$

and so the vector v in $\text{Im}(R)$ is orthogonal to $R(K)$. Conversely, if $v \in \text{Im}(R)$ is orthogonal to $R(K)$ then for any $z \in K$ we have

$$\langle v, z \rangle = \langle Rv, z \rangle = \langle v, Rz \rangle = 0,$$

which shows that v is orthogonal to K . This establishes the equality (3.2.23).

The equality

$$\langle v, Rz \rangle = \langle v, R^2z \rangle = \langle Rv, Rz \rangle \quad (3.2.27)$$

holds for all $v, z \in H$. If Rv is orthogonal to $\text{Im}(R) \cap K$ then $\langle Rv, Rz \rangle = 0$ for all $Rz \in K$ and so, by (3.2.27), v is orthogonal to $\text{Im}(R) \cap K$. Conversely, if v is orthogonal to $\text{Im}(R) \cap K$ then the first term in (3.2.27) is 0 whenever $Rz \in K$, and so by (3.2.27) it follows that Rv is orthogonal to $\text{Im}(R) \cap K$. \square

Chapter 4

Limit of Spherical Integrals

We will show in this chapter the complete proof of our main result Theorem 2.1.1 using the results in the previous two chapters.

4.1 The Limit of Spherical Integrals

In order to prove Theorem 2.1.1 we begin by proving the following result for the limit of spherical integrals described above.

Theorem 4.1.1. *Consider an affine subspace of l^2 given by $Q^{-1}(w^0)$, where $Q : l^2 \rightarrow W$ is a continuous linear surjection onto a finite-dimensional inner-product space W . Suppose that the projection $P_{(k)} : l^2 \rightarrow \mathbb{R}^k : z \mapsto z_{(k)}$ maps $\ker Q$ onto \mathbb{R}^k . Let $S_{Z_N}(a)$ be the sphere of radius a in the subspace $Z_N = \mathbb{R}^N \oplus \{0\}$ in l^2 . Let ϕ be a bounded Borel function on \mathbb{R}^k and let f be the function obtained by extending ϕ to l^2 by setting*

$$f(x) = \phi(x_1, \dots, x_k) \quad \text{for all } x \in l^2.$$

Then

$$\begin{aligned}
& \lim_{N \rightarrow \infty} \int_{S_{Z_N}(\sqrt{N}) \cap Q_N^{-1}(w^0)} f \, d\bar{\sigma} \\
&= (2\pi)^{-k/2} \int_{x \in \mathbb{R}^k} \phi(x) \exp\left(-\frac{\langle (L_0 L_0^*)^{-1}(x - z^0_{(k)}), x - z^0_{(k)} \rangle}{2}\right) \frac{dx}{\sqrt{\det(L_0 L_0^*)}},
\end{aligned} \tag{4.1.1}$$

where L_0 is the restriction of the projection $P_{(k)}$ to $\ker Q \ominus \ker P_{(k)}$ and $z^0 = Q^*(QQ^*)^{-1}(w^0)$ is the point on A closest to the origin.

As before, the notation $\ker Q \ominus \ker P_{(k)}$ means the orthogonal complement of $\ker Q \cap \ker P_{(k)}$ within $\ker Q$. Thus L_0 is the restriction of $z \mapsto z_{(k)}$ to the subspace of $\ker Q$ orthogonal to $\ker Q \cap \ker P_{(k)}$.

We will begin by using the spherical disintegration formula we constructed in chapter 3. We revisit the formula here and adapt it specifically for our purposes.

4.1.1 Integrals of functions on subspaces

We consider now a function f on $\mathbb{R}^{d+1} = \mathbb{R}^k \oplus \mathbb{R}^{d+1-k}$ that depends only on the first k components:

$$f(x, y) = \phi(x).$$

We denote by $P_{(k)}$ the projection onto the first k coordinates:

$$P_{(k)}z = z_{(k)} = (z_1, \dots, z_k) \in \mathbb{R}^k. \tag{4.1.2}$$

For convenience let us assume that $P_{(k)}$ maps $\ker Q$ onto \mathbb{R}^k . Then, applying the disintegration formula of Theorem 2.3.2, we have

$$\begin{aligned} & \int_{S^d(a) \cap Q^{-1}(w^0)} f \, d\sigma \\ &= \int_{x \in D} \phi(x) V_a(x) \frac{a_{z^0}}{\sqrt{a_{z^0}^2 - \|L_0^{-1}(x - x^0)\|^2}} \frac{dx}{|\det L_0|} \end{aligned} \quad (4.1.3)$$

wherein D is the set of all $x \in \mathbb{R}^k$ for which the term under $\sqrt{\dots}$ is positive, and

$$V_a(x) = \text{Vol} \left(S^d(a) \cap Q^{-1}(w^0) \cap P_{(k)}^{-1}(x) \right), \quad (4.1.4)$$

is the volume of the $(d - m - k)$ -dimensional sphere of radius given by (??):

$$\sqrt{a_{z^0}^2 - \|L_0^{-1}(x - x^0)\|^2}. \quad (4.1.5)$$

Using (2.3.45) we have

$$\|L_0^{-1}w^0\|^2 = \langle (L_0 L_0^*)^{-1}w^0, w^0 \rangle = \left\langle \left(I - \sum_{a=1}^m \|(u_a)_{(k)}\|^2 P_{k,a} \right)^{-1} w^0, w^0 \right\rangle. \quad (4.1.6)$$

The volume, or ‘surface area’, in the integrand on the right in (4.1.3) is therefore:

$$\begin{aligned} & \text{Vol} \left(S^d(a) \cap Q^{-1}(w^0) \cap P_{(k)}^{-1}(x) \right) \\ &= c_{d-k-m} \left[a_{z^0}^2 - \|L_0^{-1}(x - x^0)\|^2 \right]^{\frac{d-k-m}{2}} \end{aligned} \quad (4.1.7)$$

where c_{d-k-m} is the surface measure of the $(d-k-m)$ -dimensional sphere given, for all j , by the formula:

$$c_j = 2 \frac{\pi^{\frac{j+1}{2}}}{\Gamma\left(\frac{j+1}{2}\right)}. \quad (4.1.8)$$

We can then rewrite (4.1.3) as

$$\int_{S^d(a) \cap Q^{-1}(w^0)} f d\sigma = c_{d-k-m} \int_{x \in D} I'(x) \frac{dx}{|\det L_0|}, \quad (4.1.9)$$

where

$$I'(x) = \phi(x) a_{z^0} \left[a_{z^0}^2 - \|L_0^{-1}(x - x^0)\|^2 \right]^{\frac{d-k-m-1}{2}}. \quad (4.1.10)$$

The sphere $S^d(a) \cap Q^{-1}(w^0)$ has dimension $d-m$ and its volume (“surface area”) is

$$c_{d-m} a_{z^0}^{d-m}.$$

So, using the *normalized surface measure* $\bar{\sigma}$ on the sphere $S^d(a) \cap Q^{-1}(w^0)$, we have

$$\int_{S^d(a) \cap Q^{-1}(w^0)} f d\bar{\sigma} = \frac{c_{d-k-m}}{c_{d-m} a_{z^0}^{d-m}} \int_{x \in D} I'(x) \frac{dx}{|\det L_0|}, \quad (4.1.11)$$

where $I'(x)$ is as in (4.1.10).

Now to find the value of the limit of spherical integrals in Theorem 4.1.1, we first apply our disintegration theorem (4.1.11) to the integral in the left-hand side of (4.1.1) and factor $a_{z^0, N}$ out of the various terms on the right in (2.3.26) leads to

$$\int_{S_{Z_N}(a) \cap Q_N^{-1}(w^0)} f d\bar{\sigma} = \frac{c_{d-k-m}}{a_{z^0, N}^k c_{d-m}} \int_{\mathbb{R}^k} I_N(x) \frac{dx}{|\det L_{0, N}|}, \quad (4.1.12)$$

where

$$I_N(x) = \phi(x) \left\{ 1 - a_{z^0, N}^{-2} \left\| L_{0, N}^{-1}(x - z^{0, N}_{(k)}) \right\|^2 \right\}^{\frac{d-k-m-1}{2}} 1_{D_N}(x) \quad (4.1.13)$$

and the set D_N is, as before, comprised of all points x for which the term within $\{\dots\}$ is non-negative. And now we will take the limit $N \rightarrow \infty$ of the right side of (4.1.12). To do this we will look at the integral in three parts.

4.1.2 The Limit of the Constant Term

Let N_0 be a value of N for which Q_N is surjective and $P_{N, k}(\ker Q_N) = \mathbb{R}^k$. Then for $N \geq N_0$,

$$\left(N - \|z^{0, N_0}\|^2 \right)^{k/2} \leq a_{z^0, N}^k \leq \left(N - \|z^0\|^2 \right)^{k/2}, \quad (4.1.14)$$

because the point $z^{0, N}$ is at most as far from the origin as z^{0, N_0} and at least as far as z^0 . Then, with c_j as given in (4.1.8), we have, as $N \rightarrow \infty$,

$$\begin{aligned} \frac{c_{d-k-m}}{a_{z^0, N}^k c_{d-m}} &\sim \frac{c_{N-1-k-m}}{N^{\frac{k}{2}} c_{N-1-m}} = \frac{\pi^{\frac{N-k-m}{2}} \Gamma\left(\frac{N-m}{2}\right)}{N^{\frac{k}{2}} \Gamma\left(\frac{N-k-m}{2}\right) \pi^{\frac{N-m}{2}}} \\ &= \pi^{-k/2} \frac{\Gamma\left(\frac{N-m}{2}\right)}{N^{\frac{k}{2}} \Gamma\left(\frac{N-k-m}{2}\right)} \\ &\sim \pi^{-k/2} \frac{1}{N^{\frac{k}{2}}} \left(\frac{N-k-m}{2} \right)^{k/2} \\ &\sim (2\pi)^{-k/2}. \end{aligned} \quad (4.1.15)$$

4.1.3 The Limit of the Determinant

We have seen the following result earlier in (2.3.43) in the context of $H = l^2$.

Proposition 4.1.2. *Let H , W , and X be real Hilbert spaces, and $Q : H \rightarrow W$ and $\mathcal{L} : H \rightarrow X$ be continuous linear mappings. Let*

$$L = \mathcal{L}|_{\ker Q}$$

and let L_0 be the restriction of L to the orthogonal complement of $\ker L$ within $\ker Q$.

Then

$$L_0 L_0^* = \mathcal{L} P_{\ker Q} \mathcal{L}^*, \quad (4.1.16)$$

where $P_{\ker Q}$ is the orthogonal projection in H onto the subspace $\ker Q$. Here the adjoint L_0^* has domain X and codomain $D(L_0)$, the orthogonal complement of $\ker L$ in $\ker Q$.

Proof. Let

$$J : D(L_0) \rightarrow H$$

be the inclusion map. The adjoint J^* is the orthogonal projection of H onto the subspace $D(L_0)$. Then

$$J J^* : H \rightarrow H$$

is the orthogonal projection in H with image being the subspace $D(L_0)$.

Next we note that

$$L_0 = \mathcal{L} P_{\ker Q} J, \quad (4.1.17)$$

since the value of both sides for any $z \in D(L_0)$ is $\mathcal{L}z$. Then

$$L_0 L_0^* = \mathcal{L} P_{\ker Q} J J^* P_{\ker Q} \mathcal{L}^*. \quad (4.1.18)$$

Now for any $x \in X$ the element $P_{\ker Q} \mathcal{L}^* x$ is in $\ker Q$ and is orthogonal to any element $v \in \ker \mathcal{L} \cap \ker Q$ because

$$\langle P_{\ker Q} \mathcal{L}^* x, v \rangle = \langle x, \mathcal{L} P_{\ker Q} v \rangle = \langle x, \mathcal{L} v \rangle = \langle x, 0 \rangle = 0. \quad (4.1.19)$$

Thus $P_{\ker Q} \mathcal{L}^* x$ lies in $D(L_0)$. Thus, recalling our observation above about JJ^* , we have

$$(JJ^*)P_{\ker Q} \mathcal{L}^* = P_{\ker Q} \mathcal{L}^*.$$

Using this in (4.1.18) we have:

$$L_0 L_0^* = \mathcal{L} P_{\ker Q} P_{\ker Q} \mathcal{L}^* = \mathcal{L} P_{\ker Q} \mathcal{L}^*. \quad (4.1.20)$$

□

We can now determine the limit of $L_{0,N} L_{0,N}^*$ as $N \rightarrow \infty$.

Proposition 4.1.3. *Let H be a Hilbert space, and $Z_1 \subset Z_2 \subset \dots$ a sequence of finite-dimensional subspaces of H whose union is dense in H . Let $Q : H \rightarrow W$ and $\mathcal{L} : H \rightarrow X$ be surjective continuous linear functions, where X is finite-dimensional, and \mathcal{L}_N and Q_N their restrictions to Z_N :*

$$Q_N = Q|_{Z_N} \quad \text{and} \quad \mathcal{L}_N = \mathcal{L}|_{Z_N}.$$

Suppose \mathcal{L} maps $\ker Q$ surjectively onto X . Then:

(i) \mathcal{L}_N maps $\ker Q_N$ surjectively onto X for large N ;

(ii) the operators $L_{0,N}L_{0,N}^*$ on W converge to $L_0L_0^*$:

$$\lim_{N \rightarrow \infty} L_{0,N}L_{0,N}^* = L_0L_0^*, \quad (4.1.21)$$

where L_0 is the restriction of \mathcal{L} to $\ker Q \ominus \ker \mathcal{L}$, the orthogonal complement of $\ker Q \cap \ker \mathcal{L}$ within $\ker Q$, and $L_{0,N}$ is the restriction \mathcal{L} to $\ker Q_N \ominus \ker \mathcal{L}_N$.

Let us note what $L_{0,N}$ is more explicitly:

$$L_{0,N} : \ker Q_N \cap [\ker Q_N \cap (\ker \mathcal{L}_N)^\perp] \rightarrow X : z \mapsto \mathcal{L}_N z = \mathcal{L}z. \quad (4.1.22)$$

The statement that \mathcal{L}_N maps $\ker Q_N = Z_N \cap \ker Q$ surjectively onto X for large N therefore means that $L_{0,N}$ is an isomorphism for large N .

We note that, as a consequence of (4.1.21),

$$\lim_{N \rightarrow \infty} \det(L_{0,N}L_{0,N}^*) = \det(L_0L_0^*). \quad (4.1.23)$$

We will need this in working out the limit of the right hand side in the disintegration formula (4.1.37).

Proof. By Proposition 4.1.2 we have:

$$L_{0,N}L_{0,N}^* = \mathcal{L}P_{\ker Q_N}\mathcal{L}^* \quad \text{and} \quad L_0L_0^* = \mathcal{L}P_{\ker Q}\mathcal{L}^*. \quad (4.1.24)$$

Now for any $v \in H$ we have

$$\begin{aligned}
\lim_{N \rightarrow \infty} L_{0,N} L_{0,N}^* v &= \lim_{N \rightarrow \infty} \mathcal{L}(P_{\ker Q_N} \mathcal{L}^* v) \\
&= \mathcal{L}\left(\lim_{N \rightarrow \infty} P_{\ker Q_N}(\mathcal{L}^* v)\right) \\
&= \mathcal{L}P_{\ker Q}(\mathcal{L}^* v) \quad \text{by Theorem 3.2.2,} \\
&= L_0 L_0^* v \quad \text{by (4.1.24).}
\end{aligned} \tag{4.1.25}$$

Since X is finite-dimensional this pointwise convergence implies the convergence of operators which proves (4.1.21). \square

4.1.4 The Limit of the Integrand

Let N_0 be any value of N for which $L_{0,N}$ is surjective onto X . Then let us also recall that, for $N > N_0$, the integrand I_N given in (4.1.13):

$$I_N(x) = \phi(x) \left\{ 1 - a_{z^{0,N}(k)}^{-2} \|L_{0,N}^{-1}(x - z^{0,N}(k))\|^2 \right\}^{\frac{d-k-m-1}{2}} 1_{D_N}(x), \tag{4.1.26}$$

where D_N is the set of all $x \in \mathbb{R}^k$ for which the term within $\{\dots\}$ is positive.

Then we observe that

$$\begin{aligned}
\|L_{0,N}^{-1}(x - z^{0,N}(k))\|^2 &\leq \|(L_{0,N} L_{0,N}^*)^{-1}\| \|x - z^{0,N}(k)\|^2 \\
&\leq C(\|x\|^2 + \|z^{0,N}(k)\|^2) \\
&\leq C(\|x\|^2 + \|z_{(k)}^{0,N}\|^2) \\
&\leq C(\|x\|^2 + \|z^{0,N}\|^2),
\end{aligned} \tag{4.1.27}$$

where

$$C = \sup_{N \geq 1} 2 \|(L_{0,N} L_{0,N}^*)^{-1}\| < \infty, \quad (4.1.28)$$

because of the finiteness of the limit of $\|(L_{0,N} L_{0,N}^*)^{-1}\|$ as $N \rightarrow \infty$, as seen in (4.1.21).

Moreover,

$$a_{z^0, N}^2 = N - \|z^{0,N}\|^2$$

lies between $N - \|z^{0, N_0}\|^2$ and $N - \|z^0\|^2$, because, from the definition of $z^{0,N}$ as the point on $Q^{-1}(w^0) \cap Z_N$ closest to the origin we have,

$$\|z^{0, N_0}\| \leq \|z^{0,N}\| \leq \|z^0\|. \quad (4.1.29)$$

Consequently,

$$\lim_{N \rightarrow \infty} a_{z^0, N}^{-2} \|L_{0,N}^{-1}(x - z^{0,N}_{(k)})\|^2 = 0. \quad (4.1.30)$$

Hence, any given point $x \in \mathbb{R}^k$ lies in D_N for N large enough.

As we have just seen, the term within $\{\dots\}$ in I_N goes to 1 as $N \rightarrow \infty$; this implies

$$\begin{aligned} & \lim_{N \rightarrow \infty} \left\{ 1 - a_{z^0, N}^{-2} \|L_{0,N}^{-1}(x - z^{0,N}_{(k)})\|^2 \right\}^{\frac{d-k-m-1}{2}} \\ &= \lim_{N \rightarrow \infty} \left\{ 1 - a_{z^0, N}^{-2} \|L_{0,N}^{-1}(x - z^{0,N}_{(k)})\|^2 \right\}^{\frac{N}{2}}. \end{aligned} \quad (4.1.31)$$

To work out the limit on the right side of (4.1.31), let us note first that, by

dominated convergence,

$$\begin{aligned}
(1 + x_N)^N &= 1 + Nx_N + \frac{N(N-1)}{2!}x_N^2 + \dots \\
&= 1 + Nx_N + \frac{1 * (1 - N^{-1})}{2!}(Nx_N)^2 + \dots \\
&\rightarrow \exp\left(\lim_{N \rightarrow \infty} Nx_N\right), \quad \text{if } \lim_{N \rightarrow \infty} Nx_N \text{ exists and is finite.}
\end{aligned} \tag{4.1.32}$$

In the present context

$$Nx_N = -Na_{z^0, N}^{-2} \left\| L_{0, N}^{-1}(x - z^{0, N}_{(k)}) \right\|^2. \tag{4.1.33}$$

For N large enough, independent of x , this is bounded above by

$$2C(\|x\|^2 + \|z^{0, N_0}\|^2) \tag{4.1.34}$$

because

$$Na_{z^0, N}^{-2} \leq N/(N - \|z^{0, N_0}\|^2).$$

Moreover,

$$\begin{aligned}
&\lim_{N \rightarrow \infty} Nx_N \\
&= -1 * \lim_{N \rightarrow \infty} \langle (L_{0, N} L_{0, N}^*)^{-1}(x - z^{0, N}_{(k)}), (x - z^{0, N}_{(k)}) \rangle \\
&= -\langle (L_0 L_0^*)^{-1}(x - z^0_{(k)}), x - z^0_{(k)} \rangle
\end{aligned} \tag{4.1.35}$$

where we have used the limiting formula (4.1.21) as well as Proposition ?? (which implies that $z^{0, N}_{(k)} \rightarrow z^0_{(k)}$ as $N \rightarrow \infty$).

Returning to (4.1.31) we have

$$\begin{aligned} \lim_{N \rightarrow \infty} \left\{ 1 - a_{z^0, N}^{-2} \left\| L_{0, N}^{-1}(x - z^{0, N}_{(k)}) \right\|^2 \right\}^{\frac{N}{2}} 1_{D_N}(x) \\ = \exp \left(-\frac{1}{2} \langle (L_0 L_0^*)^{-1}(x - z^0_{(k)}), x - z^0_{(k)} \rangle \right). \end{aligned} \quad (4.1.36)$$

Now finally we give the proof of Theorem (4.1.1) using the limits we have found above. We break the proof into two parts first proving it for a function in L^1 and then for a bounded function.

4.1.5 Proof of Theorem 4.1.1 for $\phi \in L^1(\mathbb{R}^k)$

Looking back at (4.1.12) we have

$$\lim_{N \rightarrow \infty} \int_{S_{Z_N}(a) \cap Q_N^{-1}(w^0)} f d\bar{\sigma} = \lim_{N \rightarrow \infty} \frac{c_{d-k-m}}{a_{z^0, N}^k c_{d-m}} \int_{\mathbb{R}^k} I_N \frac{dx}{|\det L_{0, N}|}, \quad (4.1.37)$$

where

$$I_N = \phi(x) \left\{ 1 - a_{z^0, N}^{-2} \left\| L_{0, N}^{-1}(x - z^{0, N}_{(k)}) \right\|^2 \right\}^{\frac{d-k-m-1}{2}} 1_{D_N}(x). \quad (4.1.38)$$

We have already determined the limits of the constant term outside the integral (in (4.1.15)), as well as those of the full integrand on the right hand side. Moreover, we observe that

$$|I_N| \leq |\phi(x)|.$$

Thus, assuming that ϕ is integrable over \mathbb{R}^k , we can apply dominated convergence to conclude that

$$\begin{aligned} & \lim_{N \rightarrow \infty} \int_{S_{Z_N}(a) \cap Q_N^{-1}(w^0)} f \, d\bar{\sigma} \\ &= (2\pi)^{-k/2} \int_{\mathbb{R}^k} \phi(x) \exp\left(-\frac{1}{2} \langle (L_0 L_0^*)^{-1} (x - z_{(k)}^0), x - z_{(k)}^0 \rangle\right) \frac{dx}{\sqrt{\det(L_0 L_0^*)}}. \end{aligned} \quad (4.1.39)$$

4.1.6 Proof of Theorem 4.1.1 for more general ϕ

Proof. Let ϕ be any bounded Borel function on \mathbb{R}^k . Let us recall from (4.1.12) that:

$$\int_{S_{Z_N}(a) \cap Q_N^{-1}(w^0)} f \, d\bar{\sigma} = \int_{\mathbb{R}^k} \phi(x) \, d\mu_N(x), \quad (4.1.40)$$

where

$$f(x) = \phi(x_1, \dots, x_k) \quad \text{for all } x = (x_1, x_2, \dots)$$

and

$$\begin{aligned} & d\mu_N(x) \\ &= \frac{c_{d-k-m}}{a_{z^0, N}^k c_{d-m}} \left\{ 1 - a_{z^0, N}^{-2} \|L_{0, N}^{-1} (x - z_{(k)}^{0, N})\|^2 \right\}^{\frac{d-k-m-1}{2}} 1_{D_N}(x) \frac{dx}{|\det L_{0, N}|}. \end{aligned} \quad (4.1.41)$$

Taking $\phi = 1$ in (4.1.40) we see that μ_N is a probability measure. Now let μ_∞ be the Gaussian measure on \mathbb{R}^k given by

$$d\mu_\infty(x) = (2\pi)^{-k/2} \exp\left(-\frac{1}{2} \langle (L_0 L_0^*)^{-1} (x - z_{(k)}^0), x - z_{(k)}^0 \rangle\right) \frac{dx}{\sqrt{\det(L_0 L_0^*)}}. \quad (4.1.42)$$

With this notation, the result (4.1.39) says that

$$\lim_{N \rightarrow \infty} \int_{\mathbb{R}^k} \psi d\mu_N = \int_{\mathbb{R}^k} \psi d\mu_\infty \quad \text{for all } \psi \in L^1(\mathbb{R}^k). \quad (4.1.43)$$

Taking ψ to be the indicator function of any compact set B we have:

$$\lim_{N \rightarrow \infty} \mu_N(B) = \mu_\infty(B). \quad (4.1.44)$$

Since μ_N and μ_∞ are probability measures, this also implies

$$\lim_{N \rightarrow \infty} \mu_N(B^c) = \mu_\infty(B^c). \quad (4.1.45)$$

Now let $\epsilon > 0$. Then there is a compact set $B_\epsilon \subset \mathbb{R}^k$ for which

$$\mu_\infty(B_\epsilon) > 1 - \epsilon.$$

We have

$$\begin{aligned} \int_{\mathbb{R}^k} \phi d\mu_N - \int_{\mathbb{R}^k} \phi d\mu_\infty &= \int_{\mathbb{R}^k} \phi 1_{B_\epsilon} d\mu_N - \int_{\mathbb{R}^k} \phi 1_{B_\epsilon} d\mu_\infty \\ &\quad + \int_{\mathbb{R}^k} \phi 1_{B_\epsilon^c} d\mu_N - \int_{\mathbb{R}^k} \phi 1_{B_\epsilon^c} d\mu_\infty. \end{aligned} \quad (4.1.46)$$

Taking ψ to be $\phi 1_{B_\epsilon}$, which is integrable over \mathbb{R}^k , in (4.1.43), we have

$$\lim_{N \rightarrow \infty} \left[\int_{\mathbb{R}^k} \phi 1_{B_\epsilon} d\mu_N - \int_{\mathbb{R}^k} \phi 1_{B_\epsilon} d\mu_\infty \right] = 0. \quad (4.1.47)$$

Next,

$$\begin{aligned}
\limsup_{N \rightarrow \infty} \left| \int_{\mathbb{R}^k} \phi 1_{B_\epsilon^c} d\mu_N \right| &\leq \|\phi\|_{\text{sup}} \lim_{N \rightarrow \infty} \mu_N(B_\epsilon^c) \\
&= \|\phi\|_{\text{sup}} \mu_\infty(B_\epsilon^c) \\
&\leq \|\phi\|_{\text{sup}} \epsilon.
\end{aligned} \tag{4.1.48}$$

Using these observations in (4.1.46) we have

$$\limsup_{N \rightarrow \infty} \left| \int_{\mathbb{R}^k} \phi d\mu_N - \int_{\mathbb{R}^k} \phi d\mu_\infty \right| \leq 0 + 2 \|\phi\|_{\text{sup}} \epsilon. \tag{4.1.49}$$

Since $\epsilon > 0$ is arbitrary, this establishes our goal:

$$\lim_{N \rightarrow \infty} \int_{\mathbb{R}^k} \phi d\mu_N = \int_{\mathbb{R}^k} \phi d\mu_\infty. \tag{4.1.50}$$

This completes the proof of Theorem 4.1.1. □

4.2 The Proof of Theorem 2.1.1

Now in this last section we give the proof for Theorem 2.1.1. We restate the theorem here for easy reference:

Theorem 4.2.1. *Let A be a finite-codimension affine subspace in l^2 . Let k be a positive integer and suppose that the image of A under the coordinate projection $\pi_{(k)}$ is all of \mathbb{R}^k . Let ϕ be a bounded Borel function on \mathbb{R}^k . Then*

$$\lim_{N \rightarrow \infty} \int_{S_{A_N}} \phi(x_1, \dots, x_k) d\bar{\sigma}(x_1, \dots, x_N) = \int_{\mathbb{R}^k} \phi(z_{(k)}) d\mu(z), \tag{4.2.1}$$

where $\bar{\sigma}$ is the normalized standard surface area measure on S_{A_N} and μ is the probability measure on \mathbb{R}^∞ specified by the characteristic function

$$\int_{\mathbb{R}^\infty} \exp(i\langle t, x \rangle) d\mu(x) = \exp\left(i\langle t, z^0 \rangle - \frac{1}{2} \|P_0 t\|^2\right) \quad \text{for all } t \in \mathbb{R}_0^\infty, \quad (4.2.2)$$

where z^0 is the point on A closest to the origin and P_0 is the orthogonal projection in l^2 onto the subspace $A - z^0$.

Also let us recall some of the notation and definitions used so far. We have $Q : l^2 \rightarrow \mathbb{R}^m$ as a continuous linear surjection such that for some $w^0 \in \mathbb{R}^m$, $A = Q^{-1}(w^0)$. Let $J_N : \mathbb{R}^N \rightarrow l^2$ be the inclusion operator and $Q_N = QJ_N$. This gives us $A_N = Q_N^{-1}(w^0)$.

We have from Theorem 4.1.1 in the previous section:

$$\begin{aligned} & \lim_{N \rightarrow \infty} \int_{S_{Z_N}(\sqrt{N}) \cap Q_N^{-1}(w^0)} f d\bar{\sigma} \\ &= (2\pi)^{-k/2} \int_{x \in \mathbb{R}^k} \phi(x) \exp\left(-\frac{\langle (L_0 L_0^*)^{-1}(x - z_{(k)}^0), x - z_{(k)}^0 \rangle}{2}\right) \frac{dx}{\sqrt{\det(L_0 L_0^*)}}, \end{aligned} \quad (4.2.3)$$

Let μ be the measure on \mathbb{R}^∞ given in Theorem 2.1.1. Let us determine the pushforward measure $\pi_{(k)*}\mu$ of μ to \mathbb{R}^k :

$$\pi_{(k)*}\mu(S) = \mu(\pi_{(k)}^{-1}(S)), \quad \text{for all Borel } S \subset \mathbb{R}^k, \quad (4.2.4)$$

where

$$\pi_{(k)} : \mathbb{R}^\infty \rightarrow \mathbb{R}^k : z \mapsto z_{(k)} = (z_1, \dots, z_k)$$

is the projection on the first k coordinates.

Let \mathcal{L} be the restriction of $\pi_{(k)}$ to l^2 :

$$\mathcal{L} : l^2 \rightarrow X = \mathbb{R}^k : z \mapsto z_{(k)}. \quad (4.2.5)$$

Then the adjoint is

$$\mathcal{L}^* : X \rightarrow l^2 : v \mapsto (v, 0, 0, \dots).$$

The image of the orthogonal projection P_0 is the affine space $A = Q^{-1}(w^0)$. Thus

$$P_0 = P_{\ker Q}.$$

Then for any $t \in \mathbb{R}^k$, we have

$$\begin{aligned} \int_{\mathbb{R}^k} \exp(i\langle t, x \rangle) d\pi_{(k)*} \mu(x) &= \int_{\mathbb{R}^\infty} \exp(i\langle t, \pi_{(k)}x \rangle) d\mu(x) \\ &= \int_{\mathbb{R}^\infty} \exp(i\langle \mathcal{L}^*t, x \rangle) d\mu(x) \\ &= \exp\left(i\langle \mathcal{L}^*t, z^0 \rangle - \frac{1}{2}\|P_0\mathcal{L}^*t\|^2\right) \\ &= \exp\left(i\langle t, \mathcal{L}z^0 \rangle - \frac{1}{2}\langle P_{\ker Q}\mathcal{L}^*t, P_{\ker Q}\mathcal{L}^*t \rangle\right) \\ &= \exp\left(i\langle t, \mathcal{L}z^0 \rangle - \frac{1}{2}\langle \mathcal{L}P_{\ker Q}\mathcal{L}^*t, t \rangle\right). \end{aligned} \quad (4.2.6)$$

The measure μ_∞ in (4.1.42) is given by

$$d\mu_\infty(x) = (2\pi)^{-k/2} \exp\left(-\frac{1}{2}\langle (L_0L_0^*)^{-1}(x - z_{(k)}^0), x - z_{(k)}^0 \rangle\right) \frac{dx}{\sqrt{\det(L_0L_0^*)}}, \quad (4.2.7)$$

where $L_0 : \ker Q \rightarrow \mathbb{R}^k$ is the restriction of \mathcal{L} to the Hilbert space $\ker Q \subset l^2$. Its

characteristic function is given by

$$\begin{aligned}
& \int_{\mathbb{R}^k} \exp(i\langle t, x \rangle) d\mu_\infty(x) \\
&= \int_{\mathbb{R}^k} \exp\left(i\langle (L_0 L_0^*)^{1/2} t, y \rangle + i\langle t, z_{(k)}^0 \rangle (2\pi)^{-k/2} - \frac{\|y\|^2}{2}\right) dy \\
&= \exp\left(i\langle t, z_{(k)}^0 \rangle - \frac{1}{2} \|(L_0 L_0^*)^{1/2} t\|^2\right)
\end{aligned} \tag{4.2.8}$$

where, in the first line, we used the natural change of variables $x = (L_0 L_0^*)^{1/2} y + z_{(k)}^0$, and for the second line we used a standard formula for Gaussian integration. Now we recall from (4.1.20) that $L_0 L_0^*$ equals $\mathcal{L} P_{\ker Q} \mathcal{L}^*$. Thus,

$$\int_{\mathbb{R}^k} \exp(i\langle t, x \rangle) d\mu_\infty(x) = \exp\left(i\langle t, \mathcal{L} z^0 \rangle - \frac{1}{2} \langle \mathcal{L} P_{\ker Q} \mathcal{L}^* t, t \rangle\right). \tag{4.2.9}$$

This agrees exactly with the characteristic function for $\pi_{(k)*} \mu$ we obtained in (4.2.6).

Hence

$$\pi_{(k)*} \mu = \mu_\infty. \tag{4.2.10}$$

Combining this with the result of Theorem 4.1.1 given in (4.1.50), we conclude that

$$\begin{aligned}
& \lim_{N \rightarrow \infty} \int_{S_{Z_N}(a) \cap Q_N^{-1}(w^0)} \phi(x_1, \dots, x_k) d\bar{\sigma}(x_1, \dots, x_N) \\
&= \int_{\mathbb{R}^k} \phi d\mu_\infty \\
&= \int_{\mathbb{R}^k} \phi d\pi_{(k)*} \mu = \int_{\mathbb{R}^\infty} \phi \circ \pi_{(k)} d\mu.
\end{aligned} \tag{4.2.11}$$

This completes the proof of Theorem 2.1.1.

Chapter 5

Abstract Wiener Spaces

In this chapter we will prove a version of the result Theorem 2.1.1 in terms of more general Hilbert and Banach spaces. This type of infinite dimensional setting is called the Abstract Wiener Space. We will provide at the beginning of the chapter a very basic introduction to Abstract Wiener Spaces.

5.1 Introduction to Abstract Wiener Spaces

Abstract Wiener Spaces were originally developed by Gross and we reference his original paper on the topic [4] for the interested reader. We also recommend for beginners the introductory notes on the topic by Eldredge [3] and the text by Kuo [10].

Let H be a real separable Hilbert space with Hilbert norm $\|\cdot\|_H$. We define a measurable norm $|\cdot|$ on H by the property for that for any $\epsilon > 0$ there exists a finite dimensional subspace F_ϵ of H such that for any finite dimensional subspace $F' \subset H$

orthogonal to F_ϵ then

$$\gamma_{F'}\{x \in F'; |x| > \epsilon\} < \epsilon$$

where $\gamma_{F'}$ is the Gaussian measure on F' .

Now let B be the completion of H with respect to the measurable norm $|\cdot|$. Now the injection map,

$$j : (H, \|\cdot\|_H) \rightarrow (B, |\cdot|)$$

is continuous. Note the distinction between this and the usual injection map has to do with the norm with which the topology on the space is defined. Thus for any $\phi \in B^*$ we can restrict to a continuous linear function on H given by

$$\phi(j(x)) = \langle j^*\phi, x \rangle_H \quad \text{for all } x \in H.$$

for a unique element $j^*\phi \in H$. Thus we have a continuous linear injection

$$j^* : B^* \rightarrow H$$

and the image of j^* is a dense subspace of H .

Now we use the Abstract Wiener Space setting to formulate our main result in this more general Banach space.

5.2 Main Result in Abstract Wiener Space

With notation as above, let L be a closed affine subspace of H . Then, as shown in [9], there is a Borel measure μ_L on B such that every $\phi \in B^*$, viewed as a random

variable defined on B , has Gaussian distribution specified by

$$\int_B \exp(it\phi) d\mu_L = \exp\left(it\langle p_L, j^*\phi \rangle - \frac{t^2}{2} \|P_0(j^*\phi)\|^2\right) \quad \text{for all } t \in \mathbb{R}, \quad (5.2.1)$$

where p_L is the point on L closest to 0 and $P_0 : H \rightarrow H$ is the orthogonal projection onto the subspace

$$L_0 = L - p_L.$$

The linear mapping

$$j^*(B^*) \rightarrow L^2(B, \mu_L) : j^*(\phi) \mapsto \phi, \quad (5.2.2)$$

extends to a continuous linear mapping

$$I_L : H \rightarrow L^2(B, \mu_L). \quad (5.2.3)$$

Moreover,

$$\int_B \exp(iI_L(h)) d\mu_L = \exp\left(i\langle p_L, h \rangle - \frac{1}{2} \|P_0 h\|^2\right) \quad \text{for all } h \in H. \quad (5.2.4)$$

To compare with a familiar situation we observe that if H is finite-dimensional then $B = H$, and:

$$I_L(h) = \langle \cdot, h \rangle_H. \quad (5.2.5)$$

The Gaussian measure μ_L is supported on the closure \bar{L} of $j(L)$ inside B .

Now let us see how one can extend a function from a finite-dimensional subspace V of H to a function on the Banach space B . First let us note that if W is a closed

subspace of H that contains V then we have the function f_W on W given by

$$f_W = f_V \circ P_V^W, \quad (5.2.6)$$

where

$$P_V^W : W \rightarrow V$$

is the orthogonal projection onto the subspace $V \subset W$. Suppose h_1, \dots, h_k is an orthonormal basis of V . Then

$$P_V^W(w) = \sum_{r=1}^k \langle w, h_r \rangle h_r \quad \text{for all } w \in W. \quad (5.2.7)$$

Next we extend this process all the way to B . However, there is no “orthogonal projection” from B onto V . Nonetheless, by choosing an orthonormal basis h_1, \dots, h_k of V we can define

$$P_V^B = \sum_{r=1}^k I_L(h_r) h_r, \quad (5.2.8)$$

where I_L is as in (5.2.3). If H is finite-dimensional then, in view of (5.2.5), we can see that the expression for P_V^B in (5.2.8) agrees with (5.2.7). If V happens to be contained in the subspace $j^*(B^*)$ then P_V^B is given more clearly by

$$P_V^B x = \sum_{r=1}^k \phi_r(x) h_r$$

where ϕ_r is the point in B^* for which $j^*(\phi_r) = h_r$. If f is a function on V then we can extend to a μ_L -almost-everywhere defined function f_B on B by:

$$f_B = f \circ P_V^B. \quad (5.2.9)$$

(This notion was discussed in Gross [4] for $V \subset j^*(B^*)$.) With this notation, we can formulate our main result in the setting of Abstract Wiener Spaces.

Theorem 5.2.1. *Let H be a real separable Hilbert space and B the closure of H with respect to a measurable norm. Let L be a closed affine subspace of H of finite codimension. Let f be a Borel function on a finite-dimensional nonzero subspace V of H such that the orthogonal projection P_V^H maps L onto V . Suppose Z_1, Z_2, \dots are finite-dimensional subspaces of H , and*

$$V \subset Z_1 \subset Z_2 \subset \dots \subset H$$

with $\cup_{N \geq 1} Z_N$ being dense in H . Then

$$\lim_{N \rightarrow \infty} (R_{Z_N} f_{Z_N})(L \cap S_{Z_N}) = G f_B(L) \quad (5.2.10)$$

Here, on the left is the normalized surface-area integral of f_{Z_N} over the circle $L \cap S_{Z_N}$ formed by intersecting L with the sphere in Z_N of radius $\sqrt{\dim Z_N}$, and on the right is the integral of f_B over B with respect to the measure μ_L .

Proof. Let

$$L_0 = L - p_L,$$

be the subspace of H parallel to L ; here p_L is the point on L closest to 0. Since the affine subspace L is of finite codimension, there is an orthonormal basis u_1, \dots, u_m of L_0^\perp . Let

$$u_{j,N} \text{ be the orthogonal projection of } u_j \text{ onto } Z_N.$$

Since $\cup_{N \geq 1} Z_N$ is dense in H , $u_{j,N} \neq 0$ for large N (Lemma 3.1.1). Thus, for N large

enough, the orthogonal projection $u_{j,N} \neq 0$ for every $j \in \{1, \dots, m\}$.

A vector $v \in Z_N \subset H$ is orthogonal to u_1, \dots, u_m if and only if it is orthogonal to the vectors $u_{1,N}, \dots, u_{m,N}$.

A point $x \in H$ lies in L if and only if

$$\langle x, u_1 \rangle = p_1, \dots, \langle x, u_m \rangle = p_m,$$

where $p_j = \langle p_L, u_j \rangle$ for each j . Thus $L_N = L \cap Z_N$ consists of all points $x \in Z_N$ satisfying

$$\langle x, u_{j,N} \rangle = p_j \quad \text{for all } j \in \{1, \dots, m\}. \quad (5.2.11)$$

For large N the set of all such x constitutes an affine subspace in Z_N of codimension m .

Let V_0 be the orthogonal projection of L_0 on V :

$$V_0 = P_V^H(L_0).$$

If $L_0 = \{0\}$ then $P_V^H(L)$ consists of just one point, and the functions f_{Z_n} and f_B are all constant, equal to the value of f at that point. In this case our main result is true because both sides are equal to the value of f at this point. So now we assume that $V_0 \neq 0$ has dimension $k \geq 1$. Let us choose an orthonormal basis h_1, h_2, h_3, \dots of H such that the first k vectors h_1, \dots, h_k form a basis of V_0 .

Let

$$d_N = \dim Z_N.$$

Then

$$(R_{Z_N} f_{Z_N})(S_{L_N}) = \int_{S_{L_N}} f(x_1 h_1 + \dots + x_k h_k) d\bar{\sigma}(x_1, \dots, x_{d_N}), \quad (5.2.12)$$

where S_{L_N} is the circle in Z_N formed by the intersection of the sphere of radius $\sqrt{d_N}$ in \mathbb{R}^{d_N} with the affine subspace of \mathbb{R}^{d_N} comprised of all points x for which $x_1 h_1 + \dots + x_{d_N} h_{d_N}$ lies in L .

We identify H with l^2 via the orthonormal basis h_1, h_2, \dots , and denote again by L the affine subspace of l^2 that corresponds to $L \subset H$.

Then by Theorem 4.1.1

$$\begin{aligned} & \lim_{N \rightarrow \infty} (R_{Z_N} f_{Z_N})(S_{L_N}) \\ &= (2\pi)^{-k/2} \int_{\mathbb{R}^k} f(x_1 h_1 + \dots + x_k h_k) \cdot \\ & \quad \cdot \exp\left(-\frac{1}{2} \langle (L_0 L_0^*)^{-1} (x - z_{(k)}^0), x - z_{(k)}^0 \rangle\right) \frac{dx}{\sqrt{\det(L_0 L_0^*)}}, \end{aligned} \quad (5.2.13)$$

with notation as in Theorem 4.1.1. The right hand side here is equal to

$$\int_B f(I_L(h_1)h_1 + \dots + I_L(h_k)h_k) d\mu_L, \quad (5.2.14)$$

which we see by observing that

$$\begin{aligned}
& \int_B \exp(i(t_1 I_L(h_1) + \dots + t_k I_L(h_k))) d\mu_L \\
&= \int_B \exp(iI_L(t_1 h_1 + \dots + t_k h_k)) d\mu_L \\
&= \exp\left(i\langle z^0, t_1 h_1 + \dots + t_k h_k \rangle - \frac{1}{2} \|P_{L_0}(t_1 h_1 + \dots + t_k h_k)\|^2\right) \\
&= \exp\left(i\langle t, z_{(k)}^0 \rangle - \frac{1}{2} \langle (L_0 L_0^*) t, t \rangle\right).
\end{aligned} \tag{5.2.15}$$

This implies that the distribution of $(I_L(h_1), \dots, I_L(h_k))$ has the Gaussian density

$$(2\pi)^{-k/2} \exp\left(-\frac{1}{2} \langle (L_0 L_0^*)^{-1} (x - z_{(k)}^0), x - z_{(k)}^0 \rangle\right) \frac{dx}{\sqrt{\det(L_0 L_0^*)}} \tag{5.2.16}$$

that appears on the right side in (5.2.13). We have thus completed the proof, since (5.2.14) is exactly $Gf_B(L)$. \square

Chapter 6

Polynomials on High Dimensional Spheres

6.1 Introduction

Following from the work above we see there is a natural relationship between integration on the sphere and Gaussian integration. We turn now to show that using this relationship we can show a natural relationship between the basis of $L^2(S^{N-1}(\sqrt{N}), \bar{\sigma})$ and $L^2(\mathbb{R}^\infty, \mu)$ where μ is the infinite product of standard Gaussian measures. We proceed by using the L^2 inner product on the sphere with respect to spherical surface integration to orthogonalize monomials on the sphere which leads to the Hermite polynomials as a basis for $L^2(\mathbb{R}^\infty, \mu)$. This result has been shown in [16], following the works of [8], through different methods using the limiting behavior of the spherical Laplacian and a projective limit space. Here we use a more algebraic framework through polynomials and their restriction as functions to spheres of varying radius

and dimension.

6.2 Spherical and Gaussian integration

In this section we formally establish relationships between integration over spheres and integration with respect to Gaussian measure.

6.2.1 Integration of homogeneous functions

A function f on \mathbb{R}^N is said to be *homogeneous of degree d* if

$$f(tx) = t^d f(x) \quad \text{for all } x \in \mathbb{R}^N \text{ and } t \in \mathbb{R}.$$

We will work with homogeneous polynomial functions.

Let us note that the product of a homogeneous function of degree d_1 and a homogeneous function of degree d_2 is a homogeneous function of degree $d_1 + d_2$.

Proposition 6.2.1. *Let f be a Borel function on \mathbb{R}^N , homogeneous of degree d and integrable with respect to the standard Gaussian measure. Then*

$$2^{\frac{d}{2}} \frac{\Gamma\left(\frac{d+N}{2}\right)}{\Gamma\left(\frac{N}{2}\right)} \int_{S^{N-1}} f \, d\bar{\sigma} = \int_{\mathbb{R}^N} f(x) (2\pi)^{-N/2} e^{-\frac{\|x\|^2}{2}} dx, \quad (6.2.1)$$

where on the left $\bar{\sigma}$ is the uniform measure on the unit-sphere S^{N-1} , normalized to having total measure 1, and on the right we have the standard Gaussian measure on \mathbb{R}^N .

Let us note for us later that by homogeneity of f we have

$$\int_{S^{N-1}(a)} f d\bar{\sigma} = a^d \frac{\Gamma\left(\frac{N}{2}\right)}{2^{\frac{d}{2}} \Gamma\left(\frac{d+N}{2}\right)} \int_{\mathbb{R}^N} f(x) (2\pi)^{-N/2} e^{-\frac{\|x\|^2}{2}} dx, \quad (6.2.2)$$

for any radius $a > 0$.

Proof. We have the polar disintegration formula

$$\int_{\mathbb{R}^N} f(x) (2\pi)^{-N/2} e^{-\frac{\|x\|^2}{2}} dx = \int_0^\infty \left[\int_{S^{N-1}(r)} f(x) d\sigma(x) \right] (2\pi)^{-N/2} e^{-r^2/2} dr, \quad (6.2.3)$$

where σ is the standard surface measure on the sphere $S^{N-1}(r)$ of radius r (see, for example, [14, (3.10)] for proof). Then we observe by homogeneity of f that the spherical integral over the sphere of radius r is a multiple of the integral over the unit sphere:

$$\begin{aligned} \int_{S^{N-1}(r)} f(x) d\sigma(x) &= \int_{S^{N-1}} f(rx) r^{N-1} d\sigma(x) \\ &= r^{d+N-1} \int_{S^{N-1}} f(x) d\sigma(x). \end{aligned} \quad (6.2.4)$$

Using this in the Gaussian integration (6.2.3) we have

$$\begin{aligned} \int_{\mathbb{R}^N} f(x) (2\pi)^{-N/2} e^{-\frac{\|x\|^2}{2}} dx &= \int_0^\infty \left[r^{d+N-1} \int_{S^{N-1}} f d\sigma \right] (2\pi)^{-N/2} e^{-r^2/2} dr \\ &= \left[\int_{S^{N-1}} f d\sigma \right] \int_0^\infty r^{d+N-1} (2\pi)^{-N/2} e^{-r^2/2} dr \\ &= (2\pi)^{-N/2} 2^{\frac{d+N}{2}-1} \Gamma\left(\frac{d+N}{2}\right) \int_{S^{N-1}} f d\sigma, \end{aligned} \quad (6.2.5)$$

where in obtaining the Gamma function we used the substitution $y = r^2/2$ in the

integration. Taking $f = 1$ (degree 0) here gives the surface area of the sphere S^{N-1} to be

$$c_{N-1} = 2 \frac{\pi^{N/2}}{\Gamma(N/2)}. \quad (6.2.6)$$

Then, returning to (6.2.5), we have:

$$\begin{aligned} & \int_{\mathbb{R}^N} f(x) (2\pi)^{-N/2} e^{-\frac{\|x\|^2}{2}} dx \\ &= (2\pi)^{-N/2} 2^{\frac{d+N}{2}-1} \Gamma\left(\frac{d+N}{2}\right) c_{N-1} \int_{S^{N-1}} f d\bar{\sigma} \\ &= 2^{\frac{d}{2}} \frac{\Gamma\left(\frac{d+N}{2}\right)}{\Gamma\left(\frac{N}{2}\right)} \int_{S^{N-1}} f d\bar{\sigma} \end{aligned} \quad (6.2.7)$$

□

As an immediate consequence of this result, we see that the Gaussian L^2 inner-product when applied to homogeneous functions can be computed by working out the L^2 inner-product on the unit sphere, with normalized uniform measure:

Proposition 6.2.2. *Let f and g be homogeneous Borel functions on \mathbb{R}^N , of degrees d_f and d_g , respectively, and square-integrable with respect to standard Gaussian measure μ . Then*

$$\langle f, g \rangle_{L^2(\mathbb{R}^N, \mu)} = 2^d \frac{\Gamma\left(\frac{N}{2} + d\right)}{\Gamma\left(\frac{N}{2}\right)} \langle f, g \rangle_{L^2(S^{N-1}, \bar{\sigma})}, \quad (6.2.8)$$

where $d = (d_f + d_g)/2$. If $d_f + d_g$ is odd then both sides in (6.2.8) are 0. We also have

$$\langle f, g \rangle_{L^2(\mathbb{R}^N, \mu)} = a_{d,N} \langle f, g \rangle_{L^2(S^{N-1}(\sqrt{N}), \bar{\sigma})} \quad (6.2.9)$$

where

$$a_{d,N} = \prod_{j=1}^d \left(1 + 2 \left(\frac{j-1}{N} \right) \right). \quad (6.2.10)$$

Proof. The product $f\bar{g}$ is homogeneous of degree $2d$. Then, applying Proposition 6.2.1 to the function $f\bar{g}$ we obtain the formula (6.2.8). If $d_f + d_g$ is odd then $(fg)(-x) = -fg(x)$ for all $x \in \mathbb{R}^N$, and so both sides of (6.2.8) are 0.

Let us then assume that $d_f + d_g$ is even; then d is an integer. Suppose $d \geq 1$. Since

$$\Gamma(s+1) = s\Gamma(s),$$

we have

$$\Gamma(s+d) = (s+d-1)(s+d-2) \dots s\Gamma(s),$$

and so, with $s = N/2$ we have

$$\Gamma\left(\frac{N}{2} + d\right) = 2^{-d}(N+2d-2)(N+2d-4) \dots N\Gamma(N/2). \quad (6.2.11)$$

Then the formula (6.2.8) becomes

$$\langle f, g \rangle_{L^2(\mathbb{R}^N, \mu)} = (N+2d-2)(N+2d-4) \dots N \langle f, g \rangle_{L^2(S^{N-1}, \bar{\sigma})}. \quad (6.2.12)$$

If $d = 0$ then this equation follows directly from (6.2.8), provided we interpret the right hand side as just $\langle f, g \rangle_{L^2(S^{N-1}, \bar{\sigma})}$.

Using homogeneity of f and g again, we can rewrite the right hand side as an

integral over the sphere of radius \sqrt{N} :

$$\begin{aligned}
& \langle f, g \rangle_{L^2(\mathbb{R}^N, \mu)} \\
&= \left(1 + 2 \left(\frac{d-1}{N}\right)\right) \left(1 + 2 \left(\frac{d-2}{N}\right)\right) \dots * 1 * N^d \langle f, g \rangle_{L^2(S^{N-1}, \bar{\sigma})} \\
&= a_{d,N} \langle f, g \rangle_{L^2(S^{N-1}(\sqrt{N}), \bar{\sigma})}
\end{aligned} \tag{6.2.13}$$

where $a_{d,N}$ is as given in (6.2.10). □

We now show that the pairing

$$\langle p, q \rangle_{a,N} = \int_{S^{N-1}(a)} p(x) \overline{q(x)} d\bar{\sigma}(x), \tag{6.2.14}$$

where $\bar{\sigma}$ is the unit-mass uniform measure on $S^{N-1}(a)$, gives an inner-product on \mathcal{P}_k , for any positive integers $k < N$.

Lemma 6.2.3. *Let $p \in \mathcal{P}_k$ be a polynomial in k variables, let f be the function on \mathbb{R}^N , where $N > k$, given by*

$$f(x_1, \dots, x_N) = p(x_1, \dots, x_k) \quad \text{for all } (x_1, \dots, x_N) \in \mathbb{R}^N.$$

If $f = 0$ on $S^{N-1}(a)$, where $a > 0$, then the polynomial p is 0. The inner-product $\langle \cdot, \cdot \rangle_{a,N}$ restricts to an inner-product on the subspace \mathcal{P}_k for $k < N$.

Proof. Let (x_1, \dots, x_k) be any point within the open ball of radius a in \mathbb{R}^k . Then

$$p(x_1, \dots, x_k) = f(x_1, \dots, x_k, \sqrt{a^2 - \|x\|^2}, 0, \dots, 0) = 0,$$

where $\|x\|^2 = \sqrt{x_1^2 + \dots + x_k^2}$. A polynomial function in \mathbb{R}^k that is zero on an open

set is identically zero, and so p is the zero polynomial.

If $\|p\|_{a,N}^2$ is 0 then the evaluation of p at every $x \in S^{N-1}(a)$ is zero, and so p is the zero polynomial. Therefore $\langle \cdot, \cdot \rangle_{a,N}$ restricts to an inner-product on the subspace \mathcal{P}_k for $k < N$. \square

6.3 Gaussian integration as a limit of spherical integration

There is a well-known relationship between Gaussian integration in infinite dimensions and integration over large spheres (see, for example, [14, 15]). Here we focus on the special case of polynomial functions.

We use the product Gaussian measure μ on \mathbb{R}^∞ , the space of all real sequences. The measure μ is supported on much smaller subspaces of \mathbb{R}^∞ , but we do not need to bring such subspaces in for our purposes here.

Theorem 6.3.1. *Suppose p and q are polynomial functions on \mathbb{R}^k , viewed also as functions on \mathbb{R}^N for $N > k$ in terms of the first k coordinates. Then*

$$\lim_{N \rightarrow \infty} \langle p, q \rangle_{L^2(S^{N-1}(\sqrt{N}), \bar{\sigma})} = \langle p, q \rangle_{L^2(\mathbb{R}^\infty, \mu)}, \quad (6.3.1)$$

with notation as before.

Recall that by Lemma 6.2.3, $\langle \cdot, \cdot \rangle_{L^2(S^{N-1}(a))}$ is an inner-product on the space of polynomials in X_1, \dots, X_k , for any $k < N$. So, restricting to the case of $p, q \in \mathcal{P}_k^d$, the result (6.3.1) says that the inner-product $\langle \cdot, \cdot \rangle_{L^2(S^{N-1}(a))}$ converges to the Gaussian inner-product on \mathcal{P}_k^d for all integers $d \geq 0$ and $k \geq 1$.

Proof. By linearity we may assume that p and q are homogeneous, since a general polynomial is a sum of homogeneous monomials. Then using the identity (6.2.9) and observing that $\lim_{N \rightarrow \infty} a_{d,N} = 1$ we obtain (6.3.1). \square

6.4 Hermite limits for monomials over large spheres

In this section we show that monomials, suitably projected, over the sphere $S^{N-1}(\sqrt{N})$ converge to Hermite polynomials.

6.4.1 The subspaces \mathcal{P}^d and projections Π^d

We equip the space \mathcal{P} of all polynomials in variables X_1, X_2, \dots with the Gaussian L^2 inner-product:

$$\langle p(X_1, \dots, X_N), q(X_1, \dots, X_N) \rangle = \langle p, q \rangle_{L^2(\gamma_N)} \quad (6.4.1)$$

where γ_N is the standard Gaussian measure on \mathbb{R}^N :

$$d\gamma_N(x) = (2\pi)^{-N/2} e^{-\|x\|^2/2} dx.$$

Each space $\mathcal{P}_N^{\leq d}$ is finite-dimensional and there is an orthogonal projection

$$\Pi_N^{\leq d} : \mathcal{P}_N^{\leq d} \rightarrow \mathcal{P}_N^{\leq d-1}, \quad (6.4.2)$$

for all $d \geq 1$ and $N \geq 1$. We can drop the subscript N from $\Pi_{d,N}$ because of the following observation.

Lemma 6.4.1. *If $M > N$ then*

$$\Pi_M^{\leq d} | \mathcal{P}_N^{\leq d} = \Pi_N^{\leq d}. \quad (6.4.3)$$

A consequence of this equality is that $\Pi_M^{\leq d} p(X_1, \dots, X_N)$ is a polynomial in X_1, \dots, X_N , as seems natural.

Proof. Consider any $p \in \mathcal{P}_N^{\leq d}$, and any monomial $X_1^{j_1} \dots X_M^{j_M} \in \mathcal{P}_M^{\leq d-1}$; then:

$$\begin{aligned} \langle p(X_1, \dots, X_N), X_1^{j_1} \dots X_M^{j_M} \rangle \\ = \langle p(X_1, \dots, X_N), X_1^{j_1} \dots X_N^{j_N} \rangle \langle 1, X_{N+1}^{j_{N+1}} \dots X_M^{j_M} \rangle, \end{aligned} \quad (6.4.4)$$

where we have used the fact that the Gaussian measure γ_M is the product of the standard Gaussian measure γ_N and the standard Gaussian measure in the remaining $M - N$ variables:

$$\int_{\mathbb{R}^N \times \mathbb{R}^{M-N}} f(x)g(y) d\gamma_M(x, y) = \int_{\mathbb{R}^N} f d\gamma_N \int_{\mathbb{R}^{M-N}} g d\gamma_{M-N} \quad (6.4.5)$$

Next we observe that

$$\langle p(X_1, \dots, X_N), X_1^{j_1} \dots X_N^{j_N} \rangle = \langle \Pi_N^{\leq d} p(X_1, \dots, X_N), X_1^{j_1} \dots X_N^{j_N} \rangle, \quad (6.4.6)$$

by definition of the orthogonal projection $\Pi_N^{\leq d}$, keeping in mind that $X_1^{j_1} \dots X_N^{j_N}$ has degree $\leq d - 1$.

Using the product nature of γ_M again we conclude that

$$\langle p(X_1, \dots, X_N), X_1^{j_1} \dots X_M^{j_M} \rangle = \langle \Pi_N^{\leq d} p(X_1, \dots, X_N), X_1^{j_1} \dots X_M^{j_M} \rangle, \quad (6.4.7)$$

for all monomials $X_1^{j_1} \dots X_M^{j_M}$ in $\mathcal{P}_M^{\leq d-1}$. On the right, $\Pi_N^{\leq d} p(X_1, \dots, X_N)$ is in $\mathcal{P}_N^{\leq d-1}$. Since the monomials in $\mathcal{P}_M^{\leq d-1}$ form a basis of this space, we conclude from (6.4.7) that

$$\Pi_M^{\leq d} p(X_1, \dots, X_N) = \Pi_N^{\leq d} p(X_1, \dots, X_N), \quad (6.4.8)$$

which establishes (6.4.3). \square

Thus all the linear mappings $\Pi_N^{\leq d}$ combine to form one linear mapping

$$\Pi^{\leq d} : \mathcal{P}^{\leq d} \rightarrow \mathcal{P}^{\leq d-1}. \quad (6.4.9)$$

6.4.2 Hermite polynomials as orthogonal projections of monomials

Now let

$$\Pi_{\perp}^{\leq d} = I - \Pi^{\leq d}. \quad (6.4.10)$$

Then

$$\Pi_{\perp}^{\leq d} | \mathcal{P}_N^{\leq d} \text{ is the orthogonal projection onto } \mathcal{P}_N^{\leq d} \ominus \mathcal{P}_N^{\leq d-1}. \quad (6.4.11)$$

Focusing for the moment on just one variable X , the m -th Hermite polynomial $H_m(X)$ is 1 if $m = 0$ and is the projection of X^m to the subspace orthogonal to \mathcal{P}^{m-1} otherwise. Thus

$$H_m(X) = \Pi_{\perp}^{\leq m}(X^m). \quad (6.4.12)$$

We note that

$$X^m - H_m(X)$$

is the orthogonal projection of X^m on the subspace of polynomials of degree $< m$, and so is of degree $< m$.

We have next the generalization of this observation to more than one variable:

Proposition 6.4.2. *For any monomial $X_1^{m_1} \dots X_N^{m_N}$ with $m_1 + \dots + m_N = d$, we have*

$$\Pi_{\perp}^{\leq d}(X_1^{m_1} \dots X_N^{m_N}) = H_{m_1}(X_1) \dots H_{m_N}(X_N). \quad (6.4.13)$$

Proof. Writing $X_1^{m_1} \dots X_N^{m_N}$ as

$$\begin{aligned} & X_1^{m_1} \dots X_N^{m_N} \\ &= (X_1^{m_1} - H_{m_1}(X_1))X_2^{m_2} \dots X_N^{m_N} + H_{m_1}(X_1)X_2^{m_2} \dots X_N^{m_N}, \end{aligned} \quad (6.4.14)$$

we observe that the first term on the right hand side is of total degree $< d$, because $X_1^{m_1} - H_{m_1}(X_1)$ is of degree $m_1 - 1$ in X_1 . Therefore, applying the projection Π_{\perp}^d , we have

$$\Pi_{\perp}^{\leq d}(X_1^{m_1} \dots X_N^{m_N}) = \Pi_{\perp}^{\leq d}(H_{m_1}(X_1)X_2^{m_2} \dots X_N^{m_N}). \quad (6.4.15)$$

Repeating this argument with X_2, X_3, \dots , we obtain

$$\Pi_{\perp}^{\leq d}(X_1^{m_1} \dots X_N^{m_N}) = \Pi_{\perp}^{\leq d}(H_{m_1}(X_1) \dots H_{m_N}(X_N)). \quad (6.4.16)$$

Now we check that the polynomial $H_{m_1}(X_1) \dots H_{m_N}(X_N)$ is orthogonal to all polynomials of total degree $< d$: if $X_1^{j_1} \dots X_N^{j_N}$ is a monomial of total degree $< d$ then

$j_k < m_k$ for at least one k , and so

$$\begin{aligned}
& \langle H_{m_1}(X_1) \dots H_{m_N}(X_N), X_1^{j_1} \dots X_N^{j_N} \rangle \\
&= \langle H_{m_1}(X_1), X_1^{j_1} \rangle \dots \langle H_{m_N}(X_N), X_N^{j_N} \rangle \\
&= 0 \quad \text{because } \langle H_{m_k}(X_k), X_k^{j_k} \rangle = 0.
\end{aligned} \tag{6.4.17}$$

Hence

$$\Pi_{\perp}^{\leq d}(H_{m_1}(X_1) \dots H_{m_N}(X_N)) = H_{m_1}(X_1) \dots H_{m_N}(X_N),$$

and so, by (6.4.16), the result (6.4.13) follows. \square

6.4.3 The limiting orthogonal projection

We turn now to look at orthogonal projections associated to a sequence of inner-products. We will apply the following result to the case of inner-products given by integration over $S^{N-1}(\sqrt{N})$.

Proposition 6.4.3. *Let V be a finite-dimensional vector space, and $\langle \cdot, \cdot \rangle_n$ an inner-product on V for each $n \in \{1, 2, 3, \dots\}$, and suppose that there is an inner-product $\langle \cdot, \cdot \rangle$ on V which is the limit of the sequence of inner-products $\langle \cdot, \cdot \rangle_n$. Let $P_n : V \rightarrow V$ be the orthogonal projection onto a subspace $W \subset V$ relative to the inner-product $\langle \cdot, \cdot \rangle_n$. Then $P_n v \rightarrow P v$, as $n \rightarrow \infty$, for all $v \in V$, where P is the orthogonal projection onto W with respect to the inner-product $\langle \cdot, \cdot \rangle$.*

Proof. Let w_1, \dots, w_k form a basis of W . Fix $v \in V$. Then, writing

$$P_n v = \sum_{j=1}^k c_j(n) w_j, \tag{6.4.18}$$

we have

$$\langle w_i, v \rangle_n = \langle w_i, P_n v \rangle_n = \sum_{j=1}^k \langle w_i, w_j \rangle_n c_j(n). \quad (6.4.19)$$

Hence the vector $c(n)$ whose components are $c_1(n), \dots, c_k(n)$ is given by matrix inversion:

$$c(n) = \begin{bmatrix} \langle w_1, w_1 \rangle_n & \langle w_1, w_2 \rangle_n & \dots & \langle w_1, w_k \rangle_n \\ \langle w_2, w_1 \rangle_n & \langle w_2, w_2 \rangle_n & \dots & \langle w_2, w_k \rangle_n \\ \vdots & \vdots & \vdots & \vdots \\ \langle w_k, w_1 \rangle_n & \langle w_k, w_2 \rangle_n & \dots & \langle w_k, w_k \rangle_n \end{bmatrix}^{-1} \begin{bmatrix} \langle w_1, v \rangle_n \\ \langle w_2, v \rangle_n \\ \vdots \\ \langle w_k, v \rangle_n \end{bmatrix}. \quad (6.4.20)$$

Now we let $n \rightarrow \infty$ and using continuity of matrix inversion and matrix multiplication we obtain:

$$c \stackrel{\text{def}}{=} \lim_{n \rightarrow \infty} c(n) = \begin{bmatrix} \langle w_1, w_1 \rangle & \langle w_1, w_2 \rangle & \dots & \langle w_1, w_k \rangle \\ \langle w_2, w_1 \rangle & \langle w_2, w_2 \rangle & \dots & \langle w_2, w_k \rangle \\ \vdots & \vdots & \vdots & \vdots \\ \langle w_k, w_1 \rangle & \langle w_k, w_2 \rangle & \dots & \langle w_k, w_k \rangle \end{bmatrix}^{-1} \begin{bmatrix} \langle w_1, v \rangle \\ \langle w_2, v \rangle \\ \vdots \\ \langle w_k, v \rangle \end{bmatrix}. \quad (6.4.21)$$

Then, multiplying by the matrix with entries $\langle w_i, w_j \rangle$, we have:

$$\sum_{j=1}^k \langle w_i, w_j \rangle c_j = \langle w_i, v \rangle, \quad (6.4.22)$$

and this means that $Pv = \sum_{j=1}^k c_j w_j$. Looking back at (6.4.18) we conclude that $\lim_{n \rightarrow \infty} P_n v = Pv$. \square

6.4.4 Hermite polynomials from monomials on spheres

Recall that $\mathcal{P}_k^{\leq d}$ is the vector space of all polynomials of degree $\leq d$ in the variables X_1, \dots, X_k .

Proposition 6.4.4. *Let*

$$\dot{\Pi}_{k,N}^{\leq d} : \mathcal{P}_k^{\leq d} \rightarrow \mathcal{P}_k^{\leq d-1} \quad (6.4.23)$$

be the orthogonal projection using the inner-product $\langle \cdot, \cdot \rangle_N = \langle \cdot, \cdot \rangle_{L^2(S^{N-1}(\sqrt{N}), \bar{\sigma})}$. Then

$$\lim_{N \rightarrow \infty} (I - \dot{\Pi}_{k,N}^{\leq d})(X_1^{j_1} \dots X_k^{j_k}) = H_{j_1}(X_1) \dots H_{j_k}(X_k). \quad (6.4.24)$$

Proof. By Theorem 6.3.1 the limit of the inner-product $\langle \cdot, \cdot \rangle_N$ as $N \rightarrow \infty$ is the Gaussian inner-product. Then by Proposition 6.4.3 we have the limit of the projections:

$$\lim_{N \rightarrow \infty} \dot{\Pi}_{k,N}^{\leq d}(X_1^{j_1} \dots X_k^{j_k}) = \Pi^{\leq d}(X_1^{j_1} \dots X_k^{j_k}). \quad (6.4.25)$$

Hence

$$\begin{aligned} \lim_{N \rightarrow \infty} (I - \dot{\Pi}_{k,N}^{\leq d})(X_1^{j_1} \dots X_k^{j_k}) &= (I - \Pi^{\leq d})(X_1^{j_1} \dots X_k^{j_k}) \\ &= \Pi_{\perp}^{\leq d}(X_1^{j_1} \dots X_k^{j_k}). \end{aligned} \quad (6.4.26)$$

Then by Proposition 6.4.2, we have:

$$\lim_{N \rightarrow \infty} (I - \dot{\Pi}_{k,N}^{\leq d})(X_1^{j_1} \dots X_k^{j_k}) = H_{j_1}(X_1) \dots H_{j_k}(X_k). \quad (6.4.27)$$

□

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