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Teachers' Navigation of Mathematical Representations in Argumentation

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Steven LeMay, Ph.D.

University of Connecticut, 2017

ABSTRACT

Mathematical argumentation has recently received more prominent attention in K-12 classrooms which has immediate consequences in the undergraduate mathematics classroom, including the critical intersection with representing mathematical concepts. Educators' perceptions of this intersection is important to understand as they have a significant impact on the skills undergraduates bring to their mathematics classrooms. This qualitative study investigated (1) how K-5 elementary educators conceptualized argumentation, (2) the role(s) and purpose(s) they attribute to representations within argumentation, and (3) the criteria for representations they use/offer when arguing claims of generality. Eight elementary educators participated in this study. Each completed two interviews and a classroom observation. The findings indicate that (1) arguments are to be produced in a prose format, they help support and learn mathematical content, and allow for differing perspectives; (2) there are no roles for representations within arguments but they were purposed to help navigate concepts involved in a claim statement as well as supplement arguments; and (3) representations called forth to determine the truth value of a claim of generality are relevant to the claim but are not universal instantiations of those

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relevant ideas, and thus are unwarranted. Some implications of this research upon K-16 mathematics teaching and learning are discussed. This study's findings contribute to the literature about mathematical argumentation in the classroom, its relationships with representing mathematical concepts, and how elementary educators perceive both. Future lines of research to strengthen this area are offered.

Teachers' Navigation of Mathematical Representations in Argumentation

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Doctor of Philosophy

at the

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Steven LeMay

2017

APPROVAL PAGE

Doctor of Philosophy Dissertation

Teachers' Navigation of Mathematical Representations in Argumentation

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Chapter 1

Introduction

1.1 Motivation

Since the beginning of the twenty-first century, K-12 mathematics education in the United States has seen vital shifts in curricular thinking. One of these shifts occurred in the area of argumentation. In 2010, the National Governors Association Center for Best Practices, Council of Chief State School Officers (NGA-CCSSO) released the Common Core State Standards (CCSS-M, 2010b), with the intent to provide a common set of standards for all states (these were adopted by Connecticut in 2010). These standards were created by a collaboration of state education officials, educators, mathematicians, and parents in an effort to define standards addressing all parties' concerns (CCSS-M, 2010b). The CCSS mathematical objectives are divided into content standards and practice standards. The practice standards call attention to “varieties of expertise that mathematics educators at all levels should seek to develop in their students” (CCSS-M, 2010b, Standards for Mathematical Practice, para. 1).

Focusing on these varieties of expertise at the national level was new and had not previously received much attention in the classroom (Stanic & Kilpatrick, 1992). There are eight practice standards ranging from making sense of problems and persevering in finding solutions to searching for patterns in repeated reasoning. The third practice standard states that students should “construct viable arguments and critique the reasoning of others” (CCSS-M, 2010b). With this, the NGA-CCSSO highlight the need for students to engage in both writing and critically evaluating mathematical arguments. This recommendation is a significant development for school mathematics; traditionally, high school geometry was the only venue to discuss arguments and proof.

The mathematics education community has provided a strong impetus to include argumentation in the Common Core mathematical practice standards, especially over the last twenty years. For example, the National Council of Teachers of Mathematics (NCTM), a nationally renowned organization focused on K-12 mathematics pedagogy, highlight proof and reasoning as essential skills for students to master (2000). They emphasize that “[b]eing able to reason is essential to understanding mathematics. By developing ideas, exploring phenomena, justifying results, and using mathematical conjectures in all content areas and...at all grade levels, students should see and expect that mathematics makes sense” (NCTM, 2000, p. 56). The nature of mathematics, in all its richness and diversity, can be uncovered through the lens of argumentation. The National Research Council (NRC), a federal organization designed to promote research in education, published a report (NRC, 2001) laying out the mathematics for which K-12 education should be responsible. This landmark document is based on advancements in educational, cognitive, and brain research. It proposes a framework of five interwoven strands that together help students acquire mathematical profi-

ciency. In particular, one of these proficiencies is adaptive reasoning, where students justify work and prove conjectures. As the authors note, “[s]tudents need to be able to justify and explain ideas in order to make their reasoning clear, hone their reasoning skills, and improve their conceptual understanding” (NRC, 2001, p. 130; see Maher and Martino, 1996). Thus, argumentation is foundational for students to attend to the acts of justifying and explaining.

The Common Core State Standards (2010) emphasize reasoning as an important pathway to understand the nature of mathematics and to create authentic mathematical experiences. Both of these activities are viewed as crucial to learning mathematics by major organizations (NCTM, 2000; NRC, 2001). This emphasis aligns with work at the international level (Mamona-Downs & Downs, 2015) where the act of proving is seen as a paramount activity in mathematics. For example, Greek researchers Mamona-Downs and Downs (2015) establish that highlighting mathematical structure is important for student learning and that proving mathematical results is a vital venue to promote such learning. Not only is the act of proof helpful for gaining mathematical knowledge, but such knowledge can be extracted from the proof itself (e.g., Canadian researchers Hanna and Barbeau, 2008 and French researcher Rav, 1999). Students thus benefit from argumentation in terms of understanding mathematical structure and content.

The vital role argumentation plays toward the path of proof can be seen in students who have fundamental issues with the act of proving while having not been exposed to argumentation. In some instances, students find writing proofs onerous; the propositions to be proved in the classroom have already been established by the mathematics community. In these cases, the students have difficulty building an appreciation for discovering the rationale behind propositions and the mathematical

structures that support them (Mamona-Downs & Downs, 2015). In other instances, the statements that are to be proved seem obvious and not worth the effort to prove. This occurs particularly often in problems that state “prove” and not “prove or disprove.” Aside from these perceptual issues, K-16 students have difficulties formulating proofs. At times, the formal mathematical language is difficult for them to untangle (A. Selden & Selden, 2003; J. Selden & Selden, 1995). Also, most students have the generally false belief that developing a proof is a linear process. It is their impression that specific steps need to be recalled (Mamona-Downs & Downs, 2015), which agrees with their view of school mathematics in general (Schoenfeld, 1989).

Given the current momentum from the Common Core State Standards (2010b) across most American states to implement argumentation as a method to develop reasoning and justification skills for students at the K-12 level, it is necessary to understand how the dynamics of teaching and learning mathematics are affected by the introduction of argumentation, especially in light of the fact that uncovering mathematical knowledge is powerfully done in this context (NCTM, 2000; NRC, 2001). How are students’ mathematical content knowledge affected when they more actively argue? For undergraduate institutions, these curricular efforts deserve particular attention: How will the emphasis on argumentation impact the preparation of incoming undergraduate college students? Success in the field of mathematics is largely dependent upon the ability to prove conjectures; what skills will students exposed to argumentation bring with them to college? What benefits will there be to undergraduate mathematics programs with respect to teaching and learning? The potential opportunities are worthwhile and invite investigation.

1.2 Problem Statement

Because of the recent emphasis on argumentation in K-12 mathematics, it is imperative that we understand how educators take hold of these ideas, both with respect to teaching and learning. Educators are typically trained in mathematics content and are less likely to be trained in how to produce and critique mathematical arguments. Similarly, students likely will be learning about argumentation in new ways compared to their older peers.

This focus on argumentation in K-12 curricula has the potential to alter the K-16 trajectory toward teaching proof at the undergraduate level. We would like undergraduates to be able to justify results and to explain one's reasoning. Both of these skills are related to argumentation. What aspects of current K-12 classroom instruction will influence how students approach undergraduate mathematics? It is thus meaningful to first explore how teachers understand the process of argumentation. Moreover, given the centrality of mathematical representations in all areas of mathematics and especially in the argumentation process, it is important to investigate how teachers understand and use mathematical representations. A natural place to begin investigating these questions is with K-5 educators, as they are strategically positioned in the formative years of students' development of mathematical ideas.

This study will focus on how K-5 educators navigate mathematical representations when producing and critiquing arguments. Indeed, this population of educators plays a vital role in student learning of the argumentation process and the representations used to manifest arguments. Understanding how teachers make sense of these ideas is a critical step toward understanding how students handle these topics as well. Students' mastery of these skills impacts undergraduate mathematics educa-

tion when these discussions are seen in the context of the K-16 trajectory. This vital intersection of mathematical representations and argumentation offers fertile ground for investigation.

In this study then, we seek to understand K-5 educators' conceptualizations of mathematical argumentation, the possible roles and purposes this population attributes to representations within the context of argumentation, and the criteria representations have when used to argue claims involving generality.

1.3 Definitions

The following important definitions will be used throughout this study. To avoid ambiguity or confusion with other views of these terms, they are explicitly defined here.

In the literature, there are few explicit definitions of an argument in the mathematical sciences community. There are some attempts to juxtapose the idea with proof (Pedemonte, 2007; A. Selden & Selden, 2003). The working definition throughout this study will follow the one we used at a professional development program for K-12 educators designed to examine the third practice standard of the CCSS (2010b). Notice that no assumptions are made in this definition on the mathematical correctness of the statements nor the rigor to which the demonstration is made. Specifically, we define a mathematical argument as follows:

Definition. A *mathematical argument* is a sequence of reasoned statements that are provided with the aim of demonstrating the truth value of a claim.

Mathematical representations are staple items for any mathematician for they are

the means to which we grapple with mathematical ideas and help convey those ideas to others. Representations used to convey such ideas include (and are not limited to) pictures, graphs, and symbols. Goldin (1998), in his treatise on representational systems, furnishes the definition for this study:

Definition. A *mathematical representation* is any object with an implied structure that “*encode[s]*, *evoke[s]*, *produce[s]*, *stand[s] for*, *represent[s]*, or *symbolize[s]*” (Goldin, 1998, p. 144) a mathematical idea.

For the purposes of this study, the term “elementary educator” takes on a broader context than the typically inferred meaning of a general classroom teacher. In order to investigate the understandings of argumentation and mathematical representations in the elementary grades, the scope is widened as follows:

Definition. School educators at the K-5 level who either directly teach children or are involved in supporting teachers’ professional development are considered *elementary educators* for this study.

So not only do we include K-5 classroom teachers in the category of “elementary educator,” but also numeracy coaches and interventionists. Numeracy coaches support classroom teachers’ pedagogy around mathematics and interventionists work more closely with individual students (in various grades) whose mathematical fluency is desired to be improved. These specific support personnel, along with K-5 classroom teachers, constitute the sample of this study and thus are included in the definition of elementary educator.

Chapter 2

Relevant Literature and Theoretical Framework

Because of the increased focus on argumentation in K-12 mathematics classrooms (CCSS-M, 2010b), there is great interest in understanding how teachers conceptualize it. Furthermore, how argumentation is manifested in the classroom is an important line of investigation. In particular, how do teachers see the roles and purposes of mathematical representations with respect to argumentation? Understanding how teachers make sense of argumentation and its place within mathematics naturally affect how students learn argumentation and its intersection with mathematical concepts. In this chapter, I provide the relevant literature pertaining to mathematical argumentation, mathematical representations, as well as teachers' impact on student learning. Together, these provide the theoretical framework for this study.

2.1 Mathematical Argumentation

Proof is an essential activity in the quest for mathematical knowledge. It is indeed a bedrock of the mathematical discipline (Hadamard, 1945; Knuth, 2002; Mamona-Downs & Downs, 2015; Ross, 1998; Staples, Bartlo, & Thanheiser, 2012; Tall, 1995). Proof is designed to not only illustrate the truth or falsity of a conjecture, but can at times elucidate “why” the proposition is true or false (Knuth, 2002). Tall (1989) presents two essential elements to mathematical proof: “it requires *clearly formulated definitions and statements*, and the other is that it requires *agreed procedures to deduce the truth of one statement to another*” (p. 5; see also Fischbein (1982)). Mathematics, unlike other scientific disciplines, concerns itself with knowledge of truth without the need to replicate results (Fischbein, 1982). Once a statement is proven, it need not be considered an open question. There will be no underlying factors that change its truth value. Proofs have other functional roles in the mathematics community as well. For example, proofs provide a method of communicating mathematics to mathematicians (Alibert & Thomas, 1991). In a classroom setting, the proof-making process has been described as socially constructed (Bell, 1976; Davis, 1986; Yackel & Cobb, 1996).

Argumentation can be seen as an avenue in the activity to proof (Schwarz, Hershkowitz, & Prusak, 2010). Pedemonte (2007) provides functional commonalities between argumentation and proof; in particular, they both present themselves as rational justifications designed to convince an audience. However, a key distinction between them is that proof is a specific form of argumentation: Whereas an argument primarily refutes or affirms a claim, a proof provides the deductive logic required to attain the truth or falsity of a statement (Pedemonte, 2007). Weber, Alcock, and

Radu (2005) describe another distinction, which is that the level of sophistication in a proof is greater than that of an argument; when going about the act of proving, one must necessarily focus on particular structures inherent in mathematical logic, definitions, and axioms (A. Selden & Selden, 2003). In spite of the fact that there are no definitions for argumentation in the mathematical sciences community (Pedemonte, 2007), there seems to be an implicit definition involving meaning-making (Schwarz et al., 2010).

In the mathematics education community, there exists a variety of definitions for argumentation, depending upon the context in which it is studied. As a social activity (Krummheuer, 1995; Sfard & Kiernan, 2001), argumentation is viewed as a productive discourse, though this has predominantly led to a subjective characterization for researchers (Wagner, Smith, Conner, Singletary, & Francisco, 2014). NGA-CCSSO (CCSS-M, 2010b) characterizes argumentation as a skill set including “creating conjectures; reasoning using stated assumptions, definitions, and previous learning; and determining the domain for which an argument applies” (Wagner et al., 2014, p. 8). Areas of research such as proof and reasoning (Staples et al., 2012; Wagner et al., 2014) incorporate elements of what argumentation has now grown into; argumentation has recently come into the limelight of research.

Taking into account the breadth of definitions for argumentation afforded by the mathematics education community, we adopt the following definition: *A (mathematical) argument is a sequence of reasoned statements with the aim of demonstrating the truth value of a claim.* The motivation for this definition primarily comes from a prior collaborative effort with colleagues in mathematics education providing professional development to K-12 educators on argumentation. Given my background in mathematics, it is my opinion this definition harmonizes to a degree with proofs undertaken

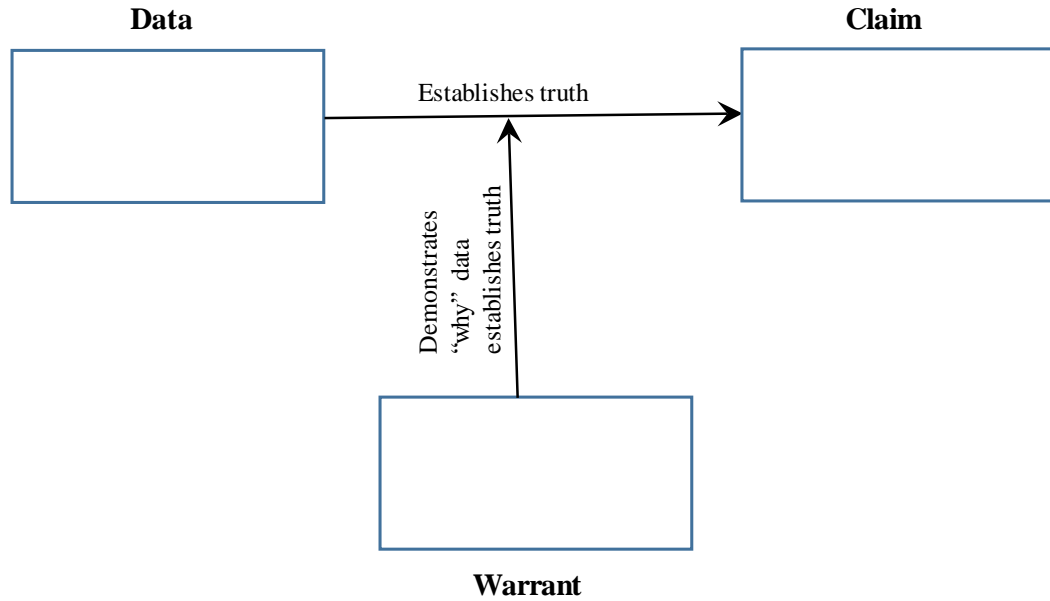
in undergraduate mathematics classrooms. This coupled with the meaning-making associated with argumentation from the mathematics community (Schwarz et al., 2010) prompted me to adopt this definition.

Toulmin (1958/2003) provides a framework for argumentation which I will use for this study. Henceforth known as Toulmin's model, his framework was set out originally to be "strictly philosophical: to criticize the assumption made by most Anglo-American academic philosophers, that any significant argument can be put in formal terms" (Toulmin, 1958/2003, p. vii). It has since been widely adopted by the mathematics education community (Hollebrands, Conner, & Smith, 2010; Krummheuer, 1995; Pedemonte & Reid, 2011; Wagner et al., 2014; Weber & Alcock, 2005) to discuss mathematical arguments.

Following Toulmin's (1958/2003) model, Wagner et al. (2014) and Weber and Alcock (2005) present the three components of argumentation as data (or evidence), warrant, and claim (Weber and Alcock (2005) refer to the latter as a conclusion). The claim is a mathematical statement whose truth value is to be decided. Data is presented in support or refutation of the claim. In order to establish the truth/falsity of the claim from the data, a warrant substantiates how the data decides the claim (J. Selden & Selden, 1995). In other words, warrants provide the justification that the data indeed imply the claim. Refer to Figure 2.1.1 for an illustration of Toulmin's model.

Toulmin's model has served as a methodological tool to investigate characteristics of argumentation (Forman, Larreamendy-Joerns, Stein, & Brown, 1998; Krummheuer, 1995; Pedemonte, 2007; Pedemonte & Reid, 2011; Wagner et al., 2014; Weber & Alcock, 2005). It has been used to study inquiry-based classrooms investigating arguments formed by groups of students. For example, Krummheuer (1995) introduced

FIGURE 2.1.1: Toulmin’s model for argumentation



the idea of collective argumentation using Toulmin’s model. The model has also been used to investigate educators’ explicit use of fostering collective argumentation (Forman et al., 1998) and how to engender its use among secondary preservice educators (Wagner et al., 2014). Focusing on arguments themselves, Pedemonte (2007) uses Toulmin’s model as a point of comparison in the structural comparisons of argumentation and proof.

Researchers focused on investigating aspects of mathematical proof have also used Toulmin’s model as a methodological tool. For example, Pedemonte and Reid (2011) implement it to analyze different abductive forms of proof. Weber and Alcock (2005) apply the model to understand the nature of warranted implications in proof, i.e., if an implication’s conclusion follows mathematically from its antecedent. In like manner, Toulmin’s model acted as a methodological tool to service the analysis of

participants' arguments in this study as discussed in more detail in Chapter 4.

2.2 Mathematical Representations

Mathematical representations are a staple for any mathematician as they are the means by which we grapple with mathematical ideas and convey these ideas to others. Representations include (and are not limited to) pictures, graphs, and symbols. Their use is ubiquitous; in fact, it is nearly impossible to be exposed to mathematics in school in the United States and not discuss the idea of mathematical modeling, where naturally-occurring phenomena are reflected as equations with variables predicting a desired unknown outcome. Mathematical topics are represented through a negotiated system of symbols between student and teacher, through a complex interaction of internal and external systems, and this is well-documented in the literature (Cobb, Yackel, & Wood, 1992; Goldin, 1998, 2002; Janvier, 1987; Pape & Tchoshanov, 2001). A bidirectional process is involved: On the one hand, mathematical ideas that are mentally constructed (internal representations) are externally documented (as external representations). On the other hand, external representations of ideas mediated by another (in particular concepts unfamiliar to the consumer) are interpreted and then internally represented. This latter direction is known as the representational view of mind (Cobb et al., 1992). This bidirectional process is mutually informative (Pape & Tchoshanov, 2001) in terms of interpretations: an external representation influences one's understanding of a concept and how to mentally image it (internal representation) and vice versa.

Goldin (1998) provides a systematic perspective on representations. He couches them within the construct of *representational systems* which is composed of *signs* and *configurations*. Examples of signs can be mathematical symbols, words in spoken and written form, grammatical syntax, and abstract mathematical ideas (such as numbers and vectors). However, this list is not exhaustive and it appears from Goldin's 1998 work that the set of signs is uncountably infinite. In it, Goldin (1998) describes signs with respect to well-defined entities where a representation's sign(s) is/are selected from a well-defined (possibly community established) set of objects. Along with signs is the notion of configurations, which are taken to be combinations of signs formed meaningfully via a set of rules. Such examples provided by Goldin are letters of an alphabet (signs) forming words (configurations), words (signs) forming sentences (configurations), digits (signs) forming multi-digit numbers (configuration), and numbers and operational signs forming mathematical expressions. Together, signs and configurations form a representational system (inheriting the rules of configurations).

Given this conception, having representational systems does not come equipped with much structure. However, Goldin (1998) admits more structure typically exists with the ability to transfer within and across systems. For example, consider the equation $5x^3 - 7x^2 - 7 + 5x = (5x - 7)(x^2 + 1)$. The equality of the two expressions is established from rules of algebra, transferring the representation $5x^3 - 7x^2 - 7 + 5x$ into $(5x - 7)(x^2 + 1)$. This is an example of a transfer within a representational system (operational symbols of algebra with properties of equality). An example of a transfer between systems is to consider the graph of the function $f(x) = 5x^3 - 7x^2 - 7 + 5x$. In this case we move from the aforementioned representational system to another, whose signs include coordinate axes and curves and whose configurations include graphing functions. These systems are different but related through this idea of transfer. To

illustrate further, Weber and Alcock (2009) use propositional calculus: the signs include letters and logical operators, the configurations include forming well-defined formulas, and rules of inference allow for the transfer from one well-defined formula to another.

The use of external representations has been closely examined by the mathematics education community and has taken its place as a crucial part of mathematical understanding (Pape & Tchoshanov, 2001). National organizations (e.g., Association of Mathematics Teacher Educators [AMTE], 2017; NCTM, 2000) recognize the influence representations have on learning. Indeed, NCTM, 2000 calls out representation as one of their process standards to have students create and cogently use them in theoretical and applied contexts. AMTE calls on preservice educators to “effectively use representations and technological tools appropriate for the mathematics content they will teach” (AMTE, 2017, p. 9). Among those representations investigated by practitioners are tables, graphs (Baltus, 2010) and strip diagrams (Cohen, 2013) to understand proportions, various representational understandings of fractions (Hunter, Bush, & Karp, 2014) and algebraic problems (Neria & Amit, 2004), to name only a few.

Researchers have also investigated students’ and teachers’ use of representations. Nistal, Van Dooren, Clarebout, Elen, and Verschaffel (2009) provide a critical review of students’ flexible representational choices with a focus on factors contributing to student success when using mathematical representations. diSessa and Sherin (2000) describe innate skills students have regarding representation (called meta-representational competence) along with those that can be learned. Multiple external representations (Ainsworth, 1999), using more than one distinct (external) representation within a given problem context, has been examined by Ainsworth (1999) with

respect to its affordances in classroom instruction who developed a taxonomy of such benefits. Morris (2009) also highlights the benefits of implementing particular representations that call attention to generalized mathematical ideas. Research on teachers' conceptions of representation has focused on pedagogical implications for student learning and not on the knowledge itself. Interviewing teachers and examining student work, Bergqvist (2005) investigated teacher beliefs of student achievement in mathematics. In a professional development setting, Taylor and Dyer (2014) investigated the goals teachers have in teaching mathematical representations as well as the dilemmas they face when teaching them; and as in Bergqvist (2005), they focus on implications of teachers' pedagogical intent in the classroom. "Teachers' conceptions of rRepresentation" (n.d.) investigated teachers' conceptions of representation through the lens of student problem-solving.

Representations provide a valuable currency to the mathematical sciences community to communicate about mathematics and it is very important that students recognize their importance. Consequently, it is also important to see how students use representations. One prominent example in that direction comes from Schifter (2009), who studied ways elementary students investigated conjectures involving generalized properties about objects in an infinite set. Observing the different ways students approached the task, she found some students using representations to decide the conjecture's truth value. From these observations she uncovered three criteria that all such representations should have:

- S1.** The meaning of the operation(s) involved is represented in diagrams, manipulatives, or story contexts.
- S2.** The representation can accommodate a class of instances (for example, all whole

numbers).

S3. The conclusion of the claim follows from the structure of the representation.
(Schifter, 2009, p. 76)

If an argument contained representations satisfying S1-S3, Schifter referred to it as a representation-based proof. The illustrative example Schifter provides of these criteria is based on arguments related to the claim that the sum of two even positive integers is even. A possible representation satisfying S1 could be to use base ten blocks (of the same type) put in pairs. The sum of two even numbers could then be signified as pairs of blocks put together. This would satisfy S2 since one could readily represent any two positive integers in this way. Lastly, S3 is satisfied as well: when putting pairs of base ten blocks together, one will only have pairs of blocks as the sum, thereby signifying an even positive integer.

Schifter goes on to draw parallels between this representation-based proof and a rigorous proof produced by a mathematician: By letting $m = 2k$ and $n = 2l$ for some $k, l \in \mathbb{Z}^+$, the person is invoking universal instantiation of even positive integers, which is analogous to the pairs of blocks for each addend. Considering $m + n = 2k + 2l = 2(k + l)$, the sum is taken and, by calling attention to the fact that addition is closed under \mathbb{Z}^+ , the person concludes the sum is even. This step is analogous to putting the pairs of blocks together and noticing that the resulting number can also be put in pairs. An elementary student likely does not have access to the level of sophistication Schifter (2009) cites, but the explanatory power of the representation-based proof is just as powerful as a mathematician's proof. In this study, these criteria S1-S3 were used as methodological tools to investigate how teachers responded to a claim involving generalized properties of objects in an infinite set. This is explained

in detail in Chapter 4.

2.3 Teacher Influence on Student Learning

The participants in this study are elementary educators. These professionals exert a great deal of influence upon student learning (Box, Skoog, & Dabbs, 2015), so it is prudent to examine this relationship. Indeed, this influence has been found to outweigh classroom and socioeconomic conditions (Fishman & Davis, 2006). These influential factors are internal (their own personal beliefs) and external (physical factors, curriculum design, etc.) (Shulman, 1986; Smith & Southerland, 2007). Teachers' epistemological beliefs affect how they frame topics (Mansour, 2013), thereby affecting their students' perceptions of them (Box et al., 2015). The contextual factors that play a role in teachers' daily decisions such as what content to deliver and how (Box et al., 2015), coupled with their abilities to prime the learning environment (William & Leahy, 2006) can have significant repercussions on students' understandings; these understandings meaningfully mold their students' knowledge schema.

Various beliefs teachers hold about mathematics have powerful consequences on how their students perceive it. As an example, Battista (1994) notes that teachers' emphasis of mathematics in the classroom inhibits the full grasp of what mathematics has to offer when it is based in computations as opposed to balancing computations with other activities such as problem-solving or generalizing. In addition, Stein, Smith, Henningsen, and Silver (2000) discuss how mathematical tasks steeped only in computation damage not only students' views of mathematics but their inability to handle cognitively demanding tasks. Ertmer, Ottenbreit-Leftwich, Sadik, Sendurur,

and Sendurur (2012) echo Battista's (Battista's 1994) concern over the emphasis of computational fluency; in their study of teachers' use of technology in the classroom, they observed in some instances teachers incorporating technology by allowing students to independently practice computational fluency using computer programs specifically designed with this in mind. In these cases, when the teacher is not more proactively involved with the students' learning, students can be quick to conclude that mathematics's primary concern is computational fluency.

Given the lack of exposure students had with argumentation prior to the Common Core (CCSS-M, 2010b), understanding teachers' conceptions of argumentation will provide powerful insights upon students' knowledge and what skills they bring when undertaking future tasks when arguing as well as regarding argumentation's natural conclusion of proving.

In science education, Czerniak, Lumpe, and Haney (1999) investigated a similar issue: classroom teachers had several preconceived notions on what would make a new approach to teaching science successful and unsuccessful, highlighting the vital role teachers play in conveying argumentation meaningfully. Feyzioğlu (2012) noticed how, when seemingly adopting new methods of delivering content, teachers returned to previous approaches. These studies show how critical it is that we gain clarity into where teachers are with respect to their own understandings so that we can better serve them, and in turn, the students.

Chapter 3

Research Questions

The momentum currently seen toward argumentation in the K-12 classroom provides a natural pathway to consider how argumentation is conceived along the K-16 trajectory, leading into the mathematical notion of proof. Furthermore, it is important to investigate how educators perceive arguing in mathematics as this has ramifications for the student population, in particular what skill set students bring to their undergraduate mathematics classrooms.

With these considerations in mind, this study is guided by the following research questions:

RQ1. How do elementary educators conceptualize argumentation?

RQ2. What role(s) and purpose(s) do elementary educators associate with representations in the context of mathematical argumentation?

RQ3. What criteria for representations do elementary educators use/offer when arguing general claims?

When investigating educators' conceptualizations of argumentation, I desired to understand what they consider attributes of a mathematical argument. With respect to the roles and purposes of representations, I considered the roles to specifically refer to the function(s) the representations have in servicing the argument, whereas I considered the purposes to specifically refer to the reason(s) the educators had in using them. Lastly, the criteria they offer in arguing general claims were investigated with respect to qualities their representations possessed and the degree to which they effectively substantiated the claim.

In Chapter 4, I outline the study design and methods used to answer the research questions above (RQ1-RQ3). The research design includes an overall description of the participants, the tools used to collect data, the data that was collected, and the methods employed during data analysis to address the questions. In Chapter 5, I describe the findings in response to these research questions. The material is presented with respect to each question and illustrative examples from the data is used to support the findings.

Chapter 4

Methods

This chapter is dedicated to outlining the aspects of research design, data collection, and data analysis in answering this study's research questions (cf. Chapter 3). In the following sections, I discuss each respectively.

4.1 Research Design

This study employs a basic qualitative design (Merriam, 2009). An assumption when conducting qualitative research is that participant meanings are socially constructed (Grbich, 2013). Mathematical activity in the classroom setting has a social dimension (Yackel & Cobb, 1996). Furthermore, I was interested in understanding teachers' conceptualizations of mathematical argumentation and what their roles and purposes for representations within argumentation were. These are not objectively derived, but rather are constructed by individuals, which warrants a qualitative investigation (Grbich, 2013).

4.2 Data Collection

After an Internal Review Board approval to conduct this research, qualitative data were collected to answer this study's research questions. In the following subsections, I describe the participants as well as the data collected.

4.2.1 Participant Context

The participants chosen for this study are elementary educators working within a district of the state of Connecticut. Of the eight educators, six teach in a classroom setting. The grade levels taught range from kindergarten to fifth (excluding fourth). Of the remaining two participants, one is the numeracy coach for a school and the other is an interventionist for the same school. The numeracy coach provides mathematics instructional support and professional development to teachers. The interventionist works with students throughout the grades who are identified as in need of additional mathematics instruction. All but one of the participants come from the same school.

Recruitment began by purposefully sampling teachers known to the researcher. These teachers suggested names of other teachers who might be willing to serve as participants. Interested teachers were e-mailed a consent form detailing the study procedures as well as a photo-video release form.

4.2.2 Data Descriptions

Event	Data Collected
Packet (Appendix A)	Participant Responses
Interview 1 (Appendix C)	Interview Responses Video Notes
Interview 2 (Appendix D)	Interview Responses Video Notes
Classroom Observation (Appendix E)	Field Notes

The table above highlights the different methods used to acquire data and what data were collected. The first three events (the packet and Interviews 1 and 2) were in sequential order. The last item in the table, classroom observations, were not necessarily done in sequential order. The following subsections detail each event in turn.

Packet

The packet contains two mathematical problems designed for participants to provide an argument (see Appendix A). The first describes two students providing an explanation behind their answer to the division $\frac{1}{0}$. Participants were asked to (1) choose which person's explanation they agree with and (2) explain why their choice

is correct. For analytical and reporting purposes, this problem will be referred to as “the division problem.” The second problem is about Damarcus’ claim that a triangle can be made whose total angle measure exceeds 180° . The participants were asked to (1) declare whether or not they agree with Damarcus and (2) convince another student if Damarcus could do it or not (depending on their declaration in part (1)). This problem will be referred to as “the triangle problem” for analytical and reporting purposes.

Participants reported taking between 5 and 20 minutes to complete the packet whenever it was convenient to them. I was not present when they were completing the problems. The arguments they provided are data points for this study, and these were part of Interview 1, which is described next.

First Interview

Interview 1 was semi-structured as some participant responses required follow-up questioning (Wengraf, 2001) to confirm inferences I made while interviewing. The interview questions were designed to elicit teachers’ use of representation in a mathematical argument and to uncover associations teachers made when selecting mathematical representations (see Appendix C). The interview can be described as having two parts. In the first part are questions directly asking about their own understandings of argumentation and representations. The other part pertained to their own responses to the packet mentioned above. In particular, each were asked to highlight the representations they made and discuss their choices. In addition, various solutions to the problems in the packet (created by me) were discussed with the participants as indicated in the interview protocol (these can be found in Appendix B). These

solutions contained different arguments for each problem with varying amounts of representations.

Given the visual nature of external configurations, both this interview and Interview 2 were videotaped, allowing me to focus on the participant's answers, rather than on keeping records of the different representations discussed during the interview. The video recorded visual references that would not be effectively captured by audiotape.

Second Interview

Interview 2 was scheduled after Interview 1 took place. I conducted a semi-structured interview (to allow for the ability to answer follow-up questions (Wengraf, 2001) so that inferences made by me could be confirmed) centered on their appraisals of mathematical representations selected student work samples (see Appendix D). Participants were asked to supply blinded student work samples which included students' arguments. Similar to Interview 1, Interview 2 can be thought of in two parts. In the first part, questions from the first part of Interview 1 are echoed to document any possible changes in the participants' meanings of mathematical argumentation and representations. The other part focuses on the student work samples and questions include asking what the argumentation piece is, what are the students' representations, and related questions on the participants' meanings of the students' representations.

This interview was also videotaped, allowing me to focus on the participants' answers, rather than on keeping records of the different representations discussed during the interview. The video recorded visual references that would not be effectively cap-

tured by audiotape.

Classroom Observation

I observed each participant during one mathematics lesson incorporating argumentation. For this observation, teachers were asked to have students engage in argumentation for at least one part of the lesson. No other instructions were provided except for minor clarifications (e.g., argumentation does not have to be an objective of the lesson).

Teachers were videotaped during parts of the lesson when the teacher engaged with students in argumentation. If the observation took place after either interview, I attended to times when the teacher likely (in the researcher's opinion) thought they were engaging in argumentation based upon their responses during the interview(s). As well, specific language centered on Toulmin's (Toulmin, 1958/2003) model such as claim, warrant, and/or evidence was documented as an instance of arguing (see 2.1 for a description of this model). Since the focus of this observation was on the teachers' engagement with representations in arguments, the students were not directly videotaped; however, interactions with students were audio recorded for the teachers' responses. Parental notification forms were sent by the teacher to parents/guardians detailing that this research study was being conducted in their classroom and that they had the option to not allow their child to be on videotape during the observation.

During the observation, I used a protocol to document incidents of argumentation during the lesson (see Appendix E). There are two main parts of this protocol. First, while the observation was taking place, I took salient field notes in corresponding boxes. "Argumentation Task Types" details how the argumentation was presented to

students. The categories are students' construction of an argument/proof in response to a claim, identifying a misconception in a given argument, dis/agreeing with a given argument, or critiquing an argument. Space is provided for inserting other types that are not listed that are observed. "Teacher Emphasis in Argumentation" documents which, if any, elements of argumentation the participants particularly focus on or use with students. Suggested categories are the format/structure of arguments, the vocabulary used, the mathematical representations that go into arguments, and the proposed length of arguments. As in "Argumentation Task Types," there is space to insert other elements that occur during the observation. Similarly, "Teacher Emphasis in Representations" categorize which representations the participant uses with students when engaging in argumentation. Suggested categories are symbols, drawings, graphs, tables, words, as well as "other" if a particular representation does not conform to any of these representations. An additional section under "Teacher Emphasis in Representations" labeled "Final Reps?" is to document whose representation (the student's or teacher's) is ultimately used when the interaction is over.

4.3 Data Analysis

Data described in the preceding section were analyzed using different qualitative analytical methods appropriately to address the research questions (see 4.2.2. Specifics of the analyses are presented in the following subsections presented by research question.

4.3.1 Analyses to Address RQ1 and RQ2

In order to answer these research questions, an inductive analysis of coding, categorizing, and themeing was conducted. Inductive analyses of qualitative data are conducted when participants' voices are desired to be heard as it is assumed they have the authority on the topics of investigation (Grbich, 2013). This is indeed the case for research questions RQ1 and RQ2; it is *their* conceptualizations of argumentation as well as *their* roles and purposes of representations that are being investigated. The data analyzed to address these questions were participant responses to the packet problems and the Interviews 1 and 2.

Two levels of analysis were conducted with the packets, one when each participant completed the packet before Interview 1, and the next occurred after Interview 1. Prior to Interview 1, the responses were analyzed to identify the participant's arguments, as well as the specific mathematical representations provided in the responses. In particular, the analysis focused on how the (perceived) representations were used, both in themselves and with respect to their argument to ascertain what the purpose(s) was/were. This allowed me to familiarize with the data and to have an initial impression of the responses before interviewing the participant.

After Interview 1, the video was analyzed in places where the participants drew direct attention to their representations. In those moments in the video, I holistically coded (Dey, 1993; Saldaña, 2013) for what the role(s)/purpose(s) was/were for their representations. Holistic coding allowed me the opportunity to focus on the representations as "self-standing units of data" (Saldaña, 2013, p. 142) so that the representations could be analyzed separately from the interviews themselves. This was particularly crucial in responding to research question RQ2 to determine the par-

ticipants' role(s) and purpose(s) of the representations. Since the packet responses' representations were also part of the interview, they were coded again in the process of analyzing the video. That is, at times the representations were multiply coded.

The analysis of the videos of Interviews 1 and 2 began with a preliminary review of the videos in order to familiarize myself with the data and to provide a catalog of events to aid further analysis (Heath & Luff, 2010). This catalog included documentation of events (e.g., "always want [students] to do arguments" and "[representations] as self-check") that transpired as well as time stamps. Intermittently, there were moments during the interview where I deemed it necessary to write (at most a few) words that served as an analytic memo (Grbich, 2013; Saldaña, 2013), reflecting on the moments I captured. Once the cataloging was complete, I viewed the videos again and coded the interviews using a process coding scheme (Saldaña, 2013). Process coding looks for participant action, whether observable or conceptual, and for the purposes of this study, process coding was useful in identifying what the participants did with representations and the ways they constructed arguments. The process coding was conspicuously helpful in answering research questions RQ1 and RQ2; by discovering the actions the participants make with their representations and arguments. This enabled me to find the participants' meanings of arguments and representations as well as the role(s) and purpose(s) attributed to their representations.

Once the coding process was complete, a thematic development of the codes (Grbich, 2013) derived the results of this study (described in Chapter 5). This development is a standard technique when conducting qualitative studies as results are to come from the data and not from preconceived notions applied by the researcher (Grbich, 2013; Merriam, 2009). The created codes were categorized by looking at their commonalities and how they were related (or unrelated) to each other. These

categories were cogently named (or renamed during construction) to express a quality unifying all the codes in that category. To move from categories to themes, a similar process was repeated where similar categories were unified. The resulting themes are described in sections 5.1 and 5.2.

4.3.2 Analyses to Address RQ3

Findings for this research question were drawn from an analysis of the packet responses, in particular to the triangle problem (see Appendix A). In it, several participants believed the claim to be false: It is not possible to draw a triangle with a total angle measure exceeding 180° , which is a claim of generality about (planar) triangles. The division problem did not present itself with such opportunities to argue a claim of generality, so the focus of this analysis was on the triangle problem.

To analyze what criteria the participants were using in their representations, Schifter’s (2009) criteria S1-S3 (see 2.2) were deductively applied (see Saldaña (2013)) to each of their representations. Deductive coding is a technique to code data where the researcher uses pre-established codes generated by colleagues in the field. This differs from the coding methods described in 4.3.1 in the sense that those codes were developed by me and informed by the data. Deductive coding is appropriate for this analysis because S1-S3 provide specific items to identify a representation’s use to argue a general claim.

For this study, the term “operation” in S1 had to be more broadly interpreted than referring to a concrete arithmetic operation. I instead interpreted the term to refer to any mathematical concept meaningfully tied to the claim. For example, drawing a triangle would satisfy S1 in the sense that triangles can (and very likely must) be

negotiated in order to decide the truth value of the claim that no triangle exceeds 180° in total angle measure. Deductive coding (Saldaña, 2013) allowed me to apply pre-established codes (Schifter's (2009) criteria) to the participants' representations. Each representation was analyzed for S1-S3 before moving on to the next representation; each of the criteria are woven into the representations in interconnected ways; analyzing S1-S3 for a single representation simultaneously ran the risk of missing important insights from each. Once this coding process was complete, a thematic development took place to formulate the findings described in Chapter 5.

4.4 Subjectivity Statement

In qualitative research, it is essential that researchers critically evaluate their research endeavours. This is a standard practice when conducting qualitative studies so that researchers are cognizant of potential bias that exists and, if so, determining ways of minimizing that bias.

In observations of elementary educators before undertaking this study, I saw how teachers would represent various mathematical concepts. Sometimes their configurations were not my own, writing what would be considered incorrect expressions according to mathematicians, such as $2 + 4 = 6 - 1 = 5$. My expertise in mathematics naturally colored my interpretations of these expressions. During this investigation, I remained more open to their representations and refrained from placing judgments on their characteristics.

For two years, I was involved in professional development offered to some of the participants of this study that was specifically focused on argumentation. Indeed,

much of my formative thinking behind argumentation came from my collaborating with researchers in mathematics education, mathematics, and the teachers. I also received insight into how some of the participants of this study implemented the knowledge they gained from their professional development experiences. For these participants especially, I endeavoured not to allow these experiences to influence my interactions with them nor my analysis of the data. The data collection tools (interview questions and observation protocol) gave homogeneity to my interactions with all the participants and it helped me avoid pursuing paths that would exploit my prior experiences with these particular educators. The data analysis allowed me the opportunity to reevaluate any initial impressions I may have had during my interactions and confirm or question those impressions. In these ways, my objectivity was preserved.

Chapter 5

Findings

This chapter is dedicated to reporting the findings of this study. They are in response to the research questions in Chapter 3. They are

RQ1 How do elementary educators conceptualize argumentation?

RQ2 What role(s) and purpose(s) do elementary educators associate with representations in the context of mathematical argumentation?

RQ3 How do elementary educators reason with representations in arguing general claims?

The findings are presented by research question. In sections 5.1 and 5.2, the primary data used to support the findings are the participants' responses to the problems in the packet and their responses during Interviews 1 and 2. Data from the student work samples (without participants' annotations) as well as the classroom observations served as secondary data sources in support of the findings. In 5.3, the data to

support those findings come from the participants' responses to the triangle problem in the packet.

5.1 Participants' Conceptualization of Arguments

The data analysis revealed three major factors that influence participants' conceptualizations of argumentation: (1) arguments are perceived as products composed of prose, (2) they are used as a support to aid the learning of mathematical topics, and (3) argumentation can allow for opinion. These findings are expounded upon in turn in the following subsections.

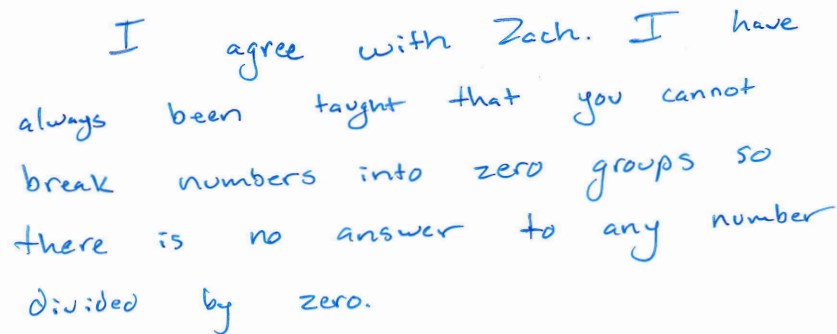
5.1.1 Argumentation as Prose

Analysis of the data showed that the participants' primary format of an argument in mathematics is as prose. That is to say, arguments are written using typical essay form. They do not have apparent characteristics that would lead someone to infer the participants are writing mathematics. Instead, the arguments follow conventional paragraph form and grammatical rules. When creating arguments in response to the problems in the packet, the participants repeatedly relied on a finished product that was, for the vast majority, bereft of representations. Instead, they provided arguments that by and large resembled essays. When appraising student work, the teachers displayed an emphasis in having their students write arguments in a similar form.

An illustrative example of this finding can be seen with Mr. Moreau, a fifth grade classroom teacher. His responses to the problem packet were completely in prose (see

for instance Figure 5.1.1, Mr. Moreau’s response to the division problem). Despite the fact that the representation $\frac{1}{0}$ is provided in the prompt of the problem, there are no mathematical representations in his argument. Mr. Moreau wrote a two-sentence response to this question (note in particular the indent at the beginning).

1. Wanda claims $\frac{1}{0} = 0$ because you are dividing one object into no groups, and there is no way to do that. Zach claims $\frac{1}{0}$ does not have an answer because there is no number that can be multiplied by 0 to get 1. Who do you agree with? Why is your choice the correct one?



I agree with Zach. I have always been taught that you cannot break numbers into zero groups so there is no answer to any number divided by zero.

FIGURE 5.1.1: Mr. Moreau’s response to the division problem

Furthermore, he preferred solutions to have prose within arguments, in particular when he did not understand the given representations. The transcript excerpt below reveals this phenomenon. Prior to this conversation, Mr. Moreau was asked to rank the solutions from the packet (refer to Appendix B to the division problem). He was allowed to rank them in any way he saw fit. After some deliberation, he provided a ranking of B, A, C (with B the highest rank).

INT. : Now why is A in the middle?

MR. M : Because he agrees with Zach who I also agree with but his representation is a little off. I don’t understand $1 = 0$.

INT. : Let’s say that student A was in your class? What would you say to that

particular student, seeing what you're seeing?

MR. M : Honestly, it's hard to say. My students wouldn't necessarily see this. See, he included a variable. (*laughs*)

INT. : OK, so granted that may be slightly advanced.

MR. M : It's a little bit advanced. I would just point to the bottom. "OK so $\frac{1}{0} = g$, $0 \times \frac{1}{0} = 0 \times g$, $1 = 0$. How do you...explain that." I would just ask them to explain it to me because I don't understand it.

INT. : So this alone isn't enough to give the argument that Zach is right.

MR. M : Yeah. (Mr. Moreau, Interview 1)

His confusions about the logic inferred by the equations (in particular the statement $1 = 0$) prompted him to ask the hypothetical student to "explain" the thought process, to go further than present the equations which logically deduced the student's claim. Speaking mathematically, Student A's argument is valid (the approach here is by contradiction). Just having the equations present would not suffice for Mr. Moreau; without prose elements, potential solutions were treated with a slight suspicion.

Another illustration of the finding that arguments are be in a prose form comes from Mrs. Joelson who is a first grade classroom teacher. When she answered the triangle problem in the packet (refer to Appendix B), she opted to write what is conspicuously seen as a paragraph response (see Figure 5.1.2). A revealing conversation ensued when asked about the representations that were present in her argument:

MRS. J : Yeah I mean you don't have to draw the entire triangle. You can just draw the angles for comparison next to each other.

2. Damarcus claims he can draw a triangle whose total angle measure exceeds 180° .

- (a) Do you agree with Damarcus that this can be done? Circle the appropriate response.

YES

NO

- (b) Convince one of the other students whether he can or cannot do this.

Damarcus cannot do this because 180° is considered a straight angle. All of the different types of angles are $< 180^\circ$, ex. acute $< 90^\circ$, right $= 90^\circ$, obtuse \rightarrow between 90° - 180° . Also, in a triangle, the ^{sum of the} 3 angles always equals 180° . So, Damarcus cannot draw a triangle measuring greater than 180° .

FIGURE 5.1.2: Mrs. Joelson's response to the triangle problem

INT : OK.

MRS. J : And then you could clearly see that 180° definitely is greater than the other types of angles. So that could be much more visual.

INT : Now is there a reason why we don't have those here? Is it just because of the whole protractor thing (*referencing a need to have a protractor to measure the three angles in a triangle*)?

MRS. J : Umm, no I would say I just didn't even think about, I didn't think about drawing them....I am also more of a writing person than a math person so I

express my thoughts...

INT : Are they different?

MRS. J : Yes and no.

INT : OK. You know I'm going to investigate both!

MRS. J : Yes! (*laughs*) Umm, yes they're different because that I think I definitely gave a good explanation through writing.

INT : OK.

MRS. J : But they're not different because through argumentation writing can be used. I don't know I've always believed people are either math people or reading/writing people. (Mrs. Joelson, Interview 1)

Later on during this conversation she categorized herself as “more reading/writing.” This perspective that people either are geared toward mathematics or reading/writing belies her proclivity for argumentation in a prose form. Her last remarks in particular invite the implication that “reading/writing” people naturally approach math problems with writing in mind. When they do, responses have structural elements inherited from conventional grammatical rules. These teachers (Mr. Moreau and Mrs. Joelson) illustrate the theme of this study that teachers' conceptualization of mathematical arguments involves a specific format: prose.

5.1.2 Arguments to Support and Learn Content

Analysis of the data uncovered that teachers view argumentation as more than verifying or refuting a claim; they use arguments to drive at learning a mathematical concept. In particular, arguments are viewed as pedagogical tools to expose

mathematical ideas. Moreover, the data analysis revealed that there is an indirect relationship between the grade level of the teacher and the amount to which this predilection exists: the higher the grade level, the less likely the teacher would use argumentation as solely a means to convey mathematical topics. In the earlier grades, less emphasis is placed upon the nature of an argument itself and more upon exposing mathematical facts. Traveling along the continuum toward the later grades, the situation reverses; although they still use arguments to reveal content, there is a tendency toward understanding the structural nature of an argument.

The classroom teachers from kindergarten through third grade all provided meanings to argumentation that alluded to their understanding as assisting to learn content. Consistently, they would describe argumentation as displaying “math thinking,” a reference to relevant calculations or showcasing a mathematical strategy. Mrs. Costa (grade 3) put it this way: argumentation involves “how you got the answer and why.” Here, “answer” is a numerical reference, so she is pointing to numerical methods of obtaining a solution. Arguments per se are exercises in displaying conceptual awareness, not simply verifying the truth or falsity of a claim. These descriptions of argumentation do not conform to the definition found in section 1.3; the emphasis is placed on procedure, not on deduction.

In grade five, a pivot is seen where the structure of an argument takes more of a central place, though the idea that arguments expose content is also present. This can be prominently seen from the classroom observations of Mr. Moreau and Mrs. Lewandowski (both grade 5 classroom teachers). In both observations, students filled out worksheets with Toulmin’s (1958/2003) components of argumentation (claim, warrant, and evidence). In both, the worksheets were divided into three parts with prompts to give a claim, warrant, and evidence. Based on these teacher-produced

tasks, their conceptions that arguments contain these elements are apparent.

Further evidence comes from Ms. Mickelson, an interventionist, and Mrs. Tavaréz, a numeracy coach, who saw the place for structure as well. In the case of Ms. Mickelson, she explicitly states that a claim, warrant, and evidence compose the body of an argument. With Mrs. Tavaréz, her structure appears slightly more open-ended: “reasons supported with evidence, both the ‘what’ that we’re doing at the moment in the task and what I know about math. So I have a belief, this is why I think this, and this is my reasoning behind it.” When pressed on her meaning on the word “belief,” she matter-of-factly stated “Oh, the claim,” indicating that the two words are synonymous. Though there are aspects of Toulmin’s (1958/2003) model, Mrs. Tavaréz does not seem as much a strict constructionist as the others. These educators attended to (in some degree) the argument itself.

The participants in this study revealed through the analysis that arguments are used to reveal mathematical content. This emphasis was found to lessen as one approaches the higher grades, where the participants look to more structural features of arguments.

5.1.3 Arguments as Viewpoints

This study’s data analysis revealed that the participants held a perspective about arguing that it can accommodate a choice in a sense that connotes more freedom than the logical rules of inference allow. Arguments are seen as capable of juxtaposing logically dissonant ideas but cushioning both simultaneously. In particular, there are instances when an argument is viewed less as an exercise in logical deduction, but rather as a chance to opine on the mathematical content of a claim.

When Mr. Moreau (grade 5) was initially exposed to the concept of argumentation, he was taken aback:

[W]hen I started here I worked with [Mrs. Tavaréz (numeracy coach)] closely and she's like 'Yeah, we really want these kids arguing in math.' I'm like 'We don't want them arguing necessarily.' They already argue on their own....I think we mean just to...just talking about the math from opposing viewpoints. I don't think we actually want kids to argue.

(Mr. Moreau, Interview 1)

His thinking shifts into a courtlike scenario: "almost like a debate. You want to debate about the math like 'I think it's this way because' and provide evidence and then 'I think it's this way'" (Mr. Moreau, Interview 1). He now likens mathematical argumentation to an arena where the notion of convincing is more concerned with persuading. This does not align with conventions in the mathematical community where convincing implies a logical deduction from given facts.

To offer another example of this finding, Mrs. Koch, a kindergarten classroom teacher, references opinions and different perspectives when arguing during Interview 1:

[Y]ou can arrive at the same answer in many different ways and arguing, argumentation gives you the chance to say those different ways and show the different ways and see how different people think and solve problems. And it's not an opinion, but it's just that person's perspective. So I think it's...If they can explain it, not that it's right or wrong. Yes, it's kind of more solid than an opinion, but you can share it and see how it's different and come to the same answer. (Mrs. Koch, Interview 1)

She emphasizes that, though arguing could be perceived as an opinion piece, it is more about displaying “your take” on the mathematics involved in the claim. Note also that, for her, explanations are emphasized more heavily than correctness when she states “[i]f they can explain it, not that it’s right or wrong” (Mrs. Koch, Interview 1).

After Interview 1, she reflected more on this, then decided on a new formulation of these thoughts, which she provided during Interview 2:

I never wanted to say ‘the right or wrong answer’ and I don’t want to discourage thinking. I don’t want them to shut down. But I was thinking ‘OK yes, we do need to get at yes, there is a right answer’ and ‘But how do we get there?’ I still have the same belief that there’s many ways to get there. There are different paths and I want to hear how they can get there so they can explain it, show me, to convince me. (Mrs. Koch, Interview 2)

The course correction in her conceptualization focuses even more on the “different paths” one can take to arrive at a sound conclusion. The student work samples gave insight into this belief. Students provided different representations for sums in the context of a story problem. Mrs. Koch had them provide the sum that answers the problem ($4+5=9$) in different ways: drawing a picture, a number line, and a number sentence. Having just one of these would satisfy the mathematical requirements of an argument, but having multiple ways of expressing the solution illustrates the importance she places on showcasing different perspectives.

Mr. Moreau and Mrs. Koch display how participants had a conceptualization of argumentation that was opinion-oriented in the sense that different viewpoints about

mathematical approaches can be couched within an argument.

5.2 Roles and Purposes of Representations

Analysis of the data uncovered two purposes that the participants had for representations in the context of arguing: (1) representations help teachers navigate the mathematical content of the claim of the argument and (2) they are used to supplement a constructed argument. The data analysis regarding the role that mathematical representations play within argumentation was surprising. Descriptions of these findings are in the proceeding subsections.

5.2.1 Navigating Mathematical Content

Through the data analysis it was uncovered that, in order to make sense of the mathematics involved in deciding the truth or falsity of a claim, the participants called forth relevant representations. The arguments they create reflect the meaning-making activities that their representations inspired and/or generated.

When writing arguments for the problems to the packet, several of the participants made use of representations that helped them negotiate the mathematical topics involved. In some instances, they were able to document their representations. This was most often the case with the triangle problem. In other cases, their external representations did not adequately reflect the concepts they wanted to convey, which was typically the case with the division problem. When examining the triangle problem, most of the participants claimed Damarcus could not draw a triangle whose angle measure sum was greater than 180° . From a mathematical perspective, in order to

successfully claim the negation, one could show how a triangle contains only 180° in total angle measure (assuming Euclidean geometry). According to the representations they provide, this is precisely what some of the participants began with: in the margins of the argument, a triangle would be drawn. From it, they would appeal to properties of triangles in Euclidean space to (attempt to) conclude their claim.

To illustrate, Figure 5.2.1 contains Mrs. Lewandowski’s (grade 5) response to the triangle problem. Here we see three different triangles, all of which match the desired properties of having total angle measure of 180° . Her representations (the three triangles) are salient to the argument, despite the difficulties they present as an illogical approach to concluding her claim (this last point is discussed in greater depth in 5.3).

With respect to the division problem, none of the participants produced external representations outside of the equation $\frac{1}{0} = 0$ or writing a multiplication statement in the manner of Zach’s explanation (e.g., $? \times 0 = 1$). Take for example Mrs. Costa’s (grade 3) response in Figure 5.2.2. Of note is that Mrs. Costa has no external representations driving toward the conclusion of her argument. However, there are indications she is meaningfully using a representation of division as “part over whole” when she states “ $\frac{1}{0}$ has the denominator as 0, meaning there is no whole or any pieces at all.” Even more, she alludes that this representation for division seems incapable of being externally represented $\frac{1}{0}$: “If you were to try and draw this you can’t because there is no whole.” Mrs. Costa acts on her internal representations to make sense of the problem at hand and to form her conclusion.

Corroborating the preceding are the participants’ encouragement to their students to display appropriate external representations relevant to their arguments. With teachers such as Mrs. Koch (kindergarten), this was explicitly done through

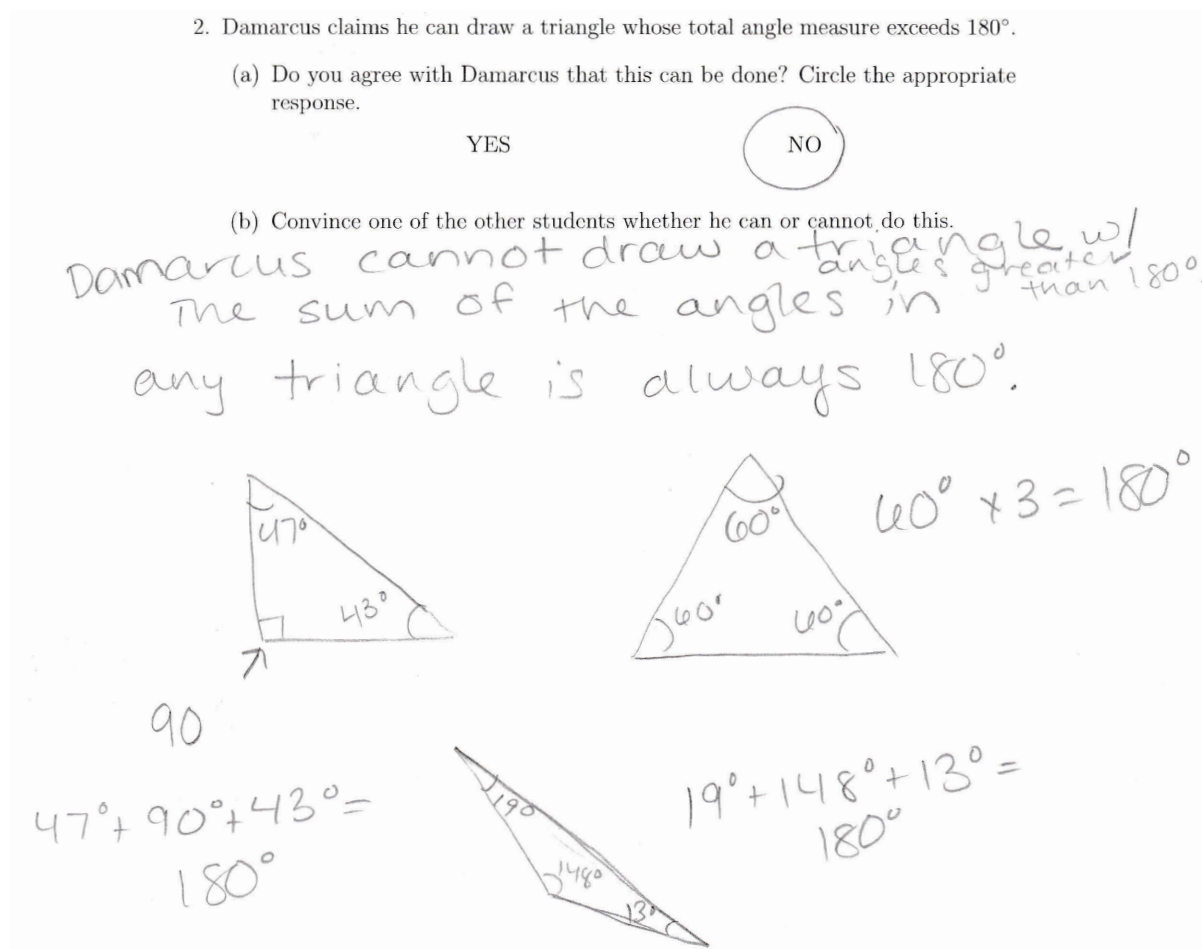


FIGURE 5.2.1: Mrs. Lewandowski's response to the triangle problem

the student tasks during the classroom observation as well as the student work samples provided during Interview 2. In those cases, students provided multiple external representations for the same particular mathematical statement (an example of which was given in subsection 5.1.3). When Ms. Mickelson (interventionist) worked with some grade 5 students for her classroom observation, different manipulatives were provided to them to make sense of the given problem (determining the number of doughnuts that could be placed in a not well-defined doughnut box). Mrs. Meri, a grade 2 classroom teacher, during her classroom observation had groups of students

Please answer the following questions as completely as possible.

1. Wanda claims $\frac{1}{0} = 0$ because you are dividing one object into no groups, and there is no way to do that. Zach claims $\frac{1}{0}$ does not have an answer because there is no number that can be multiplied by 0 to get 1. Who do you agree with? Why is your choice the correct one?

I agree w/Wanda. $\frac{1}{0}$ has the denominator as 0, meaning there is no whole or any pieces at all. If you were to try and draw this you can't because there is no whole.

FIGURE 5.2.2: Mrs. Costa's response to the division problem

working on different problems and with each group she prompted them with questions about how to “show their thinking,” whether it be with a diagram, a number sentence, or other form. Participants gave students opportunities to come up with salient representations to aid their thinking behind the mathematics involved in an arguments.

From these examples, we see (some of) the ways participants employed their representations to navigate mathematical content that is involved in deciding the truth value of a claim. These aided their abilities to provide arguments inasmuch as they helped decide the claim's truth value.

1. Wanda claims $\frac{1}{0} = 0$ because you are dividing one object into no groups, and there is no way to do that. Zach claims $\frac{1}{0}$ does not have an answer because there is no number that can be multiplied by 0 to get 1. Who do you agree with? Why is your choice the correct one?

I agree with Zach because multiplication and division are inverse operations. Zach's claim that $\frac{1}{0}$ does not have an answer is correct because b/c when you multiply by zero you'll get zero. When Wanda states that $\frac{1}{0} = 0$ b/c you dividing one into no groups she's incorrect b/c if you think of taking one of something & dividing it into zero groups you essentially can't do that. The other reason Zach is correct is b/c if you think of inverse or related operations in terms of fact families

	1×0	you
	0×1	
still	$0 \div 1$	
	$1 \div 0$	

$\frac{1}{0}$ is not an option or valid.

FIGURE 5.2.3: Ms. Mickelson's response to the division problem

5.2.2 Supplementing Arguments

The data analysis revealed that, when engaging in the argumentation process, participants employed representations exemplifying pieces of their argument and could be referenced if needed by a reader to help understand said pieces. This act of annotating arguments through the utilization of mathematical representations is also seen in their students' work as they "sideline" representations in the places where they are most useful in the argument.

Supplementing an argument in this sense does not imply any incompleteness of the argument itself. To supplement an argument for our purposes means to supply additional information for the reader to help make sense of the argument. They are not essential to the argument's ability to decide the truth or falsity of a claim. Having external representations conspicuously outside the argument proper afforded participants the chance to provide context to their argument. Indeed, when going through their arguments in the packet, several explicitly referred to their representations during the explanation in an effort to further warrant their statements. To illustrate, Ms. Mickelson (interventionist), in her discussion of the division problem during Interview 1, comes to a confusion about part of her response and navigates through it (see Figure 5.2.3). Below is an excerpt of the ensuing conversation:

Ms. M : So I underlined multiplication and division...are inverse operations and I did that because you could actually show that, like I could have actually shown that...Oh wait, I did down here! (*points to multiplication and division expressions at the bottom right*)

INT : OK, so not only could you have, but you did.

Ms. M : I did. I actually did.

INT : OK, fair enough. So what were you trying to represent there?

Ms. M : So I was representing that (*reads argument from beginning*)...wait, hold on. Oh, I was trying to show like why Zach was correct in saying that one over zero (*aside: because then I don't know why I didn't say that*) doesn't have an answer. Wait, what did I write? (*reads part of argument*) Oh. I should have been here (*points to multiplication and division expressions*), oh my God, so this didn't

even work. Oh but I like the fact that I put it there 'cause then I could see my thinking...

INT : OK.

Ms. M : So I guess that's why I used it as a representation because it helped, not only like whoever's going to read this, like me to really go through and think about what I'm trying to show here. (Mrs. Mickelson, Interview 1)

Ms. Mickelson comes to the realization that her multiplication and division expressions enhanced her ability to make sense of the argument and articulate it to others. However, it was not part of the original argument and she did not notice its power to help her explain the argument to the interviewer immediately. It is relegated to the "sidelines" of the argument as a signpost to help understand some of the conceptual ideas going on.

Student work samples and observations in the classroom support the participants' inclination to put representations to the "sidelines" of an argument. In numerous cases, representations produced by the students were used by them to create the argument but were not in the argument, instead being isolated to the side of the argument. In students' journal entries in Mrs. Joelson's (grade 1) and Mrs. Meri's (grade 2) classes, representations were set off outside of the prose which constituted their arguments. In Mr. Moreau's and Mrs. Lewandowski's (grade 5) classes, student work samples consisted of worksheets where students filled in their arguments. As in Mrs. Joelson's and Mrs. Meri's classes, the representations were found outside the argument proper; calculations that helped them come to a conclusion were put off to the side (sometimes as far as the margin of the worksheet) and not explicitly mentioned in several instances.

The preceding examples highlight how representations are “sidelined” when grappling with arguments. Their representations provided crucial annotations for them to see how the arguments establish the truth or falsity of the claim.

5.2.3 Roles for Representations

The data analysis for this study revealed that the participants did not assign any roles to representations *within* arguments. In order to perceive any roles for representations in the arguments, representations within an argument were examined for possible benefits for its use in the argument. Even though they used representations when crafting arguments as seen from their purposes (see subsections 5.2.1 and 5.2.2), they were not deemed necessary to be incorporated into them. Instead, they appear to be considered a step toward obtaining a viable argument. More specifically, their external representations are used in the process of determining what evidence will be necessary to present. Yet, these representations are not present in the argument itself, implying they were not needed as a method to convey the argument to readers. This is confirmed by examining the student work, where “the math work” done to establish the truth of a claim was not presented in the arguments.

This is not to say that there were no representations at all in their arguments. Indeed, many conventionally accepted external representations can be seen in the participants’ arguments. However, these representations are considered so ubiquitous in communicating mathematics that I did not consider their use as indicative of a representation’s role in the argument. For example, any use of numerals, despite their categorization as external representations, do not play a meaningful role in the argument since the reason(s) the participants used them was not due to anything

specific to their needs but because it is conventional and indeed expedient to substitute numerals for their language form (3 for three). Those decisions were deemed not unique to the participants. In order to determine a role for representations, their use had to be a more conscious decision and in cases such as the preceding, no such decision was made.

5.3 Criteria for Representations

When making use of representations while arguing a general claim, the data analysis uncovered that the participants offer some characteristics of representation-based proof (stated in 2.2) (Schifter, 2009). In particular, (1) their representations make light of a deeper mathematical concept, (2) most representations typically fell short of universally instantiating a representation about a set of objects with common properties, and (3) their arguments are founded on sound mathematical theory, but whose derivation relies on deeper conceptual understanding than what the participants' representations afford. Each of these findings are detailed in the following subsections.

The analyzed data which contributed to the findings in this section were participants' responses to the triangle problem. In certain instances, an argument was given to claim that no triangle can have a total angle measure exceeding 180° . None of the responses successfully accomplished this from a mathematical point of view, but the claim is of a general nature about triangles, so the criteria described by Schifter (2009) make this particular argument a candidate for a representation-based proof. Figures 5.3.1 through 5.3.4 present the representations that were analyzed.

5.3.1 Meaning of Operation

Analysis of the data uncovered that representations employed by the participants in order to reveal the truth/falsity of a claim indeed provide insight into a mathematical idea central to the claim. For the triangle problem, a representation that responds to Schifter's (2009) criteria would be a (planar) triangle whose individual angle measures are originally unknown but despite this can be deduced via parallel lines as having a sum of 180° . One participant did this (refer to Figure 5.3.1). Even though this representation does not correctly reflect the desired claim's truth value, the minor error in the drawing (the angle a in the bottom left of the drawing is not congruent to the angle a in the triangle) does not appreciably take away from the power the representation has to justify the claim.

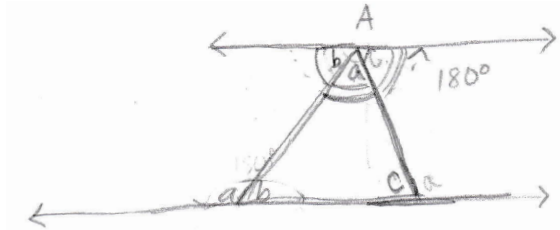


FIGURE 5.3.1: Mrs. Tavarez's representation

Of the other three participants who provided representations, Figures 5.3.2 through 5.3.4 show what representations they provided. In 5.3.2 the external representation is an equilateral triangle, which does drive at the deeper idea that the triangle problem involves triangles. It is unclear from the representation per se whether the participant realizes that the angles will be important to investigate. Figure 5.3.3 contains two external representations of note, namely, the 90- and 180-degree angles. These reflect the notion that the problem discusses angle measures. However, it is unclear whether the participant realizes (through the lens of the representations alone) whether knowl-

edge of triangles will come into play in the argument. Similar to Figure 5.3.2, Figure 5.3.4 utilizes three examples of triangles. One of these is an equilateral triangle (in the upper left), which is equivalent to Figure 5.3.2, another is a right triangle (in the upper left), and the last is a scalene triangle. All three, like Figure 5.3.2, reflect the deeper conceptual idea that the ensuing argument will require knowledge of triangles. Of these three participants, the representations in Figure 5.3.4 reflects the deepest level of conceptual understanding about the problem at hand; different types of triangles are presented (hinting at the general nature of the claim) and angle measures are provided (hinting at the realization that understanding the angle measures of a triangle is important). All of the participants present external representations relevant to the conditions given in the triangle problem.

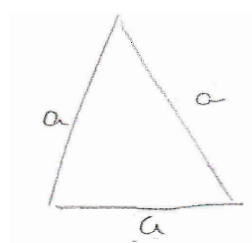


FIGURE 5.3.2: Ms. Mickelson's representation

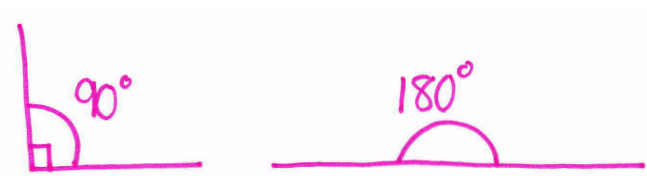


FIGURE 5.3.3: Mrs. Costa's representations

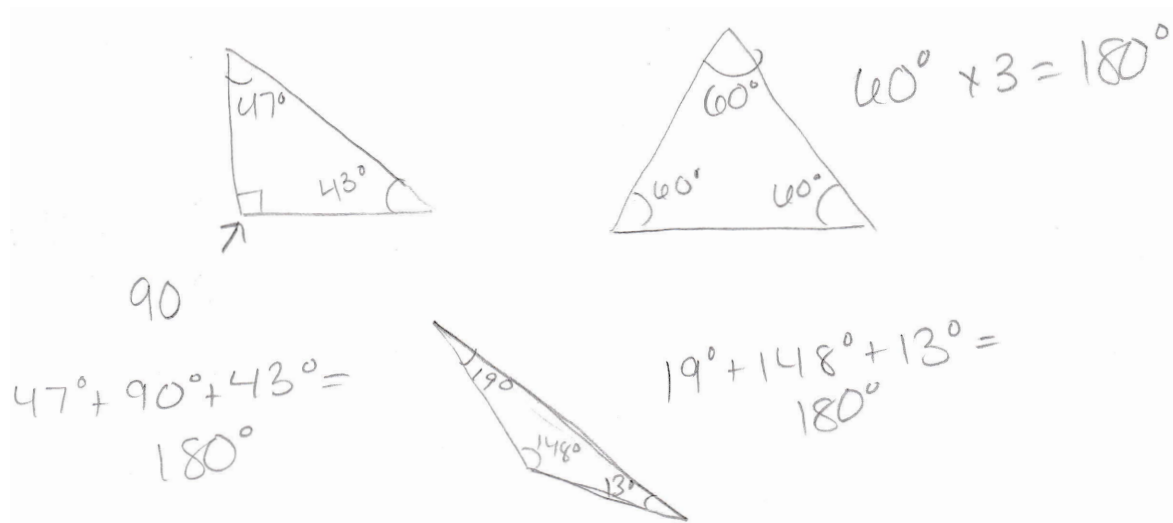


FIGURE 5.3.4: Mrs. Lewandowski's representations

5.3.2 Is the Representation a Universal Instantiation?

The data analysis revealed that Schifter's (2009) second criterion is mostly not satisfied with the participants' representations: The majority (75% of the participants) provided representations which did not reflect a generic example from an infinitely large set. The one representation that is a universal instantiation is in Figure 5.3.1. There are no indications that the angle measures are known in the given triangle. Indeed, the participant does what would be considered by the mathematical community as standard practice with respect to labeling the angles of the triangle by placing a letter within the region spanned by each angle, thereby considering the angles as unknown quantities. The properties of a generic triangle are manifested in this participant's external representation.

The other three participants provide representations that are not universal instantiations of their respective sets. Figure 5.3.2 provides a specific case of a triangle (equilateral). In fact, in the set of planar triangles, equilateral triangles present the

most abundant properties; they are particular instances of isosceles triangles with equal side length and (consequently) angle measures. Thus, an equilateral triangle does not service the need of creating a mathematically valid argument to the claim. Figure 5.3.4 also contains an equilateral triangle (the top right triangle is equiangular, thus equilateral), which present the same difficulties as those previously mentioned. The other instantiations however provide more context to the participants' designs on her argument: By giving more than one example of a triangle, the educator realizes that the claim is not about a specific triangle but needs to draw upon a larger set of triangles. The primary drawback to the other given examples is that they are specific cases of triangles as well. A universal instantiation of a triangle necessitates that individual angle measures be unknown. Each instantiation provided by the participant does represent a triangle, but only in specific respects. Figure 5.3.3 also presents difficulties with universal instantiation, with the set of objects in this case being angles in the plane. As in Figure 5.3.4, more than one example is presented, but it is less clear based on the representations alone that the participant is drawing from that larger set. The justification for this is from the nature of her instantiations: the 90- and 180-degree depictions do not, in my opinion, examine angles of more varying measure. These angles are useful in many other applications so it is less likely the participant is cognizant of the fact that there is a generic nature about the angles at hand.

To summarize, most of the representations provided by the participants did not successfully instantiate a generic case from an infinitely large set which was necessary in order to properly deduce the claim. Rather, they instantiated specific cases to help argue why triangles have exactly 180° in total angle measure.

5.3.3 Structure of Representation Tied to Claim

Upon analyzing the arguments of each participant, the majority of the arguments (75%) were not justified by the representations provided. However, Figure 5.3.1 is the exception. In this case the representation does indeed support that a (planar) triangle cannot exceed 180° in total angle measure once the preliminary assumption that the two lines provided are parallel is made. This is not indicated in the representation using standard notation, but can be inferred based on the nature of how the line through the top vertex would seemingly be parallel to the line extended by the side opposite the top vertex in the participant's drawing. Using knowledge of parallel lines, the congruent angles labeled b and c (they are alternate interior angles) readily imply equal angle measures, so $m\angle a + m\angle b + m\angle c = 180^\circ$; all three angles in total form a line incident with the line she drew through the top vertex. Her representation conspicuously contains 180° in the upper right, next to the three labeled angles, apparently intended to draw attention to the evidence of the combined angle measure. The arrow between c and 180° could also be interpreted as indicating a sum of angle measures.

The other three participants do not provide representations conducive to justifying the given claim. Figure 5.3.2 presents an equilateral triangle that cannot lay claim to statements about the set of all triangles. In a similar way, Figure 5.3.4 contain three representations of triangles, but taken individually are unable to justify the claim; each is a specific instance of a triangle. It can be inferred that the participant realized having a single triangle would not be enough in order to argue the truth of the claim, but a finite number of cases still does not provide enough evidence to claim that *every* triangle has a total angle measure of 180° . The angles given in Figure 5.3.3

also do not justify this claim. The angles are not generic nor do they characterize a universally instantiated angle from a triangle. The angles are related to parts of the claim: 90° is an angle related to a right triangle, an element of the set of triangles, and 180° appears to be a reference to the required sum of the angle measures. The representations themselves do not immediately provide a pathway to investigate the truth value of the claim.

In summary, we see that most of the offered representations were not able to successfully decide the truth value of the general claim. They did afford the participants the opportunity to attempt to decide the truth value, but from a mathematical perspective they (the representations) were inadequate to do so.

Chapter 6

Discussion

In this chapter, I first individually discuss each of the findings presented in Chapter 5. I then describe how they are intertwined and relate to each other. Some limitations to this study are presented and also taken up to describe how this study's findings can be confirmed and/or strengthened.

6.1 Conceptualizations of Argumentation

The first finding addressing RQ1 was that arguments were perceived as objects composed of prose. This does not come as very surprising given the contextual factors at play for public school educators. The Common Core State Standards now pay attention to argumentation as a critical process in mathematics (CCSS-M, 2010b) and elementary educators typically have not been equipped to bring this to bear in the mathematics classroom (see Wagner et al. (2014) for a treatment of argumentation with secondary educators). Yet, such educators are resilient and are generalists borne

from their ability to teach several subjects, so they very likely infer meaning about mathematical argumentation from other areas. A natural area to draw knowledge about argumentation from is the Common Core’s English Language Arts/Literacy Standards (CCSS-ELA, 2010a). In those standards, argument is listed as a text type and indeed is considered invaluable for students to learn in light of the “argument culture” firmly established at institutions of higher learning (CCSS-ELA, 2010a). This primacy of the interpretation of argumentation in the English Language Arts/Literacy Standards could be an influential factor in elementary educators’ choice to mathematically argue in prose.

The second finding was that arguments are used as a support to aid learning a mathematical concept. This supports the worldview of mathematics held by many which is that the goal of mathematics is memorizing a set of rules/procedures and know how to use them in context (e.g., in story problems). Evidence of that perspective comes in the following manner: many mathematics tasks given to students are categorized as low-level in cognitive demand (Stein et al., 2000). In particular, Stein et al. (2000) describe that memorization tasks and procedures without connections tasks (performing a procedure without the need to understand the context within which it is couched) do not afford the student-mathematician opportunities to see the richness that mathematics provides. Many researchers (see Knuth (2002); Mamona-Downs and Downs (2015); Tall (1989)) make the case that proof is a fundamental mathematical practice, so if argumentation is treated as a pathway toward understanding proof, the need arises to appreciate the multifaceted uses that arguments serve in the mathematics classroom.

In the upper grades, there was slightly less emphasis on arguments to aid learning a mathematical topic. A shift was observed toward structural elements of arguments.

Specifically, elements of Toulmin's (1958/2003) model were brought to the foreground. This aligns more closely with the mathematics community's perception of proof, where evidence is warranted at important steps toward proving a conjecture. This is promising to see, given the desire to have argumentation act as an antecedent for proof at the undergraduate level. If elementary students are exposed to understanding such concepts as logical rules of inference and structuring mathematically valid responses to claims, approximations can be made throughout the grades to channel students toward the ability of writing formal mathematical proofs by the time they become undergraduates.

The third finding answering RQ1 was that argumentation can allow for opinion. If we frame opinion as a socially-mediated activity, elementary educators' allowing for opinion could be seen as them engaging in collective argumentation (Krummheuer, 1995; Sfard & Kiernan, 2001), offered by Krummheuer (1995) as a group of students engaging in the formation of a single argument (see Conner, Singletary, Smith, Wagner, and Francisco (2014)). Furthermore, elementary educators provide a plethora of social activities for students in the classroom (such as circle time, rug time and warmup problems) (Chapin, O'Connor, & Anderson, 2013). One activity specific to mathematics is Number Talks (Parrish, 2011). In a Number Talk, students discuss various computational methods possible to calculate a given expression in pairs and/or as a group. They sometimes make comparisons and identify strengths and weaknesses. If done similarly in the context of arguments, students would be able to critique arguments, which is suggested as part of the third Practice Standard in the Common Core State Standards for mathematics (critique the reasoning of others) (CCSS-M, 2010b).

6.2 Roles and Purposes of Representations

There are two findings for the purposes of representations in arguments in response to RQ2: (1) they help teachers navigate the mathematical content of the claim/proposition of the argument and (2) they are used to supplement a constructed argument. The fact that elementary educators have these purposes in mind for representations does not take away from their mathematical content knowledge nor their ability to produce arguments. In particular, using representations to navigate the content is akin to abilities in problem-solving (Polya, 1957). Given that representations have powerful consequences for learning (see for example Izsák and Sherin (2003)), teachers should also explore other purposes that representations have within the context of argumentation.

The finding that the educators had no roles for representations in the arguments was surprising. Since they had meaningful purposes for their representations while engaging in the argumentation process, it is unfortunate that representations were not seen as purposeful when presenting their arguments. In the mathematics community, there are several reasons to incorporate representations into the argument, including untangling for the reader complicated mathematical thoughts and efficiently conveying mathematical ideas. The participants' representations, when present, were typically intended to clarify particular parts of their own thinking. The foundation is in place then to smoothly transition educators to incorporate their representations into their arguments meaningfully. At the same time, it is reasonable to see that external representations are not used since writing arguments in prose form is not immediately conducive to embedding such representations as symbols, graphs, and diagrams.

6.3 Criteria for Representations

This study uncovered three findings answering RQ3 aligned with Schifter’s (2009) criteria for representation-based proofs: (1) their representations make light of a deeper mathematical idea, (2) most representations typically fell short of universally instantiating a representation about an infinite set of objects, and (3) their arguments are founded on sound mathematical theory, but whose derivation relies on deeper conceptual understanding than what the participants’ representations afford. With respect to representations reflecting deeper mathematical concepts, the participants’ use of their representations is consistent with conventions in the mathematics community. It is expected that, in order to successfully surmise a proof for a given conjecture, one typically will manipulate their representations to uncover a pathway to produce a proof. The similarities found with elementary educators is promising and, if this translates with their students in the same way, future undergraduates will have many less concerns about justifying work or substantiating results.

The second finding is that most of the participants’ representations did not reflect a generic example about an infinite set of objects. Since one can reasonably infer that their assumption was that their representations would reflect the nature of the claim, a logical question to ask is why they believed their representations sufficed to give an argument. Finding an answer to that question with the available data appears difficult. A possible contributing factor is the (faulty) reasoning that arguing a claim about an infinite set cannot be done with a proof by exhaustion. Specific members of the set is not enough to logically argue a claim about all its elements; doing so amounts to promoting a false version of induction by claiming the implication “If a statement about an infinite set of objects is true for a finite subset of instances, then

the statement is also true for every instance in the set.” Another important piece related to this second finding is that, at the undergraduate level, students tend to make similar conceptual errors, relying on a few examples to generalize a proposition. This is now not surprising to see given that it would be likely this technique had been (implicitly) encouraged by their K-12 teachers.

The third finding answering RQ3 was that, though the conclusions in the participants’ arguments were valid, the theoretical foundation the representations rested upon were insufficient to argue the claim. From a mathematical perspective, the claim followed from knowledge about parallel lines and triangles, using both strategically. None of the arguments were valid based only on their representations, so it may be reasonable that representations alone do not factor into an educator’s mind when attempting to provide an argument. The data collected in this study does not completely answer this question; more research is needed in this area.

In light of the above discussion, it becomes clear that Schifter’s (2009) criteria benefit teachers’ mathematical content knowledge about using universal instantiations for claims dealing with an infinite set of objects. At the undergraduate level, having students understand how to approach proofs of this form is very beneficial, since many problems involve such sets. Moreover, they provide students with rich opportunities to “unpack” complicated mathematical language (cf. J. Selden & Selden, 1995).

6.4 Intersections of Findings

The preceding section discussed each of the individual findings. This section is dedicated to describing some of the relationships between the findings found in

this study. The relationships are provided as implications with the antecedents and consequents being my interpretations of what the causations are.

This study uncovered that arguments have a prose form when written. Some external representations do not easily conform to this. Take as an example the representations provided by the participants in the triangle problem in Figures 5.3.1-5.3.4. Representations such as these visuals do not neatly fit into a paragraph (indeed, this can be said of elements from the set of external representations of geometric objects). It then could follow that there would be no role for representations in arguments, as was uncovered by this study. Furthermore, it was uncovered that any representations that are used would not be within the argument proper. In this study, this manifested itself as representations supplementing arguments.

Another finding from this study is that arguments were used to support and learn mathematical content. Given this conceptualization of argumentation, it stands to reason that the participants would also leverage their own and their students' understandings of content with other mathematical tools, including representations. This was indeed uncovered by this study; the participants purposed representations as navigating the mathematical content of a claim when undergoing argumentation.

When deciding the truth or falsity of a general claim, this study found that the majority of participants did not universally instantiate representations related to the claim. From a mathematical perspective, not having such a representation at the ready logically inhibits an arguer's ability to decide the truth value of such a claim. It would then follow that the participants' representations were not tied to the claim in a warranted manner, which was also found in this study.

6.5 Limitations

One of the limitations of this study is that seven of the eight participants came from a single school as a consequence of the sampling methods used. While this allowed the researcher a certain control over the influence that different contexts could have on the results, it can be argued that the results need to be confirmed in studies that replicate this one but are conducted at other school sites. Moreover, having teachers from various types of schools (private, parochial, etc.) could also go far in extending the results found in this study and from more than one area of the state, or the country.

Another limitation of this study is the gender breakdown of the participants: seven females and one male. Gathering a sample with more equitable percentages of genders could be undertaken to investigate the research questions. According to the Institute of Education Services (IES), the gender breakdown in the state of Connecticut as of 2012 was 22.1% male and 77.9% female (IES). As this study's participants were 12.5% male and 87.5% female, the sample approximates the general population fairly well, yet research involving a larger percentage of male teachers would be needed to assure results that more closely represent the population.

Five of the eight educators sampled for this study also participated in a professional development program whose focus was on argumentation. Having such a variety of experiences with argumentation presents another limitation to this study. The findings were found across both these teachers and those who did not participate in the professional development program which assures reliability across experience with argumentation. Yet, other studies with only participants having little exposure to argumentation could enhance the findings found here.

One of the data collection methods was one classroom observation, whose primary role was to triangulate results found through the interviews. One of the other limitations of this study is that collecting data from one observation may not be completely indicative of what happens in the classroom. Though it does provide confirmation, it only takes a snapshot of what the participants were doing. While this study was not focused on teacher practice, observing them during other times might have offered a slightly more enhanced view of how they use representations within argumentation.

Chapter 7

Implications and Future Directions

With the findings from the data analysis (Chapter 5) and their discussion (Chapter 6), in this chapter I outline some implications of these findings on educational practice. I also point out directions for future research studies in light of the results presented here.

7.1 Implications for Educational Practice

There are three major areas where this study has significant implications on teaching: teacher education, undergraduate mathematics education, and K-12 mathematics teaching. Each of these are elaborated on in turn in the following subsections.

7.1.1 Implications on Teacher Education

Given this study's findings regarding the participants' conceptualizations of argumentation, more work needs to be done in order to support inservice teachers' understandings of mathematical argumentation, and especially at the elementary level. Though parallels exist between argumentative writing in the English language arts (CCSS-ELA, 2010a) and mathematical argumentation, elementary educators need to be made aware of the specific nature of arguing in a mathematical setting in order to enact the Common Core's (CCSS-M, 2010b) third mathematical practice standard. Professional development opportunities addressing this need could also expose teachers to multiple ways of arguing mathematically. This is not limited to showcasing proof techniques (e.g., direct or indirect proof, proof by cases or by contradiction), but also how argumentation can be implemented in the classroom setting such as in whole group or small group discussions as well as with individual students. Specifically, the characterization of argumentation as an opportunity for opinion (see 5.1.3) which was uncovered by this study needs to be augmented with the characterization that arguments adhere to the logical rules of inference and deduction.

With respect to representations, inservice educators on the whole may hold similar views as the participants in this study that representations have no role to play within arguments themselves. In line with the above, steps should be taken to display a variety of reasons why incorporating representations enriches and strengthens the mathematical quality of their arguments. Professional development opportunities for elementary educators could go even further: As this study found that representations had an annotative purpose for arguments, this understanding can be leveraged to support teachers' uses of representations while undertaking arguments; representations

can be used as guideposts for readers as well as help the arguer convey mathematical ideas.

Everything outlined in the preceding paragraphs must also occur with preservice elementary educators. Undergraduates enrolled in teacher preparation programs should more purposefully be exposed to mathematical argumentation and representations in ways this study's findings are pointing toward: argumentation should be explicitly defined and more broadly constructed and representations be explicitly presented as richly integrative with arguments.

7.1.2 Implications for Undergraduate Mathematics Education

Taking into account all the findings of this study regarding elementary educators' conceptualizations of argumentation and representations, undergraduate mathematics educators can receive new perspectives on both from their students. When seeing how elementary educators perceive these topics, at the undergraduate level we gain an appreciation for how students could also be perceiving these topics. This in turn informs pedagogy and praxis in relevant ways toward how undergraduates might approach particular aspects of the courses taught, especially how representations can be intricately connected with arguments and proof.

There are immediate implications on content courses designed for preservice elementary educators. Such courses should sharpen their focus on argumentation and representations in ways pointed out by this study's findings. As mentioned in the preceding subsection, this would include explicitly defining and more broadly constructing argumentation and representations as richly integrative with arguments.

Shifting focus to the general audience of undergraduate mathematics educators,

another implication of this study is specific to arguing claims of generality. We should be cognizant to appropriately highlight the importance of supporting students' use of representations to argue such claims. Given what this study uncovered regarding how the participants underwent this task, we can anticipate similar challenges for students and pedagogically arm ourselves with methods to help students instantiate relevant representations and act on them appropriately to argue (and when appropriate prove) claims of generality.

All these implications point toward a strong assertion, supported by the findings of this study: it is reasonable to expect that incoming undergraduates will have broader understandings of mathematical argumentation. These understandings can be a lever to provide a natural pathway for them into proof. The results of this study can inform colleagues on how students might approach various topics related to proof, including how cogent representations are instantiated. Special attention is required of those undergraduate educators teaching proof-based courses.

7.1.3 Implications for K-12 Mathematical Learning

This study sheds light upon how elementary educators conceptualize argumentation. Given the influence teachers have on student learning, it is important to identify students' needs in creating arguments and meaningful ways to address them. Those educators nuanced in mathematical argumentation, equipped with the findings from this study, would be in an ideal position to provide appropriate supports to students, helping them successfully engage in argumentation. Similarly, with the given findings regarding teachers' use of representations, those educators well-versed in representations within arguments can support students' use of representations designed to

enhance their arguments.

When ascertaining the truth or falsity of a general claim, this study uncovered various properties of the participants' representations. These results can be used to anticipate how students might approach similar situations. In particular, this study underscores the imperative that students' representations around general claims contain properties conducive to deciding the claims' truth values.

7.2 Directions for Future Research

This study has several possible future lines of research to pursue. Possible research endeavors for K-12 education are: (1) What are elementary educators' understandings of argumentation and representations from other areas? (2) What growth can be seen with their understandings on argumentation and representations? (3) What impact does professional development have on teachers' conceptions of these ideas? (4) How do those teachers receiving professional development act on those understandings? (5) How do elementary educators navigate argumentation in mathematics with arguing in the English language arts? These questions can also be examined latitudinally, namely with teachers teaching middle and high school. Furthermore, other types of studies (quantitative and mixed methods) can potentially uncover new perspectives on answering this study's research questions as well as the questions above. Studies should also be conducted with elementary and secondary students in mind to address questions such as: (1) What are students' understandings of argumentation and representation? and (2) What student growth can be seen with respect to their understandings of argumentation and representations?

This study also presents avenues of research in undergraduate mathematics. For example: (1) How do undergraduate mathematics instructors perceive the roles of representations in arguments and proofs? (2) In what ways do these roles agree or disagree with each other? (3) Similarly, how do undergraduate students (both those majoring in mathematics and not) navigate representations in arguments and proof? (4) In what ways are the perceptions these populations have about representations common and are dissimilar? and (5) Why do these commonalities and dissimilarities exist?

This study's design, methods, and findings can be used to inform, guide, and advance fundamental knowledge regarding these crucial issues. Investigations into any of the above questions concerning educators can have a research design that follows this study (or closely follows it) and the data collection tools (interviews and observations) are useful in understanding students' and educators' uses of argumentation, representations, and their intersection. More broadly speaking, this study's design and methods can be applied to answering questions related to students' and/or teachers' conceptualizations of any mathematics concept, such as functions, relations, limits, derivatives, and integrals to name only a few.

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Appendices

A Packet Problems

Please answer the following questions as completely as possible.

1. Wanda claims $\frac{1}{0} = 0$ because you are dividing one object into no groups, and there is no way to do that. Zach claims $\frac{1}{0}$ does not have an answer because there is no number that can be multiplied by 0 to get 1. Who do you agree with? Why is your choice the correct one?

2. Damarcus claims he can draw a triangle whose total angle measure exceeds 180° .

(a) Do you agree with Damarcus that this can be done? Circle the appropriate response.

YES

NO

(b) Convince one of the other students whether he can or cannot do this.

B Research-Produced Solutions to Packet Problems

Please answer the following questions as completely as possible.

1. Wanda claims $\frac{1}{0} = 0$ because you are dividing one object into no groups, and there is no way to do that. Zach claims $\frac{1}{0}$ does not have an answer because there is no number that can be multiplied by 0 to get 1.

(a) Do you agree with Wanda or Zach? Circle the appropriate response.

Wanda

Zach

- (b) Provide an argument in support of your choice in part (a). Use the space below for all scratch work as well as your argument. Please clearly identify your final answer.

A

I think Zach is right. Here's why:

$$\begin{aligned}\frac{1}{0} &= g \\ 0 \times \frac{1}{0} &= 0 \times g \\ 1 &= 0\end{aligned}$$

But 1 isn't the same as 0, so $\frac{1}{0}$ must not be a number.

B

I choose Zach. If it had an answer, then 1 would have to equal that answer times 0. Any number times 0 is 0. That would mean 1 is 0! That can't happen so Zach's right that $\frac{1}{0}$ doesn't have an answer.

C

Wanda is right because you can't take something and divide it by nothing. That's impossible to do, so there is zero ways to do $\frac{1}{0}$.

2. Damarcus claims he can draw a triangle whose total angle measure exceeds 180° .

(a) Do you agree with Damarcus that this can be done? Circle the appropriate response.

YES

NO

(b) Provide an argument in support of your choice in part (a). Use the space below for all scratch work as well as your argument. Please clearly identify your final answer.

A

I say yes, he can do it because I did too.

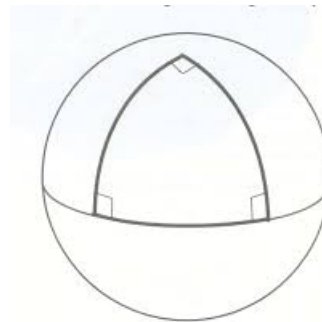
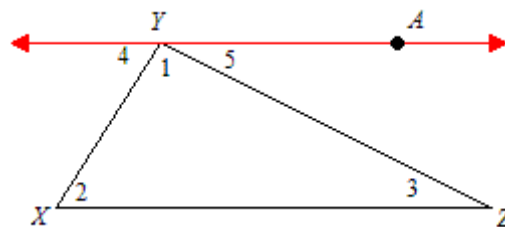


Figure 6.13 Triple-right triangle on a sphere

B

Damarcus must have done something wrong. A line has an angle measure of 180° and I can show you that all three angles of a triangle can be placed next to each other to form a line. I drew it below: Pretend the line is parallel to the bottom of this random



triangle. Angles 2 and 4 are the same size and so are angles 3 and 5. All three angles are now next to each other on the same line, which forms 180° . So Damarcus is not right and I disagree with him.

C Interview 1 Questions

Let us start by talking about your thoughts on representing mathematics and argumentation.

1. Have you received any formal or informal training about mathematical argumentation? If so, please tell me about that.
2. What do you think is meant by (or definition of – if participant had formal training) mathematical argumentation?

Supporting questions, if not addressed by the participant:

- a. What would you say are characteristics of a mathematical argument?
 - b. In your understanding, do arguments have a particular format?
 - c. Do you think a mathematical argument is made up of parts? If so, what would those parts be?
3. What does mathematical representations mean to you?
Supporting questions, if not addressed by the participant:
 - a. Can you give me some examples? (use paper as needed)
 - b. Where would these representations be found? In which mathematical contexts or settings?
 - c. Do representations have different manifestations depending on the settings (refer to the settings mentioned above)?
 - d. Do you think mathematical representations are crucial? If so, tell me when and why?
 4. Are there unique characteristics specific to argumentation that elicits representing mathematics differently from other settings?
 - a. What are characteristics of representations specific to argumentation?
 - b. Do representations have a specific place within argumentation? (i.e, can it be used to start a mathematical argument; should they be used for a specific purpose within the argument; can they be used in an argument or are they used to support an argument only; can arguments only contain representations or must other elements be there also)

Researcher's notes:

Let us now take a look at the activity/task related to the student work samples that you have brought.

1. Please describe this task for me.
 - a. What are the learning objective(s)? (See question 3, if participant's main objectives are around argumentation)
 - b. What prior knowledge is expected of the students to successfully complete it?
 - i. What exposure to argumentation is assumed?
 - ii. What exposure to mathematical representations is assumed?
 - c. What mathematical skills are the students expected to use to successfully complete it?
 - d. What is (are) the learning goal(s) after this task is completed?
 - e. Do you have an assessment plan for this task?
 - i. Tell me what it consists of.
 - ii. In your understanding, how are representations and/or arguments considered during the assessment process?
2. Why did you choose this task?
 - a. How does this task relate to mathematical argumentation?
 - b. Where do you see the argumentation piece?
 - c. What qualities do you see in the task that lead you choose it as related to mathematical argumentation?
3. In terms of mathematical argumentation,
 - a. What are the learning objective(s) specific to argumentation?
 - b. Do the objective(s) involve how the students write the argument?
 - c. What characteristics do you look for in your students' solutions in terms of how they go about crafting their arguments? Why those characteristics?
 - d. Did you have any specific learning objectives regarding representations for this task? Which ones?
 - e. What characteristics do you look for in the representations used by students? Why those characteristics? to start a mathematical argument; or used for a specific purpose within the argument; or used to support an argument only; can arguments only contain representations or must other elements be there also?

Researchers Notes:

Now I would like to focus more on the representations showcased in the students' work.

1. Please identify for me the mathematical representations in the student work.
 - a. Sample 1
 - b. Sample 2
 - c. [Interviewer will adjust according to # of samples, specific samples, and time left]
 - d. Were these representations to be expected from your students? Why?/why not?
 - e. Are these samples representative of what all students produce? Are there any that are not here that you have seen them use? Tell me about those: describe them, are those expected, (interviewer presses to get similar information as above for these).
2. When you look closely at the representations showcased in the student work samples,
 - a. What commonalities do you notice in their representations?
 - b. Why do you believe these commonalities are present?
 - c. What differences do you notice in their representations?
 - d. Why do you believe these differences are present?
 - e. How would you classify the different representations produced by students?
(Different aspects may be important to participant. Ask for classification with respect to quality, if not offered by the participant.)
3. In your opinion, what determines the clarity of a representation
 - a. With respect to the goals of this task?
 - b. With respect to the mathematics being represented here?
 - c. With respect to your mathematical preference, what helps you better understand the solution presented?
 - d. Now, if you think in general (not for this specific task), what would you say determines the clarity of mathematical representations?

Researcher's Notes:

D Interview 2 Questions

Let's first start with some questions similar to what I asked you last time in case your views or thoughts have changed and to help me better understand how you conceptualize these ideas.

1. Since the last interview, have you attempted to engage more with mathematical argumentation and/or with representations?
 - a. If you have, could you describe what you have done? I would be interested in anything you have done, regardless of whether it was done for your own pedagogy or not.
2. What do you think is meant by (or definition of—if participant had formal training) by mathematical argumentation?

Supporting questions, if not addressed by the participant:

- a. What would you say now are characteristics of a mathematical argument?
 - b. In your understanding, do arguments have a particular format?
 - c. Do you think a mathematical argument is made up of parts? If so, what would those parts be?
3. What does it mean to you to represent mathematics?

Supporting questions, if not addressed by the participant:

 - a. Can you give me some examples? (use paper as needed)
 - b. Where would these representations be found? In which mathematical contexts or settings?
 - c. Do representations have different manifestations depending on the settings (refer to the settings mentioned above)?
 - d. Do you think mathematical representations are crucial? If so, tell me when and why.
4. Are there unique characteristics specific to argumentation that elicits representing mathematics differently from other settings?

Supporting questions, if not addressed by the participant:

- a. What are characteristics of representations specific to argumentation?
 - b. Do representations have a specific place within argumentation? (i.e., can it be used to start a mathematical argument; should they be used for a specific purpose within the argument; or used only to support an argument; or can arguments only contain representations or must other elements be there also?)

Researcher's notes:

Let us now take a look at the activity/task related to the student work samples that you have brought.

1. Please describe this task for me.
 - a. What are the learning objective(s)? (See question 3, if participant's main objectives are around argumentation)
 - b. What prior knowledge is expected of the students to successfully complete it?
 - i. What exposure to argumentation is assumed?
 - ii. What exposure to mathematical representations is assumed?
 - c. What mathematical skills are the students expected to use to successfully complete it?
 - d. What is (are) the learning goal(s) after this task is completed?
 - e. Do you have an assessment plan for this task?
 - i. Tell me what it consists of.
 - ii. In your understanding, how are representations and/or arguments considered during the assessment process?
2. Why did you choose this task?
 - a. How does this task relate to mathematical argumentation?
 - b. Where do you see the argumentation piece?
 - c. What qualities do you see in the task that lead you choose it as related to mathematical argumentation?
3. In terms of mathematical argumentation,
 - a. What are the learning objective(s) specific to argumentation?
 - b. Do the objective(s) involve how the students write the argument?
 - c. What characteristics do you look for in your students' solutions in terms of how they go about crafting their arguments? Why those characteristics?
 - d. Did you have any specific learning objectives regarding representations for this task? Which ones?
 - e. What characteristics do you look for in the representations used by students? Why those characteristics? to start a mathematical argument; or used for a specific purpose within the argument; or used to support an argument only; can arguments only contain representations or must other elements be there also?

Researchers Notes:

Now I would like to focus more on the representations showcased in the students' work.

1. Please identify for me the mathematical representations in the student work.
 - a. Sample 1
 - b. Sample 2
 - c. [Interviewer will adjust according to # of samples, specific samples, and time left]
 - d. Were these representations to be expected from your students? Why?/why not?
 - e. Are these samples representative of what all students produce? Are there any that are not here that you have seen them use? Tell me about those: describe them, are those expected, (interviewer presses to get similar information as above for these).
2. When you look closely at the representations showcased in the student work samples,
 - a. What commonalities do you notice in their representations?
 - b. Why do you believe these commonalities are present?
 - c. What differences do you notice in their representations?
 - d. Why do you believe these differences are present?
 - e. How would you classify the different representations produced by students?
(Different aspects may be important to participant. Ask for classification with respect to quality, if not offered by the participant.)
3. In your opinion, what determines the clarity of a representation
 - a. With respect to the goals of this task?
 - b. With respect to the mathematics being represented here?
 - c. With respect to your mathematical preference, what helps you better understand the solution presented?
 - d. Now, if you think in general (not for this specific task), what would you say determines the clarity of mathematical representations?

Researcher's Notes:

E Observation Protocol

Argumentation Task Types					
Type of argumentation task(s) that take place during the observation. Optional column for tasks that cannot be classified as any of the typical argumentation types listed. Observer will request copy of task(s). It is expected that at most three argumentation tasks will be worked on during one class period but additional copies of this page will be on hand, in the event more tasks take place.					
	Proof/Construct own argument	Misconception	Agree/Disagree	Criticizing others' arguments	Other
TASK 1					
TASK 2					
TASK 3					

Teacher Emphasis in Argumentation					
Various aspects of arguments that teachers emphasize during the argumentation. Optional column for aspects that cannot be classified as any of the typical argumentation aspects listed. Observer will request copy of task(s). It is expected that at most three argumentation tasks will be worked on during one class period but additional copies of this page will be on hand, in the event more tasks take place.					
	Format/Structure of Argument	Vocabulary	Math Representations	Details (Sufficient Length)	Other
TASK 1					
TASK 2					
TASK 3					

Teacher Emphasis in Representations							
<p>Various aspects of representations that teachers emphasize during the argumentation. Optional column for aspects that cannot be classified as any of the typical representations listed.</p> <p>Observer will request copy of task(s). It is expected that at most three argumentation tasks will be worked on during one class period but additional copies of this page will be on hand, in the event more tasks take place.</p>							
	Symbols	Drawings	Graphs	Tables	Words	Other	Final Reps?
TASK 1							Student(s) Teacher
TASK 2							Student(s) Teacher
TASK 3							Student(s) Teacher