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# Sharp Trudinger-Moser Inequalities On Riemannian Manifolds And Heisenberg Groups

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## ABSTRACT

This dissertation contains two research directions.

The first direction is the extension of Trudinger-Moser type inequalities on complete noncompact Riemannian manifolds with certain curvature conditions. In [28], we established critical Trudinger-Moser type inequalities with sharp constants on bounded domains of the manifold. Then we applied a rearrangement-free argument developed by N. Lam and G. Lu [23, 24] to improve the local inequality to the global inequality. Inspired by a work of N. Lam, G. Lu and L, Zhang [27], we also proved the equivalence of critical and subcritical Trudinger-Moser inequalities on complete noncompact Riemannian manifolds and hence obtained subcritical Trudinger-Moser inequalities on the manifold. In [29], we studied smooth maps between manifolds and established Adams type inequalities for maps between manifolds with certain curvature assumptions.

The second direction is concentration-compactness principles of Trudinger-Moser inequalities on Heisenberg groups and Riemannian manifolds. The classical results on $\mathbb{R}^n$ rely heavily on symmetrization, which is no longer available in non-Euclidean

settings. To overcome such difficulty, in [31, 32], we developed a rearrangement free argument to prove concentration-compactness principles on both bounded and unbounded domains of Heisenberg groups and Riemannian manifolds.

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# APPROVAL PAGE

Doctor of Philosophy Dissertation

# Sharp Trudinger-Moser Inequalities on Riemannian Manifolds and Heisenberg Groups

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# Contents

# Chapter 1

# Introduction

## 1.1 Introduction to Trudinger-Moser Inequalities

### 1.1.1 Trudinger-Moser Inequalities on Bounded Domains of $\mathbb{R}^n$

Consider $u \in C_0^1(\mathbb{R}^n)$, let $1 \leq p < n$, $p^* = \frac{pn}{n-p}$, then there exists a constant $C > 0$ such that

$$||u||_{L^{p^*}(\mathbb{R}^n)} \leq C||\nabla u||_{L^p(\mathbb{R}^n)}. \tag{1.1.1}$$

This Sobolev inequality implies the Sobolev embedding from $W^{1,p}(\Omega)$ to $L^{p^*}(\Omega)$, where $\Omega$ is any open bounded domain. However, the embedding from $W^{1,n}(\Omega)$ to $L^\infty(\Omega)$ does not hold. Instead, N. S. Trudinger [56] discovered (see also S. I. Pohozaev [52] and V. I. Yudovic [22]) the emdedding of exponential type. Namely, the embedding $W_0^{1,n}(\Omega) \subset L_{\varphi_n}(\Omega)$ is continuous, where $L_{\varphi_n}(\Omega)$ is the Orlicz space associated with the Young function $\varphi_n(t) = \exp\left(\alpha\,|t|^{n/(n-1)}\right) - 1$ for some $\alpha > 0$. J. Moser [45] sharpened this embedding and proved the following inequality:

**Theorem 1.1.1.** *There is a dimensional constant $C_n > 0$ such that if $\Omega$ is an open subset of Euclidean space $\mathbb{R}^n$ $(n \geq 2)$ with finite Lebesgue measure, then*

$$\frac{1}{|\Omega|}\int_\Omega \exp\left(\alpha|u(x)|^{\frac{n}{n-1}}\right)dx \leq C_n \tag{1.1.2}$$

*for any $\alpha \leq \alpha_n$, any $u$ in the Sobolev space $W_0^{1,n}(\Omega)$, provided $||\nabla u||_{L^n(\Omega)} \leq 1$, where $\alpha_n = n\omega_{n-1}^{\frac{1}{n-1}}$ and $\omega_{n-1}$ is the area of the surface of the unit $n-$ball. Moreover, the constant $\alpha_n$ is sharp in the sense that if $\alpha$ exceeds $\alpha_n$, then the above inequality cannot hold with uniform $C_n$ independent of $u$.*

This inequality is nowadays known as the Trudinger-Moser inequality. In order to prove (1.1.2), J. Moser used the following symmetrization argument: every function $u$ is associated to a radially symmetric function $u^*$ such that the sub-level sets of $u^*$ are balls with the same area as the corresponding sub-level sets of $u$. Moreover, $u$ is a positive and non-increasing function defined on $B_R(0)$ where $|B_R(0)| = |\Omega|$. Hence, by the layer cake principle, we can have that

$$\int_\Omega f(u)\,dx = \int_{B_R(0)} f(u^*)\,dx$$

for any function $f$ that is the difference of two monotone functions. In particular, we obtain

$$\|u\|_{L^p(\Omega)} = \|u^*\|_{L^p(\Omega)}; \tag{1.1.3}$$

$$\int_\Omega \exp\left(\alpha\,|u|^{\frac{n}{n-1}}\right)dx = \int_{B_R(0)} \exp\left(\alpha\,|u^*|^{\frac{n}{n-1}}\right)dx. \tag{1.1.4}$$

Moreover, the well-known Pólya-Szegö inequality

$$\int_{B_R(0)} |\nabla u^*|^p\,dx \le \int_\Omega |\nabla u|^p\,dx \tag{1.1.5}$$

plays an important role in the proof of Trudinger-Moser inequality (1.1.2).

Since J. Moser established the best constant for the above inequality (1.1.2) in 1971, the question of whether the following supremum

$$\sup\left\{\frac{1}{|\Omega|}\int_\Omega \exp\left(\alpha_n|u(x)|^{\frac{n}{n-1}}\right)dx:\ u\in W_0^{1,n}(\Omega), ||\nabla u||_{L^n(\Omega)}\le 1\right\}$$

is attained has remained open for quite some years. In 1986, L. Carleson and S. Y. Chang proved in the celebrated paper [9] that the above supremum indeed has an extremal for the case when $\Omega$ is a ball in $\mathbb{R}^n$ for $n\ge 2$. Since the boundary value of the extremal function is zero, their result came as a surprise since it was already known that the Sobolev inequality has no extremals supported in balls for $p>1$ (see G. Talenti [55], T. Aubin [3]). In order to prove the existence of an extremal function, L. Carleson and S. Y. Chang tried to reduce the problem to a one-dimensional problem (for blow-up analysis approach, see also [34, 35, 42]). We note that the result of [9] was extended to arbitrary bounded smooth domains by M. Flucher [17] when $n=2$, and by K. Lin [36] for the case $n>2$, and more recently to existence of extremal functions on Riemannian manifolds by Y. Li [33, 34] .

Similar to the inequality (1.1.2), J. Moser also proved in [45] an analogous inequality on the two-dimensional sphere $\mathbb{S}^2$. He proved that there exists a constant $C_0$ such that $4\pi$ is the best constant for the inequality

$$\int_{\mathbb{S}^2} \exp\left(4\pi|u(x)|^2\right) d\sigma \leq C_0$$

for any smooth $u$ on $\mathbb{S}^2$ with $\int_{\mathbb{S}^2} |\nabla u|^2 d\sigma \leq 1$ and $\int_{\mathbb{S}^2} u d\sigma = 0$, where $d\sigma$ is the surface measure and $\nabla$ is the gradient on $\mathbb{S}^2$.

J. Moser further proved that there exists a constant $C_0$ such that $8\pi$ is the best constant for the inequality

$$\int_{\mathbb{S}^2} \exp\left(8\pi|u(x)|^2\right) d\sigma \leq C_0$$

for any smooth $u$ on $\mathbb{S}^2$ with $\int_{\mathbb{S}^2} |\nabla u|^2 d\sigma \leq 1$ and $\int_{\mathbb{S}^2} u d\sigma = 0$, and $u(\xi) = u(-\xi)$.

This inequality on the sphere $\mathbb{S}^2$ has been used to solve the so-called Nirenberg problem: given a function $K(x)$ on the two-dimensional sphere $\mathbb{S}^2$, is there a metric $ds^2$ which is conformally related to the standard round metric $ds_0^2$ on $\mathbb{S}^2$ such that $K(x)$ is the Gauss curvature associated with the metric $ds^2$? This is equivalent to ask if there exists a function $u$ on $\mathbb{S}^2$ so that $ds^2 = e^u ds_0^2$ and $u$ satisfies the following differential equation:

$$\triangle u + Ke^{2u} - 1 = 0, \tag{1.1.6}$$

where $\triangle$ is the Laplace-Beltrami operator on $\mathbb{S}^2$ with respect to the standard metric $ds_0^2$.

J. Moser applied the inequality (1.1.2) to this problem of the prescribing Gaussian curvature on $\mathbb{S}^2$ by considering the following functional

$$G(u) = \log\left\{\frac{1}{4\pi}\int_{\mathbb{S}^2} Ke^{2u}d\sigma\right\} - \frac{1}{4\pi}\int_{\mathbb{S}^2}|\nabla u|^2 d\sigma - \frac{1}{2\pi}\int_{\mathbb{S}^2} ud\sigma.$$

Using inequality (1.1.2) J. Moser showed that there is a positive constant $C$ such that

$$\frac{1}{4\pi}\int_{\mathbb{S}^2} e^{2u}d\sigma \leq C\exp\left\{\frac{1}{4\pi}\int_{\mathbb{S}^2}|\nabla u|^2 d\sigma - \frac{1}{2\pi}\int_{\mathbb{S}^2} ud\sigma\right\}. \tag{1.1.7}$$

This implies that the functional $G(u)$ is bounded from above. If the further assumption that $K(\xi) = K(-\xi)$ on $\mathbb{S}^2$ is imposed, J. Moser showed that the equation (1.1.18) has a solution $u$ such that $u(\xi) = u(-\xi)$ provided that $\max_{\mathbb{S}^2} K(\xi) > 0$ using the fact that the functional $G(u)$ is bounded from above.

### 1.1.2 Adams Inequalities

The higher order version of the sharp Trudinger-Moser inequality was established by D. Adams [2]. To state Adams' result, we use the symbol $\nabla^m u$, where $m$ is a positive integer, to denote the $m-$th order gradient for $u \in C^m(\Omega)$, the class of $m-$th order differentiable functions:

$$\nabla^m u = \begin{cases} \triangle^{\frac{m}{2}} u & \text{for } m \text{ even} \\ \nabla \triangle^{\frac{m-1}{2}} u & \text{for } m \text{ odd.} \end{cases} \tag{1.1.8}$$

where $\nabla$ is the usual gradient operator and $\triangle$ is the Laplacian. We use $||\nabla^m u||_{L^p}$ to denote the $L^p$ norm ($1 \leq p \leq \infty$) of the function $|\nabla^m u|$, the usual Euclidean length of the vector $\nabla^m u$. We also use $W_0^{k,p}(\Omega)$ to denote the Sobolev space which is a completion of $C_0^\infty(\Omega)$ under the norm of $||u||_{L^p(\Omega)} + ||\nabla^k u||_{L^p(\Omega)}$. Then D. Adams proved the following Adams' inequality

**Theorem 1.1.2.** *Let $\Omega$ be an open and bounded set in $\mathbb{R}^n$. If $m$ is a positive integer less than $n$, then there exists a constant $C_0 = C(n,m) > 0$ such that for any $u \in W_0^{m,\frac{n}{m}}(\Omega)$ and $||\nabla^m u||_{L^{\frac{n}{m}}(\Omega)} \leq 1$, then*

$$\frac{1}{|\Omega|}\int_\Omega \exp(\alpha|u(x)|^{\frac{n}{n-m}})dx \leq C_0 \tag{1.1.9}$$

*for all $\alpha \leq \alpha(n,m)$ where*

$$\alpha(n,m) = \begin{cases} \frac{n}{\omega_{n-1}}\left[\frac{\pi^{n/2}2^m\Gamma(\frac{m+1}{2})}{\Gamma\frac{n-m+1}{2}}\right]^{\frac{n}{n-m}} & \text{when } n \text{ is odd} \\ \frac{n}{\omega_{n-1}}\left[\frac{\pi^{n/2}2^m\Gamma(\frac{m}{2})}{\Gamma\frac{n-m}{2}}\right]^{\frac{n}{n-m}} & \text{when } n \text{ is even} \end{cases} \tag{1.1.10}$$

*Furthermore, for any $\alpha > \alpha(n,m)$, the integral can be made as large as possible.*

Note that $\alpha(n,1)$ coincides with J. Moser's value of $\alpha_n$ and $\alpha(2m,m) = 2^{2m}\pi^m\Gamma(m+1)$ for both odd and even $m$.

We remark here that J. Moser's work relies on an rearrangement argument and the so-called Pólya-Szegö's inequality. In order to adapt J. Moser's approach, one needs to establish the $L^p-$ norm preserving properties of the high order gradient functions $\nabla^m u$, which is known to be false in general for $m \geq 2$. To overcome such difficulty, D. Adams represented the function $u$ in terms of its gradient function $\nabla^m u$ by using a convolution operator and then used O'Neil's idea [49] of rearrangement of convolution of two functions. Such an argument avoids dealing with the issue of $L^{\frac{n}{m}}$-norm preserving of the gradient of the rearranged function. This idea has also been developed to derive the sharp constants for Adams' inequality involving with higher order derivatives on compact Riemannian manifolds without boundary by L. Fontana [18] and in the subelliptic setting to derive the sharp Trudinger-Moser inequality on Heisenberg groups and CR spheres by W. Cohn and G. Lu (see [14] and [15]).

### 1.1.3 Trudinger-Moser Inequalities on the whole $\mathbb{R}^n$

So far, we have only considered Trudinger-Moser inequalities on finite domains $\Omega$ in Euclidean spaces $\mathbb{R}^n$ with $|\Omega| < \infty$. There have been generalizations of Trudinger-Moser inequalities on domains $\Omega$ in Euclidean spaces $\mathbb{R}^n$ with $|\Omega| = \infty$. S. Adachi and K. Tanaka [1] proved the following subcritical Trudinger-Moser type inequality

**Theorem 1.1.3.** *For any $\alpha < \alpha_n$, there exists a positive constant $C_{n,\alpha}$ such that for any $u \in W^{1,n}(\mathbb{R}^n)$, $||\nabla u||_{L^n(\mathbb{R}^n)} \leq 1$,*

$$\int_{\mathbb{R}^n} \phi_n\left(\alpha\left|u\right|^{\frac{n}{n-1}}\right) dx \leq C_{n,\alpha}\left\|u\right\|_{L^n(\mathbb{R}^n)}^n, \tag{1.1.11}$$

*where*

$$\phi_n(t) = e^t - \sum_{j=0}^{n-2} \frac{t^j}{j!}.$$

*The constant $\alpha_n$ is sharp in the sense that the supremum is infinity when $\alpha \geq \alpha_n$.*

We note in the above theorem, we only impose the restriction $||\nabla u||_{L^n(\mathbb{R}^n)} \leq 1$. The method in [1] requires a symmetrization argument which is not available in many other non-Euclidean settings. The above inequality fails at the critical case $\alpha = \alpha_n$. So it is natural to ask when the above inequality can be true when $\alpha = \alpha_n$. This is done by Y. Li and B. Ruf in [53], [35] by using the restriction of the full norm

$||\nabla u||_{L^n(\mathbb{R}^n)} + ||u||_{L^n(\mathbb{R}^n)} \leq 1$. To be precise, Y. Li and B. Ruf proved the following critical Trudinger-Moser inequality

**Theorem 1.1.4.** *For any $u \in W^{1,n}(\mathbb{R}^n)$ satisfying $||u||_{W^{1,n}(\mathbb{R}^n)} \leq 1$, there exists $C_n > 0$ such that*

$$\int_{\mathbb{R}^n} \phi_n(\alpha_n |u|^{\frac{n}{n-1}})dx \leq C_n. \tag{1.1.12}$$

*Moreover, the constant $\alpha_n$ is sharp in the sense that if $\alpha_n$ is replaced by any larger number, the supremum becomes infinity.*

The method of proving the above two theorems in unbounded domains in Euclidean spaces (e.g., the entire Euclidean spaces) both use the symmetrization argument in the Euclidean spaces and in particular the Pólya-Szegö inequality. Such a symmetrization principle is not available for higher order derivatives or on non-Euclidean spaces (e.g. Riemannian manifolds and Heisenberg groups). To overcome such difficulty, N. Lam and G. Lu [24] developed a rearrangement-free argument and gave an easier proof of the inequality (1.1.12) (see also [23] where such method was applied to Heisenberg groups). More recently, N. Lam, G. Lu and L. Zhang [27] proved that (1.1.11) and (1.1.12) are equivalent. The new method in [24] also enabled them to establish sharp Trudinger-Moser inequalities with fractional order derivatives on $\mathbb{R}^n$ (see also [54] for the case of even order derivatives). More precisely,

**Theorem 1.1.5.**

$$\sup_{u \in W^{\gamma,n/\gamma}(\mathbb{R}^n), ||(\tau I - \triangle)^{\gamma/2} u||_{L^{n/\gamma}(\mathbb{R}^n)} \leq 1} \int_{\mathbb{R}^n} \phi_{n,\gamma}(\alpha(n,\gamma)|u|^{\frac{n}{n-\gamma}})dx < \infty \tag{1.1.13}$$

*where* $0 < \gamma < n$ *is arbitrary real number,* $(\tau I - \triangle)^{\gamma/2}$ *is defined in terms of the Bessel potential,* $\phi_{n,\gamma}(t) = \sum_{j=j_p-1}^{\infty} \frac{t^j}{j!}$*,* $j_p = [n/\gamma] + 1$ *and* $\alpha(n,\gamma)$ *is defined as*

$$\alpha(n,\gamma) = \frac{n}{\omega_{n-1}} \left[ \frac{\pi^{n/2} 2^{\gamma} \Gamma(\gamma/2)}{\Gamma((n-\gamma)/2)} \right]^{\frac{n}{n-\gamma}}.$$

*Moreover,* $\alpha(n,\gamma)$ *is sharp in the sense that if it is replaced by any larger number, the supreme will become infinity.*

Note that when $\gamma$ is integer, $\alpha(n,\gamma)$ coincides with the best constant in (1.1.9).

### 1.1.4 Trudinger-Moser Inequalities on Heisenberg Groups $\mathbb{H}^n$

Trudinger-Moser Inequalities on Heisenberg groups have also been extensively studied in the past decades (see [12, 14, 23, 26] and the references therein). Let $\Omega \subset \mathbb{H}^n$ be an open subset with finite volume. W. Cohn and G. Lu [14] proved that

**Theorem 1.1.6.**

$$\sup_{f \in HW^{1,Q}(\Omega), ||\nabla_{\mathbb{H}} f||_{L^Q(\Omega)} \leq 1} \frac{1}{|\Omega|} \int_{\Omega} e^{\alpha_Q |f|^{\frac{Q}{Q-1}}} d\xi < \infty, \tag{1.1.14}$$

*with*

$$\alpha_Q = Q \left( 2\pi^n \Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{Q-1}{2}\right) \Gamma\left(\frac{Q}{2}\right)^{-1} \Gamma(n)^{-1} \right)^{\frac{1}{Q-1}}, \tag{1.1.15}$$

*where* $Q = 2n+2$*,* $\nabla_{\mathbb{H}}$ *denotes the subelliptic gradient and* $HW^{1,Q}(\Omega)$ *is the horizontal Sobolev space. Moreover,* $\alpha_Q$ *cannot be replaced by any larger number.*

Both critical and subcritical inequalities are established on the whole space $\mathbb{H}^n$. N. Lam and G. Lu [23] proved

**Theorem 1.1.7.** *Let $0 \leq \beta < Q$. There exists a uniform constant $c$ depending only on $Q, \beta$ such that for all $\alpha \leq \alpha_{Q,\beta}$, one has*

$$\sup_{\substack{f\in HW^{1,Q}(\mathbb{H}^n)\\ \|f\|_{HW^{1,Q}(\mathbb{H}^n)}\leq 1}} \int_{\mathbb{H}^n} \frac{\Phi\left(\alpha f\left(\xi\right)^{\frac{Q}{Q-1}}\right)}{\rho\left(\xi\right)^{\beta}} d\xi < c. \tag{1.1.16}$$

*where $\alpha_{Q,\beta} = \alpha_Q\left(1-\frac{\beta}{Q}\right)$, $\Phi\left(t\right) = e^t - \sum_{j=0}^{Q-2} \frac{t^j}{j!}$. The constant $\alpha_{Q,\beta}$ is the best possible in the sense that if $\alpha > \alpha_{Q,\beta}$, then the supremum in the inequality (4.2.1) is infinite.*

N. Lam, G. Lu and H. Tang [26] proved

**Theorem 1.1.8.**

$$\sup_{f\in HW^{1,Q}(\mathbb{H}^n), ||\nabla_{\mathbb{H}} f||_{L^Q(\mathbb{H}^n)}\leq 1} \frac{1}{||f||_{L^Q(\mathbb{H}^n)}^{Q-\beta}} \int_{\mathbb{H}^n} \frac{\Phi\left(\alpha f\left(\xi\right)^{\frac{Q}{Q-1}}\right)}{\rho\left(\xi\right)^{\beta}} d\xi < \infty \tag{1.1.17}$$

*holds for any $\alpha < \alpha_{Q,\beta}$. Moreover, the constant $\alpha_{Q,\beta}$ is the best possible in the sense that if $\alpha \geq \alpha_{Q,\beta}$, the supreme will become infinity.*

### 1.1.5 Trudinger-Moser Inequalities on Riemannian Manifolds

Similar to the inequality (1.1.2) with the best constant, J. Moser also proved in [45] an analogous inequality on the two-dimensional sphere $\mathbb{S}^2$. He proved that there exists a constant $C_0$ such that $4\pi$ is the best constant for the inequality

$$\int_{\mathbb{S}^2} \exp\left(4\pi |u(x)|^2\right) d\sigma \leq C_0$$

for any smooth $u$ on $\mathbb{S}^2$ with $\int_{\mathbb{S}^2} |\nabla u|^2 d\sigma \leq 1$ and $\int_{\mathbb{S}^2} u d\sigma = 0$, where $d\sigma$ is the surface measure and $\nabla$ is the gradient on $\mathbb{S}^2$.

Moser further proved that there exists a constant $C_0$ such that $8\pi$ is the best constant for the inequality

$$\int_{\mathbb{S}^2} \exp\left(8\pi|u(x)|^2\right) d\sigma \leq C_0$$

for any smooth $u$ on $\mathbb{S}^2$ with $\int_{\mathbb{S}^2} |\nabla u|^2 d\sigma \leq 1$ and $\int_{\mathbb{S}^2} u d\sigma = 0$, and $u(\xi) = u(-\xi)$. This inequality on the sphere $\mathbb{S}^2$ has been used to solve the so-called Nirenberg problem: given a function $K(x)$ on the two-dimensional sphere $\mathbb{S}^2$, is there a metric $ds^2$ which is conformally related to the standard round metric $ds_0^2$ on $\mathbb{S}^2$ such that $K(x)$ is the Gauss curvature associated with the metric $ds^2$? This is equivalent to ask if there a function $u$ on $\mathbb{S}^2$ so that $ds^2 = e^u ds_0^2$ and $u$ satisfies the following differential equations:

$$\triangle u + Ke^{2u} - 1 = 0, \tag{1.1.18}$$

where $\triangle$ is the Laplace-Beltrami operator on $\mathbb{S}^2$ with respect to the standard metric $ds_0^2$.

Moser applied the inequality (1.1.2) to this problem of the prescribing Gaussian curvature on $\mathbb{S}^2$ by considering the following functional

$$G(u) = \log\left\{\frac{1}{4\pi}\int_{\mathbb{S}^2} Ke^{2u} d\sigma\right\} - \frac{1}{4\pi}\int_{\mathbb{S}^2} |\nabla u|^2 d\sigma - \frac{1}{2\pi}\int_{\mathbb{S}^2} u d\sigma.$$

Moser showed that there is a positive constant $C$ such that

$$\frac{1}{4\pi}\int_{\mathbb{S}^2} e^{2u} d\sigma \leq C \exp\left\{\frac{1}{4\pi}\int_{\mathbb{S}^2} |\nabla u|^2 d\sigma - \frac{1}{2\pi}\int_{\mathbb{S}^2} u d\sigma\right\} \tag{1.1.19}$$

This implies that the functional $G(u)$ is bounded from above. If the further assumption that $K(\xi) = K(-\xi)$ on $\mathbb{S}^2$ is imposed, Moser showed that the equation (1.1.18) has a solution $u$ such that $u(\xi) = u(-\xi)$ provided that $\max_{\mathbb{S}^2} K(\xi) > 0$ using the fact that the functional $G(u)$ is bounded from above.

The sharp form of the inequality (1.1.19) on $\mathbb{S}^2$ was sharpened by Onofri [50] which states that the following inequality holds:

$$\frac{1}{4\pi}\int_{\mathbb{S}^2} e^{2u} d\sigma \leq \exp\left(\frac{1}{4\pi}\int_{\mathbb{S}^2}(2u + |\nabla u|^2)d\sigma\right) \tag{1.1.20}$$

with equality if and only if $e^{2u}g_0$ is isometric to $g_0$. Onofri's inequality was also independently proved by Hong [20].

The higher dimensional Onofri's inequality was subsequently established by Beckner [5]. Let $\triangle_{\mathbb{S}}$ denote the Laplacian on $\mathbb{S}^n$. The Paneitz operator on $4-$manifolds was discovered by Paneitz [51]. It was extended for all dimensions $n \neq 2$ by Branson [7] and Beckner [5]. The important relationship between the higher dimensional Moser-Onofri inequality and the problem of prescribing $Q-$curvature on high dimensional Riemannian manifolds has been explored extensively in the work of Paneitz [51], Branson [7], Branson-Chang-Yang [8], Chang-Yang [10,11] and the references therein.

Due to the importance of the application of Trudinger-Moser inequalities to the conformal geometry, Trudinger-Moser inequalities on manifolds have attracted special attention. L. Fontana [18] established Adams inequality (1.1.9) on compact Riemannian manifolds. More precisely,

**Theorem 1.1.9.** *Let $M$ be a compact Riemannian manifold of dimension $n$ and $m$ be a positive integer strictly smaller than $n$. Then there exists a constant $C = C(m, M)$ such that for all $u \in C^m(M)$ with $\int_M u dV = 0$ and $\int_M |\nabla^m u|^{n/m} dV \leq 1$. the following inequality holds*

$$\int_M \exp(\alpha(n,m)|u|^{\frac{n}{n-m}})dV \leq C(n,M), \tag{1.1.21}$$

*where the constant $\alpha(n,m)$ is sharp in the same sense as before.*

The existence of extremal function of (1.1.21) is studied by Y. Li [34]. In the subelliptic setting, W. Cohn and G. Lu [15] proved the following sharp Trudinger-Moser inequalities on complex spheres $\mathbb{S} = \{z \in \mathbb{C}^n : |z| = 1\}$

$$\sup_{u\in C^\infty, \int_{\mathbb{S}} u=0, ||\nabla_{\mathbb{C}} u||_{L^{2n}(\mathbb{S})}\leq 1} \int_{\mathbb{S}} e^{B|u|^{\frac{2n}{2n-1}}} < \infty, \tag{1.1.22}$$

where $B = n((n-1)\pi^{-1}2^{2n-2}B(\frac{2n-1}{2}, \frac{1}{2}))^{\frac{1}{2n-1}}$ and $B(\frac{2n-1}{2}, \frac{1}{2})$ is the beta function.

It is fairly natural to ask if one can establish Trudinger-Moser inequalities on non-compact Riemannian manifolds. G. Lu and H. Tang [38, 39] proved sharp Trudinger-Moser inequalities on hyperbolic spaces, which are complete non-compact Riemannian manifolds with constant negative curvature. Trudinger-Moser inequalities with higher order derivatives (Hardy-Adams inequalities) on hyperbolic spaces have also been studied in [30, 41]. Such inequalities act as the borderline case of higher order Hardy-Sobolev-Maz'ya inequalities (see [40]). Recently, we [28] studied sharp critical and subcritical Trudinger-Moser inequalities on complete non-compact Riemannian manifolds. Our first result reads as following:

**Theorem 1.1.10.** *Let $(M, g)$ be a complete noncompact Riemannian manifold whose Ricci curvature has a lower bound, i.e. $Rc(M, g) \geq \lambda g$ for some $\lambda \in \mathbb{R}$. Moreover, assume its injectivity radius has a lower bound, i.e. $inj(M, g) \geq i > 0$. Then there exists a constant $C = C(n, M)$ such that*

$$\sup_{u\in W^{1,n}(M),||u||_{1,\tau}\le 1}\int_M \phi_n(\alpha_n|u|^{\frac{n}{n-1}})dV_g \le C, \tag{1.1.23}$$

*where* $||u||_{1,\tau}^n = \int_M |\nabla u|^n + \tau|u|^n dV_g$ *and* $\phi_n(t) = \sum_{j=n-1}^{\infty}\frac{t^j}{j!}$*. Moreover,* $\alpha_n$ *is sharp in the sense that if it is replaced by any larger number, the supremum will become infinity.*

It should be pointed out that the proof of critical inequalities (1.1.12) does not work on general Riemannian manifolds. We applied a rearrangement-free argument developed in [24]. Moreover, inspired by [27], if we further assume the manifold is equipped with bounded sectional curvature, we also proved subcritical Trudinger-Moser inequalities:

**Theorem 1.1.11.** *Let* $(M, g)$ *be a complete noncompact Riemannian manifold satisfies the same condition of* (1.1.23)*. If further assume that* $M$ *is with bounded sectional curvature, then for any* $u \in W^{1,n}(M)$ *such that* $||\nabla u||_{L^n(M)} \le 1$, $\alpha < \alpha_n = n\omega_{n-1}^{1/(n-1)}$*, there exists a constant* $C = C(n, \alpha, M)$ *such that*

$$\frac{1}{||u||_{L^n(M)}^n}\int_M \phi_n(\alpha|u|^{\frac{n}{n-1}})dV_g \le C. \tag{1.1.24}$$

*Moreover, this inequality is sharp in the sense when* $\alpha \ge \alpha_n$ *the inequality fails.*

### 1.1.6 Trudinger-Moser Inequalities for Maps Between Manifolds

There are also attempts to prove Sobolev type inequalities and inequalities with exponential growth for maps between manifolds. From now on, we denote by $(M, g)$ the m-dimensional Riemannian manifold equiped with the metric tensor $g : TM \times TM \to \mathbb{R}$, and similarly by $(N, h)$ the n-dimensional Riemannian manifold. We first consider the simpler case when $M$ is the usual Euclidean space $\mathbb{R}^m$, and the smooth map $u : \overline{\Omega} \to N$ is defined on a bounded and convex open set $\Omega \subset \mathbb{R}^m$ and maps the boundary $\partial\Omega$ to a single point, i.e. $u = u_0$ on $\partial\Omega$. If we let the function $\delta : N \to [0, \infty)$ be defined by $\delta(z) = dist(z, u_0)$ be the distance function on $N$, then $\delta$ is Lipschitz continuous with Lipschitz constant 1, and the composition function $v = \delta \circ u$ belongs to $W_0^{1,m}(\Omega)$. From (1.1.2) we instantly have

$$\sup_{u \in C^\infty(M,N), ||dv||_{L^m(\Omega)} \leq 1} \int_\Omega e^{\alpha_n |\mathrm{dist}(u,u_0)|^{\frac{m}{m-1}}} dV < \infty, \tag{1.1.25}$$

or equivalently,

$$||\mathrm{dist}(u, u_0)||_{L^{\psi_m}(\Omega)} \leq C||dv||_{L^m(\Omega)}, \tag{1.1.26}$$

where $|| \cdot ||_{L^{\psi_m}(\Omega)}$ is the Orlisz norm associated with the Young function $\psi_m(t) = e^{\alpha_m |t|^{\frac{m}{m-1}}}$. Let $du$ be the differential of $u$, i.e.

$$du = \frac{\partial u^\alpha}{\partial x^i} dx^i \otimes \frac{\partial}{\partial y^\alpha},$$

where $\{x^i\}$ and $\{y^\alpha\}$ are local coordinates of $M$ and $N$, is a section bundle $T^*M \otimes u^{-1}TN$. Then in our case, $|dv| \leq |du|$ and hence we achieve a Trudinger-Moser type inequality

$$||dist(u, u_0)||_{L^{\psi_m}(\Omega)} \leq C||du||_{L^m(\Omega)}. \tag{1.1.27}$$

With the similar argument, one can easily get the usual Sobolev inequality

$$||dist(u, u_0)||_{L^{mp/(m-p)}(\Omega)} \leq C||du||_{L^p(\Omega)} \tag{1.1.28}$$

for $1 \leq p < m$. For $p > m$, we also have the estimate for the Hölder norm. However, for the higher order case, the classical results do not always hold. Let $Ddu = \nabla(du)$ be the Hessian of $u$ which belongs to $T^*M \otimes T^*M \otimes u^{-1}TN$. R. Moser [48] proved that there exists constant $C > 0$ such that

$$||du||_{L^{mp/(m-p)}(\Omega)} \leq C||Ddu||_{L^p(\Omega)}, \tag{1.1.29}$$

for $2 \leq p < n$. Combining this with the above Sobolev inequality, we have for $2 \leq p < m/2$

$$||dist(u, u_0)||_{L^{mp/(m-2p)}(\Omega)} \leq C||Ddu||_{L^p(\Omega)}. \tag{1.1.30}$$

Denote by $\tau(u)$ the tension field of $u$ as the trace of $Ddu$, i.e., $\tau(u) = Tr(Ddu)$, which acts as the usual Laplacian for scalar-valued maps. (For details, see [21].) If we replace $Ddu$ by $\tau(u)$, there are also some inequalities in special cases, see [46, 47]. As a special case of Adams type inequality, for the second order operator $\tau(u)$, R. Moser [48] proved the following.

**Theorem 1.1.12.** *There exists a constant $C$ such that for any $u_0 \in N$ and $u \in C^\infty(\overline{\Omega}, N)$ with $u = u_0$ on $\partial\Omega$ satisfies*

$$||dist(u, u_0)||_{L^\phi(\Omega)} \leq C||\tau(u)||_{L^{m/2}(\Omega)}, \tag{1.1.31}$$

*where $\phi(t) = t^{m/(m-2)} \exp(t^{m/(m-2)})$.*

Recently we [29] generalize the above theorem to domains of manifold. The first result we have is for manifolds with bounded negative curvature. More preceisely,

**Theorem 1.1.13.** *Suppose $(M, g)$ is a complete Riemannian manifold, $dim M = m$ and its sectional curvature is bounded as*

$$(m-2)K_0 \leq K \leq K_0 \tag{1.1.32}$$

*for $K_0 < 0$. Let $\Omega$ be a convex bounded domain of $M$. Then there exists a constant $C$ such that for any $u_0 \in N$ and $u \in C^\infty(\overline{\Omega}, N)$ with $u = u_0$ on $\partial\Omega$,*

$$||dist(u, u_0)||_{L^\phi(\Omega)} \leq C||\tau(u)||_{L^{m/2}(\Omega)} \tag{1.1.33}$$

For manifolds with positive curvature, we proved the following result.

**Theorem 1.1.14.** *Suppose $(M, g)$ is a manifold with boundary and its Ricci tensor is bounded below as*

$$Ric_M \geq K_0 g, \tag{1.1.34}$$

*where $K_0 > 0$ is a constant. Suppose that $(N, h)$ is a complete manifold with nonpositive sectional curvature. Let $\Omega$ be a domain of $M$ whose boundary has nonnegative mean curvature. Then there exists a dimensional constant $C$ such that for any $u_0 \in N$ and $u \in C^\infty(\overline{\Omega}, N)$ with $u = u_0$ on $\partial\Omega$, we have*

$$||dist(u, u_0)||_{L^\phi(\Omega)} \leq C||\tau(u)||_{L^{m/2}(\Omega)}. \tag{1.1.35}$$

If we restrict the size of $\Omega$, we can obtain the same inequality on manifolds simply with bounded curvature. For simplicity of notation, we define for a given constant $c$,

$$f_c(r) = \begin{cases} \frac{\sqrt{c}r\cos(\sqrt{c}r)}{\sin(\sqrt{c}r)} & c > 0, \\ 1 & c = 0, \\ \frac{\sqrt{-c}r\cosh(\sqrt{-c}r)}{\sinh(\sqrt{-c}r)} & c < 0. \end{cases} \tag{1.1.36}$$

Denote

$$F(r) = \min\{(m-1)f_{K_1}(r) - 1, 1 + (m-2)f_{K_1}(r) - f_{K_0}(r)\}. \tag{1.1.37}$$

Then we have

**Theorem 1.1.15.** *Suppose the sectional curvature of $M$ is bounded by*

$$K_0 \leq K \leq K_1 \tag{1.1.38}$$

*where $K_0 < 0$ and $K_1 > 0$. Assume that for $R = diam(\Omega) \leq inj(M)$, we have*

$$F(R) \geq \delta > 0 \tag{1.1.39}$$

*for some $\delta > 0$. Then there exists a constant $C = C(\Omega)$ such that for any $u_0 \in N$ and $u \in C^\infty(\overline{\Omega}, N)$ with $u = u_0$ on $\partial\Omega$, the following inequality holds*

$$||dist(u, u_0)||_{L^\phi(\Omega)} \leq C||\tau(u)||_{L^{m/2}(\Omega)} \tag{1.1.40}$$

An instant corollary of (1.1.32), (1.1.35) and (1.1.40) is: when $u$ is a harmonic map, i.e. $\tau(u) = 0$, $u$ is a constant map. Hence our results actually give a proof of the maximum principle for smooth maps between manifolds. For the proof of our results, please see the following chapter.

## 1.2 Introduction to Concentration-Compactness Principles

### 1.2.1 Concentration-Compactness Principles on $\mathbb{R}^n$

A natural questions arises towards the inequality (1.1.2): is the embedding $W_0^{1,n}(\Omega) \hookrightarrow L_{\varphi_n}(\Omega)$ compact? (Recall that $\varphi_n(t) = \exp\left(\alpha |t|^{n/(n-1)}\right) - 1$)? That is equivalent to ask: for $\{u_k\}$ boudned in $W_0^{1,n}(\Omega)$, does $e^{\alpha_n |u_k|^{\frac{n}{n-1}}} \to e^{\alpha_n |u|^{\frac{n}{n-1}}}$ in $L^1(\Omega)$ for some $u$? Define $\{u_k\}$ as following

$$u_k(x) = \begin{cases} 0 & if\ |x| \geq 1 \\ k^{-\frac{1}{n}} n^{\frac{1}{n}} \omega_{n-1}^{-\frac{1}{n}} \log \frac{1}{|x|} & if\ e^{-\frac{k}{n}} \leq |x| \leq 1 \\ k^{\frac{n-1}{n}} n^{-\frac{n-1}{n}} \omega_{n-1}^{-\frac{1}{n}} & if\ |x| \leq e^{-\frac{k}{n}} \end{cases}. \tag{1.2.1}$$

It is easy to verify that $||\nabla u_k||_{L^n(\Omega)}^n = 1$ and $u_k \to 0$ a.e and weakly in $W_0^{1,n}(\Omega)$ and moreover, $|\nabla u_k|^n \rightharpoonup \delta_0$ and $e^{\alpha_n |u_k|^{\frac{n}{n-1}}} \rightharpoonup c\delta_0$ for some constant $c > 0$. This phenomenon was studied by P. L. Lions [37] and he proved the following theorem

**Theorem 1.2.1.** *Given $\{u_k\} \subset W_0^{1,n}(\Omega)$ such that $||\nabla u_k||_{L^n(\Omega)} \leq 1$. Then only one of the following cases happens:*

- *$u_k \to 0$ a.e. and weakly in $W_0^{1,n}(\Omega)$, $|\nabla u_k|^n \rightharpoonup \delta_{x_0}$, $e^{\alpha_n |u_k|^{\frac{n}{n-1}}} \rightharpoonup c\delta_{x_0}$ for some constant $c \geq 0$, $x_0 \in \Omega$.*

- *$u_k \rightharpoonup u$ weakly in $W^{1,n}(\Omega)$ and any $0 < p < M_{n,u} = \left(1 - ||\nabla u||^n_{L^n(\Omega)}\right)^{-\frac{1}{n-1}}$,*

$$\sup_k \int_\Omega e^{\alpha_n p |u_k|^{\frac{n}{n-1}}} dx < \infty. \tag{1.2.2}$$

More recently, R. Černý, A. Cianchi and S. Hencl [57] discovered a new approach to obtain and sharpen (1.2.2) as well as fill in a gap. They proved that $M_{n,u}$ is sharp in the sense that there exists a sequence $\{u_k\}$ satisfying all the conditions in Lions' theorem and the supreme in (1.2.2) is infinity if $p \geq M_{n,u}$. This approach was further extended to study the concentration-compactness principle for the whole space $\mathbb{R}^n$ by J. M. do Ó, M. de Souza, E. de Medeiros and U. Severo [16]. Their results can be stated as follows:

**Theorem 1.2.2.** *Given $\{u_k\}$ such that $u_k \rightharpoonup u \neq 0$ in $W^{1,n}(\mathbb{R}^n)$ and $||u_k||_{W^{1,n}(\mathbb{R}^n)} = 1$, then for any $0 < p < \tilde{M}_{n,u} = \frac{1}{(1-||u||^n_{W^{1,n}(\mathbb{R}^n)})^{\frac{1}{n-1}}}$,*

$$\sup_k \int_{\mathbb{R}^n} \phi_n(\alpha_n p |u_k|^{\frac{n}{n-1}}) dx < \infty \tag{1.2.3}$$

*Moreover, $\tilde{M}_{n,u}$ is sharp in the sense that for any $p > \tilde{M}_{n,u}$, there exits a sequence $\{u_k\}$ satisfying all the conditions such that that the supremum becomes infinite.*

### 1.2.2 Concentration-Compactness Principles on Heisenberg Groups $\mathbb{H}^n$

The sharp Trudinger-Moser inequality on Heisenberg groups was due to W. Cohn and G. Lu [14] and has been extended to Heisenberg type groups and Carnot groups in [4, 13] and with singular weights in [25]. Both critical and subcritical Trudinger-Moser inequalities are established on the entire Heisenberg group in [23, 26]. Then it is fairly natural to ask whether concentration-compactness principles (1.2.2) and (1.2.3) holds for the sub-elliptic settings. The main difficulty lies in the fact that the proof of both (1.2.2) and (1.2.3) rely on the Polyá-Szegö inequality on Euclidean spaces and such inequality is not available on Heisenberg groups. In [32], we gave an affirmative answer to this question.

On the Heisenberg group $\mathbb{H}^n$ with $Q = 2n + 2$ its homogeneous dimension. Let $\nabla_{\mathbb{H}}$ be the sub-gradient and $HW^{1,Q}$ be the horizontal Sobolev space (strict definitions will be given in the following chapter). Our first result concerns concentration-compactness principles on domains with finite measure on $\mathbb{H}^n$.

**Theorem 1.2.3.** *Let $0 \leq \beta < Q$. Assume that $\{u_k\}$ is a sequence in $HW_0^{1,Q}(\Omega)$ with $|\Omega| < \infty$, such that $\|\nabla_{\mathbb{H}} u_k\|_{L^Q(\Omega)} = 1$ and $u_k \rightharpoonup u \neq 0$ weakly in $HW_0^{1,Q}(\Omega)$. Then for any*

$$0 < p < M_{Q,u} = \frac{1}{\left(1 - ||\nabla_{\mathbb{H}} u||^Q_{L^Q(\Omega)}\right)^{1/(Q-1)}}, \tag{1.2.4}$$

*the following inequality holds,*

$$\sup_k \int_\Omega \frac{e^{\alpha_{Q,\beta} p u_k^{\frac{Q}{Q-1}}}}{\rho(\xi)} d\xi < \infty, \tag{1.2.5}$$

*where $\rho(\xi) = |\xi|$ is the homogeneous norm, $\alpha_{Q,\beta} = \alpha_Q(1 - \frac{\beta}{Q})$ and $\alpha_Q$ is defined as*

$$\alpha_Q = Q\left(2\pi^n \Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{Q-1}{2}\right)\Gamma\left(\frac{Q}{2}\right)^{-1}\Gamma\left(n\right)^{-1}\right)^{\frac{1}{Q-1}}. \tag{1.2.6}$$

*Moreover, $M_{Q,u}$ is sharp in the sense that there exists a sequence $\{u_k\}$ satisfying $\|\nabla_{\mathbb{H}} u\|^Q_{L^Q(\Omega)} = 1$ and $u_k \rightharpoonup u \neq 0$ in $HW_0^{1,Q}(\Omega)$ such that the supremum is infinite* *for* $p \geq M_{Q,u}$.

On the other hand, the concentration-compactness principle for the whole space $\mathbb{H}^n$ reads as following.

**Theorem 1.2.4.** *Let* $0 \leq \beta < Q$. *Assume that* $\{u_k\}$ *is a sequence in* $HW^{1,Q}(\mathbb{H}^n)$ *such that* $\|u_k\|_{HW^{1,Q}(\mathbb{H}^n)}^Q = 1$ *and* $u_k \rightharpoonup u \neq 0$ *in* $HW^{1,Q}(\mathbb{H}^n)$. *If*

$$0 < p < \tilde{M}_{Q,u} = \frac{1}{\left(1 - ||u||_{HW^{1,n}(\mathbb{H}^n)}^Q\right)^{1/(Q-1)}}, \tag{1.2.7}$$

*then*

$$\sup_k \int_{\mathbb{H}^n} \frac{\Phi\left(\alpha_{Q,\beta} p u_k^{\frac{Q}{Q-1}}\right)}{\rho(\xi)^\beta} d\xi < \infty, \tag{1.2.8}$$

*where* $\Phi(t) = e^t - \sum_{j=0}^{Q-2} \frac{t^j}{j!}$. *Furthermore,* $\tilde{M}_{Q,u}$ *is sharp in the sense that there exists a sequence* $\{u_k\}$ *satisfying* $\|u_k\|_{HW^{1,Q}(\mathbb{H}^n)}^Q = 1$ *and* $u_k \rightharpoonup u \neq 0$ *in* $HW^{1,Q}(\mathbb{H}^n)$ *such that the supremum is infinite for* $p > \tilde{M}_{Q,u}$.

It should be pointed out that in [32], the proof of (1.2.8) still partially depends on a rearrangement inequality due to [43]. In [31], we developed a method which is completely free of any rearrangement argument. In the following chapter, we will apply such method to give new proof of (1.2.8).

### 1.2.3 Concentration-compactness Principles on Riemannian Manifolds

We will now state our main results in [31] of concentration-compactness principles on Riemannian manifolds. The first theorem is the concentration-compactness principle on compact Riemannian manifolds with boundary, which can be viewed as an analogue of Theoreom 1.2.3.

**Theorem 1.2.5.** *$(M, g)$ be a compact Riemannian manifold with boundary. Suppose $\{u_k\}$ is a sequence in $W_0^{1,n}(M)$ satisfying $||\nabla u_k||_{L^n(M)} = 1$ and $u_k \rightharpoonup u$ weakly in $W_0^{1,n}(M)$. Then for any $0 < p < M_{n,u} = (1 - ||\nabla u||^n_{L^n(M)})^{-\frac{1}{n-1}}$,*

$$\sup_k \int_M e^{\alpha_n p |u_k|^{\frac{n}{n-1}}} dV_g < \infty. \tag{1.2.9}$$

*Moreover, the inequality is sharp in the sense that if $p \geq M_{n,u}$, there exists a sequence $\{u_k\}$ with $||\nabla^m u_k||_{L^n(M)} = 1$ weakly convergent to $u$ in $W_0^{1,n}(M)$ but the supreme in (1.2.9) becomes infinite.*

The second main result is the concentration-compactness principle on complete noncompact Riemannian manifolds, which reads as following.

**Theorem 1.2.6.** *Let $(M, g)$ be a complete noncompact Riemannian manifold with injectivity $inj(M) > i_0 > 0$ and the Ricci tensor $Ric \geq \lambda g$ for some constant $\lambda$. Suppose $\{u_k\}$ is a sequence in $W^{1,n}(M)$ satisfying $||u_k||_{W^{1,n}(M)} = 1$ and $u_k \rightharpoonup u$ weakly in $W^{1,n}(M)$. Then for any $0 < p < \tilde{M}_{n,u} = (1 - ||u||^n_{W^{1,n}(M)})^{-\frac{1}{n-1}}$, the following inequality holds*

$$\sup_k \int_M \phi(\alpha_n p |u_k|^{\frac{n}{n-1}}) dV_g < \infty \tag{1.2.10}$$

*where* $\phi(t) = \sum_{j=n-1}^{\infty} \frac{t^j}{j!}$*. Moreover, the inequality is sharp in the sense for any* $p > \tilde{M}_{n,u}$*, there exists a sequence* $\{u_k\}$ *with* $||u_k||_{W^{1,n}(M)} = 1$ *and weakly convergent to* $u$ *in* $W^{1,n}(M)$ *but the supreme in* (1.2.10) *becomes infinite.*

Recall that in [32], we applied a rearrangement inequality on $\mathbb{H}^n$ by Manfredi and Vera De Serio [43] to prove the concentration-compactness principle on $\mathbb{H}^n$ . We find out that our new method in [31], which is totally rearrangement-free, can also be easily adjusted to prove Theorem 1.2.5 and Theorem 1.2.6.

# Chapter 2

# Trudinger-Moser Inequalities on Riemannian Manifolds

## 2.1 Preliminary of Riemannian Geometry

In this section, we provide some preliminaries. For an n-dimensional Riemannian manifold with its metric $(M, g)$, where $g : TM \times TM \to \mathbb{R}$ is a symmetric $(2,0)$-tensor, we will denote by $\nabla u$ the covariant differentiation of u, which in the local coordinate chart can be expressed as

$$\nabla u = \nabla_i u dx^i \tag{2.1.1}$$

where $\nabla_i u$ is the i-th covariant derivative of u and we always take summation when there is the same index at both subscript and superscript.

Let $\{\frac{\partial}{\partial x^i}\}_{i=1}^n$ be the base of the tangent space $TM$, then we denote $g_{ij} = g(\frac{\partial}{\partial x}, \frac{\partial}{\partial x^j})$

and $g^{ij}$ as the component of the inverse matrix $(g_{ij})^{-1}_{n\times n}$. The Laplace-Beltrami operator is defined as

$$\triangle u = g^{ij}\nabla_i\nabla_j u \tag{2.1.2}$$

For a smooth curve on the manifold $\gamma : (a, b) \to M$, we say it is geodesic if $\nabla_{\gamma'}\gamma' = 0$, where $\gamma'$ is the tangent vector field determined by this curve. If we let $L$ be the collection of all curves connecting the points $P$ and $Q$ on manifold, then the distance between $P$ and $Q$ is defined by

$$d(P, Q) = inf\{\gamma \in L| \int_a^b |\gamma'(t)|dt\} \tag{2.1.3}$$

and for $\gamma(t)$ a geodesic curve connecting $P$ and $Q$, we say it is a minimal geodesic curve if

$$\int_a^b |\gamma'(t)dt| = d(P, Q). \tag{2.1.4}$$

Thus a complete noncompact Riemannian manifold is a Riemannian manifold which is noncompact and satisfies that for any two points on it, say $P$ and $Q$, there exists a minimal geodesic curve connecting these two points.

**Definition 2.1.1.** Let $(M, g)$ be a Riemannian n-manifold and let $x \in M$. Given $Q > 1$, $k \in \mathbb{N}$ and $\alpha \in (0, 1)$, we define the $C^{k,\alpha}$ harmonic radius at $x$ as the largest number $r_H = r_H(Q, k, \alpha)(x)$ such that on the geodesic ball $B_x(r_H)$ of center $x$ and radius $r_H$, there is a harmonic coordinate chart such that the metric tensor is $C^{k,\alpha}$

controlled in this coordinate system. Namely, if $g_{ij}$ are the components of $g$ in this coordinate system, then

1) $Q^{-1}\delta_{ij} \leq g_{ij} \leq Q\delta_{ij}$ as bilnear forms

2) $\sum_{1\leq|\beta|\leq k} r_H^{|\beta|} \sup_y |\partial_\beta g_{ij}(y)| + \sum_{|\beta|=k} r_H^{k+\alpha} \sup_{y\neq z} \frac{|\partial_\beta g_{ij}(y)-\partial_\beta g_{ij}(z)|}{d(y,z)}^{\alpha} \leq Q-1$

The harmonic radius $r_H(Q,k,\alpha)(M)$ is defined as $r_H(Q,k,\alpha)(M) = \inf_{x\in M} r_H(Q,k,\alpha)(x)$.

The next theorem (see [19]) shows that one can obtain the lower bound of harmonic radius in terms of the bounds of Ricci curvature and injectivity radius.

**Theorem 2.1.2.** *Let $\alpha \in (0,1)$, $Q > 1$, $\delta > 0$. let $(M,g)$ be a Riemannian manifold and $\Omega$ an open subset of $M$. Set*

$$\Omega(\delta) = \{x \in M \ s.t. \ d(x,\Omega) < \delta\} \tag{2.1.5}$$

*Suppose for some $\lambda \in \mathbb{R}$ and $i > 0$, we have that for all $x \in \Omega(\delta)$,*

$$Rc(M,g) \geq \lambda g \text{ and } inj(M,g) \geq i \tag{2.1.6}$$

*Then there exists a positive constant $C = C(n,Q,\alpha,\delta,\lambda,i)$ such that for any $x \in \Omega$, $r_H(Q,0,\alpha)(x) \geq C$.*

We now introduce the Sobolev space on Riemannian manifolds. First we denote the volume form as $dV_g = \sqrt{|g|}dx^1 \wedge \cdots \wedge dx^n$ where $|g|$ is the determinant of the matrix $(g_{ij})_{n\times n}$. Then the $L^p$ norm of $\nabla u$ is defined as

$$||\nabla u||_{L^p(M)} = \left(\int_M (g^{ij}\nabla_i u \nabla_j u)^{p/2} dV_g\right)^{1/p} \tag{2.1.7}$$

For function $u(P)$ which is differentiable on the manifold, we say it belongs to $C^{1,p}(M)$ if $||u||_{W^{1,p}(M)} = ||\nabla u||_{L^p(M)} + ||u||_{L^p(M)} < \infty$. The Sobolev space $W^{1,p}(M)$ is the completion of $C^{1,p}(M)$ under the norm $||u||_{W^{1,p}(M)} = ||\nabla u||_{L^p(M)} + ||u||_{L^p(M)}$. In addition, we define $C_0^{1,p}$ as the set of all functions in $C^{1,p}$ with compact support and $W_0^{1,p}$ as the completion of $C_0^{1,p}$ under the norm $||\cdot||_{W^{1,p}(M)}$.

**Theorem 2.1.3.** *(See [3]) For a complete Riemannian manifold* $W_0^{1,p}(M) = W^{1,p}(M)$.

Now we recall some results from Riemannian geometry. For $f \in C^2(M)$. We denote $Hess(f) = \nabla^2 f$ which is defined for any $X, Y \in TM$ by

$$Hess(f)(X,Y) = \nabla^2 f(X,Y) = Y(Xf) - (\nabla_Y X)f, \tag{2.1.8}$$

and

$$\triangle f = Tr(Hess(f)). \tag{2.1.9}$$

Taking geodesic polar coordinate $\{r, \theta^1, \cdots, \theta^{m-1}\}$ on $(M, g)$, the metric tensor could then be written as

$$g = dr^2 + f^2(r) g_{ij}(r,\theta) d\theta^i d\theta^j. \tag{2.1.10}$$

For the distance function $\rho$ on $M$, we have the following property which enable us to get rid of the radial direction when evaluating $Hess(\rho)$.

**Proposition 2.1.4.** *Let* $P \in M$. *For any* $X \in T_P M$, $Hess(\rho)(X, \frac{\partial}{\partial r}(P)) = 0$.

Since there is no conjugate point in $M$ under our assumption, there exists a unique Jacobi field $\tilde{X}(t)$ along the geodesic curve $\gamma : [0, r] \to M$ such that

$$\tilde{X}(0) = 0, \tilde{X}(r) = X \tag{2.1.11}$$

$$\begin{aligned} Hess(\rho)(X,X) &= \{\tilde{X}(\tilde{X}r) - (\nabla_{\tilde{X}}\tilde{X})r\}_P \\ &= \{\tilde{X}\langle\tilde{X}, \frac{\partial}{\partial r}\rangle - \langle\nabla_{\tilde{X}}\tilde{X}, \frac{\partial}{\partial r}\rangle\}_P \\ &= \langle\tilde{X}, \nabla_{\tilde{X}}\frac{\partial}{\partial r}\rangle_P \\ &= \langle\tilde{X}, \nabla_{\frac{\partial}{\partial r}}\tilde{X}\rangle_P \end{aligned} \tag{2.1.12}$$

The last equality follows from the property that $[\tilde{X}, \frac{\partial}{\partial r}] = 0$.

For a manifold with constant sectional curvature $c$, the metric tensor has the following form under geodesic polar coordinate $\{r, \theta^1, \cdots, \theta^{m-1}\}$

$$g = dr^2 + f^2(r)g_{ij}(\theta)d\theta^i d\theta^j \tag{2.1.13}$$

where $f(r)$ is as follows:

$$f(r) = \begin{cases} \frac{\sin(\sqrt{c}r)}{\sqrt{c}} & c > 0, \\ r & c = 0, \\ \frac{\sinh(\sqrt{-c}r)}{\sqrt{-c}} & c < 0. \end{cases}$$

For a manifold with constant curvature, it is well known that

$$\tilde{X}(t) = \frac{f(t)}{f(r)}A(t), \tag{2.1.14}$$

where the vector field $A(t)$ satisfies $\nabla_{\frac{\partial}{\partial t}}A = 0$, $\langle A, \frac{\partial}{\partial t}\rangle = 0$ and $A(r) = X$. Hence

$$\begin{aligned} Hess(\rho)(X,X) &= \langle \tilde{X}, \nabla_{\frac{\partial}{\partial t}} \tilde{X} \rangle \\ &= \langle A(r), \frac{f'(r)}{f(r)} A(r) \rangle \\ &= \frac{f'(r)}{f(r)} g(X,X) \end{aligned} \tag{2.1.15}$$

where $\langle X, \frac{\partial}{\partial r} \rangle = 0$. We can then get the formula of $\triangle\rho$ for the manifold with constant curvature. Denote $\{E_1, \cdots, E_{m-1}, \frac{\partial}{\partial r}\}$ as the standard orthonormal basis of $T_P M$, then

$$\begin{aligned} (\triangle\rho)(\gamma(r)) &= \textstyle\sum_{i=1}^{m-1} Hess(\rho)(E_i, E_i) \\ &= (m-1)\frac{f'(r)}{f(r)} \end{aligned} \tag{2.1.16}$$

In particular, for manifolds with constant negative curvature $K_0 < 0$, we have

$$\triangle\rho = (m-1)\sqrt{-K_0}\frac{\cosh(\sqrt{-K_0}r)}{\sinh(\sqrt{-K_0}r)} \tag{2.1.17}$$

and for manifolds with positive curvature $K_1 > 0$, we have

$$\triangle\rho = (m-1)\sqrt{K_1}\frac{\cos(\sqrt{K_1}r)}{\sin(\sqrt{K_1}r)} \tag{2.1.18}$$

**Theorem 2.1.5.** *(Hessian Comparison Principle) Let $M_k (k = 1, 2)$ be two Riemannian manifolds and $\gamma : [0, r] \to M_k$ be the geodesic curves. If the curvature of $M_1$ along $\gamma_1$ is greater than or equal to the curvature of $M_2$ along $\gamma_2$, then*

$$Hess(\rho_1)(X_1, X_1) \leq Hess(\rho_2)(X_2, X_2) \tag{2.1.19}$$

*where* $X_k \in T_{\gamma(r)}M_k$ *such that*

$$\langle X_1, \frac{\partial}{\partial r_1} \rangle_{M_1} = \langle X_2, \frac{\partial}{\partial r_2} \rangle_{M_2} \tag{2.1.20}$$

*and* $||X_1|| = ||X_2||$.

Let $X \in T^*M$ be the form $X = X_i dx^i$, denoting $divX = g^{ij}\nabla_j X_i$, $duX = g^{ij}X_i\nabla_j u^\alpha \frac{\partial}{\partial y^\alpha} \in TN$, we have the following Pohozaev identity.

**Proposition 2.1.6.** $div(|du|^2X - 2\langle duX, \nabla u\rangle) = |du|^2 divX - 2\langle du\nabla X, \nabla u\rangle - 2\langle duX, \tau(u)\rangle$

*Proof.*

$$\begin{aligned} LHS &= (\nabla|du|^2)\cdot X + |du|^2 divX - 2\nabla(duX)\cdot\nabla u - 2\langle duX, \tau(u)\rangle \\ &= |du|^2 divX + (\nabla|du|^2)\cdot X - 2\nabla duX\cdot\nabla u - 2\langle du\nabla X, \nabla u\rangle - 2\langle duX, \tau(u)\rangle \end{aligned}$$

For the second term,

$$\begin{aligned} (\nabla|du|^2)\cdot X &= g^{ij}\nabla_i|du|^2 X_j \\ &= g^{ij}\nabla_i(g^{kl}h_{\alpha\beta}\nabla_k u^\alpha \nabla_l u^\beta)X_j \\ &= g^{ij}2(g^{kl}h_{\alpha\beta}\nabla_k u^\alpha \nabla_i\nabla_l u^\beta)X_j \\ &= g^{ij}2(g^{kl}h_{\alpha\beta}\nabla_k u^\alpha \nabla_l\nabla_i u^\beta)X_j \\ &= 2\nabla duX\cdot\nabla u \end{aligned}$$

The last line follows from the property for smooth maps

$$\nabla_i \nabla_j u^\alpha = \nabla_j \nabla_i u^\alpha \tag{2.1.21}$$

Hence we get to the right hand side of the above identity.

□

**Theorem 2.1.7.** *(See [6]) Let $B_P(r)$ be the geodesic ball of center $P$ and radius $r$ in $M$, i.e. the set of points in $M$ at a distance from the point $P$ smaller than $r$. Then:*

$$Vol(B_P(r)) = \frac{\omega_{n-1}}{n} r^n (1 - \frac{1}{6(n+2)} s(P) r^2 + o(r^2)) \tag{2.1.22}$$

*where $s(P)$ is the scalar curvature ar $P$. Moreover, in normal geodesic coordinates around $P$, the volume form of $M$ is*

$$dV(\tilde{P}) = r^{n-1}(1 - \frac{1}{3} R(\theta) r^2 + o(r^2)) d\theta dr \tag{2.1.23}$$

*where $\tilde{P} = \exp_P(r\theta)$, $\theta \in S^{n-1}$ and $d\theta$ is the standard surface measure of $S^{n-1}$. $R(\theta)$ is the Ricci curvature evaluated on the vector $\theta$.*

In order to establish Adams inequality on compact Riemannian manifolds (with or without boundary), one needs to get representation formulas for $u(P)$ and T. Aubin showed the following (see [3]):

- Let $M$ be a compact $C^\infty$ Riemannian manifold without boundry. Then there exists $G(P, \tilde{P})$, a Green function of the Laplacian which has the following properties:

(a) For all functions $\phi \in C^2$

$$\phi(P) = \frac{1}{Vol(M)} \int_M \phi(\tilde{P}) dV(\tilde{P}) + \int_M G(P, \tilde{P}) \triangle \phi(\tilde{P}) dV(\tilde{P}) \tag{2.1.24}$$

(b) There exists a constant $k$ such that:

$$|G(P, \tilde{P})| < k(1 + |\log r|) \quad \text{for } n = 2 \tag{2.1.25}$$

$$|G(P, \tilde{P})| < kr^{2-n} \quad \text{for } n > 2 \tag{2.1.26}$$

$$|\nabla_{\tilde{P}} G(P, \tilde{P})| < kr^{1-n} \tag{2.1.27}$$

$$|\nabla^2_{\tilde{P}} G(P, \tilde{P})| < kr^{-n} \tag{2.1.28}$$

where $r = d(P, \tilde{P})$

- Let M be an oriented compact Riemannian manifold with boundry of clas $C^\infty$. Then there exists $G(P, \tilde{P})$, the Green function such that:

(a) For all $\phi \in C^2(\overline{M})$

$$\phi(P) = \int_M G(P, \tilde{P}) \triangle \phi(\tilde{P}) dV(\tilde{P}) - \int_{\partial M} \nu^i \nabla_{i\tilde{P}} G(P, \tilde{P}) \phi(\tilde{P}) ds(\tilde{P}) \tag{2.1.29}$$

(b) The kernel $G(P, \tilde{P})$ satisfies the same estimates as in (2.1.30)-(2.1.28).

L. Fontana [18] further obtained the representation formula with gradient:

$$u(P) = \int_M \nabla G(P, \tilde{P}) \cdot \nabla u(\tilde{P}) dV(\tilde{P}) \tag{2.1.30}$$

where $u(P)$ satisfies $\int_M u(\tilde{P})dV(\tilde{P}) = 0$ when $M$ is without boundry and $u \in C_0^2$ when $M$ has boundry of class $C^\infty$. Moreover, we can write the estimate for $\nabla_{\tilde{P}} G(P, \tilde{P})$ as

$$|\nabla_{\tilde{P}} G(P, \tilde{P})| \leq \frac{1}{\omega_{n-1}} d(P, \tilde{P})^{1-n}[1 + O(d(P, \tilde{P}))] \tag{2.1.31}$$

## 2.2 Critical Trudinger-Moser Inequality on Complete Noncompact Riemannian Manifolds - Proof of (1.1.23)

### 2.2.1 The Trudinger-Moser inequality on bounded domains of Riemannian manifolds

In this section we will establish a sharpened Trudinger-Moser inequality on any finite domains of noncompact and complete Riemannian manifolds which improve the estimate obtained in [18]. When $m = 1$, notice that (1.1.21) only shows that the constant $C$ is a constant depending on the dimension $n$ and the manifold $M$. However, how it depends on $M$ has not been studied further. We will show, by carefully going through Adams' proof again by using O'Neil's lemma, that $C(n, M)$ is a continuously increasing function with respect to the volume of $M$. This is crucial for us to carry out the argument for the sharp Trudinger-Moser inequality on noncompact and complete Riemannian manifolds.

Using Adams' argument together with O'Neil's lemma, we will prove the following

**Theorem 2.2.1.** *Let $M$ be a compact Riemannian manifold and $T$ be the operator defined by $Tf(P) = \int_M K(P, \tilde{P})f(\tilde{P})dV(\tilde{P})$, where $K(P, \tilde{P}) = d(P, \tilde{P})^{\alpha-n}(1 + ad(P, \tilde{P})^\beta)$,$a \geq 0$, $\beta > 0$, $0 < \alpha < n$. We have the following conclusions:*

*(1) Then there exists $C = C(\alpha, \beta, a, M)$ such that*

$$\sup_{f \in L^{n/\alpha}(M), \|f\|_{L^{n/\alpha}} \leq 1} \int_M \exp\{\frac{n}{\omega_{n-1}} |Tf(P)|^{\frac{n}{n-\alpha}}\} dV(P) \leq C \tag{2.2.1}$$

*where $\omega_{n-1}$ is the area of the unit sphere in $R^n$. Moreover, $\frac{n}{\omega_{n-1}}$ is sharp.*

*(2) The constant $C = C(\alpha, \beta, a, M)$ depends on the volume $M$ in a way such that it is monotone increasing with respect to the volume of $M$.*

We remark that part (1) of Theorem 2.2.1 was proved in [18]. However, part (2) was not established in [18]. Nevertheless, part (2) is the most substantial ingredient for us to establish (1.1.23). For simplicity, we will just show the proof when $\alpha = 1$. By letting $f(\tilde{P}) = \nabla u(\tilde{P})$, $K(P, \tilde{P}) = d(P, \tilde{P})^{1-n}[1 + O(1)d(P, \tilde{P})]$ and in this case, $a = O(1)$ and $\beta = 1$. Then from (2.1.30) and (2.1.31), it is easy to see

$$\sup_{f \in L^n(M), \|f\|_{L^n} \leq 1} \int_M \exp\{\frac{n}{\omega_{n-1}} |Tf(P)|^{\frac{n}{n-1}}\} dV(P) \leq C(n, M) \tag{2.2.2}$$

where $C(n, M)$ depends on the volume of $M$ and is increasing with respect to $vol(M)$.

By applying the representation formula of $u(P)$, through Green's function, we can show that the above fractional type inequality implies

$$\int_M \exp(\alpha_n |u|^{\frac{n}{n-1}}) dV(x) \leq C(n, M) \tag{2.2.3}$$

with $C(n, M)$ depending on the volume of $M$ and increasing with respect to $Vol(M)$.

To prove Theorem 2.2.1, we recall some lemmas required for the proof. The first one is O'Neil lemma regarding the rearrangement of the convolution which we refer to Lemma 3.1 in [18].

**Lemma 2.2.2.** *$\forall t > 0, f \geq 0$, there exists a constant $B$ such that*

$$(Tf)^{**} \leq \frac{\omega_{n-1}}{\alpha}\left(\frac{nt}{\omega_{n-1}}\right)^{\alpha/n}(1+Bt^{\beta/n})f^{**}(t) + \int_t^\infty f^*(s)\left(\frac{ns}{\omega_{n-1}}\right)^{\frac{\alpha-n}{n}}(1+Bs^{\beta/n})ds \tag{2.2.4}$$

*where $T$, $\alpha$, $\beta$ is defined in Theorem 2.2.1, $f^*$ is the usual non-increasing rearrangement of $|f|$ and $f^{**}$ is defined by $f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)ds$*

In the special case when $K(P,\tilde{P}) = d(P,\tilde{P})^{1-n}[1+O(1)d(P,\tilde{P})]$, by O'Neil lemma, we have

$$(Tf)^{**}(t) \leq \int_0^t K^*(s)dsf^{**}(t) + \int_t^\infty f^*(s)K^*(s)ds \tag{2.2.5}$$

where $K^*(s) = (\frac{ns}{\omega_{n-1}})^{\frac{1-n}{n}}(1+O(1)(\frac{ns}{\omega_{n-1}})^{\frac{1}{n}})$.

We want to make some comments for the $O(1)$ that appears above. $O(1)$ is a constant which depends on the manifold $M$. Later when we apply Theorem 2.2.1 to prove the Moser-Trudinger inequality on complete compact manifold $M$, we will replace $M$ in Theorem 2.2.1 by a bounded domain. However, under the assumption of Theorem 2.1.2, $O(1)$ is still a constant depending on the complete noncompact manifold $M$. Actually the constant $O(1)$ is produced during the estimate of $|\nabla_P G(P,\tilde{P})|$. And as in [18],

$$\nabla_P G(P,\tilde{P}) = \nabla_P H(P,\tilde{P}) + \int_M \nabla_P H(P,R)\sum_{i=1}^\infty J_i(R,\tilde{P})dV(R) \tag{2.2.6}$$

where $H(P,\tilde{P}) = \frac{1}{(n-2)\omega_{n-1}}d(P,\tilde{P})^{2-n}f(d(P,\tilde{P}))$, $f(r)$ is a smooth decreasing function which is 1 when $\tilde{P}$ is near $P$ and zero when $d(P,\tilde{P}) > inj(M,g)$. $J_i(P,\tilde{P}) =$

$\int_M J_{i-1}(P,R)J_1(R,\tilde{P})dV(R)$ and $J_1(P,\tilde{P}) = -\triangle_{\tilde{P}}H(P,\tilde{P}) - 1$. With the help of Theorem 2.1.2, all the integrals above are comparable with integrals in $\mathbb{R}^n$. Therefore, in the later proof of Moser-Trudinger inequality on the complete noncompact manifold $M$, $O(1)$ is a constant depending only on $M$.

Further by comparing with Lemma 2.2.2 we can easily see that in this case, the constant $B$ in the Lemma 2.2.2 is $O(1)(\frac{n}{\omega_{n-1}})^{\frac{1}{n}}$. Now we are going to review the proof of Theorem 2.2.1 for the case $\alpha = 1$. Recall that our goal is to study the relation between the constant $C$ in Theorem 2.2.1 with $Vol(M)$.

**Proof of Theorem 2.2.1**. Let $t_1 = Vol(M)$. Then using $f^* \le f^{**}$ we have

$$\begin{aligned} \int_M \exp\left\{\frac{n}{\omega_{n-1}}|Tf(P)|^{\frac{n}{n-1}}\right\} dV(P) &= \int_0^{t_1} \exp\left\{\frac{n}{\omega_{n-1}}|(Tf)^*(t)|^{\frac{n}{n-1}}\right\} dt \\ &\le \int_0^{t_1} \exp\left\{\frac{n}{\omega_{n-1}}|(Tf)^{**}(t)|^{\frac{n}{n-1}}\right\} dt. \end{aligned}$$

By applying Lemma 2.2.2,

$$\begin{aligned} (Tf)^{**} \le \omega_{n-1}\left(\frac{n}{\omega_{n-1}}\right)^{1/n}(1+Bt_1^{1/n})t^{(1-n)/n}\int_0^t f^*(s)ds \\ + \left(\frac{\omega_{n-1}}{n}\right)^{\frac{n-1}{n}}\int_t^{t_1} f^*(s)s^{(1-n)/n}(1+Bs^{1/n})ds. \end{aligned}$$

Set $C_1 = \omega_{n-1}\left(\frac{n}{\omega_{n-1}}\right)^{1/n}(1+Bt_1^{1/n})$. Then we have

$$\int_M \exp\left\{\frac{n}{\omega_{n-1}}|Tf(P)|^{\frac{n}{n-1}}\right\} dV(P)$$
$$\leq \int_0^{t_1} \exp\left\{\frac{n}{\omega_{n-1}}\left[C_1 t^{(1-n)/n}\int_0^t f^*(s)ds + \left(\frac{\omega_{n-1}}{n}\right)^{\frac{n-1}{n}}\int_t^{t_1} f^*(s)s^{(1-n)/n}(1+Bs^{1/n})ds\right]^{\frac{n}{n-1}}\right\} dt$$

Next we set

$$x = \log(1/s),\ y = \log(1/t),\ y_1 = \log(1/t_1),\ C_2 = (\frac{n}{\omega_{n-1}})^{\frac{n-1}{n}}C_1,\ \phi(x) = f^*(e^{-x})e^{-x/n}.$$

Then after a change of variables we have

$$\int_M \exp\left\{\frac{n}{\omega_{n-1}}|Tf(P)|^{\frac{n}{n-1}}\right\} dV(P)$$
$$\leq \int_{y_1}^{\infty} \exp\left\{\left[C_2\int_y^{\infty}\phi(x)e^{-\frac{n-1}{n}(x-y)}dx + \int_{y_1}^{y}(1+Be^{-x/n})\phi(x)dx\right]^{\frac{n}{n-1}} - y\right\} dy$$

Define

$$g(x,y) = \begin{cases} 1+Be^{-x/n} & y_1 \leq x \leq y \\ C_2 e^{-\frac{n-1}{n}(x-y)} & y_1 < y < x < \infty, \end{cases} \tag{2.2.7}$$

and

$$F(y) = y - \left(\int_{y_1}^{\infty} g(x,y)\phi(x)dx\right)^{\frac{n}{n-1}}. \tag{2.2.8}$$

Then the integral

$$\int_M \exp\left\{\frac{n}{\omega_{n-1}}|Tf(P)|^{\frac{n}{n-1}}\right\} dV(P)$$

becomes $\int_{y_1}^{\infty} e^{-F(y)}dy$. Since

$$1 \geq \int_M |f|^n dV = \int_0^{t_1} (f^*)^n(s)ds = \int_{y_1}^{\infty} \phi(y)^n dy \tag{2.2.9}$$

then the proof of Theorem 2.2.1 hence the dependence of the constant $C$ with respect to $Vol(M)$, is reduced to the proof of the following lemma (see [2]).

**Lemma 2.2.3.** *Suppose that $\phi : [y_1, \infty) \to \mathbb{R}^+$ satisfies $\int_y^{\infty} \phi(x)^n \leq 1$. Let $g$ and $F$ be as defined above involving $C_2$. Then*

$$\int_{y_1}^{\infty} e^{-F(y)}dy \leq C_3 < \infty \tag{2.2.10}$$

*where $C_3$ depends on $y_1$, $C_2$, but not on $\phi$.*

By applying this lemma, Theorem 2.2.1 follows. Moreover, the constant $C(n, M)$ in Theorem 2.2.1 is actually $C_3$ in Lemma 2.2.3. Thus we need to find out the relation between $C_3$ and $y_1$ and $C_2$.

**Proof of Lemma 2.2.3**. First of all,

$$\begin{aligned}
\sup_{y \geq y_1} \left( \int_y^{\infty} g(x,y)^{\frac{n}{n-1}} dx \right)^{\frac{n-1}{n}} &= \sup_{y \geq y_1} \left( \int_y^{\infty} (C_2 e^{-\frac{n-1}{n}(x-y)})^{\frac{n}{n-1}} dx \right)^{\frac{n-1}{n}} \\
&= \sup_{y \geq y_1} C_2 \left( \int_y^{\infty} e^{-(x-y)} dx \right)^{\frac{n-1}{n}} \\
&= C_2.
\end{aligned}$$

Set $E_\lambda = \{y \geq y_1 : F(y) \leq \lambda\}$. Then

$$\int_{y_1}^{\infty} e^{-F(y)} dy = \int_{-\infty}^{\infty} |E_\lambda| e^{-\lambda} d\lambda. \tag{2.2.11}$$

We will prove Lemma 2.2.3 by dividing its proof into proofs of several lemmas.

**Lemma 2.2.4.** *There exists a positive constant $C_4$ such that if $E_\lambda \neq \emptyset$, then $\lambda \geq -C_4$.*

**Proof of Lemma 2.2.4**. By the definition of $E_\lambda$, we have

$$\begin{aligned}
& y - \lambda \\
\leq & \left( \int_{y_1}^{\infty} g(x,y)\phi(x) dx \right)^{\frac{n}{n-1}} \\
\leq & \left( \int_{y_1}^{y} \phi(x)(1 + Be^{-x/n}) dx + \int_{y}^{\infty} \phi(x) C_2 e^{\frac{(y-x)(n-1)}{n}} dx \right)^{\frac{n}{n-1}} \\
\leq & \left\{ \left( \int_{y_1}^{y} \phi^n dx \right)^{\frac{1}{n}} \left( \int_{y_1}^{y} (1 + Be^{-\frac{x}{n}})^{\frac{n}{n-1}} dx \right)^{\frac{n-1}{n}} + \left( \int_{y}^{\infty} \phi^n dx \right)^{\frac{1}{n}} \left( \int_{y}^{\infty} C_2^{\frac{n}{n-1}} e^{y-x} dx \right)^{\frac{n-1}{n}} \right\}^{\frac{n}{n-1}}.
\end{aligned}$$

We estimate

$$\begin{aligned}
& \left( \int_{y_1}^{y} (1 + Be^{-\frac{x}{n}})^{\frac{n}{n-1}} dx \right)^{\frac{n-1}{n}} \\
\leq & \left( \int_{y_1}^{y} (1 + Be^{-\frac{x}{n}})^2 dx \right)^{\frac{n-1}{n}} \\
\leq & \left( \int_{y_1}^{y} (1 + 2Be^{-\frac{x}{n}} + B^2 e^{-\frac{2x}{n}}) dx \right)^{\frac{n-1}{n}} \\
\leq & (y + C_5)^{\frac{n-1}{n}},
\end{aligned}$$

where

$$C_5 = -y_1 + 2Bne^{-\frac{y_1}{n}} + \frac{n}{2}B^2 e^{-\frac{2y_1}{n}}.$$

Let $L(y) = (\int_y^\infty \phi^n dx)^{\frac{1}{n}} \le 1$. Then

$$\begin{aligned}
& y - \lambda \\
\le & \Big( (1 - L(y)^n)^{1/n} (y + C_5)^{\frac{n-1}{n}} + C_2 L(y) \Big)^{\frac{n}{n-1}} \\
\le & (y + C_5)\,(1 - L(y)^n)^{\frac{1}{n-1}} \\
& + \frac{n}{n-1} 2^{\frac{1}{n-1}} \left\{ (y + C_5)^{\frac{1}{n}} \,(1 - L(y)^n)^{\frac{1}{n(n-1)}}\, C_2 L(y) + (C_2 L(y))^{\frac{n}{n-1}} \right\} \\
\le & (y + C_5)(1 - \frac{1}{n-1} L(y)^n) + \frac{n}{n-1} 2^{\frac{1}{n-1}} (y + C_5)^{\frac{1}{n}} C_2 L(y) + \frac{n}{n-1} 2^{\frac{1}{n-1}} C_2^{\frac{n}{n-1}} \\
= & y + C_5 - \frac{1}{n-1} \Big( (y + C_5)^{\frac{1}{n}} L(y) \Big)^n + \frac{n}{n-1} 2^{\frac{1}{n-1}} (y + C_5)^{\frac{1}{n}} C_2 L(y) + \frac{n}{n-1} 2^{\frac{1}{n-1}} C_2^{\frac{n}{n-1}}.
\end{aligned}$$

Let us set

$$\sigma = (y + C_5)^{\frac{1}{n}} L(y),\ C_6 = \frac{n}{n-1} 2^{\frac{1}{n-1}} C_2,\ C_7 = \frac{n}{n-1} 2^{\frac{1}{n-1}} C_2^{\frac{n}{n-1}}.$$

Then

$$y - \lambda \le y + C_5 - \frac{1}{n-1}\sigma^n + C_6\sigma + C_7. \tag{2.2.12}$$

This implies

$$\lambda \ge \frac{1}{n-1}\sigma^n - C_6\sigma - C_5 - C_7. \tag{2.2.13}$$

Consider the function $f(\sigma) = \frac{1}{n-1}\sigma^n - C_6\sigma$. By solving $f' = 0$, we conclude that $f(\sigma)$

attains its minimum at $\sigma = (\frac{n-1}{n}C_6)^{\frac{1}{n-1}}$ and $f_{min} = \frac{1}{n-1}(\frac{n-1}{n}C_6)^{\frac{n}{n-1}} - C_6(\frac{n-1}{n}C_6)^{\frac{1}{n-1}}$. Therefore

$$\begin{aligned}\lambda &\geq \frac{1}{n-1}\left(\frac{n-1}{n}C_6\right)^{\frac{n}{n-1}} - C_6\left(\frac{n-1}{n}C_6\right)^{\frac{1}{n-1}} - C_5 - C_7 \\ &\geq -C_6^{\frac{n}{n-1}} - C_5 - C_7.\end{aligned}$$

By setting $C_4 = C_6^{\frac{n}{n-1}} + C_5 + C_7$, then we complete the proof of Lemma 2.2.4.

Next we prove the following

**Lemma 2.2.5.** *There exist positive constants $C_8$, $C_9$ such that $|E_\lambda| \leq C_8|\lambda| + C_9$*

**Proof of Lemma 2.2.5**. From

$$y - \lambda \leq y + C_5 - \frac{1}{n-1}\sigma^n + C_6\sigma + C_7 \tag{2.2.14}$$

it follows

$$\frac{1}{n-1}\sigma^n \leq \lambda + C_5 + C_7 + C_6\sigma \leq \lambda + C_5 + C_7 + \frac{\sigma^n}{n} + \frac{C_6^{\frac{n}{n-1}}}{n/(n-1)}. \tag{2.2.15}$$

Hence we have

$$\sigma^n \leq n(n-1)\lambda + n(n-1)\left(C_5 + \frac{n-1}{n}C_6^{\frac{n}{n-1}} + C_7\right) \tag{2.2.16}$$

which implies

$$\sigma \leq (n(n-1))^{\frac{1}{n}} |\lambda|^{\frac{1}{n}} + \left(n(n-1)(C_5 + \frac{n-1}{n} C_6^{\frac{n}{n-1}} + C_7)\right)^{\frac{1}{n}}. \tag{2.2.17}$$

By setting

$$C_{10} = (n(n-1))^{\frac{1}{n}} \ \ and \ C_{11} = \left(n(n-1)(C_5 + \frac{n-1}{n} C_6^{\frac{n}{n-1}} + C_7)\right)^{\frac{1}{n}},$$

we can simplify the above inequality as

$$\sigma \leq C_{10}|\lambda|^{\frac{1}{n}} + C_{11}. \tag{2.2.18}$$

Let $R$ be an arbitrary positive number such that $E_\lambda \cap [R, \infty) \neq \emptyset$. Take $r_1$, $r_2 \in E_\lambda \cap [R, \infty)$, $r_1 < r_2$, then

$$\begin{aligned}
& r_2 - \lambda \\
\leq & \left( \int_{y_1}^{r_1} g(s, r_2)\phi(s) ds + \int_{r_1}^{r_2} g(s, r_2)\phi(s) ds + C_2 L(r_1) \right)^{\frac{n}{n-1}} \\
\leq & \left\{ \left( \int_{y_1}^{r_1} g^{\frac{n}{n-1}} ds \right)^{\frac{n-1}{n}} \left( \int_{y_1}^{r_1} \phi^n ds \right)^{\frac{1}{n}} + \left( \int_{r_1}^{r_2} g^{\frac{n}{n-1}} ds \right)^{\frac{n-1}{n}} L(r_1) + C_2 L(r_1) \right\}^{\frac{n}{n-1}} \\
\leq & \left\{ (r_1 + C_5)^{\frac{n-1}{n}} + \left( (r_2 - r_1 + C_5)^{\frac{n-1}{n}} + C_2 \right) L(r_1) \right\}^{\frac{n}{n-1}} \\
\leq & r_1 + C_5 + \frac{n}{n-1} 2^{\frac{1}{n-1}} \{ (r_1 + C_5)^{\frac{1}{n}} ((r_2 - r_1 + C_5)^{\frac{n-1}{n}} + C_2) L(r_1) \\
& + ((r_2 - r_1 + C_5)^{\frac{n-1}{n}} + C_2)^{\frac{n}{n-1}} L(r_1)^{\frac{n}{n-1}} \} \\
\leq & r_1 + C_5 + \frac{n}{n-1} 2^{\frac{1}{n-1}} \left\{ \left( (r_2 - r_1 + C_5)^{\frac{n-1}{n}} + C_2 \right) \sigma + C(n) \left( (r_2 - r_1 + C_5) + C_2^{\frac{n}{n-1}} \right) L(r_1)^{\frac{n}{n-1}} \right\}
\end{aligned}$$

Let $\delta = r_2 - r_1$, then

$$\delta \leq \lambda + C_5 + \frac{n}{n-1} 2^{\frac{1}{n-1}} \left( (\delta + C_5)^{\frac{n-1}{n}} + C_2 \right) \sigma + \frac{n}{n-1} 2^{\frac{1}{n-1}} C(n)(\delta + C_5) L(r_1)^{\frac{n}{n-1}} + \frac{n}{n-1} 2^{\frac{1}{n-1}} C(n) C_2^{\frac{n}{n-1}}$$

By letting

$$C_{12} = C_5 + \frac{n}{n-1} 2^{\frac{1}{n-1}} C(n) C_2^{\frac{n}{n-1}} + \frac{n}{n-1} 2^{\frac{1}{n-1}} C(n) C_5$$

we have

$$\delta \leq \lambda + \frac{n}{n-1} 2^{\frac{1}{n-1}} C_{10} (\delta + C_5)^{\frac{n-1}{n}} |\lambda|^{\frac{1}{n}} + \frac{n}{n-1} 2^{\frac{1}{n-1}} C_{11} (\delta + C_5)^{\frac{n-1}{n}} + \frac{n}{n-1} 2^{\frac{1}{n-1}} C_2 C_{10} |\lambda|^{\frac{1}{n}} + \frac{n}{n-1} 2^{\frac{1}{n-1}} C_2 C_{11} + \frac{n}{n-1} 2^{\frac{1}{n-1}} C(n) \delta L(r_1)^{\frac{n}{n-1}} + C_{12}$$

Notice that

$$(\delta + C_5)^{\frac{n-1}{n}} |\lambda|^{\frac{1}{n}} \leq \frac{\delta + C_5}{\frac{n}{n-1}} \epsilon^{\frac{n}{n-1}} + \frac{|\lambda|}{n} \frac{1}{\epsilon^n}, \tag{2.2.19}$$

$$(\delta + C_5)^{\frac{n-1}{n}} \leq \frac{\delta + C_5}{\frac{n}{n-1}} \epsilon^{\frac{n}{n-1}} + \frac{1}{n} \frac{1}{\epsilon^n}, \tag{2.2.20}$$

$$|\lambda|^{\frac{1}{n}} \leq \frac{|\lambda|}{n} + \frac{1}{\frac{n}{n-1}}. \tag{2.2.21}$$

Putting them into the above inequality and after some simplifications we have the following estimate

$$\delta(1 - 2^{\frac{1}{n-1}}C_{10}\epsilon^{\frac{n}{n-1}} - 2^{\frac{1}{n-1}}C_{11}\epsilon^{\frac{n}{n-1}} - \frac{n}{n-1}2^{\frac{1}{n-1}}C(n)L(r_1)^{\frac{n}{n-1}})$$
$$\leq |\lambda|(1 + \frac{1}{n-1}2^{\frac{1}{n-1}}C_{10}\epsilon^{-n} + \frac{1}{n-1}2^{\frac{1}{n-1}}C_2C_{10})$$
$$+ \left(2^{\frac{1}{n-1}}C_{10}C_5\epsilon^{\frac{n}{n-1}} + 2^{\frac{1}{n-1}}C_{11}C_5\epsilon^{\frac{n}{n-1}} + \frac{1}{n-1}2^{\frac{1}{n-1}}C_{11}\epsilon^{-n} + 2^{\frac{1}{n-1}}C_2C_{10} + \frac{n}{n-1}2^{\frac{1}{n-1}}C_2C_{11} + C_{12}\right)$$

Similarly as in [2], we make $R = O(\lambda)$ as $\lambda \to \infty$ so that $\frac{n}{n-1}2^{\frac{1}{n-1}}C(n)L(r_1)^{\frac{n}{n-1}} \leq \frac{1}{2}$. Then we can choose $\epsilon$ small enough and hence complete the proof by setting

$$C_8 = \frac{1 + \frac{1}{n-1}2^{\frac{1}{n-1}}C_{10}\epsilon^{-n} + \frac{1}{n-1}2^{\frac{1}{n-1}}C_2C_{10}}{1/2 - 2^{\frac{1}{n-1}}C_{10}\epsilon^{\frac{n}{n-1}} - 2^{\frac{1}{n-1}}C_{11}\epsilon^{\frac{n}{n-1}}}, \tag{2.2.22}$$

$$C_9 = \frac{2^{\frac{1}{n-1}}C_{10}C_5\epsilon^{\frac{n}{n-1}} + 2^{\frac{1}{n-1}}C_{11}C_5\epsilon^{\frac{n}{n-1}} + \frac{1}{n-1}2^{\frac{1}{n-1}}C_{11}\epsilon^{-n} + 2^{\frac{1}{n-1}}C_2C_{10} + \frac{n}{n-1}2^{\frac{1}{n-1}}C_2C_{11} + C_{12}}{1/2 - 2^{\frac{1}{n-1}}C_{10}\epsilon^{\frac{n}{n-1}} - 2^{\frac{1}{n-1}}C_{11}\epsilon^{\frac{n}{n-1}}}. \tag{2.2.23}$$

We now continue the **Proof of Lemma 2.2.3**. By Lemma 2.2.4 and Lemma 2.2.5, we have

$$\begin{aligned}\int_{y_1}^{\infty} e^{-F(y)}dy &= \int_{-\infty}^{\infty} |E_\lambda|e^{-\lambda}d\lambda \\ &\leq \int_{-C_4}^{\infty} (C_8|\lambda| + C_9)e^{-\lambda}d\lambda \\ &= C_8(C_4e^{C_4} + 1 - e^{C_4}) + C_8 + C_9e^{C_4} \\ &\leq C_8(C_4e^{C_4} + 1) + C_8 + C_9e^{C_4}.\end{aligned}$$

Set

$$C_3 = C_8(C_4e^{C_4} + 1) + C_8 + C_9e^{C_4}.$$

Then we complete the proof of Lemma 2.2.3.

Recall that our goal is to study the relation between the constant $C(n, M)$ with $Vol(M)$, the volume of $M$. From the proof of Theorem 2.2.1, we know that this constant, which, according to the proofs of Lemmas 2.2.4 and 2.2.5, is the constant $C_3$ in Lemma 2.2.3. Thus we have very solid information for this constant. By carefully checking the definitions of all those constants appeared during our proof of in Lemma 2.2.3, we finally find out that $C_3$ is an increasing function with respect to $Vol(M)$.

### 2.2.2 Critical Trudinger-Moser Inequality on Complete Noncompact Riemannian Manifolds

To prove (1.1.23), we only need to prove this claim for all $u \in C_0^\infty(M) \setminus \{0\}$ s.t $u \geq 0$ and $\|u\|_{1,\tau} \leq 1$. This comes from the following density argument: $\forall u \in W^{1,n}(M)$, $|u| = u^+ + u^-$, then $|u|^{\frac{n}{n-1}} = |u^+|^{\frac{n}{n-1}} + |u^-|^{\frac{n}{n-1}}$. So we can reduce the problem to nonnegative functions.$\forall u \in W^{1,n}(M)$ and $u \geq 0$, since $C_0^\infty(M)$ is dense in $W^{1,n}(M)$, we can find a series of nonnegative functions $\{u_m\}_{m=1}^\infty$ in $C_0^\infty$ s.t $u_m \to u$ in $W^{1,n}(M)$. Then there exists subsequence $\{u_{m_k}\}$ s.t $u_{m_k} \to u$ a.e. From Fatou's Lemma,

$$\int_M \liminf_{k\to\infty} \phi\left(\alpha_n |u_{m_k}|^{\frac{n}{n-1}}\right) dV_g \leq \liminf_{k\to\infty} \int_M \phi\left(\alpha_n |u_{m_k}|^{\frac{n}{n-1}}\right) dV_g \qquad (2.2.24)$$

Therefore we have

$$\int_M \phi\left(\alpha_n|u|^{\frac{n}{n-1}}\right)dV_g \leq \sup_{u\in C_0^\infty(M),u\geq 0,\|u\|_{1,\tau}\leq 1}\int_M \phi\left(\alpha_n|u|^{\frac{n}{n-1}}\right)dV_g \tag{2.2.25}$$

Now in order to show $\int_M \phi\left(\alpha|u|^{\frac{n}{n-1}}\right)dV_g \leq C(n,\tau)$ for $u \in C_0^\infty(M)\setminus\{0\}$, we set $A(u) = 2^{-\frac{1}{n(n-1)}}\tau^{\frac{1}{n}}\|u\|_{L^n}$, $\Omega(u) = \{x \in M; u(x) \geq A(u)\}$. Since $\|u\|_{1,\tau} \leq 1$, we have $\tau^{\frac{1}{n}}\|u\|_{L^n(M)} \leq 1$, so $A(u) \leq 1$. Since we have the following

$$\int_M |u|^n dV_g \geq \int_{\Omega(u)} |u|^n dV_g \geq \int_{\Omega(u)} |A(u)|^n dV_g \geq |\Omega(u)| 2^{-\frac{1}{n-1}}\tau\|u\|_{L^n(M)}^n \tag{2.2.26}$$

so $|\Omega(u)| \leq 2^{\frac{1}{n-1}}\tau^{-1}$ and $\Omega(u)$ is compact. Since $M$ is complete, $\Omega(u)$ is further bounded. Now we split the integral as

$$\begin{aligned}\int_M \phi\left(\alpha_n|u|^{\frac{n}{n-1}}\right)dV_g &= \int_{\Omega(u)} \phi\left(\alpha_n|u|^{\frac{n}{n-1}}\right)dV_g + \int_{M\setminus\Omega(u)} \phi\left(\alpha_n|u|^{\frac{n}{n-1}}\right)\\ &\equiv I_1 + I_2\end{aligned}$$

For $I_2$

$$\begin{aligned}
I_2 &\le \int_{\{x:u(x)<1\}} \sum_{k=n-1}^{\infty} \frac{\alpha_n^k |u|^{\frac{kn}{n-1}}}{k!} dV_g \\
&\le \sum_{k=n-1}^{\infty} \frac{\alpha_n^k}{k!} \int_M |u|^n dV_g \\
&= \sum_{k=n-1}^{\infty} \frac{\alpha_n^k}{k!} \|u\|_{L^n(M)}^n \\
&\le \frac{1}{\tau} \sum_{k=n-1}^{\infty} \frac{\alpha_n^k}{k!} \\
&\equiv C(\tau, n)
\end{aligned}$$

For $I_1$: Let $v(x) = u(x) - A(u)$ in $\Omega(u)$ so $v \in W_0^{1,n}(\Omega(u))$ and

$$\begin{aligned}
|u|^{n'} &= (|v| + A(u))^{n'} \\
&\le |v|^{n'} + n'2^{n'-1} \left( |v|^{n'-1} A(u) + A(u)^{n'} \right) \\
&\le |v|^{n'} + n'2^{n'-1} \left( \frac{(|v|^{n'-1} A(u))^n}{n} + \frac{1}{n'} \right) + n'2^{n'-1} A(u)^{n'} \\
&= |v|^{n'} \left( 1 + \frac{2^{\frac{1}{n-1}}}{n-1} |A(u)|^n \right) + n'2^{n'-1} \left( \frac{1}{n'} + |A(u)|^{n'} \right) \\
&\le |v|^{n'} \left( 1 + \frac{2^{\frac{1}{n-1}}}{n-1} |A(u)|^n \right) + C(n)
\end{aligned}$$

where $n' = \frac{n}{n-1}$. We can bound the second term by a constant $C = C(n)$ because $A(u) \le 1$.

Let $w(x) = v(x) \left( 1 + \frac{2^{\frac{1}{n-1}}}{n-1} |A(u)|^n \right)^{\frac{n}{n-1}}$, then $w \in W_0^{1,n}(M)$ and $|u|^{n'} \le w^{n'} +$

$C(n)$, so for $I_1$,

$$I_1 \leq \int_{\Omega(u)} \exp\{\alpha_n |u|^{n'}\} dV_g \leq e^{\alpha_n C(n)} \int_{\Omega(u)} \exp\{\alpha_n |w|^{n'}\} dV_g \tag{2.2.27}$$

In order to apply Theorem 2.2.1 for compact manifold, we need to verify that $\|\nabla w\|_{L^n(M)} \leq 1$

$$\begin{aligned}\int_{\Omega(u)} |\nabla w|^n dV_g &= \left(1 + \frac{2^{\frac{1}{n-1}}}{n-1}|A(u)|^n\right)^{n-1} \int_{\Omega(u)} |\nabla v|^n dV_g \\ &\leq \left(1 + \frac{2^{\frac{1}{n-1}}}{n-1}|A(u)|^n\right)^{n-1} (1 - \tau \int_M |u|^n dV_g)\end{aligned}$$

$$\begin{aligned}\left(\int_{\Omega(u)} |\nabla w|^n dV_g\right)^{\frac{1}{n-1}} &\leq \left(1 + \frac{2^{\frac{1}{n-1}}}{n-1}|A(u)|^n\right) (1 - \tau \int_M |u|^n dV_g)^{\frac{1}{n-1}} \\ &\leq \left(1 + \frac{\tau}{n-1} \int_M |u|^n dV_g\right) \left(1 - \frac{\tau}{n-1} \int_M |u|^n dV_g\right) \\ &\leq 1\end{aligned}$$

so $I_1 \leq e^{\alpha_n C(n)} C(n, \Omega(u))$ from (1.1.21), where $C(n, \Omega(u))$ is a constant depending on $n$ as well as the damain $\Omega(u)$ we choose. Since $|\Omega(u)| \leq 2^{\frac{1}{n-1}} \tau^{-1}$, from the argument at the end of the last section, $|\Omega(u)|$ is uniformly bounded, independent of the choice of $u$. Hence we prove the theorem.

For sharpness, we consider the following sequence, which originally comes from [45],

$$u_k(d(P,\tilde{P})) = \begin{cases} \omega_{n-1}^{-\frac{1}{n}}(\frac{k}{n})^{\frac{n-1}{n}} & \mathrm{d}(P,\tilde{P}) \le e^{-\frac{k}{n}} \\ -\omega_{n-1}^{-\frac{1}{n}}(\frac{k}{n})^{-\frac{1}{n}} \log d(P,\tilde{P}) & \mathrm{e}^{-\frac{k}{n}} \le d(P,\tilde{P}) \le 1 \\ 0 & \mathrm{d}(P,\tilde{P}) \ge 1 \end{cases} \tag{2.2.28}$$

where $P$ is fixed on $M$. Since in our assumption, locally the integral on manifold is comparable to the integral on $\mathbb{R}^n$, following a similar computation as in [24] and [18], we get that the sequence $\{\frac{u_k}{||u_k||_{1,\tau}}\}$ will fail the inequality (1.1.23).

## 2.3 Subcritical Trudinger-Moser Inequality on Complete Noncompact Riemannian Manifolds - Proof of (1.1.24)

Consider the integral

$$\frac{1}{||u||^n_{L^n(M)}} \int_M \phi(\alpha|u|^{\frac{n}{n-1}})dV_g$$

for $\alpha < \alpha_n = n\omega_{n-1}^{1/(n-1)}$. It is defined as the sum of some subintegral on a covering of the manifold. We need the following covering lemma (see [3]).

**Lemma 2.3.1.** *let $M$ be a Riemannian manifold with injectivity radius $\delta_0 > 0$ and bounded curvature; then there exists, if $\delta$ is small enough, a uniformly locally finite covering of $M$ by a family of open balls $B_{P_i}(\delta)$*

Uniformly locally finite means : there exists a constant $k$, which may depend on $\delta$ such that for each point $P \in M$ has a neighborhood whose intersection with each $B_{P_i}(\delta)$, at most $k$, is empty.

Now for such covering we have local chart with harmonic coordinate $\{P_i, B_{P_i}(\delta), x^j\}_{i\in\Lambda}$ and correspondingly the partition of unity $\{\alpha_i(P)\}_{i\in\Lambda}$ (here we abuse the notation of $\alpha$, when we write $\alpha_i$, we mean the partition of unity and when we write $\alpha_n$, we mean the best constant in the Moser-Trudinger inequality) such that

$$\frac{1}{||u||^n_{L^n(M)}}\int_M \phi(\alpha|u|^{\frac{n}{n-1}})dV_g = \frac{1}{||u||^n_{L^n(M)}}\sum_i \int_{B_{P_i}(\delta)} \alpha_i(P)\phi(\alpha|u(P)|^{\frac{n}{n-1}})dV_g$$

Now we will do rescaling to $u(P)$. For $b$ large enough, $\{B_{P_i}(\frac{\delta}{b})\}$ will not have intersection since they are uniformly locally finite. $\forall P \in M$, either there exists $i$ such that $P \in B_{P_i}(\frac{\delta}{b})$, or $P \in M \setminus \cup_i B_{P_i}(\frac{\delta}{b})$. In the first case, for $P = (x^1, \cdots, x^n)$, we define $v(P) = au(bx)$, where $a$ and $b$ are constants left to be chosen; in the second case, we simply let $v(P) = 0$. From the above arguement, it is well-defined.

We want to choose $a$ and $b$ such that $||v||_{1,\tau} \leq 1$.

$$\begin{aligned}
\int_M |\nabla v|^n dV_g &= \sum_i \int_{B_{P_i}(\delta/b)} \alpha_i(bx)a^n b^n |(\nabla u)(bx)|^n \sqrt{|g(x)|}dx \\
&\leq Q^{n/2}\sum_i \int_{B_0(\delta)} \alpha_i(x)a^n|(\nabla u)(x)|^n dx \\
&\leq Q^n \int_M |\nabla u|^n dV_g
\end{aligned}$$

We use Theorem 2.1.2 that for $\delta$ small enough, we have the pinching estimate for the volume element

$$Q^{-n/2}dx \leq dV_g \leq Q^{n/2}dx$$

On the other hand

$$\begin{aligned}
\int_M |v|^n dV_g &= \sum_i \int_{B_{(\delta/b)}} \alpha_i(bx) a^n |u(bx)|^n \sqrt{|g(x)|} dx \\
&\leq Q^{n/2} \frac{a^n}{b^n} \sum_i \int_{B_0(\delta)} \alpha_i(x) |u(x)|^n dx \\
&\leq Q^n \frac{a^n}{b^n} \int_M |u|^n dV_g
\end{aligned}$$

Remember that we want

$$\begin{aligned}
||v||_{1,\tau}^n &= \int_M |\nabla v|^n + \tau |v|^n dV_g \\
&\leq Q^n a^n \int_M |\nabla u|^n dV_g + \tau Q^n \frac{a^n}{b^n} \int_M |u|^n dV_g \\
&\leq 1
\end{aligned}$$

Therefore we let

$$a = \frac{(1-\epsilon)^{1/n}}{Q}$$

$$b = \left( \frac{(1-\epsilon)\tau}{\epsilon} \right)^{1/n} ||u||_n$$

where $\epsilon$ is some positive constant. Then

$$\begin{aligned}\frac{1}{||u||_n^n}\int_M \phi(\alpha|u|^{\frac{n}{n-1}})dV_g &= \frac{1}{||u||_n^n}\sum_i \int_{B_0(\delta)} \alpha_i(x)\phi(\alpha|u(x)|^{\frac{n}{n-1}})\sqrt{|g(x)|}dx \\ &= \frac{1-\epsilon}{\epsilon}\tau\sum_i \int_{B_0(\delta)} \alpha_i(x)\phi(\alpha|u(x)|^{\frac{n}{n-1}})\sqrt{|g(x)|}d(\frac{x}{b}) \\ &\leq Q^{n/2}\frac{1-\epsilon}{\epsilon}\tau\sum_i \int_{B_0(\delta/b)} \alpha_i(bx)\phi(\frac{\alpha|au(bx)|^{\frac{n}{n-1}}}{a^{\frac{n}{n-1}}})dx \\ &\leq Q^n\frac{1-\epsilon}{\epsilon}\tau\int_M \phi(\alpha'|v|^{\frac{n}{n-1}})dV_g \\ &< \infty\end{aligned}$$

where $\alpha' = \alpha/a^{\frac{n}{n-1}} \leq \alpha_n$ when $\epsilon$ is small enough and $Q$ is very close to 1 , thus we can apply (1.1.12).

For sharpness, we consider the following sequence of functions, which originally comes from [1]

$$u_k(d(P,\tilde{P})) = \begin{cases} \omega_{n-1}^{-\frac{1}{n}}(\frac{k}{n})^{\frac{n-1}{n}} & \mathrm{d}(P,\tilde{P}) \leq e^{-\frac{k}{n}} \\ -\omega_{n-1}^{-\frac{1}{n}}(\frac{k}{n})^{-\frac{1}{n}}\log d(P,\tilde{P}) & \mathrm{e}^{-\frac{k}{n}} \leq d(P,\tilde{P}) \leq 1 \\ 0 & \mathrm{d}(P,\tilde{P}) \geq 1 \end{cases} \tag{2.3.1}$$

The following calculation depends on Theorem 2.1.2

$$\begin{aligned}
\int_M |\nabla \frac{u_k}{\sqrt{Q}}|^n dV_g &= Q^{-n/2}\int_{B_Q(1)} |\nabla u_k|^n r^{n-1}\sqrt{|g|}drd\theta \\
&= Q^{-n/2}\int_{B_Q(1)\setminus B_Q(e^{-k/n})} |-\omega_{n-1}^{-1/n}(\frac{k}{n})^{-1/n}\frac{1}{r}|^n r^{n-1}\sqrt{|g|}drd\theta \\
&\leq Q^{-n/2}Q^{n/2}\omega_{n-1}^{-1}(\frac{k}{n})^{-1}\int_{e^{-k/n}}^1\int_{S_{n-1}}\frac{1}{r}drd\theta \\
&\leq 1
\end{aligned}$$

hence $||\nabla \frac{u_k}{\sqrt{Q}}||_{L^n(M)} \leq 1$. Next we evaluate $||\frac{u_k}{\sqrt{Q}}||_{L^n(M)}$,

$$\begin{aligned}
\int_M |\frac{u_k}{\sqrt{Q}}|^n dV_g &= Q^{-n/2}\int_{B_Q(e^{-k/n})} |\omega_{n-1}^{-1/n}(\frac{k}{n})^{\frac{n-1}{n}}|^n dV_g \\
&+ Q^{-n/2}\int_{B_Q(1)\setminus B_Q(e^{-k/n})} |-\omega_{n-1}^{-1/n}(\frac{k}{n})^{-1/n}\log d(P,\tilde{P})|^n dV_g \\
&\leq Q^{-n/2}\int_0^{e^{-k/n}}\int_{S_{n-1}} |\omega_{n-1}^{-1/n}(\frac{k}{n})^{\frac{n-1}{n}}|^n r^{n-1}\sqrt{|g|}drd\theta \\
&+ Q^{-n/2}\omega_{n-1}^{-1}(\frac{k}{n})^{-1}\int_{e^{-k/n}}^1\int_{S_{n-1}} |\log r|^n r^{n-1}\sqrt{|g|}drd\theta \\
&\leq \omega_{n-1}^{-1}(\frac{k}{n})^{n-1}e^{-k}\omega_n + \omega_{n-1}^{-1}(\frac{k}{n})^{-1}\int_{e^{-k/n}}^1\int_{S_{n-1}} |\log r|^n r^{n-1}drd\theta
\end{aligned}$$

Both terms tend to zero as $k \to \infty$, hence $||\frac{u_k}{\sqrt{Q}}||_{L^n(M)} \to 0$ as $k \to \infty$

$$
\begin{aligned}
\int_M \phi(\alpha_n|\frac{u_k}{\sqrt{Q}}|^{\frac{n}{n-1}})dV_g &= \int_{B_Q(1)\backslash B_Q(e^{-k/n})} \exp\left\{\alpha_n'| - \omega_{n-1}^{-1/n}(k/n)^{-1/n}\log d(P,Q)|^{\frac{n}{n-1}}\right\} dV_g \\
&- \int_{B_Q(1)\backslash B_Q(e^{-k/n})} \sum_{j=0}^{n-2} \frac{\alpha_n'^j \left(-\omega_{n-1}^{-1/n}(k/n)^{-1/n}\log d(P,\tilde{P})\right)^{\frac{n}{n-1}j}}{j!} dV_g \\
&+ \int_{B_Q(e^{-k/n})} \phi\left(\alpha_n'(\omega_{n-1}^{-1/n}(k/n)^{\frac{n-1}{n}})^{\frac{n}{n-1}}\right) dV_g \\
&\geq -\sum_{j=0}^{n-2} k^{-j\frac{1}{n-1}} \int_{B_Q(1)} \frac{n^{j\frac{1}{n-1}}}{j!} |\log d(P,Q)|^{j\frac{n}{n-1}} dV_g \\
&+ (e^{-k/n})^n \omega_n \phi\left(\alpha_n'(\omega_{n-1}^{-\frac{1}{n-1}}\frac{k}{n})\right)
\end{aligned}
$$

Where $\alpha_n' = \alpha_n/\sqrt{Q}$. It is easy to see that the first term tends to zero and hence we have

$$
\frac{1}{||\frac{u_k}{\sqrt{Q}}||^n_{L^n(M)}} \int_M \phi\left(\alpha_n|\frac{u_k}{\sqrt{Q}}|^{\frac{n}{n-1}}\right) dV_g \to \infty \tag{2.3.2}
$$

# Chapter 3

# Trudinger-Moser Inequalities for Maps Between Manifolds

In this chapter, we will give detailed proof of (1.1.32), (1.1.35) and (1.1.40).

## 3.1 Parametrix of gradient field

Let $v(x) = dist(u(x), u_0) \in W_0^{1,2}(\Omega)$, since $|dvX| \leq |duX|$ holds for any $X \in T^*M$. We can take the trivial extension of $v(x)$ to the whole manifold $M$ and still denote it as $v$. For our purpose, we need to obtain a representation formula of $v(x)$ in terms of its first order derivative. In the case when $\Omega \subset \mathbb{R}^m$, one has for any $x \in \Omega$, $diam\Omega \leq R$

$$v(x) = -\frac{1}{\omega_{m-1}} \int_0^R r^{-m} \int_{\partial B_\sigma(x)} dv(y-x) ds(y) dr \tag{3.1.1}$$

which heavily depends on the symmetries of the Euclidean space. For general case, we construct a parametrix to express a function in terms of its gradient.

**Lemma 3.1.1.** *For every $v \in C^1(\Omega)$, $v\mid_{\partial\Omega}=0$, denote $B_\sigma$ as the geodesic ball centered at $x$ with radius $\sigma$, here $\sigma$ is any positive number less than the injective radius of all points on $\Omega$. Then we have the following parametrix*

$$\begin{aligned} v(x) &= \int_{B_\sigma} \Gamma_1(x,y)\nabla v(y)dV(y) + \sum_{j=2}^k \int_\Omega \Gamma_j(x,y)\nabla v(y)dV(y) \\ &\quad + \int_\Omega R_k(x,y)v(y)dV(y) \end{aligned} \tag{3.1.2}$$

*with $k=[\frac{m}{2}]$, $\Gamma_1(x,y)=\nabla_y \frac{f(d(x,y))}{(m-2)\omega_{m-1}d(x,y)^{m-2}}$, where $f(r)$ is a smooth function such that $f\equiv 1$ near on $[-\frac{\sigma}{2},\frac{\sigma}{2}]$, $suppf \subset [-\sigma,\sigma]$. Moreover, $\Gamma_j$ are vector fields satisfies $|\Gamma_j(x,y)| \lesssim d^{2j-1-m}(x,y) \lesssim d^{3-m}(x,y)$, $2\le j\le k$, and $|R_k(x,y)| \lesssim d^{-1}(x,y)$.*

**Proof**: Define

$$H(x,y) = \frac{f(d(x,y))}{(m-2)\omega_{m-1}d(x,y)^{m-2}} \tag{3.1.3}$$

where $f(r)$ is a smooth nonincreasing function such that $f\equiv 1$ near 0, $supp\{f\}\subset B_\sigma$. Then

$$\begin{aligned} \int_{B_\sigma} \nabla_y H(x,y)\nabla v(y)dV(y) &= \lim_{\sigma\to 0}\Big(\int_{\partial\Omega} - \int_{\partial B_\sigma}\Big)(\overrightarrow{n}\cdot\nabla_y H(x,y)v(y)d\sigma(y) \\ &\quad - \int_\Omega \triangle_y H(x,y)v(y)dV(y)) \\ &= v(x) - \int_\Omega \triangle_y H(x,y)v(y)dV(y) \end{aligned}$$

where we use the boundary condition for $v$ and the definition of $f(r)$ near 0. So we have the Green formula,

$$v(x) = \int_{B_\sigma} \nabla_y H(x,y) \nabla v(y) dV(y) + \int_\Omega \triangle_y H(x,y) v(y) dV(y) \tag{3.1.4}$$

We then do iteration for $v(y)$ in the second term of the above Green formula and get

$$\begin{aligned} v(x) &= \int_{B_\sigma} \nabla_y H(x,y) \nabla v(y) dV(y) \\ &+ \int_\Omega \triangle_y H(x,y) (\int_\Omega \nabla_z H(y,z) \nabla v(z) dV(z) + \int_\Omega \triangle_z H(y,z) v(z) dV(z)) dV(y) \\ &= \int_{B_\sigma} \nabla_y H(x,y) \nabla v(y) dV(y) + \int_\Omega (\int_\Omega \triangle_y H(x,y) \nabla_z H(y,z) dV(y)) \nabla v(z) dV(z) \\ &+ \int_\Omega (\int_\Omega \triangle_y H(x,y) \triangle_z H(y,z) dV(y)) v(z) dV(z) \\ &= \int_{B_\sigma} \nabla_y H(x,y) \nabla v(y) dV(y) + \int_\Omega (\int_\Omega \triangle_z H(x,z) \nabla_y H(z,y) dV(z)) \nabla v(y) dV(y) \\ &+ \int_\Omega (\int_\Omega \triangle_z H(x,z) \triangle_y H(z,y) dV(y)) v(y) dV(y) \end{aligned}$$

where we apply Fubini theorem and a change of variable. For simplicity of notation, we define

$$\Gamma_1(x,y) = \nabla_y H(x,y) \tag{3.1.5}$$

$$\Gamma_2(x,y) = \int_\Omega \triangle_z H(x,z) \Gamma_1(z_1,z) dV(z) \tag{3.1.6}$$

$$R_2(x,y) = \int_\Omega \triangle_z H(x,z) \triangle_z H(z,y) dV(z) \tag{3.1.7}$$

then the representation formula reads as

$$\begin{aligned}v(x) = &\int_{B_\sigma} \Gamma_1(x,y)\nabla v(y)dV(y) + \int_\Omega \Gamma_2(x,y)\nabla v(y)dV(y)\\ &+ \int_\Omega R_2(x,y)v(y)dV(y)\end{aligned}$$

If we continue to do iteration for $v(y)$ in the third term above, after $k$-step, we would have the following,

$$\begin{aligned}v(x) = &\int_{B_\sigma} \Gamma_1(x,y)\nabla v(y)dV(y) + \sum_{j=2}^{k}\int_\Omega \Gamma_j(x,y)\nabla v(y)dV(y)\\ &+ \int_\Omega R_k(x,y)v(y)dV(y)\end{aligned}$$

where

$$\Gamma_j(x,y) = \int_\Omega \triangle_z H(x,z)\Gamma_{j-1}(z,y)dV(z) \tag{3.1.8}$$

$$R_k = \int_{\Omega\times\cdots\times\Omega} \prod_{i=0}^{k-1} \triangle_{z_{i+1}} H(z_i, z_{i+1})dV(z_1)\cdots dV(z_{k-1}) \tag{3.1.9}$$

where $z_0 = x$ and $z_k = y$. To finish the proof, we need to analyze the behavior of $\Gamma_1$, $\Gamma_j$ and $R_k$ near the diagonal. For our purpose, we need the following lemma,

**Lemma 3.1.2.** *Let $\Omega$ be a bounded open set and $X(x,y)$ and $Y(x,y)$ be continuous functions defined on $\Omega \times \Omega \setminus diag$ such that*

$$|X(x,y)| \leq Const \times d^{\alpha-m}(x,y) \tag{3.1.10}$$

$$|Y(x,y)| \leq Const \times d^{\beta-m}(x,y) \tag{3.1.11}$$

*for some* $\alpha, \beta \in (0,m)$. *Then*

$$Z(x,y) = \int_{\Omega} X(x,z)Y(z,y)dV(z) \tag{3.1.12}$$

*satisfies*

$$|Z(x,y)| \leq Const \times d^{\alpha+\beta-m}(x,y) \quad if \;\; \alpha+\beta < m \tag{3.1.13}$$

$$|Z(x,y)| \leq Const \times (1 + |\log d(x,y)|) \quad if \;\; \alpha+\beta = m \tag{3.1.14}$$

$$|Z(x,y)| \leq Const \quad if \;\; \alpha+\beta > m \tag{3.1.15}$$

*Proof.* Let $2\rho = d(x,y)$ be small enough, then we split the integral into three parts

$$Z(x,y) = (\int_{B_x(\rho)} + \int_{B_y(3\rho)\backslash B_x(\rho)} + \int_{\Omega\backslash B_y(3\rho)})X(x,z)Y(z,y)dV(z) \tag{3.1.16}$$

and each part can be controlled as following,

$$\begin{aligned} \left|\int_{B_x(\rho)} X(x,z)Y(z,y)dV(z)\right| &\lesssim \int_{B_x(\rho)} d^{\alpha-m}(x,z)\rho^{\beta-m}dV(z) \\ &\lesssim \rho^{\alpha+\beta-m} \end{aligned}$$

$$\begin{aligned} \left|\int_{B_y(3\rho)\setminus B_x(\rho)} X(x,z)Y(z,y)dV(z)\right| &\lesssim \int_{B_y(3\rho)\setminus B_x(\rho)} \rho^{\alpha-m}d^{\beta-m}(z,y)dV(z) \\ &\lesssim \rho^{\alpha+\beta-m} \end{aligned}$$

For the third part, notice that $\Omega$ is bounded, $d(P,R) \geq \rho$, $d(z,y) \geq 3\rho$ and $|d(P,R) - d(z,y)| \leq 2\rho$, hence there exists a constant $C$ such that $d(P,R) \geq Cd(z,y)$, then

$$\begin{aligned} \left|\int_{\Omega\setminus B_y(3\rho)} X(x,z)Y(z,y)dV(z)\right| &\lesssim \int_{\Omega\setminus B_y(3\rho)} (Cd(z,y))^{\alpha-m}d(z,y)^{\beta-m}dV(z) \\ &\lesssim \int_{\Omega\setminus B_y(3\rho)} d^{\alpha+\beta-2m}(z,y)dV(z) \\ &\lesssim \rho^{\alpha+\beta-m} \quad if\ \alpha+\beta<m \end{aligned}$$

Also, when $\alpha+\beta=m$, we have

$$\left|\int_{\Omega\setminus B_y(3\rho)} X(x,z)Y(z,y)dV(z)\right| \lesssim 1+|\log\rho| \tag{3.1.17}$$

when $\alpha+\beta>m$,

$$|\int_{\Omega\setminus B_y(3\rho)} X(x,z)Y(z,y)dV(z)| \le C \tag{3.1.18}$$

□

For $\Gamma_1(x,y)$, we only need to consider when $d(x,y)$ is very small,

$$\begin{aligned}|\Gamma_1(x,y)| &= |\nabla_y H(x,y)| \\ &= |(\frac{1}{(m-2)\omega_{m-1}r^{m-2}})'| \\ &\lesssim d^{1-m}(x,y)\end{aligned}$$

For $\triangle_y H(x,y)$, from the definition of Laplacian-Beltrami operator, we know

$$|\triangle_y H(x,y)| \tag{3.1.19}$$

$$= |\frac{1}{(m-2)\omega_{m-1}r^{m-1}}((m-3)f' - rf'' + ((m-2)f - rf')\partial_r \log\sqrt{|g|})| \tag{3.1.20}$$

$$\lesssim d^{2-m}(x,y) \tag{3.1.21}$$

where we use the property that under the curvature assumption of $M$ , we have (see [3]),

$$|\partial_r \log\sqrt{|g|}| \lesssim r \tag{3.1.22}$$

Then by the above lemma, we get following estimate for $\Gamma_2(x,y)$,

$$
\begin{aligned}
|\Gamma_2(x,y)| &= |\int_\Omega \triangle_z H(x,z)\Gamma_1(z,y)dV(z)| \\
&\lesssim d^{3-m}(x,y)
\end{aligned}
$$

Generally, we have for $\Gamma_j(x,y)$, $2 \leq j \leq k = \frac{m-1}{2}$

$$
|\Gamma_j(x,y)| \lesssim d^{2j-1-m}(x,y) \lesssim d^{3-m}(x,y) \tag{3.1.23}
$$

And for $R_k(x,y)$,

$$
\begin{aligned}
|R_k| &= |\int_{\Omega\times\cdots\times\Omega} \prod_{i=0}^{k-1} \triangle_{z_{i+1}} H(z_i, z_{i+1}) dV(z_1)\cdots dV(z_{k-1})| \\
&\lesssim d^{-1}(x,y)
\end{aligned}
$$

Specially, with out any confusion, we denote $r = d(x,y)$ and let $\Omega_r = \Omega \bigcap B_r(x)$, then we have

**Corollary 3.1.3.** *For $v$ as above,*

$$
|v(x)| \leq \frac{1}{\omega_{m-1}} \int_{\Omega_\sigma} r^{1-m} |\frac{\partial v}{\partial r}(y)| dV(y) + C_\sigma (\int_\Omega r^{3-m} |\nabla v(y)| dV(y) + \int_\Omega r^{-1} |v(y)| dV(y)) \tag{3.1.24}
$$

## 3.2 Local and Global estimates

**Lemma 3.2.1.** *Assume there exist a constant* $A > 0$ *such that for any* $u_0 \in N$ *and* $u \in C^\infty(\overline{\Omega}, N)$ *with* $u = u_0$ *on* $\partial\Omega$*, the following inequality holds*

$$||du||_{L^2(\Omega)} \leq A||\tau(u)||_{L^2(\Omega)} \tag{3.2.1}$$

*then there exist a constant* $C > 0$ *such that for any* $u_0 \in N$ *and* $u \in C^\infty(\overline{\Omega}, N)$ *with* $u = u_0$ *on* $\partial\Omega$*, the following inequality holds*

$$||dist(u, u_0)||_{L^\phi(\Omega)} \leq C||\tau(u)||_{L^{m/2}(\Omega)} \tag{3.2.2}$$

*Proof.* Since our calculations will be local, without loss of generality, we will use the usually Euclidean volume form as our integration volume form. Recall that

$$\begin{aligned} v(x) &\lesssim \int_{\Omega_\sigma} r^{1-m}|\frac{\partial v}{\partial r}| + \int_\Omega r^{3-m}|\nabla v| + \int_\Omega r^{-1}|v| \\ &= I + II + III \end{aligned}$$

where $\sigma > 0$ will be chosen later. We start from the estimate for the first term $I$. Consider the maximal function

$$g(x) = \sup_{r>0} \frac{1}{B_r(x)} \int_{B_r(x)} |\nabla v| \tag{3.2.3}$$

then we split the first term into two parts,

$$I = \int_0^\rho r^{1-m} \int_{\partial\Omega_r} |\frac{\partial v}{\partial r}| d\sigma dr + \int_\rho^\sigma r^{1-m} \int_{\partial\Omega_r} |\frac{\partial v}{\partial r}| d\sigma dr$$
$$= I_1 + I_2$$

For the first part,

$$I_1 = \int_0^\rho r^{1-m} (\frac{d}{dr} \int_{\Omega_r} |\frac{\partial v}{\partial r}|) dr$$
$$= (r^{1-m} \int_{\Omega_r} |\frac{\partial v}{\partial r}|) \, |_0^\rho - \int_0^\rho (1-m) r^{-m} (\int_{\Omega_r} |\frac{\partial v}{\partial r}|) dr$$
$$= \rho^{1-m} \int_{B_\rho} |\frac{\partial v}{\partial r}| + (m-1) \int_0^\rho (r^{-m} \int_{\Omega_r} |\frac{\partial v}{\partial r}|) dr$$
$$\lesssim \rho g(x)$$

This holds for almost every $P \in \Omega$ since we apply Lebesgue differential theorem on the third line.

For the second part, recall that $X = r\nabla r$

$$I_2 = \int_{\Omega \backslash B_\rho} \frac{|dvX|}{r^m}$$
$$\leq (\int_{B_\sigma \backslash B_\rho} \frac{1}{r^m})^{1/2} (\int_{\Omega_R \backslash B_\rho} \frac{|dvX|^2}{r^m})^{1/2}$$

We can control the second term in the right hand side as

**Proposition 3.2.2.**

$$\int_{\Omega\backslash B_\rho} \frac{|dvX|^2}{r^m} \lesssim (1+(\log\frac{R}{\rho})^{1-4/m})||\tau(u)||^2_{L^{m/2}} \tag{3.2.4}$$

The following lemma will be very important in the proof of this proposition.

**Proposition 3.2.3.** *Suppose the sectional curvature $K$ of $M$ satisfies*

$$K_0 \leq K \leq K_1 \tag{3.2.5}$$

*for some constant $K_0$ and $K_1$. Without loss of generality, we assume $K_0 \leq 0$. For all $r \leq inj(\Omega)$, defining $X = r\nabla r$, then it holds*

$$|du|^2 divX - 2\langle du\nabla X, \nabla u\rangle \geq F(r)|du|^2 \tag{3.2.6}$$

*Proof.* For the left hand side,

$$\begin{aligned} |du|^2 divX &= |du|^2|\nabla r|^2 + |du|^2 r\triangle r &(3.2.7)\\ &= |du|^2 + |du|^2 r\triangle r &(3.2.8) \end{aligned}$$

$$\langle du\nabla X, \nabla u\rangle = \langle du(\nabla r \otimes \nabla r), \nabla u\rangle + \langle du(r\nabla^2 r), \nabla u\rangle \tag{3.2.9}$$

For a fixed point $P \in M$, without loss of generality, we can assume $h_{\alpha\beta}(u(P)) = \delta_{\alpha\beta}$. Let $\{\frac{\partial}{\partial r}, E_1, \cdots, E_{m-1}\}$ be an orthonormal basis at $P$ such that $Hess(r)(E_i, E_j)$ is diagonal under the choice of such basis. Let $E^i \in T^*M$ be the dual of $E_i$, then we can write

$$\nabla u = u_0^\alpha dr \otimes \frac{\partial}{\partial y^\alpha} + u_i^\alpha E^i \otimes \frac{\partial}{\partial y^\alpha} \tag{3.2.10}$$

$$|du|^2 = \sum_\alpha |u_0^\alpha|^2 + \sum_{i,\alpha} |u_i^\alpha|^2 \tag{3.2.11}$$

$$\langle du\nabla X, \nabla u\rangle = \sum_\alpha |u_0^\alpha|^2 + \sum_{i,\alpha} rHess(r)(E_i, E_i)|u_i^\alpha|^2 \tag{3.2.12}$$

Thus

$$\begin{aligned} |du|^2 divX - 2\langle du\nabla X, \nabla u\rangle &= |du|^2 + |du|^2 r \sum_i Hess(r)(E_i, E_i) - 2\sum_\alpha |u_0^\alpha|^2 \\ &- 2\sum_{i,\alpha} rHess(r)(E_i, E_i)|u_i^\alpha|^2 \\ &= |du|^2 - 2\sum_\alpha |u_0^\alpha|^2 \\ &+ \sum_i rHess(r)(E_i, E_i)(|du|^2 - 2\sum_\alpha |u_i^\alpha|^2) \end{aligned}$$

Case 1: $|du|^2 - 2\sum_\alpha |u_i^\alpha|^2 \geq 0$

From Hessian comparison principle and since

$$Hess(\rho)(E_i, E_i) = \frac{f'(r)}{f(r)} \tag{3.2.13}$$

we have

$$
\begin{aligned}
|du|^2 divX - 2\langle du\nabla X, \nabla u\rangle &\geq |du|^2 - 2\sum_{\alpha}|u_0^\alpha|^2 \\
&+ \sum_i r\frac{\sqrt{K_1}\cos(\sqrt{K_1}r)}{\sin(\sqrt{K_1}r)}(|du|^2 - 2\sum_{\alpha}|u_i^\alpha|^2) \\
&= |du|^2 - 2\sum_{\alpha}|u_0^\alpha|^2 \\
&+ f_{K_1}(r)((m-1)|du|^2 - 2|du|^2 + 2\sum_{\alpha}|u_0^\alpha|^2) \\
&= |du|^2(1 + f_{K_1}(r)(m-3)) + (2f_{K_1}(r) - 2)\sum_{\alpha}|u_0^\alpha|^2 \\
&\geq |du|^2(1 + f_{K_1}(r)(m-3)) + (2f_{K_1}(r) - 2)|du|^2 \\
&= |du|^2((m-1)f_{K_1}(r) - 1)
\end{aligned}
$$

where we use the property that $f_{K_1}(r) \leq 1$ as $r \geq 0$.

Case 2: $|du|^2 - 2\sum_\alpha |u_i^\alpha|^2 < 0$ for some $i$

For $j \neq i$, if $|du|^2 - 2\sum_\alpha |u_j^\alpha|^2 < 0$, then

$$
2|du|^2 < 2\sum_{\alpha}(|u_i^\alpha|^2 + |u_j^\alpha|^2) \leq 2|du|^2 \tag{3.2.14}
$$

Thus we conclude that $|du|^2 - 2\sum_\alpha |u_j^\alpha|^2 \geq 0$ for all $j \neq i$.

$$
\begin{aligned}
&|du|^2 divX - 2\langle du\nabla X, \nabla u\rangle \\
&\geq |du|^2 - 2\sum_\alpha |u_0^\alpha|^2 + f_{K_1}(r)((m-2)|du|^2 - 2|du|^2 + 2\sum_\alpha |u_i^\alpha|^2 + 2\sum_\alpha |u_0^\alpha|^2) \\
&+ f_{K_0}(r)(|du|^2 - 2\sum_\alpha |u_i^\alpha|^2) \\
&= |du|^2(1 + f_{K_1}(r)(m-2) - f_{K_0}(r)) + 2(f_{K_1}(r) - 1)\sum_\alpha |u_0^\alpha|^2 \\
&+ 2(f_{K_0}(r) - f_{K_1}(r))(|du|^2 - \sum_\alpha |u_i^\alpha|^2) \\
&\geq |du|^2(1 + f_{K_1}(r)(m-2) - f_{K_0}(r)) + 2(f_{K_0}(r) - 1)|u_0^\alpha|^2 \\
&\geq |du|^2(1 + f_{K_1}(r)(m-2) - f_{K_0}(r))
\end{aligned}
$$

For the last line, we use the property that $f_{K_0}(r) \geq 1$ as $K_0 < 0$ and $r \geq 0$.

□

*Proof.* From the proof of Proposition 3.2.3, we have

$$|du|^2 divX - 2\langle du\nabla X, \nabla u\rangle \geq F(r)|du|^2 \tag{3.2.15}$$

where $F(r) = \min\{|du|^2((m-1)f_{K_1}(r)-1), |du|^2(1+(m-2)f_{K_1}(r) - f_{K_0}(r))\}$. Hence

$$
\begin{aligned}
\int_{\Omega_r} F(r)|du|^2 - 2\langle duX, \tau(u)\rangle &\leq \int_{\partial\Omega_r} |du|^2 X\cdot\overrightarrow{n} - 2\langle duX, \nabla u\rangle\overrightarrow{n} \\
&= \int_{\partial B_\sigma\cap\Omega} r|du|^2 - \frac{2}{r}|duX|^2 - \int_{\partial\Omega\cap B_\sigma} |du|^2 X\cdot\overrightarrow{n} \\
&\leq \int_{\partial B_\sigma\cap\Omega} r|du|^2 - \frac{2}{r}|duX|^2
\end{aligned}
$$

where we use the property that on $\partial B_\sigma$, we have $\overrightarrow{n} = X/r$, the convexity of $\Omega$ and on $\Omega$ $u$ is a constant map. This is equivalent to

$$\int_{\partial B_\sigma \cap \Omega} r|du|^2 \geq \int_{\Omega_r} F(r)|du|^2 - 2\langle duX, \tau(u)\rangle + \int_{\partial B_\sigma \cap \Omega} \frac{2}{r}|duX|^2 \tag{3.2.16}$$

Define the function

$$\Phi(r) = r^{2-m} \int_{\Omega_r} |du|^2 - \frac{2}{m-2}\langle duX, \tau(u)\rangle \tag{3.2.17}$$

then

$$\begin{aligned}
\Phi'(r) &= r^{1-m}\{\int_{\Omega_r} (2-m)|du|^2 + 2\langle duX, \tau(u)\rangle + r\int_{\partial\Omega_r} |du|^2 - \frac{2}{m-2}\langle duX, \tau(u)\rangle\} \\
&= r^{1-m}(\int_{\Omega_r} -F(r)|du|^2 + 2\langle duX, \tau(u)\rangle - \int_{\Omega_r} (m-2-F(r))|du|^2 \\
&+ r\int_{\partial B_\sigma \cap \Omega} |du|^2 - \frac{2}{m-2}\langle duX, \tau(u)\rangle) \\
&\geq r^{1-m}(\int_{\partial B_\sigma \cap \Omega} -r|du|^2 + \frac{2}{r}|duX|^2 + r\int_{\partial B_\sigma \cap \Omega} |du|^2 - \frac{2}{m-2}\langle duX, \tau(u)\rangle \\
&- \int_{\Omega_r} (m-2-F(r))|du|^2) \\
&= 2\int_{\partial B_\sigma \cap \Omega} \frac{|duX|^2}{r^m} - \frac{\langle duX, \tau(u)\rangle}{(m-2)r^{m-2}} - r^{1-m}\int_{\Omega_r} (m-2-F(r))|du|^2
\end{aligned}$$

Take integration from $\rho$ to $\sigma$, we have the following monotonicity formula

$$\begin{aligned}\Phi(\sigma) \quad &\geq \Phi(\rho) + 2\textstyle\int_{\Omega\setminus B_\rho} \frac{|duX|^2}{r^m} - \frac{\langle duX, \tau(u)\rangle}{(m-2)r^{m-2}} \\ &- \textstyle\int_\rho^\sigma r^{1-m}\int_{\Omega_r}(m-2-F(r))|du|^2 dr\end{aligned} \tag{3.2.18}$$

Using Young's inequality,

$$2\int_{\Omega_\sigma\setminus B_\rho} \frac{\langle duX, \tau(u)\rangle}{(m-2)r^{m-2}} \leq \int_{\Omega\setminus B_\rho} \frac{|duX|^2}{r^m} + \frac{1}{(m-2)^2}\int_{\Omega_\sigma\setminus B_\rho} \frac{|\tau(u)|^2}{r^{m-4}} \tag{3.2.19}$$

Also from the definition of $\Phi$, we have

$$\Phi(\sigma) \leq \sigma^{2-m}\int_{\Omega_R} 2|du|^2 + \frac{r^2|\tau(u)|^2}{(m-2)^2} \tag{3.2.20}$$

$$\Phi(\rho) \geq -\rho^{2-m}\int_{B_\rho} \frac{r^2|\tau(u)|^2}{(m-2)^2} \geq -\rho^{4-m}\int_{B_\rho} \frac{|\tau(u)|^2}{(m-2)^2} \tag{3.2.21}$$

Bring all these back to (3.2.18), we get

$$\begin{aligned}\int_{\Omega_\sigma\setminus B_\rho} \frac{|duX|^2}{r^m} \leq \sigma^{2-m}\int_{\Omega_\sigma} 2|du|^2 + \frac{\sigma^2|\tau(u)|^2}{(m-2)^2} + \rho^{4-m}\int_{B_\rho} \frac{|\tau(u)|^2}{(m-2)^2} \\ + \frac{1}{(m-2)^2}\int_{\Omega_\sigma\setminus B_\rho} \frac{|\tau(u)|^2}{r^{m-4}} + \int_\rho^\sigma r^{1-m}(\int_{\Omega_r}(m-2-F(r))|du|^2)dr\end{aligned}$$

For the first four terms of right hand side above,

$$\sigma^{2-m}\int_{\Omega_\sigma} 2|du|^2 \leq A\sigma^{2-m}\int_\Omega ||\tau(u)||^2$$

$$\int_{\Omega_\sigma \setminus B_\rho} \frac{|\tau(u)|^2}{r^{m-4}} \leq (\int_{B_\sigma \setminus B_\rho} \frac{1}{r^m})^{1-4/m} ||\tau(u)||^2_{L^{m/2}(\Omega)}$$
$$\lesssim (\log \frac{\sigma}{\rho})^{1-4/m} ||\tau(u)||^2_{L^{m/2}(\Omega)}$$

$$\rho^{4-m} \int_{B_\rho} \frac{|\tau(u)|^2}{(m-2)^2} \lesssim ||\tau(u)||^2_{L^{m/2}(\Omega)}$$

$$R^{4-m} \int_{B_\sigma} \frac{|\tau(u)|^2}{(m-2)^2} \lesssim ||\tau(u)||^2_{L^{m/2}(\Omega)}$$

Notice for $F(r)$, the following holds

$$m - 2 - F(r) = O(r^2) \tag{3.2.22}$$

Then we have

$$\int_\rho^\sigma r^{1-m} (\int_{\Omega_r} (m - 2 - F(r))|du|^2) dr \leq \int_\rho^\sigma r^{1-m} (\int_{\Omega_r} O(r^2)|du|^2) dr$$
$$\lesssim \int_\rho^\sigma r^{3-m} \int_{\Omega_r} |du|^2 dr$$

To deal with this term, we need to use the above technique again, first define

$$\Psi(r) = r^{4-m} \int_{\Omega_r} |du|^2 - \frac{2}{m-4} \langle duX, \tau(u) \rangle \tag{3.2.23}$$

Taking derivative with respect to $r$, similarly, by taking $\sigma$ small enough and $r \leq \sigma$, we will get

$$\begin{aligned}\Psi'(r) = r^{3-m} &\int_{\Omega_r} (4-m)|du|^2 \\ &+ 2\langle duX, \tau(u)\rangle + r^{4-m} \int_{\partial B_\sigma \cap \Omega} |du|^2 - \frac{2}{m-4}\langle duX, \tau(u)\rangle \\ &\geq r^{3-m} \int_{\Omega_r} (F(r) - m + 4)|du|^2 + \int_{\partial B_\sigma \cap \Omega} \frac{2|duX|^2}{r^{m-2}} - \frac{2\langle duX, \tau(u)\rangle}{(m-4)r^{m-4}} \\ &\geq r^{3-m} \int_{\Omega_r} |du|^2 - C \int_{\partial B_\sigma \cap \Omega} \frac{|\tau(u)|^2}{r^{m-6}}\end{aligned}$$

Thus we get a monotonicity formula,

$$\Psi(\rho) + \int_\rho^\sigma r^{3-m} \int_{\Omega_r} |du|^2 dr - C \int_\rho^\sigma \int_{\partial B_\sigma \cap \Omega} \frac{|\tau(u)|^2}{r^{m-6}} \leq \Psi(\sigma) \tag{3.2.24}$$

Since

$$\begin{aligned}\Psi(\sigma) &\leq R^{4-m} \int_{\Omega_\sigma} 2|du|^2 + \frac{R^2}{(m-4)^2}|\tau(u)|^2 \\ &\lesssim ||\tau(u)||^2_{L^{m/2}(\Omega)}\end{aligned}$$

with

$$\begin{aligned}\Psi(\rho) &\geq -\rho^{4-m}\int_{\Omega_\rho}\frac{\rho^2}{(m-4)^2}|\tau(u)|^2\\ &\gtrsim -||\tau(u)||^2_{L^{m/2}(\Omega)}\end{aligned}$$

plus

$$\begin{aligned}\int_{\Omega_\sigma\setminus B_\sigma}\frac{|\tau(u)|^2}{r^{m-6}} &\leq (\int_{B_\sigma\setminus B_\sigma}\frac{1}{r^{m(m-6)/(m-4)}})||\tau(u)||^2_{L^{m/2}(\Omega)}\\ &\lesssim ||\tau(u)||^2_{L^{m/2}}\end{aligned}$$

We have

$$\int_\rho^\sigma r^{3-m}\int_{\Omega_r}|du|^2 \lesssim ||\tau(u)||^2_{L^{m/2}} \quad (3.2.25)$$

Notice that $|dvX| \leq |duX|$, therefore we get

$$\int_{\Omega_\sigma\setminus B_\rho}\frac{|dvX|^2}{r^m} \lesssim (1+(\log\frac{\sigma}{\rho})^{1-4/m})||\tau(u)||^2_{L^{m/2}} \quad (3.2.26)$$

□

From this lemma, we can estimate $I_2$ as following,

$$\begin{aligned}I_2 &\le (\int_{B_\sigma\setminus B_\rho}\frac{1}{r^m})^{1/2}(\int_{\Omega_\sigma\setminus B_\rho}\frac{|dvX|^2}{r^m})^{1/2}\\&\lesssim (\int_{B_\sigma\setminus B_\rho}\frac{1}{r^m})^{1/2}(1+(\log\frac{\sigma}{\rho})^{1-4/m})^{1/2}||\tau(u)||_{L^{m/2}}\\&\lesssim ||\tau(u)||_{L^{m/2}}(\log\frac{\sigma}{\rho}+(\log\frac{\sigma}{\rho})^{2-4/m})^{1/2}\\&\lesssim ||\tau(u)||_{L^{m/2}}(1+(\log\frac{\sigma}{\rho})^{1-2/m})\end{aligned}$$

So far we obtain the estimate for $I$ as

$$I\lesssim \rho g(x)+||\tau(u)||_{L^{m/2}}(1+(\log\frac{\sigma}{\rho})^{1-2/m}) \tag{3.2.27}$$

For $II$, we use a similar argument. First we split it into three parts,

$$\begin{aligned}II &= \int_0^\sigma r^{3-m}\int_{\partial\Omega_r}|\nabla v|d\sigma dr+\int_{\Omega\setminus B_\sigma}r^{3-m}|\nabla v|\\&=\int_0^\rho r^{3-m}\int_{\partial\Omega_r}|\nabla v|d\sigma dr+\int_\rho^\sigma r^{3-m}\int_{\partial\Omega_r}|\nabla v|d\sigma dr+\int_{\Omega\setminus B_\sigma}r^{3-m}|\nabla v|\\&=II_1+II_2+II_3\end{aligned}$$

For the third term, it is easy to see that

$$II_3\le R^{3-m}|\Omega|^{1/2}||du||_{L^2}\lesssim ||\tau(u)||_{L^{m/2}} \tag{3.2.28}$$

For the first term, we have

$$\begin{aligned}
II_1 &= \int_0^\rho r^{3-m}\frac{d}{dr}(\int_{\Omega_r}|\nabla v|) \\
&= (r^{3-m}\int_{\Omega_r}|\nabla v|)\,|_0^\rho - \int_0^\rho (3-m)r^{2-m}(\int_{\Omega_r}|\nabla v|) \\
&= \rho^{3-m}\int_{B_\rho}|\nabla v| + (m-3)\int_0^\rho r^{2-m}(\int_{\Omega_r}|\nabla v|) \\
&\lesssim \rho^3 g(x) \\
&\lesssim \rho g(x)
\end{aligned}$$

For the second term, we have

$$\begin{aligned}
II_2 &= \int_{\Omega_\sigma\backslash B_\rho} r^{3-m}|\nabla v| \\
&\le (\int_{B_\sigma\backslash B_\rho}\frac{1}{r^{m-1}})^{1/2}(\int_{\Omega_\sigma\backslash B_\rho}\frac{|\nabla v|^2}{r^{m-5}})^{1/2}
\end{aligned}$$

Again we can control it by applying the monotonicity formula technique, we have

$$\int_{\Omega_\sigma\backslash B_\rho}\frac{|du|^2}{r^{m-5}} \lesssim ||\tau(u)||^2_{L^{m/2}} \tag{3.2.29}$$

Therefore for $II$

$$II \lesssim \rho g(x) + ||\tau(u)||_{L^{m/2}}(1+(\log\frac{\sigma}{\rho})^{1-2/m}) \tag{3.2.30}$$

For $III$, we have

$$\int_\Omega r^{-1}|v| \lesssim (\int_{B_\sigma} r^{-2})^{1/2}(\int_\Omega |v|^2)^{1/2} \lesssim ||\tau(u)||_{L^{m/2}} \tag{3.2.31}$$

Combine (3.2.27), (3.2.30) and (3.2.31) together, we conclude there exist constant $C_1$ and $C_2$ such that

$$|v(x)| \leq C_1\rho g(x) + C_2\Lambda(1 + (\log\frac{\sigma}{\rho})^{1-2/m}) \tag{3.2.32}$$

Redefine $\tilde{v} = v/C_1$ and $c = C_2/C_1$ ,we have

$$|\tilde{v}(x)| \leq \rho g(x) + c\Lambda(1 + (\log\frac{\sigma}{\rho})^{1-2/m}) \tag{3.2.33}$$

Define the function $f = \max\{0, |\tilde{v}| - c\Lambda\}$, then

$$f(x) \leq \rho g(x) + c\Lambda(\log\frac{\sigma}{\rho})^{1-2/m} \tag{3.2.34}$$

for almost every $x \in \Omega$ and every $\rho \in (0, \sigma]$. We choose $\rho$ such that $\rho g(x) = c\Lambda(\log\frac{\sigma}{\rho})^{1-2/m}$, then we have

$$\rho = \sigma\{\mu(\frac{m}{m-2}(\frac{Rg(x)}{c\Lambda})^{\frac{m}{m-2}})\}^{\frac{2-m}{m}} \tag{3.2.35}$$

where $\mu = \lambda^{-1}$ and $\lambda(t) = t\log t$ for $t \geq 1$. Therefore,

$$f(x) \leq 2c\Lambda(\log\frac{\sigma}{\rho})^{1-2/m} \tag{3.2.36}$$

Thus we have

$$f(x)\exp\{(\frac{f(x)}{2c\Lambda})^{\frac{m}{m-2}}\} \leq 2\sigma g(x) \tag{3.2.37}$$

Since that Hardy-Littlewood maximal operator is $(p,p)$ operator, i.e. $||g||_{L^p} \lesssim ||dv||_{L^p}$ for $1 < p < \infty$, finally there exists a constant $C$ depend only on $n$ and $\Omega$, such that

$$\int_\Omega f^{\frac{m}{m-2}} \exp\{\frac{m}{m-2}(\frac{f}{2c\Lambda})^{\frac{m}{m-2}}\} \leq C \int_\Omega |dv|^{\frac{m}{m-2}} \tag{3.2.38}$$

From the definition of $v(x)$, we achieve the following

$$||u||_{L^\phi(\Omega)} \leq C(||\tau(u)||_{L^{m/2}} + ||du||_{L^{\frac{m}{m-2}}}) \tag{3.2.39}$$

Applying Hölder inequality, $||du||^{L^2} \lesssim ||\tau(u)||_{L^2}$, we get

$$||u||_{L^\phi(\Omega)} \leq C||\tau(u)||_{L^{n/2}} \tag{3.2.40}$$

where $C$ is a constant depend on $n$ and $\Omega$.

□

## 3.3 Proof of (1.1.32) and (1.1.40)

If we restrict to manifolds with negative bounded curvature, we will have the following proposition:

**Proposition 3.3.1.** *Suppose the sectional curvature $K$ of $M$ satisfies*

$$(m-2)K_0 \leq K \leq K_0 \tag{3.3.1}$$

*where $K_0 < 0$. Defining $X = r\nabla r$, it holds*

$$|du|^2 divX - 2\langle du\nabla X, \nabla u\rangle \geq (m-2)|du|^2 \tag{3.3.2}$$

*Proof.* Let $h(r) = \frac{r\cosh r}{\sinh r}$, $c = \sqrt{m-2}$, we then need to show

$$1 + h(r)(n-2) - h(cr) \geq m - 2 \tag{3.3.3}$$

It is equivalent to prove

$$m - 2 \leq \frac{h(cr) - 1}{h(r) - 1} \tag{3.3.4}$$

From the definition of $h(r)$, it suffices to prove

$$\frac{cr\cosh cr - \sinh cr}{r\cosh r - \sinh r} \cdot \frac{\sinh r}{\sinh cr} \leq c^2 \tag{3.3.5}$$

Now after we isolate $r$ from the fraction the above inequality is equivalent to

$$F(r) \doteq r - \frac{(c^2-1)\sinh r \sinh cr}{c^2\cosh r \sinh cr - c\cosh cr \sinh r} \geq 0 \tag{3.3.6}$$

Since $F(0) = 0$, we claim that $F'(r) \geq 0$ for $r \geq 0$ and hence the inequality is justified. For simplicity, we first define

$$l(r) = (c^2-1)\sinh r \sinh cr \tag{3.3.7}$$

and

$$h(r) = c^2\cosh r \sinh cr - c\cosh cr \sinh r \tag{3.3.8}$$

then we only need to show

$$1 \geq (\frac{l}{h})' \tag{3.3.9}$$

or

$$h^2 \geq l'h - lh' \tag{3.3.10}$$

For (3.3.10),

$$\begin{aligned} LHS = & c^4(\cosh r)^2(\sinh cr)^2 - 2c^3 \sinh r \cosh r \sinh cr \cosh cr \\ & + c^2(\sinh r)^2(\cosh cr)^2 \end{aligned}$$

and

$$RHS = (c^4 - c^2)((\cosh r)^2(\sinh cr)^2 - (\sinh r)^2(\cosh cr)^2) \tag{3.3.11}$$

Then

$$\begin{aligned} LHS - RHS &= c^2(\cosh r)^2(\sinh cr)^2 - 2c^3 \sinh r \cosh r \sinh cr \cosh cr \\ &\quad + c^4(\sinh r)^2(\cosh cr)^2 \\ &= c^2(c \sinh r \cosh cr - \cosh r \sinh cr)^2 \\ &\geq 0 \end{aligned}$$

Therefore (3.3.10) is proved.

□

In our setting, $X = r\nabla r$ is always well-defined on $\Omega$ since $M$ is equipped with negative curvature. Moreover, from Proposition 3.3.1, we already have

$$|du|^2 divX - 2\langle du\nabla X, \nabla u\rangle \geq (m-2)|du|^2 \tag{3.3.12}$$

Since $u$ is constant on $\partial\Omega$, we have $\nabla u = du(\overrightarrow{n})$, where $\overrightarrow{n} \in T^*M$ is the outer normal vector field of $\partial\Omega$ and thus

$$\begin{aligned}\langle duX, \nabla u\rangle \overrightarrow{n} &= \langle du(X^T + (X \cdot \overrightarrow{n})\overrightarrow{n}), du(\overrightarrow{n})\rangle \\ &= (X \cdot \overrightarrow{n})|du(\overrightarrow{n})|^2 \\ &= X \cdot \overrightarrow{n}|du|^2\end{aligned}$$

Take integration to (2.1.6), we have

$$\int_\Omega (m-2)|du|^2 - 2\langle duX, \tau(u)\rangle + \int_{\partial\Omega} X \cdot \overrightarrow{n}|du|^2 \leq 0 \tag{3.3.13}$$

Use the convexity condition for $\Omega$, $X \cdot \overrightarrow{n} \geq 0$, we have

$$\int_\Omega |du|^2 \leq \frac{2}{m-2}\int_\Omega \langle duX, \tau(u)\rangle \tag{3.3.14}$$

By Hölder's inequality,

$$||du||_{L^2(\Omega)} \leq \frac{2diam(\Omega)}{m-2}||\tau(u)||_{L^2(\Omega)} \tag{3.3.15}$$

Combine this theorem with Lemma 3.2.1, (1.1.32) is hence proved. (1.1.40) is proved similarly.

Now we prove (1.1.35). We need the following Reilly's formula for smooth maps (see [44]):

**Lemma 3.3.2.** *For any smooth map $u : (M, g) \to (N, h)$:*

$$\int_M ||\tau(u)||^2 = \int_M |\nabla du|^2 + \int_M R_{M,N}(du) + \int_{\partial M} B(u) \tag{3.3.16}$$

*where*

$$\begin{aligned} R_{M,N}(du) &= \langle du \circ Ric_M - Ric_N, du\rangle \\ &= \sum_{i=1}^{m} \langle du \circ Ric_M(e_i), du(e_i)\rangle \\ &- \sum_{i,j=1}^{m} R_N(du(e_i), du(e_j), du(e_i), du(e_j)) \end{aligned}$$

*and*

$$B(u) = \langle du \circ A - \nabla du(du(\overrightarrow{n}))\rangle_{\partial M} - \langle \triangle_{\partial M} u, du(\overrightarrow{n}) + H|du(\overrightarrow{n})| \tag{3.3.17}$$

*were $A$ is the Weingarten operator $A_x : T_x\partial M \to T_x\partial M$, at $x \in \partial M$ is then given by $A_x v = \nabla_v \overrightarrow{n}$. Then mean curvature of $\partial M$ is $H = \langle Tr(A), \overrightarrow{n}\rangle$.*

*In our case, $u = u_0$ on $\partial\Omega$, thus $du \circ A = 0$ and $\triangle_{\partial\Omega} u = 0$, which gives $B(u) \geq 0$.*

*For* $R_{M,N}$,

$$\begin{aligned} R_{M,N}(du) &= \sum_{i=1}^{m} \langle du \circ Ric_M(e_i), du(e_i)\rangle \\ &- \sum_{i,j=1}^{m} R_N(du(e_i), du(e_j), du(e_i), du(e_j)) \\ &\geq K_0|du|^2 \end{aligned}$$

*Thus*

$$\begin{aligned} \int_\Omega ||\tau(u)||^2 &= \int_\Omega |\nabla du|^2 + \int_\Omega R_{M,N}(du) + \int_{\partial\Omega} B(u) \\ &\geq K_0 \int_\Omega |du|^2 \end{aligned}$$

*By lemma 3.2.1, we finish the proof of* (1.1.35).

# Chapter 4

# Concentration-Compactness Principles on Heisenberg Groups $\mathbb{H}^n$ and Riemannian Manifolds

## 4.1 Background on Heisenberg Groups

Let $\mathbb{H}^n = \mathbb{C}^n \times \mathbb{R}$ be the $n$-dimensional Heisenberg group, whose group structure is given by

$$(x,t) \circ (x',t') = (x + x', t + t' + 2\mathrm{im}\,(x \cdot \bar{x}'))\,.$$

The Lie algebra of $\mathbb{H}^n$ is generated by the left invariant vector fields

$$X_i = \frac{\partial}{\partial x_i} + 2y_i \frac{\partial}{\partial t}, Y_i = \frac{\partial}{\partial y_i} - 2x_i \frac{\partial}{\partial t}, T = \frac{\partial}{\partial t},$$

for $i = 1, \ldots, n$. These generators satisfy the non-commutative relationship $[X_i, Y_i] = 4\delta_{ij}T$. Moreover, all the commutators of length greater than two vanish, and thus this is a nilpotent, graded, and stratified group of step two.

For each real number $r \in \mathbb{R}$, there is a dilation naturally associated with the Heisenberg group structure which is usually denoted as $\delta_r(x,t) = (rx, r^2 t)$. The Jacobian determinant of $\delta_r$ is $r^Q$, where $Q = 2n+2$ is the homogeneous dimension of $\mathbb{H}^n$.

We will use $\xi = (x,t)$ to denote any point $(x,t) \in \mathbb{H}^n$, then the anisotropic dilation structure on $\mathbb{H}^n$ introduces a homogeneous norm $|\xi| = \left(|x|^4 + t^2\right)^{1/4}$. Let

$$B_r = \{\xi : |\xi| < r\}$$

be the metric ball of center 0 and radius $r$ in $\mathbb{H}^n$. Since the Lebesgue measure in $\mathbb{R}^{2n+1}$ is the Haar measure on $\mathbb{H}^n$, one has (writing $|A|$ for the measure of $A$)

$$|B_r| = \omega_Q r^Q,$$

where $\omega_Q$ is a positive constant only depending on $Q$ (see [14]).

We write $|\nabla_{\mathbb{H}} u|$ to express the norm of the subelliptic gradient of the function $u : \mathbb{H}^n \to \mathbb{R}$ :

$$|\nabla_{\mathbb{H}} u| = \sqrt{\sum (X_i u)^2 + (Y_i u)^2}.$$

Let $\Omega$ be an open set in $\mathbb{H}^n$ and $p > 1$. We define the Horizontal Sobolev Spaces

$$HW^{1,p}(\Omega) = \left\{ u \in L^p(\Omega) : \|u\|_{HW^{1,p}(\Omega)} < \infty \right\}$$

with the norm

$$\|u\|_{HW^{1,p}(\Omega)} = \left( \int_{\Omega} \left( |\nabla_{\mathbb{H}} u(z,t)|^p + |u(z,t)|^p \right) dx dt \right)^{1/p}.$$

Also, we define the space $HW_0^{1,p}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ in the norm of $HW^{1,p}(\Omega)$.

## 4.2 Some useful known results on Heisenberg groups

In this subsection, we collect some known results which will be used in the following. The following lemma was proved in [14],

**Lemma 4.2.1.** *Let $\rho = |\xi|$ be the homogeneous norm of the element $\xi = (x,t) \in \mathbb{H}^n$, and $g(\xi) = g(\rho)$ be a $C^1$ radial function on $\mathbb{H}^n$. Then*

$$|\nabla_{\mathbb{H}} g(\xi)| = \frac{g'(\rho)}{\rho} |x|.$$

The following lemma was proved in [25],

**Lemma 4.2.2.** *Let $\alpha_Q = Q\left(2\pi^n \Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{Q-1}{2}\right) \Gamma\left(\frac{Q}{2}\right)^{-1} \Gamma(n)^{-1}\right)^{\frac{1}{Q-1}}$, $0 \le \beta < Q$. There exists a uniform constant $c$ depending only on $Q, \beta$ such that for all $\Omega \subset \mathbb{H}^n$ with $|\Omega| < \infty$ and $\alpha \le \alpha_{Q,\beta} = \alpha_Q\left(1 - \frac{\beta}{Q}\right)$, one has*

$$\sup_{\substack{u \in HW_0^{1,Q}(\Omega) \\ \|\nabla_{\mathbb{H}} u\|_{L^Q} \le 1}} \int_{\Omega} \frac{\exp\left(\alpha u(\xi)^{\frac{Q}{Q-1}}\right)}{\rho(\xi)^\beta} d\xi < c. \tag{4.2.1}$$

*The constant $\alpha_{Q,\beta}$ is the best possible in the sense that if $\alpha > \alpha_{Q,\beta}$, then the supremum above is infinite.*

The following lemma was proved in [23],

**Lemma 4.2.3.** *Let $0 \leq \beta < Q$. There exists a uniform constant $c$ depending only on $Q, \beta$ such that for all $\alpha \leq \alpha_{Q,\beta}$, one has*

$$\sup_{\substack{f \in HW^{1,Q}(\mathbb{H}^n) \\ \|f\|_{HW^{1,Q}(\mathbb{H}^n)} \leq 1}} \int_{\mathbb{H}^n} \frac{\Phi\left(\alpha f(\xi)^{\frac{Q}{Q-1}}\right)}{\rho(\xi)^{\beta}} d\xi < c. \tag{4.2.2}$$

*where $\Phi(t) = e^t - \sum_{j=0}^{Q-2} \frac{t^j}{j!}$. The constant $\alpha_{Q,\beta}$ is the best possible in the sense that if $\alpha > \alpha_{Q,\beta}$, then the supremum in the inequality (4.2.1) is infinite.*

## 4.3 Concentration-compactness Principles for Bounded Domains in $\mathbb{H}^n$ - Proof of Theorem 1.2.3

Since $\|\nabla_{\mathbb{H}} u\|_Q \leq \lim_k \|\nabla_{\mathbb{H}} u_k\|_Q = 1$, we split the proof into two cases.

Case 1: $\|\nabla_{\mathbb{H}} u\|_Q < 1$. We assume by contradiction for some $p_1 < M_{Q,u}$, we have

$$\sup_k \int_{\Omega} \frac{\exp\left(\alpha_{Q,\beta} p_1 u_k^{\frac{Q}{Q-1}}\right)}{\rho(\xi)^{\beta}} d\xi = +\infty.$$

Set

$$\Omega_L^k = \{\xi \in \Omega : u_k(\xi) \geq L\},$$

where $L$ is some constant. Let $v_k = u_k - L$. Then for any $\varepsilon > 0$, one has

$$u_k^{\frac{Q}{Q-1}} \leq (1+\varepsilon) v_k^{\frac{Q}{Q-1}} + C(\varepsilon, Q) L^{\frac{Q}{Q-1}}. \tag{4.3.1}$$

Since $0 \leq \beta < Q$, we have

$$\begin{aligned}\int_{\Omega} \frac{\exp\left(\alpha_{Q,\beta} p_1 u_k^{\frac{Q}{Q-1}}\right)}{\rho\left(\xi\right)^{\beta}} d\xi &= \int_{\Omega_L^k} \frac{\exp\left(\alpha_{Q,\beta} p_1 u_k^{\frac{Q}{Q-1}}\right)}{\rho\left(\xi\right)^{\beta}} d\xi + \int_{\Omega \setminus \Omega_L^k} \frac{\exp\left(\alpha_{Q,\beta} p_1 u_k^{\frac{Q}{Q-1}}\right)}{\rho\left(\xi\right)^{\beta}} d\xi \\ &\leq \int_{\Omega_L^k} \frac{\exp\left(\alpha_{Q,\beta} p_1 u_k^{\frac{Q}{Q-1}}\right)}{\rho\left(\xi\right)^{\beta}} d\xi + c \exp\left(\alpha_{Q,\beta} p_1 L^{\frac{Q}{Q-1}}\right) \int_{\Omega} \frac{1}{\rho\left(\xi\right)^{\beta}} d\xi \\ &\leq \int_{\Omega_L^k} \frac{\exp\left(\alpha_{Q,\beta} p_1 u_k^{\frac{Q}{Q-1}}\right)}{\rho\left(\xi\right)^{\beta}} d\xi + c\left(L, Q, |\Omega|, \beta\right),\end{aligned}$$

and then

$$\sup_k \int_{\Omega_L^k} \frac{\exp\left(\alpha_{Q,\beta} p_1 u_k^{\frac{Q}{Q-1}}\right)}{\rho\left(\xi\right)^{\beta}} d\xi = \infty.$$

By (4.3.1) we have

$$\int_{\Omega_L^k} \frac{\exp\left(\alpha_{Q,\beta} p_1 u_k^{\frac{Q}{Q-1}}\right)}{\rho\left(\xi\right)^{\beta}} d\xi \leq \exp\left(\alpha_{Q,\beta} p_1 C\left(\varepsilon, Q\right) L^{\frac{Q}{Q-1}}\right) \cdot \int_{\Omega_L^k} \frac{\exp\left(\left(1+\varepsilon\right) \alpha_{Q,\beta} p_1 v_k^{\frac{Q}{Q-1}}\right)}{\rho\left(\xi\right)^{\beta}} d\xi.$$

Thus

$$\sup_k \int_{\Omega_L^k} \frac{\exp\left(\bar{p}_1 \alpha_{Q,\beta} v_k^{\frac{Q}{Q-1}}\right)}{\rho\left(\xi\right)^{\beta}} d\xi = \infty,$$

where $\bar{p}_1 = \left(1+\varepsilon\right) p_1 < M_{Q,u}$.

Now, we define

$$T^L\left(u\right) = \min\left\{L, u\right\} \text{ and } T_L\left(u\right) = u - T^L\left(u\right)$$

and choose $L$ such that

$$\frac{1-\|\nabla_{\mathbb{H}}u\|_Q^Q}{1-\|\nabla_{\mathbb{H}}T^Lu\|_Q^Q} > \left(\frac{\bar{p}_1}{M_{Q,u}}\right)^{Q-1}. \tag{4.3.2}$$

We claim that

$$\limsup_k \int_{\Omega_L^k} |\nabla_{\mathbb{H}}v_k|^Q \, d\xi < \left(\frac{1}{\bar{p}_1}\right)^{Q-1}.$$

If not, then up to a subsequence, one has

$$\int_{\Omega_L^k} |\nabla_{\mathbb{H}}v_k|^Q \, d\xi = \int_{\Omega} |\nabla_{\mathbb{H}}T_Lu_k|^Q \, d\xi \geq \left(\frac{1}{\bar{p}_1}\right)^{Q-1} + o_k(1). \tag{4.3.3}$$

Thus,

$$\begin{aligned}\left(\frac{1}{\bar{p}_1}\right)^{Q-1} + \int_{\Omega} \left|\nabla_{\mathbb{H}}T^Lu_k\right|^Q d\xi + o_k(1) &\leq \int_{\Omega} |\nabla_{\mathbb{H}}T_Lu_k|^Q \, d\xi + \int_{\Omega\setminus\Omega_L^k} |\nabla_{\mathbb{H}}u_k|^Q \, d\xi \\ &= \int_{\Omega_L^k} |\nabla_{\mathbb{H}}u_k|^Q \, d\xi + \int_{\Omega\setminus\Omega_L^k} |\nabla_{\mathbb{H}}u_k|^Q \, d\xi = 1.\end{aligned}$$

For $L > 0$ fixed, $T^Lu_k$ is also bounded in $HW^{1,Q}(\Omega)$. Hence, up to a subsequence, $T^Lu_k \rightharpoonup T^Lu$ in $HW^{1,Q}(\Omega)$ and $T^Lu_k \to T^Lu$ almost everywhere in $\Omega$. By the lower semicontinuity of the norm in $HW^{1,Q}(\Omega)$ and the above inequality, we have

$$\bar{p}_1 \geq \frac{1}{\left(1-\liminf_{k\to\infty}\|\nabla_{\mathbb{H}}T^Lu_k\|_Q^Q\right)^{\frac{1}{Q-1}}} \geq \frac{1}{\left(1-\|\nabla_{\mathbb{H}}T^Lu\|_Q^Q\right)^{\frac{1}{Q-1}}},$$

combining with (4.3.2), we derive

$$\bar{p}_1 \geq \frac{1}{\left(1-\|\nabla_{\mathbb{H}}T^Lu\|_Q^Q\right)^{\frac{1}{Q-1}}} > \frac{\bar{p}_1}{M_{Q,u}} \frac{1}{\left(1-\|\nabla_{\mathbb{H}}u\|_Q^Q\right)^{\frac{1}{Q-1}}} = \bar{p}_1,$$

which is a contradiction. Therefore

$$\limsup_{k} \int_{\Omega_L^k} |\nabla_{\mathbb{H}} v_k|^Q \, d\xi < \left(\frac{1}{\bar{p}_1}\right)^{Q-1}.$$

By the Trudinger-Moser inequality (4.2.1), we derive

$$\sup_k \int_{\Omega_L^k} \frac{\exp\left(\bar{p}_1 \alpha_{Q,\beta} v_k^{\frac{Q}{Q-1}}\right)}{\rho(\xi)^\beta} d\xi < \infty,$$

which is also a contradiction. The proof is finished in this case.

Case 2: $\|\nabla_{\mathbb{H}} u\|_Q = 1$. We can iterate the procedure as in Case 1 and get

$$\sup_k \int_{\Omega_L^k} \frac{\exp\left(\bar{p}_1 \alpha_{Q,\beta} v_k^{\frac{Q}{Q-1}}\right)}{\rho(\xi)^\beta} = \infty,$$

where $\bar{p}_1 = (1+\varepsilon) p_1$. Then we have

$$\limsup_{k} \int_{\Omega_L^k} |\nabla_{\mathbb{H}} v_k|^Q \, d\xi = \limsup_{k} \int_{\Omega} |\nabla_{\mathbb{H}} T_L u_k|^Q \, d\xi \geq \left(\frac{1}{\bar{p}_1}\right)^{Q-1},$$

thus,

$$\left\|\nabla_{\mathbb{H}} T^L u\right\|_Q^Q \leq \liminf_{k} \int_{\Omega} \left|\nabla_{\mathbb{H}} T^L u_k\right|^Q d\xi = 1 - \limsup_{k} \int_{\Omega} |\nabla_{\mathbb{H}} T_L u_k|^Q \, d\xi \leq 1 - \left(\frac{1}{\bar{p}_1}\right)^{Q-1}.$$

On the other hand, since $\|\nabla_{\mathbb{H}} u\|_Q = 1$, we can take $L$ large enough such that

$$\left\|\nabla_{\mathbb{H}} T^L u\right\|_Q^Q > 1 - \frac{1}{2}\left(\frac{1}{\bar{p}_1}\right)^{Q-1},$$

which is contradiction, and the proof is finished in this case.

Now, we prove the sharpness of $M_{Q,u}$. For some $r>0$, we defined $\omega_k(\xi)$ by

$$\omega_k(\xi)=\begin{cases} Q^{\frac{1-Q}{Q}}(c_Q)^{-\frac{1}{Q}}k^{\frac{Q-1}{Q}} & \text{if } |\xi|\in\left[0,re^{-\frac{k}{Q}}\right] \\ Q^{\frac{1}{Q}}(c_Q)^{-\frac{1}{Q}}\log\left(\frac{r}{|\xi|}\right)k^{-\frac{1}{Q}} & \text{if } |\xi|\in\left[re^{-\frac{k}{Q}},r\right] \\ 0 & \text{if } |\xi|\in[r,\infty], \end{cases} \tag{4.3.4}$$

where $c_Q=\int_\Sigma |x^*|^Q d\xi$, $x^*=\frac{x}{|\xi|}$ and $\Sigma$ is the unit sphere on $\mathbb{H}^n$.

We can verify that $\omega_k(\xi)\in HW_0^{1,Q}(\Omega)$. Actually, from Lemma 4.2.1 we have

$$\begin{aligned}\int_\Omega |\nabla_{\mathbb{H}}\omega_k(\xi)|^Q d\xi &= \int_\Sigma\int_{re^{-\frac{k}{Q}}}^r \left|Q^{\frac{1}{Q}}(c_Q)^{-\frac{1}{Q}}k^{-\frac{1}{Q}}\frac{|x^*|}{\rho(\xi)}\right|^Q \rho(\xi)^{Q-1}d\rho(\xi)\,d\mu(x^*) \\ &= \frac{Q}{c_Q}\frac{1}{k}c_Q\int_{re^{-\frac{k}{Q}}}^r \rho^{-1}d\rho = 1\end{aligned}$$

and $\omega_k(\xi)\rightharpoonup 0$ in $HW_0^{1,Q}(\Omega)$.

Now, for $R=3r$, we define

$$u=\begin{cases} A & \text{if } |\xi|\in\left[0,\frac{2R}{3}\right] \\ 3A-\frac{3A}{R}|\xi| & \text{if } |\xi|\in\left[\frac{2R}{3},R\right] \\ 0 & \text{if } |\xi|\in[R\ ,+\infty], \end{cases} \tag{4.3.5}$$

where $A>0$ is chosen in such a way that $\|\nabla_{\mathbb{H}}u\|_{L^Q(\Omega)}=\delta<1$. Defining

$$u_k=u+\left(1-\delta^Q\right)^{1/Q}\omega_k. \tag{4.3.6}$$

Observing that $\nabla_{\mathbb{H}} u$ and $\nabla_{\mathbb{H}} \omega_k$ have disjoints supports, we have

$$\begin{aligned}\|\nabla_{\mathbb{H}} u_k\|_{L^Q(\Omega)}^Q &= \|\nabla_{\mathbb{H}} u\|_{L^Q(\Omega)}^Q + \left(1-\delta^Q\right)\\ &= 1,\end{aligned}$$

moreover, $u_k \rightharpoonup u$ in $HW_0^{1,Q}(\Omega)$.

Consequently,

$$\begin{aligned}
\int_\Omega \frac{\exp\left(\alpha_{Q,\beta} M_{Q,u} u_k^{\frac{Q}{Q-1}}\right)}{\rho(\xi)^\beta} d\xi &= \int_\Omega \frac{\exp\left(\frac{\alpha_{Q,\beta} u_k^{\frac{Q}{Q-1}}}{(1-\delta^Q)^{1/(Q-1)}}\right)}{\rho(\xi)^\beta} d\xi \\
\geq \int_{B_{r\exp\left(-\frac{k}{Q}\right)}} \frac{\exp\left(\frac{\alpha_{Q,\beta}\left(A+\left(1-\delta^Q\right)^{1/Q}\omega_k\right)^{\frac{Q}{Q-1}}}{(1-\delta^Q)^{1/(Q-1)}}\right)}{\rho(\xi)^\beta} d\xi \\
= \int_{B_{r\exp\left(-\frac{k}{Q}\right)}} \frac{\exp\left(\alpha_{Q,\beta}\left(C+\omega_k\right)^{\frac{Q}{Q-1}}\right)}{\rho(\xi)^\beta} d\xi \\
\geq \exp\left(\left(C' + \left(\left(1-\frac{\beta}{Q}\right)k\right)^{\frac{Q-1}{Q}}\right)^{\frac{Q}{Q-1}}\right) \int_{B_{r\exp\left(-\frac{k}{Q}\right)}} \frac{1}{\rho(\xi)^\beta} d\xi \\
\geq C'' \exp\left(\left(C' + \left(\left(1-\frac{\beta}{Q}\right)k\right)^{\frac{Q-1}{Q}}\right)^{\frac{Q}{Q-1}} - \left(1-\frac{\beta}{Q}\right)k\right) \to \infty,
\end{aligned}$$

for some positive constant $C, C', C''$, and the theorem is finished.

## 4.4 Concnetration-compactness Principle on the Whole Space $\mathbb{H}^n$ - Proof of Theorem 1.2.4

In this section, we will apply the new method developed in [31] to proof Theorem 1.2.4.

*Proof.* Set $A(u_k) = 2^{-\frac{1}{Q(Q-1)}}||u_k||^Q_{HW^{1,Q}(\mathbb{H}^n)}$ and $\Omega(u_k) = \{\xi \in \mathbb{H}^n : \ u_k(\xi) > A(u_k)\}$. It is easy to see that

$$A(u_k) < 1 \ and \ |\Omega(u_k)| \leq 2^{\frac{1}{Q}}. \tag{4.4.1}$$

Now we write

$$\int_{\mathbb{H}^n} \frac{\Phi(\alpha_{Q,\beta} p|u_k|^{\frac{Q}{Q-1}})}{\rho(\xi)^\beta} d\xi = \Big( \int_{\Omega(u_k)} + \int_{\mathbb{H}^n \setminus \Omega(u_k)} \Big) \frac{\Phi(\alpha_{Q,\beta} p|u_k|^{\frac{Q}{Q-1}})}{\rho(\xi)^\beta} d\xi. \tag{4.4.2}$$

Similar to the proof in [32], we can show that

$$\int_{\mathbb{H}^n \setminus \Omega(u_k)} \frac{\Phi(\alpha_{Q,\beta} p|u_k|^{\frac{Q}{Q-1}})}{\rho(\xi)^\beta} d\xi \leq C(p, Q, \beta). \tag{4.4.3}$$

Therefore it suffices to show that

$$\sup_k \int_{\Omega(u_k)} \frac{\Phi(\alpha_{Q,\beta} p|u_k|^{\frac{Q}{Q-1}})}{\rho(\xi)^\beta} d\xi < \infty. \tag{4.4.4}$$

Let $v_k = u_k - L$ defined on $\Omega_L^k = \{\xi \in \Omega(u_k) : \ |u_k| > L\}$, where $L$ is left to be chosen. Then for any $\epsilon > 0$, one has

$$|u_k|^{\frac{Q}{Q-1}} \leq (1+\epsilon)|v_k|^{\frac{Q}{Q-1}} + C(\epsilon) L^{\frac{Q}{Q-1}}. \tag{4.4.5}$$

Then we have

$$\int_{\Omega(u_k)} \frac{\Phi(\alpha_{Q,\beta} p|u_k|^{\frac{Q}{Q-1}})}{\rho(\xi)^\beta} d\xi \leq e^{\alpha_{Q,\beta} p C(\epsilon) L^{\frac{Q}{Q-1}}} \int_{\Omega_L^k} \frac{\exp((1+\epsilon)\alpha_{Q,\beta} p|v_k|^{\frac{Q}{Q-1}})}{\rho(\xi)^\beta} d\xi \tag{4.4.6}$$

$$+\Phi(\alpha_{Q,\beta} p L^{\frac{Q}{Q-1}}) \int_{\Omega(u_k)} \frac{1}{\rho(\xi)^\beta}. \tag{4.4.7}$$

Therefore it suffices to show that

$$\sup_k \int_{\Omega_L^k} \frac{\exp((1+\epsilon)\alpha_{Q,\beta} p|v_k|^{\frac{Q}{Q-1}})}{\rho(\xi)^\beta} d\xi < \infty. \tag{4.4.8}$$

Define $T^L(u_k) = \min\{u_k, L\}$, we have

$$1 = ||u_k||^Q_{HW^{1,Q}(\mathbb{H}^n)} = \int_{\mathbb{H}^n} |u_k|^Q d\xi + \int_{\mathbb{H}^n} |\nabla_{\mathbb{H}} T^L u_k|^Q d\xi + \int_{\Omega_L^k} |\nabla_{\mathbb{H}} v_k|^Q d\xi \tag{4.4.9}$$

For any $\epsilon > 0$, when $k$ is large enough, since $u_k \rightharpoonup u$ weakly in $HW^{1,Q}(\mathbb{H}^n)$, we have

$$\int_{\mathbb{H}^n} |u_k|^Q d\xi > \int_{\mathbb{H}^n} |u|^Q d\xi - \epsilon \tag{4.4.10}$$

as well as

$$\int_{\mathbb{H}^n} |\nabla_{\mathbb{H}} T^L u_k|^Q d\xi > \int_{\mathbb{H}^n} |\nabla_{\mathbb{H}} T^L u|^Q d\xi - \epsilon. \tag{4.4.11}$$

Also choose $L$ large enough such that

$$\int_{\mathbb{H}^n} |\nabla_{\mathbb{H}} T^L u|^Q d\xi > \int_{\mathbb{H}^n} |\nabla_{\mathbb{H}} u|^Q d\xi - \epsilon. \tag{4.4.12}$$

Bring them back to (4.4.9), we have

$$\int_{\mathbb{H}^n} |u|^Q d\xi + \int_{\mathbb{H}^n} |\nabla_{\mathbb{H}} u|^Q d\xi - 3\epsilon + \int_{\Omega_L^k} |\nabla_{\mathbb{H}} v_k|^Q d\xi < 1 \tag{4.4.13}$$

When $||u||_{HW^{1,Q}(\mathbb{H}^n)} < 1$, we have

$$\frac{\int_{\Omega_L^k} |\nabla_{\mathbb{H}} v_k|^Q d\xi}{1 - ||u||^Q_{HW^{1,Q}(\mathbb{H}^n)} + 3\epsilon} < 1. \tag{4.4.14}$$

For any $p < \tilde{M}_{Q,u}$, we can pick $\epsilon$ small enough such that the following holds

$$p(1+\epsilon) < \frac{1}{(1 - ||u||^Q_{HW^{1,Q}(\mathbb{H}^n)} + 3\epsilon)^{\frac{1}{Q-1}}} < \tilde{M}_{Q,u}. \tag{4.4.15}$$

One can then easily verify that $||\nabla_{\mathbb{H}}(p(1+\epsilon))^{\frac{Q-1}{Q}} v_k||_{L^Q(\mathbb{H}^n)} \leq 1$. Then we can conclude from Lemma 4.2.2.

For the case when $||u||_{HW^{1,Q}(\mathbb{H}^n)} = 1$, we have

$$\frac{\int_{\Omega_L^k} |\nabla_{\mathbb{H}} v_k|^Q d\xi}{3\epsilon} < 1. \tag{4.4.16}$$

For any $p < \infty$, we can choose $\epsilon$ small enough such that

$$p(1+\epsilon) < \frac{1}{(3\epsilon)^{\frac{1}{Q-1}}} \tag{4.4.17}$$

Then it is easy to check that $||\nabla_{\mathbb{H}}(p(1+\epsilon))^{\frac{Q-1}{Q}} v_k||_{L^Q(\mathbb{H}^n)} \leq 1$, hence we conclude again from Lemma 4.2.2.

□

## 4.5 Concentration-compactness Principle on Riemannian Manifolds

We first give the proof of Theorem 1.2.5.

*Proof.* Without loss of generality, we only need to consider $u_k \in C_0^\infty(M)$ taking positive values. First define the level set $\Omega_L^k = \{x \in M \mid |u_k(x)| > L\}$, where $L > 0$ is left to be chosen. Apparently $\Omega_L^k$ is bounded domain with finite measure $|\Omega_L^k| \leq \mathrm{Vol}(M)$. Let $v_k = u_k - L$, then for any $0 < p < M_{n,u}$, we have

$$\begin{aligned}
\int_M e^{\alpha_n p|u_k|^{\frac{n}{n-1}}} dV_g &= \int_{M\setminus\Omega_L^k} e^{\alpha_n p|u_k|^{\frac{n}{n-1}}} dV_g + \int_{\Omega_L^k} e^{\alpha_n p|u_k|^{\frac{n}{n-1}}} dV_g && (4.5.1)\\
&\leq e^{\alpha_n p L^{\frac{n}{n-1}}} \mathrm{Vol}(M) + \int_{\Omega_L^k} e^{\alpha_n p|v_k+L|^{\frac{n}{n-1}}} dV_g && (4.5.2)\\
&\leq C + e^{\alpha_n p C(\epsilon) L^{\frac{n}{n-1}}} \int_{\Omega_L^k} e^{\alpha_n p(1+\epsilon)|v_k|^{\frac{n}{n-1}}} dV_g, && (4.5.3)
\end{aligned}$$

where at the last line we use the fundamental inequality that for any small $\epsilon > 0$, there exists a constant $C(\epsilon)$ such that

$$|v_k + L|^{\frac{n}{n-1}} \leq (1+\epsilon)|v_k|^{\frac{n}{n-1}} + C(\epsilon)L^{\frac{n}{n-1}}. \qquad (4.5.4)$$

Now it suffices to show that

$$\sup_k \int_{\Omega_L^k} e^{\alpha_n p(1+\epsilon)|v_k|^{\frac{n}{n-1}}} dV_g < \infty. \qquad (4.5.5)$$

Since $u_k \rightharpoonup u$ weakly in $W_0^{1,n}(M)$, we have $||\nabla u||_{L^n(M)} \leq \liminf_k ||\nabla u_k||_{L^n(M)} = 1$. We consider the first case when $||\nabla u||_{L^n(M)} < 1$. Define the truncation function

$T^L(u_k) = \min\{u_k, L\}$, then we have

$$1 = \int_{\Omega_L^k} |\nabla u_k|^n dV_g + \int_{M\setminus\Omega_L^k} |\nabla u_k|^n dV_g \tag{4.5.6}$$

$$= \int_{\Omega_L^k} |\nabla v_k|^n dV_g + \int_M |\nabla T^L(u_k)|^n dV_g. \tag{4.5.7}$$

For a fixed $L$, we have $\nabla(T^L(u_k)) \rightharpoonup \nabla(T^L u)$ weakly in $L^n(M)$. That implies for any small $\epsilon > 0$, when $k$ is large enough, one always has $\int_M |\nabla T^L u|^n dV_g - \epsilon < \int_M |\nabla u_k|^n dV_g$. Then from (4.5.7) we derive the following

$$\int_{\Omega_L^k} |\nabla v_k|^n dV_g + \int_M |\nabla T^L u|^n dV_g - \epsilon < 1 \tag{4.5.8}$$

which implies

$$\frac{\int_{\Omega_L^k} |\nabla v_k|^n dV_g}{1 - \int_M |\nabla T^L u|^n dV_g + \epsilon} < 1. \tag{4.5.9}$$

Now we can always pick $\epsilon$ small enough and $L$ large enough such that the following holds

$$p(1+\epsilon) \leq \frac{1}{(1 - \int_M |\nabla T^L u|^n dV_g + \epsilon)^{\frac{1}{n-1}}} < \frac{1}{(1 - ||\nabla u||^n_{L^n(M)})^{\frac{1}{n-1}}}. \tag{4.5.10}$$

By the choice of $\epsilon$ and $L$, the Dirichlet norm of $(p(1+\epsilon))^{\frac{n-1}{n}} v_k$ is bounded by one, actually

$$||\nabla(p(1+\epsilon))^{\frac{n-1}{n}}v_k||^n_{L^n(M)} = (p(1+\epsilon))^{n-1}\int_{\Omega_L^k}|\nabla v_k|^n dV_g \tag{4.5.11}$$
$$\leq \frac{\int_{\Omega_L^k}|\nabla v_k|^n dV_g}{1-\int_M |\nabla T^L u|^n dV_g+\epsilon} \tag{4.5.12}$$
$$< 1. \tag{4.5.13}$$

Then from (1.1.23), inequality (4.5.5) is proved.

When $||\nabla u||_{L^n(M)} = 1$, we have

$$1 = \int_{M\setminus\Omega_L^k}|\nabla u_k|^n dV_g + \int_{\Omega_L^k}|\nabla u_k|^n dV_g \tag{4.5.14}$$
$$= \int_M |\nabla T^L u_k|^n dV_g + \int_{\Omega_L^k}|\nabla v_k|^n dV_g. \tag{4.5.15}$$

Then for any small $\epsilon > 0$, when $k$ is large enough, one always has $\int_M |\nabla T^L u|^n dV_g - \epsilon < \int_M |\nabla T^L u_k|^n dV_g$. Thus from inequality (4.5.37), we have

$$\int_M |\nabla T^L u|^n dV_g - \epsilon + \int_{\Omega_L^k}|\nabla v_k|^n dV_g < 1. \tag{4.5.16}$$

On the other hand, for a fixed $\epsilon$, one can pick $L$ large enough such that

$$1 = \int_M |\nabla u|^n dV_g < \int_M |\nabla T^L u|^n dV_g + \epsilon. \tag{4.5.17}$$

Thus we have

$$1 - 2\epsilon + \int_{\Omega_L^k} |\nabla v_k|^n dV_g < 1 \tag{4.5.18}$$

which implies

$$\frac{\int_{\Omega_L^k} |\nabla v_k|^n dV_g}{2\epsilon} < 1 \tag{4.5.19}$$

For any $p < \infty$, we can pick $\epsilon > 0$ small enough such that $p(1+\epsilon) < (2\epsilon)^{-\frac{1}{n-1}}$. It is easy to check that the Dirichlet norm of $(p(1+\epsilon))^{\frac{n-1}{n}} v_k$ satisfies the condition of the Trudinger-Moser inequality, i.e.

$$||\nabla (p(1+\epsilon))^{\frac{n-1}{n}} v_k||^n_{L^n(M)} < 1 \tag{4.5.20}$$

Again from (1.1.23), inequality (4.5.5) is proved.

For sharpness, we fix a point $P \in M$ and we will work in the geodesic ball $B_P(\delta)$ and when $\delta$ is small enough, the exponential map is a diffeomorphism. Notice that without loss of generality, we can assume $\delta = 3$. Everything works with proper obvious modification, for any $\delta$. We first introduce the following Moser's sequence ( see [45] ),

$$v_k(x) = \begin{cases} 1 & \text{dist}(P,x) < \frac{1}{k} \\ (\log k)^{-1} \log \frac{1}{dist(P,x)} & \frac{1}{k} \le \text{dist}(P,x) < 1 \cdot \\ 0 & \text{elsewhere} \end{cases} \tag{4.5.21}$$

Following easy computation we get $||\nabla v_k||_{L^n(M)} = (\log k)^{-\frac{n-1}{n}} \omega_{n-1}^{\frac{1}{n}} (1 + O(\log k)^{-1})$.

Then denote the normalization of $u_k$ as $\tilde{v}_k = \frac{v_k}{||\nabla v_k||_{L^n(M)}}$. From [18, 45], we know that $\{\tilde{v_k}\}$ is the sequence such that

$$\sup_k \int_{B_P(\frac{1}{k})} e^{\alpha|\tilde{v}_k|^{\frac{n}{n-1}}} = \infty. \tag{4.5.22}$$

It is not difficult to check that $\tilde{v_k} \rightharpoonup 0$ weakly in $W_0^{1,n}(M)$. Now we define the function $u$ as following,

$$u(x) = \begin{cases} A & \text{dist}(P,x) \leq 2 \\ 3A - Adist(P,x) & 2 < \text{dist}(P,x) \leq 3 \cdot \\ 0 & \text{elsewhere} \end{cases} \tag{4.5.23}$$

where $A$ is chosen to make $||\nabla u||_{L^n(M)} = \lambda < 1$. By defining $u_k = u + (1-\lambda^n)^{\frac{1}{n}}\tilde{v}k$, we know $u_k \rightharpoonup u$ weakly in $W_0^{1,n}(M)$ and $||\nabla u_k||_{L^n(M)} = 1$. But now the integral

$$\int_M \exp(\alpha_n M_{n,u}|u_k|^{\frac{n}{n-1}})dV_g \tag{4.5.24}$$

$$\geq \int_{B_P(\frac{1}{k})} \exp\left(\alpha_n \frac{|A + (1-\lambda^n)^{\frac{1}{n}}\tilde{v_k}|^{\frac{n}{n-1}}}{(1-\lambda^n)^{\frac{1}{n-1}}}\right) dV_g \tag{4.5.25}$$

$$= \int_{B_P(\frac{1}{k})} \exp\left(\alpha_n|C + \tilde{v_k}|^{\frac{n}{n-1}}\right)dV_g \tag{4.5.26}$$

$$= \int_{B_P(\frac{1}{k})} e^{n\left(C' + (\log k)^{\frac{n-1}{n}}(1+O(\log k)^{-1})\right)^{\frac{n}{n-1}}} dV_g \tag{4.5.27}$$

$$\geq C'' k^{-n} e^{n\left(C' + (\log k)^{\frac{n-1}{n}}\right)^{\frac{n}{n-1}}} \to \infty \tag{4.5.28}$$

for some positive constants $C, C', C''$.

□

We now give the proof of Theorem 1.2.6.

*Proof.* Without loss of generality, we only work with nonnegative functions $u_k \in C_0^\infty(M)$. Let $A(u_k) = 2^{-\frac{1}{n(n-1)}}||u_k||_{L^n(M)}$ and $\Omega(u_k) = \{x \in M : \ u_k(x) > A(u_k)\}$. Since $||u_k||_{W^{1,n}(M)} = 1$, $||u_k||_{L^n(M)} \leq 1$ and hence $A(u_k) < 1$. For the measure of $\Omega(u_k)$, we have

$$\mathrm{Vol}(\Omega(u_k)) \leq \int_M \frac{|u_k|^n}{A(u_k)^n} \leq 2^{\frac{1}{n-1}}. \tag{4.5.29}$$

We can then split the integral as following

$$\int_M \phi(\alpha_n p|u_k|^{\frac{n}{n-1}})dV_g \tag{4.5.30}$$

$$= \int_{\Omega(u_k)} \phi(\alpha_n p|u_k|^{\frac{n}{n-1}})dV_g + \int_{M\setminus\Omega(u_k)} \phi(\alpha_n p|u_k|^{\frac{n}{n-1}})dV_g \tag{4.5.31}$$

$$\leq \int_{\Omega(u_k)} \phi(\alpha_n p|u_k|^{\frac{n}{n-1}})dV_g + \int_{M\setminus\Omega_{u_k}} \sum_{j=n-1}^{\infty} \frac{(\alpha_n p)^j}{j!}|u_k|^n dV_g \tag{4.5.32}$$

$$\leq \int_{\Omega(u_k)} \phi(\alpha_n p|u_k|^{\frac{n}{n-1}})dV_g + \sum_{j=n-1}^{\infty} \frac{(\alpha_n p)^j}{j!}. \tag{4.5.33}$$

The second term of (4.5.33) is a constant. For the first term, define $\Omega_L^k = \{x \in \Omega(u_k) : \ |u_k| > L\}$, where $L > 0$ is some positive number left to be chosen. In $\Omega_L^k$, define $v_k = u_k - L$. Similar as the proof of Theorem 1.2.5, we have

$$\begin{aligned} & \int_{\Omega(u_k)} \phi(\alpha_n p|u_k|^{\frac{n}{n-1}})dV_g & (4.5.34) \\ = & \int_{\Omega_L^k} \phi(\alpha_n p|u_k|^{\frac{n}{n-1}})dV_g + \int_{\Omega(u_k)\setminus\Omega_L^k} \phi(\alpha_n p|u_k|^{\frac{n}{n-1}})dV_g & (4.5.35) \\ \leq & e^{\alpha_n p C(\epsilon) L^{\frac{n}{n-1}}} \int_{\Omega_L^k} e^{\alpha_n p(1+\epsilon)|v_k|^{\frac{n}{n-1}}} + \phi(\alpha_n p L^{\frac{n}{n-1}})\mathrm{Vol}(\Omega(u_k)). & (4.5.36) \end{aligned}$$

From (4.5.29), it suffices to show that $\sup_k \int_{\Omega_L^k} e^{\alpha_n p(1+\epsilon)|v_k|^{\frac{n}{n-1}}} dV_g < \infty$. Define $T^L(u_k) = \min\{L, u_k\}$. We first consider the case when $||u||_{W^{1,n}(M)} < 1$. We have

$$1 = \int_M |u_k|^n dV_g + \int_{\Omega_L^k} |\nabla v_k|^n dV_g + \int_M |\nabla T^L u_k|^n dV_g. \tag{4.5.37}$$

For any small $\epsilon > 0$, when $k$ is large enough, one has

$$\int_M |u|^n dV_g - \epsilon < \int_M |u_k|^n dV_g, \tag{4.5.38}$$

$$\int_M |\nabla T^L u|^n dV_g - \epsilon < \int_M |\nabla T^L u_k|^n dV_g. \tag{4.5.39}$$

From (4.5.37), we have

$$\int_M |u|^n + |\nabla T^L u|^n dV_g + \int_{\Omega_L^k} |\nabla v_k|^n dV_g - 2\epsilon < 1, \tag{4.5.40}$$

which implies

$$\frac{\int_{\Omega_L^k} |\nabla v_k|^n}{1 - \int_M |u|^n + |\nabla T^L u|^n dV_g + 2\epsilon} < 1. \tag{4.5.41}$$

For any $p < \tilde{M}_{n,u}$, we can pick $\epsilon$ small enough and $L$ large enough such that

$$p(1+\epsilon) < \frac{1}{(1 - \int_M |u|^n + |\nabla T^L u|^n dV_g + 2\epsilon)^{\frac{1}{n-1}}} < \frac{1}{(1 - ||u||^n_{W^{1,n}(M)})^{\frac{1}{n-1}}}. \quad (4.5.42)$$

It is not difficult to check that $||\nabla(p(1+\epsilon))^{\frac{n-1}{n}} v_k||_{W^{1,n}(M)} \leq 1$. From (1.1.23), we have $\sup_k \int_{\Omega_L^k} e^{\alpha_n p(1+\epsilon)|v_k|^{\frac{n}{n-1}}} < \infty$.

For the case when $||u||_{W^{1,n}(M)} = 1$, again we have the inequality (4.5.40). Since for any $\epsilon > 0$, there exists $L > 0$ such that

$$1 - \epsilon = ||u||^n_{W^{1,n}(M)} - \epsilon \leq \int_M |u|^n + |\nabla T^L u|^n dV_g. \quad (4.5.43)$$

Bring this back to (4.5.40) we have

$$\frac{\int_{\Omega_L^k} |\nabla v_k|^n dV_g}{3\epsilon} < 1. \quad (4.5.44)$$

Now for any $p < \infty$, we can hence choose $\epsilon$ small enough and $L$ large enough such that the following holds

$$p(1+\epsilon) < \frac{1}{(3\epsilon)^{\frac{1}{n-1}}}. \quad (4.5.45)$$

It is easy to verify that $||\nabla(p(1+\epsilon))^{\frac{n-1}{n}} v_k||_{L^n(M)} \leq 1$. Then from (1.1.23), we have $\sup_k \int_{\Omega_L^k} e^{\alpha_n p(1+\epsilon)|v_k|^{\frac{n}{n-1}}} dV_g < \infty$.

For sharpness, we recall the definitions of the sequence $\{u_k\}$ and $u$ in the proof of Theorem 1.2.5, and here we let $||u||_{W^{1,n}(M)} = \lambda < 1$,

$$u_k = u + (1-\lambda^n)^{\frac{1}{n}}\tilde{v}_k. \tag{4.5.46}$$

For the Dirichlet norm of $u_k$, we have

$$\int_M |\nabla u_k|^n dV_g = \int_M |\nabla u|^n dV_g + (1-\lambda^n). \tag{4.5.47}$$

On the other hand,

$$\int_M |u_k|^n dV_g = \int_M |u + (1-\lambda^n)^{\frac{1}{n}}\tilde{v}_k|^n dV_g. \tag{4.5.48}$$

Since $\tilde{v}_k \rightharpoonup 0$ weakly in $W^{1,n}(M)$, we have $\int_M |u_k|^n dV_g \to \int_M |u|^n dV_g$ and combine with (4.5.47), we know

$$||u_k||_{W^{1,n}(M)} \to 1. \tag{4.5.49}$$

Define the normalization of $u_k$ as $\tilde{u}_k = \frac{u_k}{||u_k||_{W^{1,n}(M)}}$, then for any $p = (1+\epsilon_0)\tilde{M}_{n,u}$, we have for $k$ large enough,

$$\int_M \phi(\alpha_n p|u_k|^{\frac{n}{n-1}})dV_g = \int_M \phi\Big(\frac{\alpha_n(1+\epsilon_0)|\tilde{u_k}|^{\frac{n}{n-1}}}{(1-\lambda^n)^{\frac{1}{n-1}}}\Big)dV_g \tag{4.5.50}$$

$$\geq \int_M \phi\Big(\frac{\alpha_n(1+\epsilon_0/2)|u_k|^{\frac{n}{n-1}}}{(1-\lambda^n)^{\frac{1}{n-1}}}\Big)dV_g \tag{4.5.51}$$

(since for $k$ large enough, $||u_k||_{W^{1,n}(M)}$ can be absorbed by $1+\epsilon_0$)

$$\geq \int_{B_P(\frac{1}{k})} \phi\Big(\frac{\alpha_n(1+\epsilon_0/2)|A+(1-\lambda^n)^{\frac{1}{n}}\tilde{v_k}|^{\frac{n}{n-1}}}{(1-\lambda^n)^{\frac{1}{n-1}}}\Big)dV_g \tag{4.5.52}$$

$$\geq \int_{B_P(\frac{1}{k})} \phi\Big(\alpha_n(1+\epsilon_0/2)|C+\tilde{v_k}|^{\frac{n}{n-1}}\Big)dV_g \tag{4.5.53}$$

$$\geq C''k^{-n}e^{(1+\epsilon_0/2)|C'+n^{\frac{n-1}{n}}(\log k)^{\frac{n-1}{n}}|^{\frac{n}{n-1}}}dV_g \to \infty \tag{4.5.54}$$

for some constants $C, C', C''$. The estimate of the last line is nothing different from (4.5.24). Hence the inequality (1.2.10) fails when $p > \tilde{M}_{n,u}$.

□

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