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Second-Order Modal Logic

Andrew Parisi
University of Connecticut - Storrs, andrew.parisi@uconn.edu

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Abstract: This dissertation develops an inferentialist theory of meaning. It takes as a starting point that the sense of a sentence is determined by the rules governing its use. In particular, there are two features of the use of a sentence that jointly determine its sense, the conditions under which it is coherent to assert that sentence and the conditions under which it is coherent to deny that sentence. From this starting point the dissertation develops a theory of quantification as marking coherent ways a language can be expanded and modality as the means by which we can reflect on the norms governing the assertion and denial conditions of our language. If the view of quantification that is argued for is correct, then there is no tension between second-order quantification and nominalism. In particular, the ontological commitments one can incur through the use of a quantifier depend wholly on the ontological commitments one can incur through the use of atomic sentences. The dissertation concludes by applying the developed theory of meaning to the metaphysical issue of necessitism and contingentism. Two objections to a logic of contingentism are raised and addressed. The resulting logic is shown to meet all the requirement that the dissertation lays out for a theory of meaning for quantifiers and modal operators.

Second-Order Modal Logic

Andrew Parisi

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M.A., University of Connecticut, **2013**

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Presented by
Andrew Parisi, B.A., M.A.

Major Advisor ______________________________

Marcus Rossberg

Associate Advisor ______________________________

Lionel Shapiro

Associate Advisor ______________________________

David Ripley

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Preface

This dissertation takes as its starting point the view that an account of the meaning of sentences must ultimately be given in terms of when it is incoherent to assert or deny those sentences. In this sense the work is an expansion on a bilateralist theory of meaning proposed by Rumfitt [59] and later Restall [50]. It, however, goes beyond these theories in many respects. One of the guiding lights of the dissertation is that assertion and denial alone are not enough to generate a theory of meaning for modal and quantifier expressions. These need to be supplemented with an account of how language can reflect on the norms governing assertion and denial in the case of modality and an account of assertion and denial-like uses of names in the case of quantification. Ultimately, this dissertation takes its cue from Dummett's *The Logical Basis of Metaphysics*: it attempts to shed light on difficult metaphysical questions by an examination of a theory of the meaning of the expressions used in discussion of those metaphysical questions.

Chapter 1 lays out what things are taken for granted in this dissertation. In particular, it is taken for granted that the use of an expression determines its meaning in the way discussed above. It also holds the Fregean thesis that expressions in a

language have both a sense and a reference. I argue from these assumptions to various constraints on an adequate theory of the meaning of an expression. Of particular importance for this work is that these constraints on a theory of meaning can be formalized. This sets a criterion of adequacy over all the work done in this dissertation. Any set of rules that does not meet the formal constraints on a theory of meaning cannot underwrite a theory of meaning.

Chapters 2 and 3 apply the results of chapter 1 to the case of modal operators. It is shown that standard accounts of modality merely in terms of assertion and denial are not formally adequate. I propose an alternative. A set of assertions and denials can be thought of as a description of how things are. Descriptions in a language are not independent of one another. It is possible to reflect on which descriptions are coherent and which are not. This forms the basis of the theory of modal expressions on offer. This account of modality is formalized by the notion of a hypersequent.[1] Chapter 2 shows that hypersequent calculi are formally adequate for a range of modal logics.

Chapter 3 discusses modality in the particular case of the operator 'it ought to be the case that…'. It is taken for granted that the appropriate logic for this operator is the modal logic D. The hypersequent account of that logic is shown in that chapter to be formally adequate. The chapter concludes by offering a preliminary interpretation of that formal system as a theory of meaning. It is coherent to deny 'It ought to be the case that φ' only if it is coherent to deny 'φ' in the relatively morally ideal description of the world and given a relatively morally ideal description of the world

[1] In some chapters, e.g. chapter 7 hypersequents are referred to as 'hyperpositions'.

wherein it is coherent to deny 'φ' it is coherent to deny 'It ought to be the case that φ'.

Chapters 4 and 5 discuss a theory of meaning for quantifiers. Chapter 4 clarifies arguments made by Sellars [63, 67] that not all quantification is ontologically significant. In particular, one can assert sentences of the form 'a is something' ($\exists f fa$) without incurring commitment to there being some thing which the entity denoted by 'a' is. Sellars's argument for this conclusion leads him to gesture towards a formal account of quantification that he never fully fleshes out. Chapter 4 argues that the account of quantification that is required by Sellars is one on which it is coherent to assert 'something is red' iff it is there is an expansion of my language by a term 't' such that it is coherent to assert 't is red'.

Chapter 5 gives a formal account of ontological commitment to show why someone adopting the standard account of quantification in terms of a satisfaction relation would be forced to hold that all quantification is ontologically significant. In other words, on the standard account of quantification if one asserts the sentence 'a is something' one is committed to there being some thing which the entity denoted by 'a' is. It then shows formally that on the account of quantification introduced in chapter 4 this result can be avoided. The chapter concludes by discussing the upshots of this approach for ontology generally. In particular, the ontological commitments one can incur in a language depend only tangentially on what forms of quantification that language adopts. The primary locus of ontological commitment is located in what one is committed to there being by commitment to atomic sentences.

Chapter 6 applies the lessons of the previous chapters on ontological commitment

to the case of first-order quantification. It argues that, as in the modal case, the bare notions of assertion and denial are insufficient to characterize all that is done with quantification and names. In addition to those uses of sentences there are also uses of names. Names can be accepted or rejected. These uses of names bear the same relation to denotation and lack thereof as assertion and denial of sentences bear to truth and falsity. In particular, it is shown that there is an adequate formal account of the first-order quantifier under consideration.

The dissertation concludes by combing all the work of the previous chapters and applying it to the debate between necessitism and contingentism. Williamson [74] argues that the proper account of second-order logic that is paired with the account of first-order quantification discussed in chapter 6 is one on which various important principles of second-order modal logic would fail to hold. This chapter shows that even if the analogous account of second-order quantification is adopted those principles do not fail. It also argues against Prior [43] who held that maintaining the principle that a thing exists if and only if there is something true of it requires denying the modal rule of necessitation and the interdefinability of necessity and possibility. The logic that results from these arguments is a Predicative Higher-order Intensional Logic (PHIL). PHIL is a logic for alethic modality that is developed from the accounts of modality developed in chapters 2 and 3 and quantification developed in chapters 4 to 6. The dissertation concludes with a proof that PHIL is an adequate formal account of a theory of meaning for higher-order modal logic.

The main upshot of the dissertation is that there is a viable inferentialist account of modality and quantification. Furthermore, the theory of meaning presented is not

only adequate by its own lights, but can be shown to be extensionally equivalent to most higher-order logics that are applied in linguistics. This suggests that an inferentialist philosophical account of meaning can be made consistent with current work in linguistics. While this conclusion is far from established, the work done in this dissertation shows that it is a possibility. More generally the dissertation offers an inferentialist account of meaning that does not differ from standard referentialist accounts in terms of the inferences that are validated. The two theories, as far as the inferential data is concerned, agree.

Finally, the dissertation suggests a novel approach to issues in ontology and metaphysics. These fields have been dominated by Quine's [47] claim that "to be assumed as an entity is, purely and simply, to be reckoned as the value of a variable." If this dissertation is correct then that account of ontological commitment is not the only coherent one available. The final chapter shows that the account of meaning developed in this dissertation can be used to motivate a plausible logic for contingentism. A logic whose expressive powers are that of a higher-order modal logic, but whose ontological commitments can be taken to be only objects (as opposed to properties, attributes, etc.) which exist in the actual world (as opposed to every possible world). The implication of this is that investigations into language are not necessarily metaphysically neutral. A study of language can exonerate views that may be implausible because other theories of meaning are taken as given.

Contents

Chapter 1

Theories of Meaning

Abstract. This chapter begins by singling out the objects of study of this dissertation as human languages: those languages that a human being could come to use in the practice of describing. It sets out what the constraints on offering a theory of meaning for such a language is and discusses those constraints in the context of classical propositional logic.

Keywords. Inferentialism, Sense and Reference, Theory of Meaning, Uniqueness, Cut-Admissibility

1.1 The Object of Study

This work takes language to be a broad category. The objects of study are the pure symbolic systems used for offering descriptions of things.[1]

These systems are called pure because one and the same symbolic system can be embodied in different media. One could write the system of natural numbers in base ten notation or base twelve notation without any loss of content. One could speak English in a variety of ways or opt for a vow of silence and only write English. As Sellars pointed out, there are many ways to play the same game of chess, one could play with wooden pieces on a standard board or on the counties of Texas with cars. These are embodiments of the same game. This dissertation is not concerned with the particular embodiments that a language might take but with the symbolic structures themselves.

The objects of study are also not necessarily languages that are what linguists call 'natural languages', or languages that a person could learn as their first language. In principle mathematicians can, as Frege hoped, converse with one another in formal languages. There are sentences of a formal language, for instance the Gödel sentence, that it is unlikely a person could learn when learning a natural language in the linguists sense but that mathematicians nonetheless call true. These sentences belong to languages that have some role in describing things, but it is far from clear that they are languages that a human being could learn as their first language.

The objects of study for this investigation are those languages that human beings

[1] The word 'things' is used here merely as a way of generalizing over the object of description. There is no presumption here that a thing described must be of a certain ontological category or have a metaphysical status.

could use to describe things. They need not be languages that a person could learn first but they are languages that humans can employ in giving a description of how things are.

This requirement itself imposes some restrictions on the objects of study. Humans could not use a language that is not recursively definable. Without a finite way of characterizing the good inferences of a language there is no way a human could come to be competent in that language.

Language, in the sense described above, is the vehicle by which humans describe the world. A theory of meaning for a language or set of languages offers a clearer view of the relationship between language and the world. The goal of this dissertation is to offer a theory of meaning for expressions that have something like the sense of "something", "everything", "It might have been the case that", "It is necessarily the case that", and related expressions in English.

1.2 Sense and Reference

A theory of meaning for a language like English would explain the difference between the sentence[2] "It is snowing but it is not cold" and "It is snowing and it is not cold". In so doing it would say what contribution 'but' makes to the meaning of sentences in which it occurs and how that is different from the contribution which 'and' makes. While the meaning of the above two sentences is different there is an important aspect of their meaning that they share. In any situation where it would not be wrong to

[2]Here and throughout I use 'sentence' as synonymous with 'declarative sentence'. A complete theory of meaning would have to account for the meanings of imperative, interrogative, and other uses of language. Those are set to the side in this work.

respond to an assertion of one with “That is correct” it would not be wrong to respond in the same way to the other.[3] Similarly if it would not be wrong to respond to one with “That’s wrong” it would not be wrong to respond to the other in the same way. The situations in which one would not be wrong in asserting or denying either sentence are exactly the same. The aspect of meaning that determines this feature of these sentences is called their sense.

The sense of a sentence is the part of the meaning of a sentence that is most closely related to the descriptive content of that sentence. The sense of a sentence determines what ways the world would have to be if that sentence were true or false. A second part of the meaning of a sentence – where ‘meaning’ is taken as a broad category – is the truth value of that sentence. Call this aspect of a sentence its *reference*. The reference of a sentence is that aspect of its meaning that a person might still be ignorant of even if they knew perfectly well how to use that sentence. Call any feature of the meaning of a sentence that is not its sense but relates to its proper use its *tone*. The sense and reference of “It is snowing but it is not cold” and “It is snowing and it is not cold” are thus the same. These two sentences differ only in tone. A person can know both the sense and the tone of the sentence “It is raining in Washington D.C. on August 1, 2016” without knowing the reference of that sentence.

The main assumption of this work is that the sense of a sentence along with the contribution of the world determines the reference of that sentence. The sense of a sentence determines the way things would have to be in order for that sentence to

[3] It is important that the sense of a sentence is not thereby to be identified with the situations in which it would not be inappropriate to call a sentence true or false.

be true or false. An account of what the senses of a class of sentences are thus lays bare what it would take for those sentences to be true or false. This assumption explains how a theory of meaning can be philosophically enlightening. Physicists, mathematicians, ethicists, etc. offer descriptions of the way things are. A physicist offers a description of physical reality, a mathematician offers a descriptions of mathematical reality, an ethicist offers a description of ethical reality, etc.

The specific focus of this work is to offer an account of the sense of quantifiers and modal expressions. A quantifier is an expression that is used to generalize over some syntactic category. Some paradigm cases of quantifiers are the English words ‘something’, ‘everything’, and ‘nothing’. These expressions can be used to generalize over the syntactic category of singular terms. For instance, “Something is a dog”, “Nothing is a dog”, and “Everything is a dog” are all sentences that have a quantifier in a place where a name could be, e.g. “Clifford is a dog”. It is this feature of quantifiers that accounts for Polyphemus’s misfortune when he was blinded by Odysseus. Odysseus tells Polyphemus that his name is ‘No one’, so when Polyphemus exclaims ‘No one has blinded me’ he does not communicate that he has been blinded. There are many quantifiers in English, e.g. ‘exactly three’, ‘a few’, and ‘most’. It has long been philosophical orthodoxy that the only genuine quantifiers of a language – where ‘language’ has the meaning discussed in section 1.1 – are ones that generalize over the syntactic category of singular terms. One of the results of this work is that there is nothing in principle wrong with generalizing over syntactic categories other than singular terms.

Modal operators, unlike ‘and’ and ‘or’ are not truth-functional. In general, the

truth value of a sentence whose main operator is a modal expression does not depend on the truth values of its component sentences. For example, the expression ‘Phoebe believes that’ is the main operator of the sentence “Phoebe believes that it is raining”. That sentence may be true if the sentence “It is raining” is true and even if it is not. Other modal expressions include, ‘it ought to be the case that’, ‘it was the case that’, and ‘it is possible that’.

For expressions which are not sentences their senses are the contribution that they make to sentences in which they occur. ‘And’ and ‘but’ have same sense because, as discussed above, substituting one for the other in a sentence will not change the sense of that sentence. It is this feature of language that explains how it is possible that language users can understand the senses of novel sentences. Many of the sentences in the above paragraphs have never been spoken or written before. Despite this fact they are intelligible to someone who understands the senses of the words they feature. If the contribution that an expressions makes to sentences in which it occurs is systematic, it is possible to offer an explanation of this important feature of language users. They need only grasp a finite number of senses as well as the rules for how to combine those senses to be able to grasp the sense of novel expressions. This aspect of the theory of meaning is called *compositionality*. Any account of the sense of expressions must be compositional. An account of meaning that is not compositional must provide an alternative explanation of how language users could come to understand novel sentences.

1.3 Assertion, Denial, and Positions

An assumption of this dissertation is that the meaning of an expression is determined by its use. The paradigm way in which language describes the world is through assertion and denial. It is these uses of sentences that determines their sense. The conditions under which it is not wrong to assert or deny a sentence are determined by the sense of that sentence. A position is a group of assertions and denials. Positions themselves can be thought of as descriptions. A person making assertions and denials on a particular subject is offering a position on that subject.

In a narrow sense, there may be only one position which is not wrong, the position that asserts all of the truths and denies all of the falsehoods. More broadly though, some positions can be shown to be wrong on the basis of the senses of the sentences that they assert or deny alone. For instance, the position that asserts and denies the same sentence is wrong. If a person asserts and denies the same sentence either the asserted sentence does not have the same sense as the denied sentence or that person has made a rational mistake. Another way of putting this point is that assertion and denial are exclusive speech acts. If a position is wrong in this way, it is called *incoherent*. A position that is not wrong in this way is called *coherent*. Positions other than the one that asserts and denies the same sentence are incoherent, e.g. the position that asserts 'It is raining and it is snowing' and denies 'It is snowing'. The incoherence of those positions, however, is recursively specified by the rules that fix the sense of expressions occurring in them and the incoherence of asserting and denying the same thing.

A controversial question in the philosophy of language is whether denial can be

explained in terms of assertion or not (see e.g. Dummett [13], or Rumfitt [59]). Dummett [13] offers a theory of meaning that takes assertion to be the primitive notion. He then claims that to deny a sentence is to assert its negation. Such a theory is called *unilateralist*. Part of the framework for the theory of meaning offered here is that there are two primitive notions required to determine the sense of a sentence. A view of this sort is called *bilateralist*. Following Rumfitt [59] and Restall [50] the two primitive uses of a sentence that determine its sense are the rules governing its assertion and denial. Position contain room for assertion and room for denial neither of which can be explained in terms of the other.

The challenge for the unilateralist is to explain denial in terms of other speech acts. Traditionally, unilateralists attempt to explain the denial of a sentence 'S' in terms of an assertion of the negation of 'S'. But in order for someone asserting 'not-S' to be denying 'S', the connective 'not' must have a feature that explains why an assertion of 'not-S' entails that 'S' ought to be denied.

Negation in unilateralist theories therefore plays a special role. In general to deny a sentence 'S' is to assert the negation of 'S'. The challenge then is to substantiate this claim. One way to show why this is the case is to hold that the logical form of 'not-S' is 'if S then $\perp$', where '$\perp$' is a special sentences. '$\perp$' is special because it is supposed to come with ought-to-be-deniedness built in. It would be unhelpful for a unilateralist to take this feature of '$\perp$' to be primitive. That either pushes the question back – a unilateralist must explain why '$\perp$' ought to be denied in terms of assertion – or violates the unilateralist principle that only one speech act is taken as primitive – that '$\perp$' ought to be denied is not explained in terms of assertion.

Philosophers such as [13] opt for the former option. They attempt to explain why '$\perp$' ought to be denied in terms of assertion. The strategy is to say that '$\perp$' ought to be denied because an assertion of '$\perp$' commits one to the truth of every sentence of the language. But this cannot explain why $\perp$ ought to be denied unless the conjunction of every sentence also ought to be denied. Perhaps the response on the part of the unilateralist should be that the conjunction of every sentence ought to be rejected because it cannot be true. But this is only a feature of a language once it already has negation or a conditional – there is nothing wrong with asserting the conjunction of all atomic sentences. Thus, there must be something about negation or the conditional that explains why '$\perp$' ought to be denied. It is unclear what features of these connectives can offer an explanation of denial in terms of assertion. If 'S and not-S' ought to be denied and 'not-S' is explained as 'if S then $\perp$' then we are back where we began: that sentence ought to be denied because it entails '$\perp$' and '$\perp$' ought to be denied.

1.4 The Senses of the Sentential Connectives

A sentential connective is an expression that produces a sentence when combined with other sentences. Common examples in English are 'and', 'or', and 'it is not the case that'. A sentence is just the sort of thing that can be asserted or denied. A person cannot assert a table, a chair, or a number. Similarly a person cannot assert a name or predicate.[4] A person can assert 'It is raining'. Since 'It is raining' is a

[4]While a person cannot assert a name, it is argued in chapter 6 that there is an analogous use of names to that of assertion for sentences. These uses of names are called 'acceptance' and 'rejection'. Accepting a name is bears the same relation to the fact that that name denotes as assertion of a

sentence and 'it is not the case that' is a one-place sentential connective 'It is not the case that it is raining' is a sentence. Certain sentential connectives play a crucial role in offering a theory of meaning for modal and quantificational expressions. In particular the correlates to the English expressions 'and', 'or', 'it is not the case that', and 'if ... then ...' lay the foundation for investigation launched in this work. These are called the extensional sentential connectives. This section discusses the theory of the meaning for these connectives.

Formal tools are helpful in making clear and precise any theory. A theory of meaning is no exception to this rule. The notion of a sentence, a position, and other notions discussed above can be formalized.

The language under consideration $\mathscr{L}$ consists of an infinite set of *atomic* sentences $p_1, q_1, r_1, p_2, q_2, \ldots$. The translation of atomic sentences into English is arbitrary. Some translations (perhaps all) of atomic sentences into English make it the case that it is incoherent to assert Γ and deny Δ where Γ and Δ are sets of atomic sentences. For the purposes of this work this feature of some specific translation is ignored. The senses of atomic sentences are assumed as given and are assumed not to rule any assertion or denial of other atomic sentences out. The set of sentential connectives that will be considered are '$\neg$', '$\wedge$', '$\vee$', and '$\rightarrow$' (translated roughly as 'it is not the case that', 'and', 'or', and 'if ... then _ _ _'). The sentences of $\mathscr{L}$ are generated according to the recursive clause

- For any atomic sentence p, p is an element of $\mathscr{L}$.

sentence does to the fact that that sentence is true. Similarly, to reject a name is to take it that there is nothing that that name denotes. Acceptance and rejection of names are however different speech acts than assertion and denial of sentences.

- For any sentences φ and ψ, $\neg\varphi$, $(\varphi \wedge \psi)$, $(\varphi \vee \psi)$, and $(\varphi \rightarrow \psi)$ are elements of $\mathscr{L}$.

- Nothing else is an element of $\mathscr{L}$.

If Γ and Σ are sets of sentences then $\Gamma \Rightarrow \Sigma$ is the position that one would take up by asserting all of Γ and denying all of Σ. If a position is incoherent this is indicated by prefixing it with a turnstile, $\vdash$. For instance $\vdash \Gamma \Rightarrow \Sigma$ indicates that the position $\Gamma \Rightarrow \Sigma$ is incoherent.

1.4.1 Structural Rules

Some facts about which positions are incoherent follow immediately from the features of assertion and denial. It is noted above that assertion and denial are exclusive speech acts. It is incoherent to assert and deny the same sentence. If a person asserts and denies the same sentence then either they are not using those sentences univocally or they are making a rational mistake. Put formally, any position of the form $\varphi \Rightarrow \varphi$ is incoherent, i.e. $\vdash \varphi \Rightarrow \varphi$.

A calculus offers a systematic way of relating positions to one another. In particular incoherent positions are linked with other incoherent positions in statements of the form "If $\Gamma_1 \Rightarrow \Sigma_1, \ldots, \Gamma_n \Rightarrow \Sigma_n$ are incoherent then $\Delta \Rightarrow \Lambda$ is incoherent". To keep notation short and clear this conditional statement is replaced by a horizontal bar with the positions of the antecedent above the bar and the conclusion below the bar, e.g.

$$\frac{\Gamma_1 \Rightarrow \Sigma_1 \qquad \ldots \qquad \Gamma_n \Rightarrow \Sigma_n}{\Delta \Rightarrow \Lambda}$$

The fact that assertion and denial are rationally exclusive acts is written in this new notation as

$$\overline{\Gamma, \varphi \Rightarrow \varphi, \Sigma}$$

which says that no matter what is incoherent $\Gamma, \varphi \Rightarrow \varphi, \Sigma$ is.

A *deduction* of a position $\Gamma \Rightarrow \Sigma$ relative to a calculus C is a tree whose leaves all have the form $\Delta, \varphi \Rightarrow \varphi, \Lambda$, each of whose nodes results from the nodes above it in accordance with a rule of C, and whose root is $\Gamma \Rightarrow \Sigma$. Since all instances of $\Delta, \varphi \Rightarrow \varphi, \Lambda$ are incoherent and the rules of a calculus preserve incoherence if there is a deduction of $\Gamma \Rightarrow \Sigma$ in a calculus C then $\Gamma \Rightarrow \Sigma$ is incoherent according to C. For each calculus that is considered the converse that if there is no deduction of a position then that position is coherent is also assumed. In the present context this amounts to the assumption that the only incoherent sets of assertions and denials that are not given by rules of a calculus are positions of the form $\Delta, \varphi \Rightarrow \varphi, \Lambda$.

A rule is R is *derivable* in a calculus C iff deductions of the premises of R can be transformed into a deduction of the conclusion of R by applications of the rules of C. A rule R is admissible for a calculus C iff whenever the premises of R are deducible then so is its conclusion. All derivable rules are admissible but the converse does not necessarily hold.

That $\Gamma, \varphi \Rightarrow \varphi, \Sigma$ is incoherent is given by the rule of Id in fig. 1.1. Other structural rules of the calculus can be justified on the basis that they are features of the practice of assertion and denial. For instance, if it is incoherent to assert all of Γ and deny all of Σ then it is incoherent assert all of Γ, deny all of Σ, and assert (deny) φ. This corresponds to the rules of TL(R) of fig. 1.1.

Figure 1.1: Propositional Logic

STRUCTURAL RULES

$$\text{Id}\ \frac{}{\varphi \Rightarrow \varphi} \qquad \text{Cut}\ \frac{\Gamma \Rightarrow \varphi, \Sigma \qquad \Gamma, \varphi \Rightarrow \Sigma}{\Gamma \Rightarrow \Sigma}$$

$$\text{TL}\ \frac{\Gamma \Rightarrow \Sigma}{\Gamma, \varphi \Rightarrow \Sigma} \qquad \text{TR}\ \frac{\Gamma \Rightarrow \Sigma}{\Gamma \Rightarrow \varphi, \Sigma}$$

OPERATIONAL RULES

$$\text{L}\neg\ \frac{\Gamma \Rightarrow \varphi, \Sigma}{\Gamma, \neg\varphi \Rightarrow \Sigma} \qquad \text{R}\neg\ \frac{\Gamma, \varphi \Rightarrow \Sigma}{\Gamma \Rightarrow \neg\varphi, \Sigma}$$

$$\text{L}\wedge\ \frac{\Gamma, \varphi, \psi \Rightarrow \Sigma}{\Gamma, \varphi \wedge \psi \Rightarrow \Sigma} \qquad \text{R}\wedge\ \frac{\Gamma \Rightarrow \varphi, \Sigma \qquad \Gamma \Rightarrow \psi, \Sigma}{\Gamma \Rightarrow \varphi \wedge \psi, \Sigma}$$

$$\text{L}\vee\ \frac{\Gamma, \varphi \Rightarrow \Sigma \qquad \Gamma, \psi \Rightarrow \Sigma}{\Gamma, \varphi \vee \psi \Rightarrow \Sigma} \qquad \text{R}\vee\ \frac{\Gamma \Rightarrow \varphi, \psi, \Sigma}{\Gamma \Rightarrow \varphi \vee \psi, \Sigma}$$

$$\text{L}\to\ \frac{\Gamma \Rightarrow \varphi, \Sigma \qquad \Gamma, \psi \Rightarrow \Sigma}{\Gamma, \varphi \to \psi \Rightarrow \Sigma} \qquad \text{R}\to\ \frac{\Gamma, \varphi \Rightarrow \psi, \Sigma}{\Gamma \Rightarrow \varphi \to \psi, \Sigma}$$

The validity of Cut

$$\text{Cut}\ \frac{\Gamma, \varphi \Rightarrow \Sigma \qquad \Gamma \Rightarrow \varphi, \Sigma}{\Gamma \Rightarrow \Sigma}$$

entails that any coherent position can be expanded to a maximally coherent position. Cut read from top to bottom indicates that if $\Gamma \Rightarrow \Sigma$ is coherent then for any sentence φ either $\Gamma, \varphi \Rightarrow \Sigma$ is coherent or $\Gamma \Rightarrow \varphi, \Sigma$ is coherent. A maximal coherent position is a coherent position that either asserts or denies every sentence of the language. The validity of Cut guarantees that for any coherent position there is a maximally

coherent position that asserts and denies at least all the same things as the original position. Maximal coherent positions play a crucial role in chapter 5.

One of the fundamental assumptions of this work is that the sense of a sentence determines its reference. The contribution that an expression – for now the only ones under consideration are the sentential connectives – makes to the sense of sentences in which it occurs is given by its operational rules. The sense of atomic sentences are throughout taken as given. A theory of the senses of atomic sentences is left for future work.[5]

If one knew the sense of a sentence one might still not know the reference of that sentence. What the reference of a sentence is, its truth value, is a product of the sense of that sentence and the way things are. The sense of a sentence determines its reference in the following way. The sense of a sentence determines the set of coherent positions given an assertion or denial of that sentence. If one knew how the world was with respect to those positions or some subset of those positions then one could determine the truth value of a sentence. If one knew the sense of a sentence and the way the world was then one would know the reference of that sentence.

The senses of sentences in a position determine complete stories of the world, i.e. maximal coherent positions, that are extensions of that position. This is made clear during the construction of a completeness proof such as theorem 1.6.2 where the operational rules governing sentences are used to expand a position. Complete stories of the world therefore are tightly linked with the notions of sense and reference. If a person knew all of the facts (independently of how they are expressed in a language)

[5] As is apparent in chapter 5 a theory of the senses of atomic sentences is required for a full account of how the ontological commitments of a position are to be determined.

and knew the senses of every sentence in a language then they would be able to determine the reference of each of those sentences. The senses of sentences determine which complete positions are complete stories of the world, that is, they determine which positions that assert or deny every sentence of the language are coherent.

In actual practice, the sense of a sentence determines a set of complete stories of the world. One position $\Delta \Rightarrow \Lambda$ is an extension of another $\Gamma \Rightarrow \Sigma$ iff $\Gamma \subseteq \Delta$ and $\Sigma \subseteq \Lambda$. Let $\Gamma \Rightarrow \Sigma$ be a coherent position that asserts $\neg\neg p$, i.e. $\neg\neg p \in \Gamma$. It follows from L$\neg$ and R$\neg$ that any complete story of the world that is an extension of $\Gamma \Rightarrow \Sigma$ will assert the sentence p. If a person knew enough of the story of the world that they knew that all of Γ were correctly assertible and all of Σ were correctly deniable then they would be able to determine that the complete story of the world asserted p given their knowledge of what $\neg$ contributes to the sense of a sentence.

As mentioned above the cut rule guarantees that every coherent position can be extended to a complete story of the world. If the cut rule were invalid then there would be a sentence φ and position $\Gamma \Rightarrow \Sigma$ such that $\Gamma, \varphi \Rightarrow \Sigma$ and $\Gamma \Rightarrow \varphi, \Sigma$ are both incoherent even though $\Gamma \Rightarrow \Sigma$ is coherent. In this case the coherent position $\Gamma \Rightarrow \Sigma$ cannot be extended to a complete story of the world. Since $\Gamma \Rightarrow \Sigma$ is coherent the world might be such that all of Γ is true and all of Σ is false. In such a scenario there could be no complete story of the world. It would be incoherent, given that one accepted all of Γ and rejected all of Σ, to assert φ or to deny φ. Put referentially φ could neither be true nor false, it would have no referent. This violates the principle that, for sentences at least, their sense determines their reference.[6]

[6] It is important to note that 'reference' here does not mean the same as 'stands for' or 'bears a significant relation to'. The term 'reference' in this context is only relational in its surface grammar.

The sense of each of the connectives is given by the rules governing their assertion and denial conditions. The R$\neg$ rule says that if it is incoherent to assert φ then it is incoherent to deny $\neg\varphi$. Contrapositively, R$\neg$ says that if it is coherent to deny $\neg\varphi$ then it is coherent to assert φ.

Similarly the sense of the conditional is given by L$\rightarrow$ and R$\rightarrow$. Read from bottom to top, L$\rightarrow$ says that if $\varphi \rightarrow \psi$ is coherent to assert while asserting all of Γ and denying all of Σ then either it is coherent to assert ψ while asserting all of Γ and denying all of Σ or it is coherent to deny φ while asserting all of Γ and denying all of Σ. The R$\rightarrow$ rule says that if it is coherent to deny $\varphi \rightarrow \psi$ then it is coherent to assert φ and deny ψ while asserting all of Γ and denying all of Σ.

There is debate amongst philosophers as to whether or not the conditional whose sense is given in the preceding paragraph should be translated as 'if ... then ...'. This work takes no side in this debate. This should not be taken to suggest that the formal analysis of language has no bearing on natural language. Conditionals in natural language are as various as modal vocabulary. Whether or not there is a sense of 'if φ then ψ' in English that is adequately translated as $\varphi \rightarrow \psi$ is an open question. If the structural rules of fig. 1.1 mirror the practices of assertion and denial then there is no reason to suppose that a sense could not be given to such a connective.

The targets of the theory of meaning of this work are quantifiers and modal

The claim that 'S refers to the True' is more perspicuously written as 'S is true'. Similarly for names, which do bear an important 'standing for' relation to object. The claim that 'S refers to o' is more perspicuously written as 'S is a name of o'. Names that stand for nothing do not therefore have no reference in the sense being used here. Throughout I will use the term 'denote' for the relation that names have to objects for which they stand.

expressions. A theory of the senses of these expressions is given compositionally. A full exposition of the theory of meaning here proposed would require an account of the senses of atomic sentences. While something is said about this in chapter 4 and chapter 5 much of the work is left to the side. The theory of meaning presented takes for granted a theory of the senses of atomic sentences. It assumes nothing about them except that they are determined by the rules governing their assertion and denial.

1.5 Constraints on a Theory of Sense

Prior [44] makes it clear that not every set of rules can determine the meaning of an expression. Consider the two-place connective ※ whose 'sense' is determined by L※ and R※.

$$\text{L※}\ \frac{\Gamma, \varphi, \psi \Rightarrow \Sigma}{\Gamma, \varphi\text{※}\psi \Rightarrow \Sigma} \qquad\qquad \text{R※}\ \frac{\Gamma \Rightarrow \varphi, \psi, \Sigma}{\Gamma \Rightarrow \varphi\text{※}\psi, \Sigma}$$

If ※ is introduced into the framework given by fig. 1.1 then all positions are incoherent. The following deduction establishes this for an arbitrary position $\Gamma \Rightarrow \Sigma$.

$$
\begin{prooftree}
\AxiomC{$p \Rightarrow p$}
\LeftLabel{TL}
\UnaryInfC{$p, q \Rightarrow p$}
\LeftLabel{L※}
\UnaryInfC{$p\text{※}q \Rightarrow p$}
\AxiomC{$q \Rightarrow q$}
\LeftLabel{TR}
\UnaryInfC{$q \Rightarrow p, q$}
\LeftLabel{R※}
\UnaryInfC{$q \Rightarrow p\text{※}q$}
\LeftLabel{cut}
\BinaryInfC{$q \Rightarrow p$}
\LeftLabel{L$\neg$}
\UnaryInfC{$q, \neg p \Rightarrow$}
\LeftLabel{L※}
\UnaryInfC{$q\text{※}\neg p \Rightarrow$}
\AxiomC{$q \Rightarrow q$}
\LeftLabel{TL}
\UnaryInfC{$q, p \Rightarrow q$}
\LeftLabel{L※}
\UnaryInfC{$q\text{※}p \Rightarrow q$}
\AxiomC{$p \Rightarrow p$}
\LeftLabel{TR}
\UnaryInfC{$p \Rightarrow q, p$}
\LeftLabel{R※}
\UnaryInfC{$p \Rightarrow q\text{※}p$}
\LeftLabel{cut}
\BinaryInfC{$p \Rightarrow q$}
\LeftLabel{R$\neg$}
\UnaryInfC{$\Rightarrow q, \neg p$}
\LeftLabel{R※}
\UnaryInfC{$\Rightarrow q\text{※}\neg p$}
\LeftLabel{Cut}
\BinaryInfC{$\Rightarrow$}
\LeftLabel{TL/TR}
\UnaryInfC{$\Gamma \Rightarrow \Sigma$}
\end{prooftree}
$$

Since there is no language in which every position is incoherent L※ and R※ cannot determine the sense of any connective. This work does not attempt to state necessary

and sufficient conditions for what sets of rules determine a sense. The assumptions made so far about sense, however, entail two important features of rules on their being meaning conferring.

The sense of a sentence is determined by the rules governing the assertion and denial of that sentence. This entails the uniqueness constraint on what sets of rules can by themselves determine a single sense. Let S be a set of rules proposed as determining the meaning of the n-place connective $\downharpoonleft$. $S[\upharpoonright / \downharpoonleft]$ is the result of replacing every occurrence of $\downharpoonleft$ in the rules of S by $\upharpoonright$. Let $\upharpoonright$ be an expression whose meaning is said to be determined by $S[\upharpoonright / \downharpoonleft]$. The rules S are said to uniquely characterize the sense of $\downharpoonleft$ iff the following two conditions obtain in the calculus expanded by S and $S[\upharpoonright / \downharpoonleft]$

U1 $\vdash \Gamma, \downharpoonleft \psi_1, \ldots, \psi_n \Rightarrow \Sigma$ iff $\vdash \Gamma, \upharpoonleft \psi_1, \ldots, \psi_n \Rightarrow \Sigma$

U2 $\vdash \Gamma \Rightarrow \downharpoonleft \psi_1, \ldots, \psi_n, \Sigma$ iff $\vdash \Gamma \Rightarrow \upharpoonright \psi_1, \ldots, \psi_n, \Sigma$.

If one of U1 or U2 were to fail then the sense of $\downharpoonleft$ could not be said to be determined by the rules governing its assertion and denial. Suppose for instance that U1 failed and that there were a position $\Gamma \Rightarrow \Sigma$ such that $\vdash \Gamma, \downharpoonleft \psi_1, \ldots, \psi_n \Rightarrow \Sigma$ and $\nvdash \Gamma, \upharpoonright \psi_1, \ldots, \psi_n \Rightarrow \Sigma$. In such a case there is a complete story of the world that asserts all of Γ and $\downharpoonleft \psi_1, \ldots, \psi_n$ and denies all of Σ but there is no corresponding story that asserts $\upharpoonright \psi_1, \ldots, \psi_n$. Since these two sentences might differ in their reference, for instance in the case that it is correct to assert Γ and correct to deny Σ, they cannot coincide in sense. Since, abstracting away from the particular symbols '$\downharpoonleft$' and '$\upharpoonright$', S and $S[\upharpoonright / \downharpoonleft]$ are the sets of rules, this set of rules cannot determine a single sense. If

uniqueness fails for a set of rules S then that set of rules does not *determine* a single sense for the expressions that they govern. It follows that uniqueness is a necessary condition for a set of rules to determine the sense of an expression.

Uniqueness is not a sufficient condition for a set of rules to determine a single meaning. L$\divideontimes$ and R$\divideontimes$ uniquely determine the 'sense' of $\divideontimes$. Let $\doubleplus$ be introduced with the rules $\{L\divideontimes, R\divideontimes\}[\doubleplus/\divideontimes]$. The following two deductions establish along with applications of cut establish U1 and U2 for $\divideontimes$ and $\doubleplus$.

$$
\begin{prooftree}
\AxiomC{}
\LeftLabel{Id}
\UnaryInfC{$\varphi \Rightarrow \varphi$}
\LeftLabel{TL}
\UnaryInfC{$\varphi, \psi \Rightarrow \varphi$}
\LeftLabel{L$\divideontimes$}
\UnaryInfC{$\varphi \divideontimes \psi \Rightarrow \varphi$}
\LeftLabel{TR}
\UnaryInfC{$\varphi \divideontimes \psi \Rightarrow \varphi, \psi$}
\LeftLabel{R$\doubleplus$}
\UnaryInfC{$\varphi \divideontimes \psi \Rightarrow \varphi \doubleplus \psi$}
\LeftLabel{TL/TR}
\UnaryInfC{$\Gamma, \varphi \divideontimes \psi \Rightarrow \varphi \doubleplus \psi, \Sigma$}
\end{prooftree}
\qquad
\begin{prooftree}
\AxiomC{}
\LeftLabel{Id}
\UnaryInfC{$\varphi \Rightarrow \varphi$}
\LeftLabel{TL}
\UnaryInfC{$\varphi, \psi \Rightarrow \varphi$}
\LeftLabel{L$\doubleplus$}
\UnaryInfC{$\varphi \doubleplus \psi \Rightarrow \varphi$}
\LeftLabel{TR}
\UnaryInfC{$\varphi \doubleplus \psi \Rightarrow \varphi, \psi$}
\LeftLabel{R$\divideontimes$}
\UnaryInfC{$\varphi \doubleplus \psi \Rightarrow \varphi \divideontimes \psi$}
\LeftLabel{TL/TR}
\UnaryInfC{$\Gamma, \varphi \doubleplus \psi \Rightarrow \varphi \divideontimes \psi, \Sigma$}
\end{prooftree}
$$

Another important feature of a calculus is that the rule of cut be admissible. This feature is not necessarily indispensable for a set of rules to determine a sense for a sentence but it entails some interesting properties for the senses of the expressions being studied.

The admissibility of cut is a property of a calculus not a set of rules, and so it is difficult to say what it would take for a set of rules to be cut-admissible except when that set of rules is considered as a calculus in its own right. If cut is admissible then the rules of the calculus alone guarantee that if a position is coherent then there is a coherent maximal extension of that position.

The expressions that are being investigated are those for which operational rules are being offered. The cut admissibility of the calculi in this work entails that each of the expressions being studied conservatively extends the rest of the language. If

the position $\Gamma \Rightarrow \Sigma$ is incoherent and does not contain the expression ε then there is a deduction of $\Gamma \Rightarrow \Sigma$ that does not make use of any of the rules governing ε. Contrapositively if $\Gamma \Rightarrow \Sigma$ is incoherent then there is a deduction of $\Gamma \Rightarrow \Sigma$ that makes use only of the rules governing vocabulary appearing in $\Gamma \cup \Sigma$. This provides evidence that the sense of an expression ε is determined completely by the rules governing its use.

The structural rules of a language do not contribute to the individual senses of a sentence. If a structural rule contributes anything to the sense of a sentence it contributes the same thing to the sense of every sentence. Structural rules govern the way any sentence can behave in a position. Structural rules cannot then help to explain what is the shared between the two sentences 'It is snowing and it is not cold' and 'It is snowing but its not cold'. Structural rules could also not help to explain what the difference between 'It is raining or it is snowing' and 'It is raining and it is snowing'. Since sense was posited to explain those features of sentences it would be superfluous to hold that the structural rules of a language make a contribution to the senses of sentences of that language.

If cut is admissible then the calculus in question is also consistent. Consider the calculus of fig. 1.1. If cut is admissible then for any deduction δ of $\Gamma \Rightarrow \Sigma$ there is a deduction δ' of $\Gamma \Rightarrow \Sigma$ that does not use the rule of cut. Consider the calculus of fig. 1.1 without the rule of cut. Suppose that the position $\quad \Rightarrow \quad$ were derivable from $\Gamma \Rightarrow \Sigma$. It would either have to be an axiom or follow from some rule. Axioms contain sentences and rules only add sentences to a calculus. Therefore, $\quad \Rightarrow \quad$ is not derivable. To say that a calculus is consistent is to say that $\nvDash \quad \Rightarrow \varphi \wedge \neg\varphi$. In

this setting $\vdash \quad \Rightarrow \varphi \wedge \neg\varphi$ iff $\vdash \quad \Rightarrow \quad$ as is proved by the following two deductions.

$$\text{TR}\;\frac{\Rightarrow}{\Rightarrow \varphi \wedge \neg\varphi} \qquad\qquad \text{cut}\;\frac{\Rightarrow \varphi \wedge \neg\varphi \qquad \text{L}\wedge\;\dfrac{\text{L}\neg\;\dfrac{\text{Id}\;\dfrac{}{\varphi \Rightarrow \varphi}}{\varphi, \neg\varphi \Rightarrow}}{\varphi \wedge \neg\varphi \Rightarrow}}{\Rightarrow}$$

Since there is no cut-free deduction of $\quad\Rightarrow\quad$, there is no cut-free deduction of $\Rightarrow \varphi \wedge \neg\varphi$. If cut is admissible and there is no cut-free deduction of $\quad \Rightarrow \varphi \wedge \neg\varphi$ then there is no deduction of $\quad \Rightarrow \varphi \wedge \neg\varphi$ with the cut rule. Thus, if cut is admissible then the calculus in question is consistent.

Definition 1 (Sub-formula)**.** The *sub-formulas* of a formula φ, written $sub(\varphi)$, are given by induction on the complexity of φ.

- If φ is atomic then $sub(\varphi) = \{\varphi\}$.
- If φ is $\neg\psi$ then $sub(\varphi) = sub(\psi) \cup \{\varphi\}$.
- If φ is $\psi \odot \theta$ then $sub(\varphi) = sub(\psi) \cup sub(\theta) \cup \{\varphi\}$, for $\odot \in \{\wedge, \vee, \rightarrow\}$.

If Γ is a set of formulas then $sub(\Gamma) = \bigcup\{sub(\gamma) : \gamma \in \Gamma\}$.

A calculus has the sub-formula property when for any deduction δ of $\Gamma \Rightarrow \Sigma$ there is a deduction δ' of $\Gamma \Rightarrow \Sigma$ such that if γ appears in any position in δ' then $\gamma \in sub(\Gamma \cup \Sigma)$. Let δ' be a cut-free deduction of $\Gamma \Rightarrow \Sigma$. Since every rule of fig. 1.1, save cut, generates formulas out of their sub-formulas or is an instance of Id, any formula appearing in δ' must be a sub-formula of $\Gamma \cup \Sigma$. If a calculus is cut-admissible and δ entails $\Gamma \Rightarrow \Sigma$ then there is a cut-free deduction of $\Gamma \Rightarrow \Sigma$ with the above property.

In order to show that the calculus of fig. 1.1 is cut-admissible some lemmas are proved first.

Lemma 1.1. *All connectives can be defined in terms of* $\wedge$ *and* $\neg$.

Proof. This is shown by showing that if φ is of the form $\psi \odot \theta$ for $\odot \in \{\rightarrow, \vee\}$ then there is a sentence γ such that

1. $\vdash \Gamma, \gamma \Rightarrow \varphi, \Sigma$

2. $\vdash \Gamma, \varphi \Rightarrow \gamma, \Sigma$

for any $\Gamma \cup \Sigma$.

Case 1 ($\odot$ is $\rightarrow$). In this case γ is the sentence $\neg(\psi \wedge \neg\theta)$. (1) and (2) are established by the following deductions.

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{}\LeftLabel{Id}
\UnaryInfC{$\psi \Rightarrow \psi$}
\LeftLabel{TL}
\UnaryInfC{$\varphi, \psi \Rightarrow \psi$}
\AxiomC{}
\LeftLabel{Id}
\UnaryInfC{$\varphi \Rightarrow \varphi$}
\LeftLabel{TR}
\UnaryInfC{$\varphi \Rightarrow \varphi, \psi$}
\LeftLabel{L$\rightarrow$}
\BinaryInfC{$\varphi, \varphi \rightarrow \psi \Rightarrow \psi$}
\LeftLabel{L$\neg$}
\UnaryInfC{$\varphi, \neg\psi, \varphi \rightarrow \psi \Rightarrow$}
\LeftLabel{L$\wedge$}
\UnaryInfC{$\varphi \wedge \neg\psi, \varphi \rightarrow \psi \Rightarrow$}
\LeftLabel{R$\neg$}
\UnaryInfC{$\varphi \rightarrow \psi \Rightarrow \neg(\varphi \wedge \neg\psi)$}
\LeftLabel{TL/TR}
\UnaryInfC{$\Gamma, \varphi \rightarrow \psi \Rightarrow \neg(\varphi \wedge \neg\psi), \Sigma$}
\end{prooftree}
$$

$$
\begin{prooftree}
\AxiomC{}
\LeftLabel{Id}
\UnaryInfC{$\varphi \Rightarrow \varphi$}
\LeftLabel{TR}
\UnaryInfC{$\varphi \Rightarrow \psi, \varphi$}
\AxiomC{}
\LeftLabel{Id}
\UnaryInfC{$\psi \Rightarrow \psi$}
\LeftLabel{TL}
\UnaryInfC{$\psi, \varphi \Rightarrow \psi$}
\LeftLabel{R$\neg$}
\UnaryInfC{$\varphi \Rightarrow \psi, \neg\psi$}
\LeftLabel{R$\wedge$}
\BinaryInfC{$\varphi \Rightarrow \psi, \varphi \wedge \neg\psi$}
\LeftLabel{L$\neg$}
\UnaryInfC{$\neg(\varphi \wedge \neg\psi), \varphi \Rightarrow \psi$}
\LeftLabel{R$\rightarrow$}
\UnaryInfC{$\neg(\varphi \wedge \neg\psi) \Rightarrow \varphi \rightarrow \psi$}
\LeftLabel{TL/TR}
\UnaryInfC{$\Gamma, \neg(\varphi \wedge \neg\psi) \Rightarrow \varphi \rightarrow \psi, \Sigma$}
\end{prooftree}
$$

Case 2 ($\odot$ is $\vee$). In this case γ is $\neg(\neg\varphi \wedge \neg\psi)$. The following two deductions establish this case.

$$
\begin{prooftree}
\AxiomC{}
\LeftLabel{Id}
\UnaryInfC{$\varphi \Rightarrow \varphi$}
\LeftLabel{TR}
\UnaryInfC{$\varphi \Rightarrow \varphi, \psi$}
\AxiomC{}
\LeftLabel{Id}
\UnaryInfC{$\psi \Rightarrow \psi$}
\LeftLabel{TR}
\UnaryInfC{$\psi \Rightarrow \varphi, \psi$}
\LeftLabel{L$\vee$}
\BinaryInfC{$\varphi \vee \psi \Rightarrow \varphi, \psi$}
\LeftLabel{L$\neg$}
\UnaryInfC{$\neg\varphi, \varphi \vee \psi \Rightarrow \psi$}
\LeftLabel{L$\neg$}
\UnaryInfC{$\neg\varphi, \neg\psi, \varphi \vee \psi \Rightarrow$}
\LeftLabel{L$\wedge$}
\UnaryInfC{$\neg\varphi \wedge \neg\psi, \varphi \vee \psi \Rightarrow$}
\LeftLabel{R$\neg$}
\UnaryInfC{$\varphi \vee \psi \Rightarrow \neg(\neg\varphi \wedge \neg\psi)$}
\LeftLabel{TL/TR}
\UnaryInfC{$\Gamma, \varphi \vee \psi \Rightarrow \neg(\neg\varphi \wedge \neg\psi), \Sigma$}
\end{prooftree}
$$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\varphi \Rightarrow \varphi}\,\text{Id}}{\varphi \Rightarrow \varphi, \psi}\,\text{TR}}{\Rightarrow \varphi, \psi, \neg\varphi}\,\text{R}\neg \qquad \dfrac{\dfrac{\dfrac{}{\psi \Rightarrow \psi}\,\text{Id}}{\psi \Rightarrow \varphi, \psi}\,\text{TR}}{\Rightarrow \varphi, \psi, \neg\psi}\,\text{R}\neg}{\Rightarrow \varphi, \psi, \neg\varphi \wedge \neg\psi}\,\text{R}\wedge}{\neg(\neg\varphi \wedge \neg\psi) \Rightarrow \varphi, \psi}\,\text{L}\neg}{\neg(\neg\varphi \wedge \neg\psi) \Rightarrow \varphi \vee \psi}\,\text{R}\vee}{\Gamma, \neg(\neg\varphi \wedge \neg\psi) \Rightarrow \varphi \vee \psi, \Sigma}\,\text{TL/TR}
$$

□

The proof of lemma 1.1 did not rely on cut. Therefore, if cut is admissible for the calculus that contains only rules governing $\neg$ and $\wedge$, it is admissible for the full calculus of fig. 1.1.

1.6 Models

The calculus of fig. 1.1 corresponds to a set of models. Models are used here and elsewhere as a tool. They have merely instrumental and no explanatory value. As is discussed in chapter 2 they can in principle be dispensed with for philosophical purposes.

The notion of a model is expanded throughout this work just as the notion of a position is expanded to capture different aspects of the practice of offering a description of the way things are. A model in this case is a function $v : \mathscr{L} \longrightarrow \{T, F\}$ such that

- If φ is atomic then $v(\varphi) \in \{T, F\}$
- If φ is $\neg\psi$ then $v(\varphi) = T$ iff $v(\psi) = F$.

- If φ is $\psi \wedge \theta$ then $v(\varphi) = T$ iff $v(\psi) = v(\theta) = T$.

A model is a *counter-example* to a position $\Gamma \Rightarrow \Sigma$ iff for all $\gamma \in \Gamma$ $v(\gamma) = T$ and for all $\sigma \in \Sigma$ $v(\sigma) = F$. If there are no counter-examples to $\Gamma \Rightarrow \Sigma$ this is written as $\Gamma \models \Sigma$.

Theorem 1.6.1 (Soundness). *If* $\vdash \Gamma \Rightarrow \Sigma$ *then* $\Gamma \models \Sigma$.

Proof. This is proved by induction on the length deductions in fig. 1.1. Let δ be a deduction whose last inference is I. Since the smallest deductions are instances of Id that is the base case. There are no counter-examples to $\varphi \Rightarrow \varphi$ because all models are functions. The inductive hypothesis (IH) is that if δ' is a deduction smaller than δ then there are no counter-examples to the end sequent of δ'. There are seven other cases to consider.

Case 1 (I is TL). In this case δ has the form

$$\text{TL}\,\dfrac{\dfrac{\begin{matrix}\delta\\ \vdots\end{matrix}}{\Gamma \Rightarrow \Sigma}}{\Gamma, \varphi \Rightarrow \Sigma}$$

Suppose that there is a model v that is a counter-example to $\Gamma, \varphi \Rightarrow \Sigma$. For all $\gamma \in \Gamma \cup \{\varphi\}$ $v(\gamma) = T$. It follows that for all $\gamma \in \Gamma$ $v(\gamma) = T$. Since $v(\sigma) = F$ for all $\sigma \in \Sigma$ v is a counter-example to $\Gamma \Rightarrow \Sigma$. This contradicts IH which entails that there is no counter-example to $\Gamma \Rightarrow \Sigma$.

Case 2 (I is TR). This case is similar to the case where I is TL.

Case 3 (I is cut). For this case it is necessary to establish that for any v and φ $v(\varphi) = T$ or $v(\varphi) = F$. This is proved by induction on the complexity of φ. If φ

is atomic this is built into the definition of a model. Suppose that φ is $\neg\psi$. By the inner inductive hypothesis (IIH) $v(\psi) = T$ or $v(\psi) = F$. In the first case $v(\varphi) = T$ and in the second $v(\varphi) = F$. Suppose that φ is $\psi \wedge \theta$. By IIH either $v(\psi) = T$ or $v(\psi) = F$ and either $v(\theta) = T$ or $v(\theta) = F$. This generates four sub-cases to consider.

c1 $v(\psi) = T$ and $v(\theta) = T$.

c2 $v(\psi) = T$ and $v(\theta) = F$.

c3 $v(\psi) = F$ and $v(\theta) = T$.

c4 $v(\psi) = F$ and $v(\theta) = F$.

In c2 – c4 it is not the case that $v(\psi) = v(\theta) = T$. It follows that $v(\varphi) = F$ and so $v(\varphi) = F$ or $v(\varphi) = T$. In c1 $v(\psi) = v(\theta) = T$. So $v(\varphi) = T$ and thus that $v(\varphi) = T$ or $v(\varphi) = F$.

In this case δ has the form

$$\text{cut}\ \dfrac{\dfrac{\begin{array}{c}\delta_1\\ \vdots\end{array}}{\Gamma \Rightarrow \varphi, \Sigma} \qquad \dfrac{\begin{array}{c}\delta_2\\ \vdots\end{array}}{\Gamma, \varphi \Rightarrow \Sigma}}{\Gamma \Rightarrow \Sigma}$$

Applying IH to δ_1 and δ_2 yields that there are no counter-examples to $\Gamma \Rightarrow \varphi, \Sigma$ and there are no counter-examples to $\Gamma, \varphi \Rightarrow \Sigma$. Suppose for reductio that there is a counter-example v to $\Gamma \Rightarrow \Sigma$. By the above proof either $v(\varphi) = T$ or $v(\varphi) = F$. In the first case v is a counter-example to $\Gamma, \varphi \Rightarrow \Sigma$ which is impossible. In the second case v is a counter-example to $\Gamma \Rightarrow \varphi, \Sigma$.

Case 4 (I is L$\neg$). In this case δ has the form

$$\dfrac{\dfrac{\delta}{\vdots}}{\dfrac{\Gamma \Rightarrow \varphi, \Sigma}{\Gamma, \neg\varphi \Rightarrow \Sigma}}\ \text{L}\neg$$

By IH there is no counter-example to $\Gamma \Rightarrow \varphi, \Sigma$. For reductio let v be a counter-example to $\Gamma, \neg\varphi \Rightarrow \Sigma$. In particular, $v(\neg\varphi) = T$ and so $v(\varphi) = F$. It follows that v is a counter-example to $\Gamma \Rightarrow \varphi, \Sigma$.

Case 5 (I is R$\neg$). This case is similar to the case where I is L$\neg$.

Case 6 (I is L$\wedge$). In this case δ has the form

$$\dfrac{\dfrac{\delta}{\vdots}}{\dfrac{\Gamma, \varphi, \psi \Rightarrow \Sigma}{\Gamma, \varphi \wedge \psi \Rightarrow \Sigma}}\ \text{L}\wedge$$

By IH there is no counter-example to $\Gamma, \varphi, \psi \Rightarrow \Sigma$. Suppose for reductio that there is a counter-example v to $\Gamma, \varphi \wedge \psi \Rightarrow \Sigma$. In particular $v(\varphi \wedge \psi) = T$ so $v(\varphi) = v(\psi) = T$. v is also a counter-example to $\Gamma, \varphi, \psi \Rightarrow \Sigma$.

Case 7 (I is R$\wedge$). In this case δ has the form

$$\text{R}\wedge\ \dfrac{\dfrac{\dfrac{\delta_1}{\vdots}}{\Gamma \Rightarrow \varphi, \Sigma} \qquad \dfrac{\dfrac{\delta_2}{\vdots}}{\Gamma \Rightarrow \psi, \Sigma}}{\Gamma \Rightarrow \varphi \wedge \psi, \Sigma}$$

By IH there are no counter-examples to $\Gamma \Rightarrow \varphi, \Sigma$ and $\Gamma \Rightarrow \psi, \Sigma$. Suppose for reductio that v is a counter-example to $\Gamma \Rightarrow \varphi \wedge \psi, \Sigma$. In particular, $v(\varphi \wedge \psi) = F$. Either $v(\varphi) = F$ or $v(\psi) = F$. In the first case v is a counter-example to $\Gamma \Rightarrow \varphi, \Sigma$

which is impossible. In the second case v is a counter-example to $\Gamma \Rightarrow \psi, \Sigma$ which is also impossible.

$\square$

Theorem 1.6.2 requires the definition of several structures about which some important lemmas are proved. The first such is a tree τ constructed from a position $\Gamma \Rightarrow \Sigma$ written $\tau(\Gamma \Rightarrow \Sigma)$. At the root of the tree is $\Gamma \Rightarrow \Sigma$. The tree leaves of the tree at any stage are positions. A leaf $\Gamma \Rightarrow \Sigma$ is open iff $\Gamma \cap \Sigma = \varnothing$. A leaf is called 'closed' if it is not open. A branch is open when it contains no closed leaves. Let $\varphi_1, \varphi_2, \ldots$ be a list of the sentences of $\mathscr{L}$. At each stage of construction i do the following for each open leaf $\Gamma \Rightarrow \Sigma$ in the construction:

1. If φ_i is atomic then do nothing.

2. If φ_i is $\neg\psi$ then

   - if $\varphi_i \in \Gamma$ then expand the branch of the tree under consideration by

$$\frac{\Gamma \Rightarrow \psi, \Sigma}{\Gamma \Rightarrow \Sigma}$$

   - If $\varphi_i \in \Sigma$ then expand the branch of the tree under consideration by

$$\frac{\Gamma, \psi \Rightarrow \Sigma}{\Gamma \Rightarrow \Sigma}$$

3. If φ_i is $\psi \wedge \theta$ then

   - If $\varphi_i \in \Gamma$ then expand the branch of the tree under consideration by

$$\frac{\Gamma, \psi, \theta \Rightarrow \Sigma}{\Gamma \Rightarrow \Sigma}$$

- If $\varphi_i \in \Sigma$ then expand the branch of the tree under consideration by

$$\frac{\Gamma \Rightarrow \psi, \Sigma \qquad \Gamma \Rightarrow \theta, \Sigma}{\Gamma \Rightarrow \Sigma}$$

4. Repeat steps (1-3) for φ_j where $j < i$.

Let $\beta = \Gamma \Rightarrow \Sigma, \Gamma_1 \Rightarrow \Sigma_1, \ldots$ be an open branch in $\tau(\Gamma \Rightarrow \Sigma)$. $f(\beta)$ is the position

$$\cup_i \Gamma_i \Rightarrow \cup_i \Sigma_i$$

If β is an open branch where $\Delta_j \Rightarrow \Lambda_j$ occurs before $\Delta_k \Rightarrow \Lambda_k$ then $\Delta_j \subseteq \Delta_k$ and $\Lambda_j \subseteq \Lambda_k$. The set $\curlyvee(\Gamma \Rightarrow \Sigma)$ is the set of all $f(\beta)$ for open branches β in $\tau(\Gamma \Rightarrow \Sigma)$. If a position $\Gamma \Rightarrow \Sigma$ is such that $\nvdash_{cf} \Gamma \Rightarrow \Sigma$ then $\curlyvee(\Gamma \Rightarrow \Sigma) \neq \varnothing$ where $\vdash_{cf} \Gamma \Rightarrow \Sigma$ indicates that $\Gamma \Rightarrow \Sigma$ is provable in the calculus of fig. 1.1 without the rule of cut. If $\curlyvee(\Gamma \Rightarrow \Sigma) = \varnothing$ then there are no branches in $\tau(\Gamma \Rightarrow \Sigma)$ that are open but in that case $\tau(\Gamma \Rightarrow \Sigma)$ is a cut-free deduction of $\Gamma \Rightarrow \Sigma$.

Lemma 1.2. *For any position $\Delta \Rightarrow \Lambda \in \curlyvee(\Gamma \Rightarrow \Sigma)$ if $\neg\varphi \in \Delta$ then $\varphi \in \Lambda$ and if $\neg\varphi \in \Lambda$ then $\varphi \in \Delta$.*

Proof. Let β be a branch such that $f(\beta) = \Delta \Rightarrow \Lambda$. Let φ_j be $\neg\varphi$. At stage j (or later, without loss of generality let it be j) there is a position $\Delta' \Rightarrow \Lambda'$ at which φ_j is considered. Let $\Delta'' \Rightarrow \Lambda''$ be the position in β considered at stage $j + 1$. At that stage if $\neg\varphi \in \Delta'$ then $\varphi \in \Lambda''$ and if $\neg\varphi \in \Lambda'$ then $\varphi \in \Delta'$ by case 2 of the construction of $\tau(\Gamma \Rightarrow \Sigma)$. Since $\Delta'' \subseteq \Delta$ and $\Lambda'' \subseteq \Lambda$ those facts hold for $\Delta \Rightarrow \Lambda$ too. $\square$

Lemma 1.3. *For any position $\Delta \Rightarrow \Lambda \in \curlyvee(\Gamma \Rightarrow \Sigma)$ if $\varphi \wedge \psi \in \Delta$ then $\varphi \in \Delta$ and $\psi \in \Delta$ and if $\varphi \wedge \psi \in \Lambda$ then either $\varphi \in \Lambda$ or $\psi \in \Lambda$.*

Proof. Let β be a branch such that $f(\beta) = \Delta \Rightarrow \Lambda$. Let φ_j be $\neg\varphi$. At stage j (or later, without loss of generality let it be j) there is a position $\Delta' \Rightarrow \Lambda'$ at which φ_j is considered. Let $\Delta'' \Rightarrow \Lambda''$ be the position in β considered at stage $j+1$. At that stage if $\varphi_j \in \Delta'$ then $\varphi \in \Delta''$ and $\psi \in \Delta''$ by case 3 of the construction of $\tau(\Gamma \Rightarrow \Sigma)$. Similarly, if $\varphi_j \in \Lambda'$ then at stage j by clause 3 the $\tau(\Gamma \Rightarrow \Sigma)$ branches. β either passes through the left branch or the right branch. In the first case $\varphi \in \Lambda''$. In the second case $\psi \in \Lambda''$. □

Theorem 1.6.2 (Completeness)**.** *If* $\Gamma \models \Sigma$ *then* $\vdash_{cf} \Gamma \Rightarrow \Sigma$.

Proof. Suppose that $\not\vdash \Gamma \Rightarrow \Sigma$. Let $\Delta \Rightarrow \Lambda \in \curlyvee(\Gamma \Rightarrow \Sigma)$. $\Delta \Rightarrow \Lambda$ is guaranteed to exist because otherwise $\Gamma \Rightarrow \Sigma$ would have a deduction. Let v be the model that assigns all the atomic sentences in Γ T and all other atomic sentences F.

Claim 1. *For all* $\varphi \in \Delta$ $v(\varphi) = T$ *and for all* $\varphi \in \Lambda$ $v(\varphi) = F$.

Proof of Claim 1. This is proved by induction on the complexity of φ. The base case is trivial. The inductive hypothesis (IH) is that for all formula ψ of complexity less than φ if $\psi \in \Delta$ then $v(\psi) = T$ and if $\psi \in \Lambda$ then $v(\psi) = F$. There are four cases to consider

Case 1 (φ is $\neg\psi$ and $\varphi \in \Delta$). By lemma 1.2 $\psi \in \Lambda$. By IH, $v(\psi) = F$. It follows that $v(\varphi) = T$.

Case 2 (φ is $\neg\psi$ and $\varphi \in \Lambda$). This case is similar to the first one.

Case 3 (φ is $\psi \wedge \theta$ and $\varphi \in \Delta$). By lemma 1.3 $\psi \in \Delta$ and $\theta \in \Delta$. By IH $v(\psi) = T$ and $v(\theta) = T$. So $v(\varphi) = T$.

Case 4 (φ is $\psi \wedge \theta$ and $\varphi \in \Lambda$). By lemma 1.3 either $\psi \in \Lambda$ or $\theta \in \Lambda$. In the first case by IH $v(\psi) = F$ so $v(\varphi) = F$. In the second case $v(\theta) = F$ so $v(\varphi) = F$.

It follows from claim 1 that v is a counter-example to $\Delta \Rightarrow \Lambda$. Since $\Gamma \subseteq \Delta$ and $\Sigma \subseteq \Lambda$ v is also a counter-example to $\Gamma \Rightarrow \Sigma$. □

Theorem 1.6.3 (Cut Admissibility). *If* $\vdash \Gamma \Rightarrow \Sigma$ *then* $\vdash_{cf} \Gamma \Rightarrow \Sigma$.

Proof. Suppose that $\vdash \Gamma \Rightarrow \Sigma$. By theorem 1.6.1 $\Gamma \models \Sigma$. By theorem 1.6.2 $\vdash_{cf} \Gamma \Rightarrow \Sigma$. □

1.7 Cut Elimination

Let R be the rule that proceeds from premises $S_1, \ldots, S_n$ to S_c. R is said to be *admissible* for a calculus when if $S_1, \ldots, S_n$ are derivable then so is S_c. In particular theorem 1.6.3 shows that the rule of cut is admissible for the logic of fig. 1.1. A rule R is *eliminable* when it is admissible and there is an algorithm for transforming a deduction using rule R to one that makes no use of the rule R. Most of the proofs of this work are proofs that cut is admissible for a particular calculus. But because one calculus is proved to be cut eliminable it is worthwhile to see the shape of a cut elimination proof in the simple case of fig. 1.1.

The shape of a cut-elimination proof used here proceeds by showing that the rules for the logical connectives are invertible. Using that feature of the calculus it is shown that a deduction ending in a cut over a complex formula can be transformed into a deduction with cuts only over formulas that are less complex. Finally it is shown that cuts over atomic sentences can be eliminated.

Lemma 1.4. *If* $\vdash_{cf} \Gamma, \neg\varphi \Rightarrow \Sigma$ *then* $\vdash_{cf} \Gamma \Rightarrow \varphi, \Sigma$

Proof. This is proved by induction on the length of deductions. Let δ be deduction of $\Gamma, \neg\varphi \Rightarrow \Sigma$. If δ is an instance of Id, then either $\neg\varphi \in \Sigma$ or there is a $\psi \in \Gamma \cap \Sigma$. In the former case $\Gamma, \neg\varphi \Rightarrow \Sigma$ can be rewritten as $\Gamma, \neg\varphi \Rightarrow \neg\varphi, \Sigma$. The following is a deduction of the desired sequent

$$\dfrac{\dfrac{}{\Gamma, \varphi \Rightarrow \varphi, \Sigma}\,\mathrm{Id}}{\Gamma \Rightarrow \varphi, \neg\varphi, \Sigma}\,\mathrm{R}\neg$$

In the second case $\Gamma \Rightarrow \varphi, \Sigma$ is also an instance of Id. For the inductive cases the inductive hypothesis is that if there is a deduction δ' of $\Delta, \neg\varphi \Rightarrow \Lambda$ such that the length of δ' is less than the length of δ then there is a deduction δ'' of $\Delta \Rightarrow \varphi, \Lambda$. Let I be the last inference of δ. This leaves six cases to consider.

Case 1 (I is TL). If $\neg\varphi$ is the formula introduced by TL, then the appropriate instance of TR gives the result. If not, δ has the form

$$\dfrac{\dfrac{\begin{array}{c}\delta\\ \vdots\end{array}}{\Gamma', \neg\varphi \Rightarrow \Sigma}}{\Gamma, \neg\varphi \Rightarrow \Sigma}\,\mathrm{TL}$$

By IH there is a deduction δ' of $\Gamma' \Rightarrow \varphi, \Sigma$. An application of TL to this deduction gives the result.

Case 2 (I is TR). This is similar to case where I is TL.

Case 3 (I is L$\neg$). If $\neg\varphi$ is the formula introduced by I then the deduction ending in I maybe with an application IH gives the result. If not then an application of IH and L$\neg$ gives the result.

Case 4 (I is R$\neg$ or L$\wedge$). These cases are similar to the above.

Case 5 (I is R$\wedge$). In this case δ has the form

$$
\mathrm{R}\wedge\ \dfrac{\dfrac{\begin{matrix}\delta_1\\ \vdots\end{matrix}}{\Gamma, \neg\varphi \Rightarrow \psi, \Sigma} \qquad \dfrac{\begin{matrix}\delta_2\\ \vdots\end{matrix}}{\Gamma, \neg\varphi \Rightarrow \theta, \Sigma}}{\Gamma, \neg\varphi \Rightarrow \psi \wedge \theta, \Sigma}
$$

Applications of IH to δ_1 and δ_2 and an application of R$\wedge$ yields the following deduction

$$
\mathrm{R}\wedge\ \dfrac{\dfrac{\begin{matrix}\delta_1'\\ \vdots\end{matrix}}{\Gamma \Rightarrow \varphi, \psi, \Sigma} \qquad \dfrac{\begin{matrix}\delta_2'\\ \vdots\end{matrix}}{\Gamma \Rightarrow \varphi, \theta, \Sigma}}{\Gamma \Rightarrow \varphi, \psi \wedge \theta, \Sigma}
$$

□

Lemma 1.5. *If* $\vdash_{cf} \Gamma \Rightarrow \neg\varphi, \Sigma$ *then* $\vdash_{cf} \Gamma, \varphi \Rightarrow \Sigma$

Proof. The proof of this case is similar to lemma 1.4. □

Lemma 1.6. *If* $\vdash_{cf} \Gamma, \varphi \wedge \psi \Rightarrow \Sigma$ *then* $\vdash_{cf} \Gamma, \varphi, \psi \Rightarrow \Sigma$

Proof. This again is proved by induction on the length of deductions. Let δ be a deduction of $\Gamma, \varphi \wedge \psi \Rightarrow \Sigma$ and I be the last inference of δ. If I is Id then either $\varphi \wedge \psi \in \Sigma$ or $\Gamma \cap \Sigma \neq \varnothing$. The latter case is as above. The following deduction establishes the result for the former case

$$
\mathrm{R}\wedge\ \dfrac{\mathrm{TL}\ \dfrac{\mathrm{Id}\ \dfrac{}{\Gamma, \varphi \Rightarrow \varphi, \Sigma}}{\Gamma, \varphi, \psi \Rightarrow \varphi, \Sigma} \qquad \mathrm{TL}\ \dfrac{\mathrm{Id}\ \dfrac{}{\Gamma, \psi \Rightarrow \psi, \Sigma}}{\Gamma, \varphi, \psi \Rightarrow \psi, \Sigma}}{\Gamma, \varphi, \psi \Rightarrow \varphi \wedge \psi, \Sigma}
$$

There are six inductive cases to consider.

Case 1 (I is an instance of TL). In this case either $\varphi \wedge \psi$ is the formula being introduced or not. If it is, then two applications of TL suffice. If it is not then an application IH + TL suffices.

Case 2 (I is an instance of TR). This is similar to the second case above.

Case 3 (I is an instance of L$\neg$ or R$\neg$). In this case an application of IH and I gives the result.

Case 4 (I is an instance of L$\wedge$). If $\varphi \wedge \psi$ is not the main connective then the result follows from IH and I. If it is then the deduction immediately above I gives the result maybe with an application of IH.

Case 5 (I is R$\wedge$). This case if given by an application of IH and I

$\square$

Lemma 1.7. *If* $\vdash_{cf} \Gamma \Rightarrow \varphi \wedge \psi, \Sigma$ *then* $\vdash_{cf} \Gamma \Rightarrow \varphi, \Sigma$ *and* $\vdash_{cf} \Gamma \Rightarrow \psi, \Sigma$.

Proof. This is again proved by induction on the length of deductions. If the deduction is an instance of Id and $\varphi \wedge \psi$ is not the main connective then the desired sequent is also an instance of Id. The following two deductions suffice for the case where the δ is an instance of Id and $\varphi \wedge \psi$ is the main sentence.

$$\dfrac{\dfrac{\dfrac{}{\Gamma, \varphi \Rightarrow \varphi, \Sigma}\,\text{Id}}{\Gamma, \varphi, \psi \Rightarrow \varphi, \Sigma}\,\text{TL}}{\Gamma, \varphi \wedge \psi \Rightarrow \varphi, \Sigma}\,\text{L}\wedge \qquad \dfrac{\dfrac{\dfrac{}{\Gamma, \varphi \Rightarrow \psi, \Sigma}\,\text{Id}}{\Gamma, \varphi, \psi \Rightarrow \psi, \Sigma}\,\text{TL}}{\Gamma, \varphi \wedge \psi \Rightarrow \psi, \Sigma}\,\text{L}\wedge$$

2 There are six other cases to consider. All of them except possibly the case of R$\wedge$ follow as above. That case is the only one that is explicitly considered. If $\varphi \wedge \psi$

is the sentence introduced by I in that case then the result follows from the two deductions immediately preceding I. If not then the following deduction is given

$$\dfrac{\dfrac{\begin{array}{c}\delta_1\\ \vdots\end{array}}{\Gamma \Rightarrow \varphi \wedge \psi, \gamma, \Sigma} \quad \dfrac{\begin{array}{c}\delta_2\\ \vdots\end{array}}{\Gamma \Rightarrow \varphi \wedge \psi, \theta, \Sigma}}{\Gamma \Rightarrow \varphi \wedge \psi, \gamma \wedge \theta, \Sigma}\ \mathrm{R}\wedge$$

Applying IH to δ_1 gives deductions δ_1' of $\Gamma \Rightarrow \varphi, \gamma, \Sigma$ and δ_1'' of $\Gamma \Rightarrow \psi, \gamma, \Sigma$. Applying IH to δ_2 gives deductions δ_2' of $\Gamma \Rightarrow \varphi, \theta, \Sigma$ and δ_2'' of $\Gamma \Rightarrow \psi, \theta, \Sigma$. The following deductions are sufficient for this case

$$\dfrac{\dfrac{\begin{array}{c}\delta_1'\\ \vdots\end{array}}{\Gamma \Rightarrow \varphi, \gamma, \Sigma} \quad \dfrac{\begin{array}{c}\delta_2'\\ \vdots\end{array}}{\Gamma \Rightarrow \varphi, \theta, \Sigma}}{\Gamma \Rightarrow \varphi, \gamma \wedge \theta, \Sigma}\ \mathrm{R}\wedge \qquad \dfrac{\dfrac{\begin{array}{c}\delta_1''\\ \vdots\end{array}}{\Gamma \Rightarrow \psi, \gamma, \Sigma} \quad \dfrac{\begin{array}{c}\delta_2''\\ \vdots\end{array}}{\Gamma \Rightarrow \psi, \theta, \Sigma}}{\Gamma \Rightarrow \psi, \gamma \wedge \theta, \Sigma}\ \mathrm{R}\wedge$$

□

Theorem 1.7.1. *Cut is eliminable from the calculus given by fig. 1.1.*

Proof. It is proved that if δ of $\Gamma \Rightarrow \Sigma$ is a deduction whose last inference is cut and whose only instance of cut is that one, then there is a deduction δ' of $\Gamma \rightarrow \Sigma$ that is cut-free. Call that the *Main Claim.* Given the Main Claim, and a deduction that features multiple cuts each cut can be eliminated by replacing the deduction whose last inference is cut by the deduction given by Main Claim. *Proof of Main Claim.* This is proved by induction on the complexity of the formula featured in the cut rule. Let that formula be φ. In which case the deduction δ in question has the form

$$\mathrm{R}\wedge \frac{\dfrac{\begin{array}{c}\delta_1\\ \vdots\end{array}}{\Gamma \Rightarrow \psi, \Sigma} \qquad \dfrac{\begin{array}{c}\delta_2\\ \vdots\end{array}}{\Gamma, \psi \Rightarrow \Sigma}}{\Gamma \Rightarrow \Sigma}$$

There are three cases to consider.

Case 1 (φ is atomic). This case requires a sub-induction over the combined length of δ_1 and δ_2. For the base case it is supposed that one is an instance of Id. Without loss of generality suppose that is it $\Gamma \Rightarrow \psi, \Sigma$. In this case $\Gamma \Rightarrow \psi, \Sigma$ can be rewritten as $\Gamma, \psi \Rightarrow \psi, \Sigma$. In this case δ_2 is the required deduction. The only sub-inductive case where ψ could be the main formula is an instance of TL or TR. But then the result is the deduction immediately preceding that instance of TL or TR. In all the other cases the result follows from an application of S-IH and I.

The inductive hypothesis is that if δ' is a deduction of $\Delta \Rightarrow \Lambda$ that features cuts only over formulas of complexity less than φ then there is a cut-free deduction of $\Delta \Rightarrow \Lambda$

Case 2 (φ is $\neg\psi$). In this case δ has the form

$$\mathrm{cut}\ \frac{\begin{array}{c}\delta_1\\ \Gamma \Rightarrow \neg\psi, \Sigma\end{array} \qquad \begin{array}{c}\delta_2\\ \Gamma, \neg\psi \Rightarrow \Sigma\end{array}}{\Gamma \Rightarrow \Sigma}$$

An application of lemma 1.5 to δ_1 gives a deduction of $\Gamma, \psi \Rightarrow \Sigma$. An application of lemma 1.5 to δ_2 yields a deduction of $\Gamma \Rightarrow \psi, \Sigma$. Cutting over those deductions yields a deduction of $\Gamma \Rightarrow \Sigma$ with cuts only over formulas of complexity less than φ. The result is given by an application of IH.

Case 3 (φ is $\psi \wedge \theta$). In this case δ has the form

$$\text{cut}\ \frac{\begin{array}{c}\delta_1\\ \Gamma \Rightarrow \psi \wedge \theta, \Sigma\end{array} \qquad \begin{array}{c}\delta_2\\ \Gamma, \psi \wedge \theta \Rightarrow \Sigma\end{array}}{\Gamma \Rightarrow \Sigma}$$

Applying lemma 1.7 and lemma 1.6 to δ_1 and δ_2 respectively yields deductions δ_1', δ_1'', and δ_2' of $\Gamma \Rightarrow \psi, \Sigma$, $\Gamma \Rightarrow \theta, \Sigma$, and $\Gamma, \psi, \theta \Rightarrow \Sigma$. The following deduction δ' cuts only over formulas of complexity less than φ

$$\text{cut}\ \frac{\text{cut}\ \dfrac{\text{TL}\ \dfrac{\begin{array}{c}\delta_1'\\ \Gamma \Rightarrow \psi\Sigma\end{array}}{\Gamma, \theta \Rightarrow \psi, \Sigma} \qquad \begin{array}{c}\delta_2'\\ \Gamma, \psi, \theta \Rightarrow \Sigma\end{array}}{\Gamma, \theta \Rightarrow \Sigma} \qquad \begin{array}{c}\delta_1''\\ \Gamma \Rightarrow \theta, \Sigma\end{array}}{\Gamma \Rightarrow \Sigma}$$

Since δ' contains cuts only over formulas less complex than φ there is a cut-free deduction of $\Gamma \Rightarrow \Sigma$. $\square$

Chapter 2

A Hypersequent Approach to Modal Logic

Abstract. This chapter develops a novel approach to model logic using hypersequent calculi. Calculi are presented for the modal logics K, D, T, $S4$, B, and $S5$. The rules governing the behavior of the modal expression $\Box$ do not change from calculus to calculus. Different modal logics are captured by manipulating the external structural rules of the hypersequent calculus. The rules governing the modal expression $\Box$ are explicit, separated, symmetrical, and non-circular. It is shown that each calculus uniquely characterizes the modal expression $\Box$. Two of the hypersequent calculi are known to be cut admissible. The chapter concludes by rehearsing a proof of cut admissibility for the hypersequent calculus for $S5$ found in Restall [53].

Keywords. Modal Logic, Hypersequent Calculus, Uniqueness, Cut Admissibility

2.1 Introduction

This chapter develops a framework for exploring modal logics in a proof-theoretic setting. The motivation for this work is a broad concern with the plausibility of sequent calculi to underwrite a proof-theoretic approach to modality. Because deductive calculi for modal logic in the standard setting fail to have desirable structural properties there have been many attempts at altering the structure of deductions or the objects in a deduction (for a survey see Wansing [73] or Poggiolesi [39]). Some take what are normally considered to be model-theoretic notions, e.g. relations between wordls, as primitive objects in the deductive calculi (see, e.g. Negri [34]). Others have expanded the structure of the sequent (see Mints [33], Došen [12], and Avron [1]). These latter, however, have until recently only captured a small range of the modal logics that are considered by philosophical logicians. More recently several philosophers have attempted to capture different modal logics using hypersequents (see for instance Lahav [23][1]). This chapter develops a new framework that makes use of lists of sequents, or hypersequents.

Section 2.2 presents sequent accounts of modal logic that can be found in Wansing [73] or Poggiolesi [39]. This section also discusses some constraints that philosophers have suggested on what sorts of rules can count as meaning determining. In section 2.3, the hypersequent calculi for the modal logics are presented. That section develops sound model theories for the various logics and shows in what sense they capture the familiar modal calculi. Section 2.4 proves some important results about

[1] The approach of this chapter is markedly different from the calculi developed in Lahav [23]. The focus in both cases was to develop cut-free hypersequent calculi, whereas the main goal of this chapter is to develop hypersequent calculi that uniquely characterize modal expresssions.

the calculi introduced in the previous section.

Let $\{p_1, p_2, \ldots\}$ be an denumerably infinite set of atomic sentences. Fix a formal language $\mathscr{L}$ by

$$\varphi := p|\neg\varphi|(\varphi \to \psi)| \Box \varphi$$

If Γ and Σ are sets of sentences then $\Gamma \Rightarrow \Sigma$ is a sequent. If $S_1, \ldots, S_n$ are sequents then the list $(S_1); (S_2); \ldots; (S_n)$ is a hypersequent.[2]

2.2 Modal Logic

2.2.1 Model Theory

A Kripke frame $\mathscr{F} = \langle W_{\mathscr{F}}, R_{\mathscr{F}} \rangle$ is an ordered pair of a non-empty set of worlds, $W_{\mathscr{F}}$, and a relation, $R_{\mathscr{F}}$, that holds between elements of $W_{\mathscr{F}}$. Call the set of all such frames $\mathbb{K}$. A Kripke model $M_{\mathscr{F}} = \langle \mathscr{F}, I_{M_{\mathscr{F}}} \rangle$ is an ordered pair of a Kripke frame, $\mathscr{F}$, and an interpretation function, $I_{M_{\mathscr{F}}}$. $I_{M_{\mathscr{F}}}$ is a function from worlds and sentences into $\{1, 0\}$. Let $w \in W_{\mathscr{F}}$ and $\varphi \in \mathscr{L}$. $I_{M_{\mathscr{F}}}(w, \varphi)$ is defined inductively by the following clauses.

- If φ is atomic then $I_{M_{\mathscr{F}}}(w, \varphi) \in \{1, 0\}$.
- If φ is $\neg\psi$ then $I_{M_{\mathscr{F}}}(w, \varphi) = 1$ iff $I_{M_{\mathscr{F}}}(w, \psi) = 0$.

[2] Sequents are separated off by parentheses and semicolons to improve readability. In general, lower-case Greek letters are used as schematic letters for sentences of $\mathscr{L}$, upper case Greek letter are used as schema for sets of sentences, S with decorations are used as schema for sequents, and upper case Latin letters are used as schema for hypersequents.

- If φ is $\psi \to \theta$ then $I_{M_{\mathscr{F}}}(w,\varphi) = 1$ iff $I_{M_{\mathscr{F}}}(w,\psi) = 0$ or $I_{M_{\mathscr{F}}}(w,\theta) = 1$.
- If φ is $\Box\psi$ then $I_{M_{\mathscr{F}}}(w,\varphi) = 1$ iff for all $v \in W_{\mathscr{F}}$ such that $wR_{\mathscr{F}}v$, $I_{M_{\mathscr{F}}}(v,\psi) = 1$.

If Γ is a set of sentences, $I_M(w,\Gamma) = 1(0)$ is shorthand for the claim that for all $\gamma \in \Gamma$, $I_{M_{\mathscr{F}}}(w,\gamma) = 1(0)$.

Definition 2 (Counter-Model)**.** A model $M_{\mathscr{F}}$ is a *counter-model* to a sequent $\Gamma \Rightarrow \Sigma$ at a world w iff $I_{M_{\mathscr{F}}}(w,\Gamma) = 1$ and $I_{M_{\mathscr{F}}}(w,\Sigma) = 0$.

A sequent $\Gamma \Rightarrow \Sigma$ is *valid* for a class of frames $\mathbb{S}$ iff there is no model $M_{\mathscr{F}}$ with frame $\mathscr{F} \in \mathbb{S}$ such that $M_{\mathscr{F}}$ is a counter-example to $\Gamma \Rightarrow \Sigma$. If $\Gamma \Rightarrow \Sigma$ is valid with respect to $\mathbb{S}$ this is indicated by $\models_S \Gamma \Rightarrow \Sigma$.

The modal logics considered in this chapter are K, D, T, $S4$, B, and $S5$. The modal logic K is the set of valid sequents for the whole class of frames, $\mathbb{K}$. The modal logic D is the set of valid sequents for the class of serial frames, i.e. $\mathbb{D} := \{\mathscr{F} : \forall x \in W_{\mathscr{F}}, \exists y \in W_{\mathscr{F}}(xR_{\mathscr{F}}y)\}$. The modal logic T is the set of valid sequents for the class of reflexive frames, i.e. $\mathbb{T} := \{\mathscr{F} : \forall y \in W_{\mathscr{F}}(yR_{\mathscr{F}}y)\}$. The modal logic $S4$ is the set of valid sequents for the class of reflexive and transitive frames, i.e. $\mathbb{TR} := \mathbb{T} \cap \{\mathscr{F} : \forall xyz \in W_{\mathscr{F}}, (xR_{\mathscr{F}}y \ \& \ yR_{\mathscr{F}}z \supset xR_{\mathscr{F}}z)\}$. The modal logic B is the set of valid sequents for the class of symmetric frames, i.e. $\mathbb{B} = \{\mathscr{F} : \forall xy \in W_{\mathscr{F}}(xR_{\mathscr{F}}y \supset yR_{\mathscr{F}}x)\}$. Finally, the modal logic $S5$ is given by the class of universal frames, i.e. $\mathbb{L} := \{\mathscr{F} : \forall xy \in W_M(xR_{\mathscr{F}}y)\}$. This is summed up in the table below.

Class of Frames	Restriction on R
$\mathbb{K}$	
$\mathbb{D}$	`serial`
$\mathbb{T}$	`reflexive`
$\mathbb{TR}$	`reflexive` & `transitive`
$\mathbb{B}$	`symmetrical`
$\mathbb{L}$	`universal`

2.2.2 Sequents

A sequent calculus is a set of rules and axioms by which deductions may be generated. A deduction δ of a sequent $\Gamma \Rightarrow \Sigma$ relative to a calculus C is a tree whose leaves are axioms of C, is such that each node in the tree is generated from its predecessors by means of a rule of C, and whose root is $\Gamma \Rightarrow \Sigma$. If δ is a deduction of $\Gamma \Rightarrow \Sigma$ relative to calculus C this is indicated by $\delta \vdash_c \Gamma \Rightarrow \Sigma$. More generally $\vdash_c \Gamma \Rightarrow \Sigma$ indicates that there is a deduction of $\Gamma \Rightarrow \Sigma$ relative to C.

The sequent calculus for Classical Propositional Logic, *cpl*, is given in fig. 2.1. This calculus serves as the basis for all the modal sequent calculi considered in this chapter.

The modal logics described in section 2.2.1 are captured by adding various combinations of the rules from fig. 2.2 to *cpl* .[3]

Definition 3 (Adequacy)**.** A class of frames $\mathbb{S}$ is *adequate* for a sequent calculus C iff for any sequent $\Gamma \Rightarrow \Sigma$, $\models_S \Gamma \Rightarrow \Sigma$ iff $\vdash_C \Gamma \Rightarrow \Sigma$.

[3] There are many different ways of generating sequent calculi for the modal logics described in section 2.2.1. Many of these sequent calculi can be found in Wansing [73] and Poggiolesi [39].

Figure 2.1: Classical Propositional logic

Structural Rules

$$\text{Id}\ \frac{}{\varphi \Rightarrow \varphi} \qquad \text{Cut}\ \frac{\Gamma \Rightarrow \varphi, \Sigma \qquad \Gamma, \varphi \Rightarrow \Sigma}{\Gamma \Rightarrow \Sigma}$$

$$\text{TL}\ \frac{\Gamma \Rightarrow \Sigma}{\Gamma, \varphi \Rightarrow \Sigma} \qquad \text{TR}\ \frac{\Gamma \Rightarrow \Sigma}{\Gamma \Rightarrow \varphi, \Sigma}$$

Logical Rules

$$\text{L}\neg\ \frac{\Gamma \Rightarrow \varphi, \Sigma}{\Gamma, \neg\varphi \Rightarrow \Sigma} \qquad \text{R}\neg\ \frac{\Gamma, \neg\varphi \Rightarrow \Sigma}{\Gamma \Rightarrow \varphi, \Sigma}$$

$$\text{L}\rightarrow\ \frac{\Gamma \Rightarrow \psi, \Sigma \qquad \Gamma, \varphi \Rightarrow \Sigma}{\Gamma, \varphi \rightarrow \psi \Rightarrow \Sigma} \qquad \text{R}\rightarrow\ \frac{\Gamma, \varphi \Rightarrow \psi, \Sigma}{\Gamma \Rightarrow \varphi \rightarrow \psi, \Sigma}$$

Each of class of frames discussed in section 2.2.1 above is adequate with respect to a calculus definable from figs. 2.1 and 2.2. Each calculus adopts all of the rules of fig. 2.1 and some subset of the rules of fig. 2.2. The calculi that are adequate for each class of frames are given in the table below. The table also contains citations where the proofs of adequacy can be found.

Figure 2.2: Modal Sequent Rules

$$k\,\frac{\Gamma \Rightarrow \varphi}{\Box\Gamma \Rightarrow \Box\varphi} \qquad d\,\frac{\Gamma \Rightarrow}{\Box\Gamma \Rightarrow}$$

$$t\,\frac{\Gamma, \varphi \Rightarrow \Sigma}{\Gamma, \Box\varphi \Rightarrow \Sigma} \qquad b\,\frac{\Gamma \Rightarrow \varphi, \Box\Sigma}{\Box\Gamma \Rightarrow \Box\varphi, \Sigma}$$

$$4\,\frac{\Box\Gamma \Rightarrow \varphi}{\Box\Gamma \Rightarrow \Box\varphi} \qquad 5\,\frac{\Box\Gamma \Rightarrow \varphi, \Box\Sigma}{\Box\Gamma \Rightarrow \Box\varphi, \Box\Sigma}$$

$$\Box\Delta := \{\Box\delta : \delta \in \Delta\}$$

Class of Frames	Set of Rules	Deducibility Relation	Reference
$\mathbb{K}$	k	$\vdash_k$	Sambin and Valentini [61]
$\mathbb{D}$	$k + d$	$\vdash_d$	Valentini [71]
$\mathbb{T}$	$k + t$	$\vdash_t$	Ohnishi and Matsumoto [35]
$\mathbb{TR}$	$k + t + 4$	$\vdash_{s4}$	Ohnishi and Matsumoto [35]
$\mathbb{B}$	$k + b$	$\vdash_b$	Takano [70]
$\mathbb{L}$	$k + t + 4 + 5$	$\vdash_{s5}$	Ohnishi and Matsumoto [35]

One of the philosophical motivations for exploring modal logics is to see to what extent the meaning of modal expressions can be explained by the inferences in which they feature. It has been proposed, by e.g. Belnap [4], Dummett [13], or Poggiolesi [39], that in order for a set of rules to determine the meaning of an expression the set must meet several constraints. Some of the features that philosophers have discussed are the following

1. *Explicit:* A rule R is explicit when only one occurrence of the expression in question occurs essentially in the conclusion. A set of rules that has this feature is not in danger of defining more than one status of a sentence at once. For instance, if the left of a sequent corresponds to truth and the right to falsity, the rule k offers an explanation of the truth of $\Box\varphi$ inextricably from an explanation of its falsity.

2. *Separated:* A rule R is separated when the only expression that features essentially in that rule is the expression whose meaning is being explained. If a rule features two expressions essentially then it runs the risk of offering an explanation of those two expressions at once as opposed offering an explanation of just one of them.

3. *Symmetrical:* A set of rules S is symmetrical when for each rule $R \in S$ R either introduces a connective on the left or on the right and there is at least one rule for each. If the left and right side of a sequent correspond to some status a sentence may have, then symmetrical rules guarantee that for any status there is a way of determining whether a particular sentence has that status. Thus the status of a sentence would not, in principle, be underdetermined by the rules of S.

4. *Non-Circularity:* A rule R is non-circular if there is no essential occurrence of the connective being defined in the premises of R. A circular rule leaves open the possibility that there is no way to understand how to use a particular expression without already having some antecedent grasp of that expression.

As is noted by Dummett [13] ([13, pg.257]), in some cases this may not be problematic. But it is a good feature of a rule if it avoids the possibility of circularity altogether.

5. *Uniqueness:* Let S be a set of rules, and $S[\varepsilon'/\varepsilon]$ be the result of replacing ε' for ε everywhere in S. S is said to uniquely chracterize an expression ε it introduces iff both

   (a) $\vdash \Gamma, \varphi \Rightarrow \Sigma$ iff $\vdash \Gamma, \varphi[\varepsilon'/\varepsilon] \Rightarrow \Sigma$

   (b) $\vdash \Gamma \Rightarrow \varphi, \Sigma$ iff $\vdash \Gamma \Rightarrow \varphi[\varepsilon'/\varepsilon], \Sigma$.

   Proponents of proof-theoretic semantics generally hold some version of the claim that meaning is use. This could be stated in a weak form as the claim that the rules governing the behavior of an expression determine the meaning of the expression. In the case above, the purported rules that govern the meaning of ε are S. Suppose that uniqueness fails but that S is said to determine the meaning of ε. If uniqueness fails then there is a sequent, e.g. $\Gamma, \varphi \Rightarrow \Sigma$, that is provable though $\Gamma, \varphi[\varepsilon'/\varepsilon] \Rightarrow \Sigma$ is not. The only difference between $\Gamma, \varphi \Rightarrow \Sigma$ and $\Gamma, \varphi[\varepsilon'/\varepsilon] \Rightarrow \Sigma$ is that ε' replaces ε in the second. ε and ε' cannot therefore have the same meaning. Since ε and ε' have the same use (S and $S[\varepsilon'/\varepsilon]$ respectively) S does not *determine* the meaning of ε.

6. *Cut Admissibility:* Let C be a calculus and C^{cf} be the calculus that is exactly like C except that it lacks the Cut rule. A logic is *cut admissible* when $\vdash_C \Gamma \Rightarrow \Sigma$ iff $\vdash_{C^{cf}} \Gamma \Rightarrow \Sigma$. Cut admissibility is a desirable property for a calculus. In general, it shows that $\nvdash \ \Rightarrow \ $, and so serves as proof of consistency. In most

settings it also establishes that only rules governing the expressions occurring in a sequent are used to determine whether or not there is a deduction of that sequent.[4]

7. *Sub-formula Property:* A calculus has the sub-formula property iff whenever $\vdash \Gamma \Rightarrow \Sigma$ there is a deduction δ of $\Gamma \Rightarrow \Sigma$ such that if $\Delta \Rightarrow \Lambda$ occurs in δ then any sentence $\varphi \in \Delta \cup \Lambda$ is a sub-formula of a sentence in $\Gamma \cup \Sigma$. In most settings a calculus that is cut admissible has the sub-formula property.

None of the logics characterized above have explicit rules governing $\Box$. Every logic described above requires the k rule to govern the behavior of $\Box$, but this rule is not explicit. The rules 4, b, and 5 are also circular. Let C be a modal calculus characterized above with rules R governing $\Box$. If $\boxdot$ is introduced to the calculus using the rules $R[\boxdot/\Box]$ the sequent $\Box p \Rightarrow \boxdot p$ is not deducible even though the sequent $\Box p \Rightarrow \Box p$ is. Thus, all of these logics fail to uniquely characterize the modal expression $\Box$. It was shown by Ohnishi and Matsumoto [35] that the calculus for S5 is not cut admissible.

2.3 Hypersequent Modal Logics

This section presents the various hypersequent calculi for modal logics. The logics are presented in order from weakest to strongest beginning with System K. The bulk of the model theoretic work is done in section 2.3.1. Proceeding from modal logic to

[4] Cut is said to be *admissible* when it is known that for any deducible sequent there is a cut-free deduction of that sequent. Cut is said to be eliminable when an algorithm is known for transforming deductions with cut to deductions without cut.

modal logic requires only small changes in the overall approach. Hypersequent system K is given in fig. 2.3. The various structural rules for extending the hypersequent calculi are discussed in sections 2.3.2 to 2.3.6

A hypersequent calculus contains both structural and logical rules. However, the structural rules of a hypersequent calculus are further divided into internal and external structural rules. The internal structural rules of a hypersequent calculus allow for the manipulation of sentences within a sequent in a hypersequent. The external structural rules of a hypersequent calculus allow for the manipulation of sequents within a hypersequent. One of the novelties of this approach to modal proof-theory is that the logical rules governing the modal operator $\Box$ are constant between the various modal systems. The different modal logics are characterized by adding or removing external structural rules from a hypersequent calculus.

Let $\mathscr{F}$ be a Kripke frame. A *branch of worlds* in $\mathscr{F}$ is a list $w_1, \ldots, w_n$ such that for each $1 \leqslant i \leqslant n$, $w_i R_{\mathscr{F}} w_{i-1}$.

Definition 4 (Counter-Example)**.** A model $M_{\mathscr{F}}$ is a *counter-example* to a hypersequent $(S_1), \ldots, (S_n)$ iff there is a branch $w_1, \ldots, w_n$ such that $M_{\mathscr{F}}$ is a counter-model to S_i at w_i for $1 \leqslant i \leqslant n$.

If there is a deduction of a hypersequent G according to the rules of fig. 2.3 this is indicated by $\vdash_{hk} G$.

2.3.1 System K

The calculus of fig. 2.3 captures $\mathbb{K}$ in the sense that $\vdash_k \Gamma \Rightarrow \Sigma$ iff $\vdash_{hk} (\Gamma \Rightarrow \Sigma)$. This is proved in theorem 2.3.1.

Figure 2.3: Hypersequent System K

STRUCTURAL RULES

$$\text{Id}\,\frac{}{(p \Rightarrow p)} \qquad \text{W}\,\frac{G}{G; (\quad \Rightarrow \quad)}$$

$$\text{TL}\,\frac{G; (\Gamma \Rightarrow \Sigma); H}{G; (\Gamma, \varphi \Rightarrow \Sigma); H} \qquad \text{TR}\,\frac{G; (\Gamma \Rightarrow \Sigma); H}{G; (\Gamma \Rightarrow \varphi, \Sigma); H}$$

$$\text{Cut}\,\frac{G; (\Gamma \Rightarrow \varphi, \Sigma); H \qquad G; (\Gamma, \varphi \Rightarrow \Sigma); H}{G; (\Gamma \Rightarrow \Sigma); H}$$

OPERATIONAL RULES

$$\text{L}\neg\,\frac{G; (\Gamma \Rightarrow \varphi, \Sigma); H}{G; (\Gamma, \neg\varphi \Rightarrow \Sigma); H} \qquad \text{R}\neg\,\frac{G; (\Gamma, \varphi \Rightarrow \Sigma); H}{G; (\Gamma \Rightarrow \neg\varphi, \Sigma); H}$$

$$\text{L}\rightarrow\,\frac{G; (\Gamma \Rightarrow \varphi, \Sigma); H \qquad G; (\Gamma, \psi \Rightarrow \Sigma); H}{G; (\Gamma, \varphi \rightarrow \psi \Rightarrow \Sigma); H} \qquad \text{R}\rightarrow\,\frac{G; (\Gamma, \varphi \Rightarrow \psi, \Sigma); H}{G; (\Gamma \Rightarrow \varphi \rightarrow \psi, \Sigma); H}$$

$$\text{L}\Box\,\frac{G; (\Gamma, \varphi \Rightarrow \Sigma); (\Delta \Rightarrow \Lambda); H}{G; (\Gamma \Rightarrow \Sigma); (\Delta, \Box\varphi \Rightarrow \Lambda); H} \qquad \text{R}\Box\,\frac{(\quad \Rightarrow \varphi); (\Gamma \Rightarrow \Sigma); H}{(\Gamma \Rightarrow \Box\varphi, \Sigma); H}$$

The rule Id of fig. 2.3 holds only over atomic sentences. The rule can be proved to hold for all sentences of the language.

Definition 5 (Rank)**.** The *rank* of a sentence φ is defined inductively by

- If φ is atomic then $rk(\varphi) = 0$.
- If φ is $\neg\psi$ or $\Box\psi$ then $rk(\varphi) = rk(\psi) + 1$.
- If φ is $\psi \to \theta$ then $rk(\varphi) = rk(\psi) + rk(\theta) + 1$.

Lemma 2.1. $\vdash_{hk} (\varphi \Rightarrow \varphi)$

Proof. This is proved by induction on the rank of φ. The base case is given by Id. For the other three cases, the inductive hypothesis (IH) is that the lemma holds for formulas of lower rank. The following three deductions then establish the lemma.

$$\text{R}\neg\ \dfrac{\text{L}\neg\ \dfrac{\dfrac{\text{IH}}{(\varphi \Rightarrow \varphi)}}{(\varphi, \neg\varphi \Rightarrow)}}{(\neg\varphi \Rightarrow \neg\varphi)}
\qquad\qquad
\text{R}{\to}\ \dfrac{\text{L}{\to}\ \dfrac{\dfrac{\text{IH}}{(\varphi \Rightarrow \varphi)} \quad \dfrac{\text{IH}}{(\psi \Rightarrow \psi)}}{(\varphi, \varphi \to \psi \Rightarrow \psi)}}{(\varphi \to \psi \Rightarrow \varphi \to \psi)}$$

$$\text{R}\Box\ \dfrac{\text{L}\Box\ \dfrac{\text{W}\ \dfrac{\dfrac{\text{IH}}{(\varphi \Rightarrow \varphi)}}{(\varphi \Rightarrow \varphi); (\ \Rightarrow\ )}}{(\ \Rightarrow \varphi); (\Box\varphi \Rightarrow\ )}}{(\Box\varphi \Rightarrow \Box\varphi)}$$

$\square$

The other hypersequent calculi result from the addition of rules to the calculus of fig. 2.3. It follows that lemma 2.1 holds for all those systems as well. Lemma 2.1 will often be invoked simply as Id in what follows.

In order to prove lemma 2.2 a measure on the length of deductions is required. This is given by definition 6.

Definition 6 (Length of a Deduction)**.** The *length* of a deduction δ, $l(\delta)$ is defined inductively:

- If δ is an instance of Id, then $l(\delta) = 1$.
- If δ has $\delta_1, \ldots, \delta_n$ as premises then $l(\delta) = (\sum_{1 \leqslant i \leqslant n} l(\delta_1)) + 1$.

Lemma 2.2. *If $\vdash_{hk} G$ then there is no model $M_{\mathscr{F}}$ such that $\mathscr{F} \in \mathbb{K}$ and $M_{\mathscr{F}}$ is a counter-example to G.*

Proof. This is proved by induction on the length of deductions. Let δ be a deduction of G and I be the last inference of δ. The smallest deduction is an instance of Id. Suppose that there were a model $M_{\mathscr{F}}$ such that there is a world $w \in W_{M_{\mathscr{F}}}$ such that $I_{M_{\mathscr{F}}}(w, p) = 1$ and $I_{M_{\mathscr{F}}}(w, p) = 0$. This is impossible, so there are no counter-examples to Id.

Case 1 (I is W). Let $M_{\mathscr{F}}$ be a counter-example to $(S_1); \ldots; (S_n); (S_{n+1})$. There is a branch of worlds $w_1, \ldots, w_n, w_{n+1}$ in $\mathscr{F}$ such that $w_i R_{\mathscr{F}} w_{i-1}$ and $M_{\mathscr{F}}$ is counter-model to S_i at w_i for $1 \leqslant i \leqslant n+1$. But then there is a branch $w_n, \ldots, w_1$ in $\mathscr{F}$ such that $w_i R_{\mathscr{F}} w_{i-1}$ and $M_{\mathscr{F}}$ is a counter-model to S_i at w_i for $1 \leqslant i \leqslant n$.

Case 2 (I is TL). Let $M_{\mathscr{F}}$ be a counter-example to $(S_1); \ldots; (\Gamma_j, \varphi \Rightarrow \Sigma_j); \ldots; (S_n)$. So there is a branch of worlds $w_1, \ldots, w_n$ in $\mathscr{F}$ such that $w_i R_{\mathscr{F}} w_{i-1}$ and $M_{\mathscr{F}}$ is a counter-model to S_i at w_i for $1 \leqslant i \leqslant n$. In particular, $M_{\mathscr{F}}$ is a counter-model to $\Gamma_j, \varphi \Rightarrow \Sigma_j$ at w_j. It follows that $I_{M_{\mathscr{F}}}(w_j, \Gamma_j \cup \{\varphi\}) = 1$. So $I_{M_{\mathscr{F}}}(w_j, \Gamma_j) = 1$

and $M_{\mathscr{F}}$ is a counter-model to $\Gamma_j \Rightarrow \Sigma_j$ at w_j. So $M_{\mathscr{F}}$ is a counter-example to $(S_1); \ldots; (\Gamma_j \Rightarrow \Sigma_j); \ldots; (S_n)$.

Case 3 (I is TR). This case is similar to the above.

Case 4 (I is Cut). Let $M_{\mathscr{F}}$ be a counter-example to $(S_1); \ldots; (\Gamma_j \Rightarrow \Sigma_j); \ldots; (S_n)$. So there is a branch of worlds $w_1, \ldots, w_n$ in $\mathscr{F}$ such that $w_i R_{\mathscr{F}} w_{i-1}$ and $M_{\mathscr{F}}$ is a counter-model to S_i at w_i for $1 \leqslant i \leqslant n$. In particular, $M_{\mathscr{F}}$ is a counter-model to $\Gamma_j \Rightarrow \Sigma_j$ at w_j. Either $I_M(w_j, \varphi) = 1$ or $I_M(w_j, \varphi) = 0$. In the former case, $M_{\mathscr{F}}$ is a counter-model to $\Gamma_j, \varphi \Rightarrow \Sigma_j$ at w_j. So $M_{\mathscr{F}}$ is a counter-example to $(S_1); \ldots; (\Gamma_j, \varphi \Rightarrow \Sigma_j); \ldots; (S_n)$. In the latter case, $M_{\mathscr{F}}$ is a counter-model to $\Gamma_j \Rightarrow \varphi, \Sigma_j$ at w_j, and thus $M_{\mathscr{F}}$ is a counter-example to $(S_1); \ldots; (\Gamma_j \Rightarrow \varphi, \Sigma_j); \ldots; (S_n)$.

Case 5 (I is L$\neg$). Let $M_{\mathscr{F}}$ be a counter-example to $(S_1); \ldots; (\Gamma_j, \neg\varphi \Rightarrow \Sigma_j); \ldots; (S_n)$. So there is a branch of worlds $w_1, \ldots, w_n$ in $\mathscr{F}$ such that $w_i R_{\mathscr{F}} w_{i-1}$ and $M_{\mathscr{F}}$ is a counter-model to S_i at w_i for $1 \leqslant i \leqslant n$. In particular, $M_{\mathscr{F}}$ is a counter-model to $(\Gamma_j, \neg\varphi \Rightarrow \Sigma_j)$ at w_j. Since $I_{M_{\mathscr{F}}}(w_j, \neg\varphi) = 1$, $I_{M_{\mathscr{F}}}(w_j, \varphi) = 0$. So $M_{\mathscr{F}}$ is a counter-model to $(\Gamma_j \Rightarrow \varphi, \Sigma_j)$ at w_j and $M_{\mathscr{F}}$ is a counter-example to $(S_1); \ldots; (\Gamma_j \Rightarrow \varphi, \Sigma_j); \ldots; (S_n)$.

Case 6 (I is R$\neg$). This is similar to the above case.

Case 7 (I is R$\rightarrow$ or L$\rightarrow$). These cases, as with the case of L$\neg$, follow from manipulation of what $I_{M_{\mathscr{F}}}$ assigns to a formula of the form $\varphi \rightarrow \psi$ at a world w_j.

Case 8 (I is L$\Box$). Let $M_{\mathscr{F}}$ be a counter-example to $(S_1); \ldots; (\Gamma_{j-1} \Rightarrow \Sigma_{j-1}); (\Gamma_j, \Box\varphi \Rightarrow \Sigma_j); \ldots; (S_n)$. There is a branch of worlds, $w_1, \ldots, w_n$ in $\mathscr{F}$ such that $w_i R w_{i-1}$ and $M_{\mathscr{F}}$ is a counter-model to S_i at w_i for $1 \leqslant i \leqslant n$. Since $M_{\mathscr{F}}$ is a counter-model to $\Gamma_j, \Box\varphi \Rightarrow \Sigma_j$ at w_j, $I_{M_{\mathscr{F}}}(w_j, \Box\varphi) = 1$. For any world, v, such that $w_j R v$, $I_M(v, \varphi) = 1$.

Since $w_j R w_{j-1}$, $I_M(w_{j-1}, \varphi) = 1$. $M_{\mathscr{F}}$ is a counter-model to $\Gamma_{j-1}, \varphi \Rightarrow \Sigma_{j-1}$ at w_{j-1} and so $M_{\mathscr{F}}$ is a counter-example to $(S_1); \ldots; (\Gamma_{j-1}, \varphi \Rightarrow \Sigma_{j-1}); (\Gamma_j \Rightarrow \Sigma_j); \ldots; (S_n)$.

Case 9 (I is R$\Box$). Let $M_{\mathscr{F}}$ be a counter-example to $(\Gamma_1 \Rightarrow \Box\varphi, \Sigma_1); \ldots; (S_n)$. There is a branch, $w_1, \ldots, w_n$ in $\mathscr{F}$ such that $w_i R_{\mathscr{F}} w_{i-1}$ and $M_{\mathscr{F}}$ is a counter-model to (S_i) at w_i. It follows that $M_{\mathscr{F}}$ is a counter-model to $(\Gamma_1 \Rightarrow \Box\varphi, \Sigma_1)$ at w_1. In particular, $I_M(w_1, \Box\varphi) = 0$, so there is a world w_0 such that $w_1 R_{\mathscr{F}} w_0$ and $I_{M_{\mathscr{F}}}(w_0, \varphi) = 0$. But then $M_{\mathscr{F}}$ is a counter-model to $(\quad \Rightarrow \varphi)$ at w_0. $\mathscr{F}$ contains a branch $w_0, \ldots, w_n$ such that $w_i R_{\mathscr{F}} w_{i-1}$ and $M_{\mathscr{F}}$ is a counter-model to S_i at w_i for $1 \leqslant i \leqslant n$. It follows that $M_{\mathscr{F}}$ is a counter-example to $(\quad \Rightarrow \varphi); (\Gamma_1 \Rightarrow \Sigma_1); \ldots; (S_n)$.

$\square$

Theorem 2.3.1. $\vdash_k \Gamma \Rightarrow \Sigma$ *iff* $\vdash_{hk} (\Gamma \Rightarrow \Sigma)$.

Proof. For the left to right direction let $\delta \vdash_k \Gamma \Rightarrow \Sigma$. It is proved by induction on the length of δ that $\vdash_{hk} \Gamma \Rightarrow \Sigma$. All instances of Id are the same for both systems. If the last inference of δ is TL then δ has the form

$$\begin{array}{c} \delta \\ \vdots \\ \text{TL}\ \dfrac{\Gamma \Rightarrow \Sigma}{\Gamma, \varphi \Rightarrow \Sigma} \end{array}$$

By the IH there is a deduction, δ' of $(\Gamma \Rightarrow \Sigma)$. An application of TL to δ' yields a deduction of $(\Gamma, \varphi \Rightarrow \Sigma)$. TR, Cut, L$\neg$, R$\neg$, R$\rightarrow$, and L$\rightarrow$ all follow in a similar way. Let the last inference of δ be k. In this case δ has the form

$$\begin{array}{c} \delta \\ \vdots \\ k\ \dfrac{\Gamma \Rightarrow \varphi}{\Box\Gamma \Rightarrow \Box\varphi} \end{array}$$

By IH there is a deduction δ' of $(\Gamma \Rightarrow \varphi)$ according to the rules of fig. 2.3. Let the cardinality of Γ be n. In this case the following deduction establishes the result

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\delta' \\ \vdots \\ (\Gamma \Rightarrow \varphi)\end{matrix}}{(\Gamma \Rightarrow \varphi);(\ \Rightarrow\ )}\,\mathrm{W}}{(\ \Rightarrow \varphi);(\Box\Gamma \Rightarrow\ )}\,\mathrm{L}\Box \times n}{(\Box\Gamma \Rightarrow \Box\varphi)}\,\mathrm{R}\Box$$

For the left to right direction suppose that $\nvdash_k \Gamma \Rightarrow \Sigma$ by the fact that sequent system k is adequate for $\mathbb{K}$ there is a model $M_{\mathscr{F}}$ with a world w such that $M_{\mathscr{F}}$ is a counter-model to $\Gamma \Rightarrow \Sigma$ at w. There is a branch w in $\mathscr{F}$ such that $M_{\mathscr{F}}$ is a counter-model to $\Gamma \Rightarrow \Sigma$ at w. So $M_{\mathscr{F}}$ is a counter-example to $(\Gamma \Rightarrow \Sigma)$. □

The remainder of the hypersequent calculi for modal logics are generated merely by manipulation of structural rules. The rules L□ and R□ are the only rules explicitly governing modality in any of the hypersequent systems presented.

2.3.2 System D

The calculus that results from adding

$$\mathrm{Drop}\,\dfrac{(\ \Rightarrow\ );G}{G}$$

to the calculus of fig. 2.3 captures the modal logic System D. If there is a deduction of G in this calculus this is indicated by $\vdash_{hd} G$. This calculus is sound for the class of serial models given by frames in $\mathbb{D}$. This is result is proved in lemma 2.3.

Lemma 2.3. *If* $\vdash_{hd} G$ *then there is no model* $M_{\mathscr{F}}$ *such that* $\mathscr{F} \in \mathbb{D}$ *and* $M_{\mathscr{F}}$ *is a counter-example to* G.

Proof. The proof that all of the rules of fig. 2.3 preserve validity is similar to the proof of lemma 2.2. This leaves only the proof that Drop preserves validity over the class of models with serial frames. Let δ be a deduction ending with inference Drop. Let $M_{\mathscr{F}}$ be a counter-example to $(S_1); \ldots; (S_n)$. There is a branch of worlds $w_1, \ldots, w_n$ in $\mathscr{F}$ such that $w_i R w_{i-1}$ and $M_{\mathscr{F}}$ is a counter-model to S_i at w_i. Since $\mathscr{F}$ is serial, there is a world w_0 such that $w_1 R w_0$ and $M_{\mathscr{F}}$ is a counter-model to $(\quad \Rightarrow \quad)$ at w_0. So $M_{\mathscr{F}}$ is a counter-example to $(\quad \Rightarrow \quad); (S_1); \ldots; (S_n)$. $\square$

Theorem 2.3.2. $\vdash_d \Gamma \Rightarrow \Sigma$ *iff* $\vdash_{hd} (\Gamma \Rightarrow \Sigma)$.

Proof. For the left to right direction let $\vdash_d \Gamma \Rightarrow \Sigma$. It is proved by induction on the length of deductions that $\vdash_{hd} (\Gamma \Rightarrow \Sigma)$. Every proof that does not involve the rule d has been considered above. Let d be the last inference of a deduction δ. δ has the form

$$\begin{array}{c} \delta \\ \vdots \\ d\,\dfrac{\Gamma \Rightarrow \qquad}{\Box\Gamma \qquad} \end{array}$$

Let the cardinality of Γ be n. By IH there is a deduction δ' ending in the hypersequent $(\Gamma \Rightarrow \quad)$. The following deduction establishes the left to right direction of the lemma.

$$
\begin{array}{c}
\delta' \\
\vdots \\
\mathrm{Drop}\,\dfrac{\mathrm{L}\square \times n\,\dfrac{\mathrm{W}\,\dfrac{(\Gamma \Rightarrow \quad)}{(\Gamma \Rightarrow \quad);(\quad \Rightarrow \quad)}}{(\quad \Rightarrow \quad);(\square\Gamma \Rightarrow \quad)}}{(\square\Gamma \Rightarrow \quad)}
\end{array}
$$

For the right to left direction suppose that $\nvdash_d \Gamma \Rightarrow \Sigma$. Since the sequent calculus for system D is adequate with respect to $\mathbb{D}$ there is a counter-example $M_{\mathscr{F}}$ to $\Gamma \Rightarrow \Sigma$ at a world w built on a frame $\mathscr{F} \in \mathbb{D}$. So w is a branch in $M_{\mathscr{F}}$ such that $M_{\mathscr{F}}$ is a counter-model to $\Gamma \Rightarrow \Sigma$ at w. It follows that $M_{\mathscr{F}}$ is a counter-example to $(\Gamma \Rightarrow \Sigma)$. $\square$

2.3.3 System T

The calculus obtained by adding the rule

$$
\mathrm{EC}\,\frac{G;(S_i);(S_i);H}{G;(S_i);H}
$$

to the calculus of fig. 2.3 captures the modal logic System T. If a hypersequent G is derivable according to those rules this is indicated by $\vdash_{ht} G$. This rule is sound for the class of models with reflexive frames, i.e. $\mathbb{T}$.

Lemma 2.4. *If $\vdash_{ht} G$ then there is no model $M_{\mathscr{F}}$ such that $\mathscr{F} \in \mathbb{T}$ and $M_{\mathscr{F}}$ is a counter-example to G.*

Proof. As above, the proof that the rules of fig. 2.3 are sound for the set of models with frames in $\mathbb{T}$ is similar to the proof of lemma 2.2. This only leaves it to be shown that the rule EC preserves validity over models with reflexive frames. Let $M_{\mathscr{F}}$ be a

counter-model to $(S_1);\ldots;(S_i);\ldots;(S_n)$. There is branch of worlds $w_1,\ldots,w_n$ in $\mathscr{F}$ such that $w_j R_{\mathscr{F}} w_{j-1}$ and $M_{\mathscr{F}}$ is a counter-model to S_j at w_j for $1 \leqslant j \leqslant n$. Since $\mathscr{F}$ is reflexive $w_i R_{\mathscr{F}} w_i$. So there is a branch, $w_1,\ldots,w_i,w_i,\ldots,w_n$ in $\mathscr{F}$ such that $M_{\mathscr{F}}$ is a counter-model to S_j at w_j. It follows that $M_{\mathscr{F}}$ is a counter-exmaple to $(S_1);\ldots;(S_i);(S_i);\ldots;(S_n)$. □

Theorem 2.3.3. $\vdash_t \Gamma \Rightarrow \Sigma$ *iff* $\vdash_{ht} (\Gamma \Rightarrow \Sigma)$.

Proof. For the left to right direction let $\vdash_t \Gamma \Rightarrow \Sigma$. It is proved by induction on the length of deductions that $\vdash_{ht} (\Gamma \Rightarrow \Sigma)$. All the cases except for an application of t have been considered above. Let δ be the deduction in question. If the last inference of δ is t then δ has the form

$$\dfrac{\dfrac{\begin{matrix}\delta\\ \vdots\end{matrix}}{\Gamma,\varphi \Rightarrow \Sigma}}{\Gamma,\Box\varphi \Rightarrow \Sigma}\ t$$

By IH there is a deduction δ' of $(\Gamma,\varphi \Rightarrow \Sigma)$. The following deduction completes this half of the theorem.

$$\cfrac{\cfrac{\cfrac{\cfrac{\begin{matrix}\delta'\\ \vdots\\ (\Gamma,\varphi \Rightarrow \Sigma)\end{matrix}}{(\Gamma,\varphi \Rightarrow \Sigma);(\ \Rightarrow\ )}\ \mathrm{W}}{(\Gamma \Rightarrow \Sigma);(\Box\varphi \Rightarrow\ )}\ \mathrm{L}\Box}{(\Gamma,\Box\varphi \Rightarrow \Sigma);(\Gamma,\Box\varphi \Rightarrow \Sigma)}\ \mathrm{TL{+}TR}}{(\Gamma,\Box\varphi \Rightarrow \Sigma)}\ \mathrm{EC}$$

For the right to left direction let $\nvdash_t \Gamma \Rightarrow \Sigma$. Since the sequent calculus for system T is adequate with respect to $\mathbb{T}$ there is a model $M_{\mathscr{F}}$ with a reflexive frame $\mathscr{F}$ and world $w \in W_{\mathscr{F}}$ such that $I_{M_{\mathscr{F}}}(w,\Gamma) = 1$ and $I_{M_{\mathscr{F}}}(w,\Sigma) = 0$. w is a branch in $\mathscr{F}$ such that $M_{\mathscr{F}}$ is a counter-example to $\Gamma \Rightarrow \Sigma$ at w. By lemma 2.4, $\nvdash_{ht} (\Gamma \Rightarrow \Sigma)$. □

2.3.4 System S4

The calculus obtained by adding the rules

$$\text{EW}\ \frac{G; H}{G; (\quad \Rightarrow \quad); H}$$

and EC to the calculus of fig. 2.3 captures the modal logic System S4. If a hypersequent G is derivable according to those rules this is indicated by $\vdash_{hs4} G$. Both rules are sound with respect to the class of models with reflexive and transitive frames, i.e. $\mathbb{TR}$.

Lemma 2.5. *If* $\vdash_{hs4} G$ *then there is no model* $M_{\mathscr{F}}$ *such that* $\mathscr{F} \in \mathbb{TR}$ *and* $M_{\mathscr{F}}$ *is a counter-example to* G*.*

Proof. All the cases with the exception of EW have been considered above. This leaves only the task of showing that EW preserves soundness. Let $M_{\mathscr{F}}$ be a counter-example to $(S_1); \ldots; (S_i); (\quad \Rightarrow \quad); (S_{i+2}); \ldots (S_n)$. So there is a branch $w_1, \ldots, w_n$ in $\mathscr{F}$ such that for each w_i, $M_{\mathscr{F}}$ is a counter-model to S_i at w_i. Since $R_{\mathscr{F}}$ is transitive $w_1, \ldots, w_i, w_{i+2}, \ldots, w_n$ also forms a branch in $\mathscr{F}$. It follows that $M_{\mathscr{F}}$ is a counter-example to $(S_1); \ldots; (S_i); (S_{i+2}); \ldots; (S_n)$. □

Lemma 2.6. $\vdash_{hs4} (\Box\varphi \Rightarrow \Box\Box\varphi)$

Proof. This lemma is proved by the following deduction.

$$\text{R}\Box\ \cfrac{\text{R}\Box\ \cfrac{\text{EW}\ \cfrac{\text{L}\Box\ \cfrac{\text{EW}\ \cfrac{\text{Id}\ \cfrac{}{(\varphi \Rightarrow \varphi)}}{(\varphi \Rightarrow \varphi); (\quad \Rightarrow \quad)}}{(\quad \Rightarrow \varphi); (\Box\varphi \Rightarrow \quad)}}{(\quad \Rightarrow \varphi); (\quad \Rightarrow \quad); (\Box\varphi \Rightarrow \quad)}}{(\quad \Rightarrow \Box\varphi); (\Box\varphi \Rightarrow \quad)}}{(\Box\varphi \Rightarrow \Box\Box\varphi)}$$

$\square$

Theorem 2.3.4. $\vdash_{s4} \Gamma \Rightarrow \Sigma$ *iff* $\vdash_{hs4} (\Gamma \Rightarrow \Sigma)$.

Proof. For the left to right direction let $\delta \vdash_{s4} \Gamma \Rightarrow \Sigma$. It is proved by induction on the length of δ that $\vdash_{hs4} \Gamma \Rightarrow \Sigma$. As above, the only case that has not been covered is the case where the last inference of δ is an application of EW. Let δ have the form

$$\cfrac{\begin{array}{c}\delta\\ \vdots\\ \Box\Gamma \Rightarrow \varphi\end{array}}{\Box\Gamma \Rightarrow \Box\varphi}\ \text{EW}$$

By IH there is a deduction δ' of $(\Box\Gamma \Rightarrow \varphi)$. Call the following deduction $\hat{\delta}$ and let the cardinality of $\Box\Gamma$ be n.

$$\begin{array}{rc}
 & \delta' \\
 & \vdots \\
\text{EW} & \underline{(\Box\Gamma \Rightarrow \varphi)} \\
\text{L}\Box \times n & \underline{(\Box\Gamma \Rightarrow \varphi); (\quad \Rightarrow \quad)} \\
\text{R}\Box & \underline{(\quad \Rightarrow \varphi); (\Box\Box\Gamma \Rightarrow \quad)} \\
\text{TL}\times n & \underline{(\Box\Box\Gamma \Rightarrow \Box\varphi)} \\
 & (\Box\Gamma, \Box\Box\Gamma \Rightarrow \Box\varphi)
\end{array}$$

By lemma 2.6, TL and TR, there is a deduction of $(\Box\Box\Gamma\backslash\{\Box\Box\gamma_i\}, \Box\gamma_i \Rightarrow \Box\Box\gamma_i, \Box\varphi)$ for each $\gamma_i \in \Gamma$. Applying Cut to $\hat{\delta}$ for each such γ_i generates a deduction of $(\Box\Gamma \Rightarrow \Box\Box\varphi)$.

For the right to left direction let $\nvdash_{s4} \Gamma \Rightarrow \Sigma$. $\mathbb{TR}$ is adequate for the sequent calculus formulation of S4, so there is a model $M_{\mathscr{F}}$ with a world $w \in W_{\mathscr{F}}$ such that $M_{\mathscr{F}}$ is a counter-model to $\Gamma \Rightarrow \Sigma$ at w. Since w is also a branch in $\mathscr{F}$, $M_{\mathscr{F}}$ is a counter-example to the hypersequent $(\Gamma \Rightarrow \Sigma)$. By lemma 2.5, $\nvdash_{hs4} (\Gamma \Rightarrow \Sigma)$. $\square$

2.3.5 System B

The calculus generated by adding the rule

$$\mathrm{SYM}\ \frac{(S_1);(S_2);\ldots;(S_{n-1});(S_n)}{(S_n);(S_{n-1});\ldots;(S_2);(S_1)}$$

to fig. 2.3 captures the modal logic B. If there is a deduction of a hypersequent G according to those rules this is indicated by $\vdash_{hb} G$. This calculus is sound for the class of models with symmetric frames.

Lemma 2.7. *The rule*

$$\Box E_1\ \frac{(\Gamma \Rightarrow \Box\varphi, \Sigma)}{(\Gamma \Rightarrow \Sigma);(\quad \Rightarrow \varphi)}$$

is derivable.

Proof. The following deduction establishes this lemma.

$$\mathrm{Cut}\ \dfrac{\mathrm{TR}\ \dfrac{\mathrm{W}\ \dfrac{(\Gamma \Rightarrow \Box\varphi, \Sigma)}{(\Gamma \Rightarrow \Box\varphi, \Sigma);(\quad \Rightarrow \quad)}}{(\Gamma \Rightarrow \Box\varphi, \Sigma);(\quad \Rightarrow \varphi)} \qquad \mathrm{TL/TR}\ \dfrac{\mathrm{SYM}\ \dfrac{\mathrm{L}\Box\ \dfrac{\mathrm{W}\ \dfrac{\mathrm{Id}\ \dfrac{}{(\varphi \Rightarrow \varphi)}}{(\varphi \Rightarrow \varphi);(\quad \Rightarrow \quad)}}{(\quad \Rightarrow \varphi);(\Box\varphi \Rightarrow \quad)}}{(\Box\varphi \Rightarrow \quad);(\quad \Rightarrow \varphi)}}{(\Gamma, \Box\varphi \Rightarrow \Sigma);(\quad \Rightarrow \varphi)}}{(\Gamma \Rightarrow \Sigma);(\quad \Rightarrow \varphi)}$$

□

Lemma 2.8. *The rule*

$$\Box E_2\ \frac{(\Gamma \Rightarrow \Box\varphi, \Sigma);(\Delta \Rightarrow \Lambda)}{(\Gamma \Rightarrow \Sigma);(\Delta \Rightarrow \varphi, \Lambda)}$$

is derivable.

Proof. The following deduction establishes this lemma.

$$
\begin{prooftree}
\AxiomC{$(\Gamma \Rightarrow \Box\varphi, \Sigma); (\Delta \Rightarrow \Lambda)$}
\LeftLabel{TR}
\UnaryInfC{$(\Gamma \Rightarrow \Box\varphi, \Sigma); (\Delta \Rightarrow \varphi, \Lambda)$}
\AxiomC{}
\LeftLabel{Id}
\UnaryInfC{$(\varphi \Rightarrow \varphi)$}
\LeftLabel{W}
\UnaryInfC{$(\varphi \Rightarrow \varphi); (\ \Rightarrow \ )$}
\LeftLabel{L$\Box$}
\UnaryInfC{$(\ \Rightarrow \varphi); (\Box\varphi \Rightarrow \ )$}
\LeftLabel{SYM}
\UnaryInfC{$(\Box\varphi \Rightarrow \ ); (\ \Rightarrow \varphi)$}
\LeftLabel{TL/TR}
\UnaryInfC{$(\Gamma, \Box\varphi \Rightarrow \Sigma); (\Delta \Rightarrow \varphi, \Lambda)$}
\LeftLabel{Cut}
\BinaryInfC{$(\Gamma \Rightarrow \Sigma); (\Delta \Rightarrow \varphi, \Lambda)$}
\end{prooftree}
$$

□

Lemma 2.9. *If $\vdash_{hb} G$ then there is no model $M_{\mathscr{F}}$ such that $\mathscr{F} \in \mathbb{B}$ and $M_{\mathscr{F}}$ is a counter-example to G.*

Proof. Let $\delta \vdash_{hb} G$. The lemma is proved by induction over the length of δ. As above most of the cases to be considered are similar to cases already rehearsed in lemma 2.2. This leaves only the case where the last inference of δ is an instance of SYM. Let δ have the form

$$
\begin{array}{c}
\delta \\
\vdots \\
\text{SYM}\ \dfrac{(S_1); (S_2); \ldots; (S_{n-1}); (S_n)}{(S_n); (S_{n-1}); \ldots; (S_2); (S_1)}
\end{array}
$$

Let $M_{\mathscr{F}}$ be a counter-example to $(S_n); (S_{n-1}); \ldots; (S_2); (S_1)$. There is a branch $w_n, w_{n-1}, \ldots, w_2, w_1$ in $W_{\mathscr{F}}$ such that $w_{i+1} R_{\mathscr{F}} w_i$ and $M_{\mathscr{F}}$ is a counter-model to S_i at w_i for each $1 \leqslant i \leqslant n$. Since $\mathscr{F}$ is symmetrical, for each w_i and w_{i+1}, $w_{i+1} R w_i$. $w_1, w_2, \ldots, w_{n-1}, w_n$ is such that $M_{\mathscr{F}}$ is a counter-model to S_i at w_i. It follows that $M_{\mathscr{F}}$ is a counter-example to $(S_1); (S_2); \ldots; (S_{n-1}); (S_n)$. □

Theorem 2.3.5. $\vdash_b \Gamma \Rightarrow \Sigma$ *iff* $\vdash_{hb} (\Gamma \Rightarrow \Sigma)$.

Proof. For the left to right direction let $\delta \vdash_b \Gamma \Rightarrow \Sigma$. It is proved by induction on the length of δ that $\vdash_{hb} (\Gamma \Rightarrow \Sigma)$. The only case that has not been addressed above is the case where the last inference of δ is an instance of b. In this case δ has the following form.

$$\dfrac{\begin{array}{c}\delta \\ \vdots \\ \Gamma \Rightarrow \varphi, \Box\Sigma\end{array}}{\Box\Gamma \Rightarrow \Box\varphi, \Sigma}\,\mathrm{b}$$

By IH there is a deduction $\hat{\delta}$ of $(\Gamma \Rightarrow \varphi, \Box\Sigma)$. Let the cardinality of Σ be n and the cardinality of Γ be m. The following deduction establishes the left to right direction.

$$\begin{array}{rc}
 & \hat{\delta} \\
 & \vdots \\
 & (\Gamma \Rightarrow \varphi, \Box\Sigma) \\
\Box\mathrm{E}_1 & \overline{(\Gamma \Rightarrow \varphi, \Box\Sigma/\{\sigma_1\}); (\quad \Rightarrow \sigma_1)} \\
\Box\mathrm{E}_2 \times n & \overline{(\Gamma \Rightarrow \varphi); (\quad \Rightarrow \Sigma)} \\
\mathrm{L}\Box \times m & \overline{(\quad \Rightarrow \varphi); (\Box\Gamma \Rightarrow \Sigma)} \\
\mathrm{R}\Box & \overline{(\Box\Gamma \Rightarrow \Box\varphi, \Sigma)}
\end{array}$$

For the right to left direction let $\nvdash_b \Gamma \Rightarrow \Sigma$. Because the class of models with frames in $\mathbb{B}$ is adequate for the calculus that results from fig. 2.1 along with k and b, there is a model $M_{\mathscr{F}}$ with a world $w \in W_{\mathscr{F}}$ such that $M_{\mathscr{F}}$ is a counter-model to $\Gamma \Rightarrow \Sigma$ at w. It follows that $M_{\mathscr{F}}$ is a counter-example to $(\Gamma \Rightarrow \Sigma)$. By lemma 2.9, $\nvdash_{hb} (\Gamma \Rightarrow \Sigma)$. □

2.3.6 System S5

The calculus that results from adding the rule

$$\text{EE}\ \frac{G; (S_1); (S_2); H}{G; (S_2); (S_1); H}$$

to the calculus for hypersequent S4 generates a calculus that captures the modal logic S5. Hypersequents for logics weaker than S5 are lists of sequents. Because of the structural rules that have been added hypersequents at this point behave as though they are sets of sequents. By EC the number of repeated sequents does not make a difference to what hypersequents are deducible and by EE the order of the sequents occurring in a hypersequent does not make a difference to what hypersequents are deducible. If a hypersequent G is deducible according to the rules of this calculus this is indicated by $\vdash_{hs5} G$. Systems equivalent to this one have appeared in Restall [51, 53, 54].

Lemma 2.10. *The rule*

$$GE_1\ \frac{G; (\Gamma \Rightarrow \Box\varphi, \Sigma)}{G; (\Gamma \Rightarrow \Sigma); (\quad \Rightarrow \varphi)}$$

is derivable.

Proof. Let the cardinality of G be n, the cardinality of the set of sentences appearing in G be m, and the cardinality of $\Gamma \cup \Sigma$ be j. This lemma is proved by the following deduction.

$$\text{Cut}\ \dfrac{\text{TR}\ \dfrac{\text{EW}\ \dfrac{G; (\Gamma \Rightarrow \Box\varphi, \Sigma)}{G; (\Gamma \Rightarrow \Box\varphi, \Sigma); (\quad \Rightarrow \quad)}}{G; (\Gamma \Rightarrow \Box\varphi, \Sigma); (\quad \Rightarrow \varphi)} \qquad \text{TL/TR}\times m+j\ \dfrac{\text{EW}\times n\ \dfrac{\text{EE}\ \dfrac{\text{L}\Box\ \dfrac{\text{EW}\ \dfrac{\text{Id}\ \dfrac{}{(\varphi \Rightarrow \varphi)}}{(\varphi \Rightarrow \varphi); (\quad \Rightarrow \quad)}}{(\quad \Rightarrow \varphi); (\Box\varphi \Rightarrow \quad)}}{(\Box\varphi \Rightarrow \quad); (\quad \Rightarrow \varphi)}}{(\quad \Rightarrow \quad); \ldots; (\Box\varphi \Rightarrow \quad); (\quad \Rightarrow \Box\varphi)}}{G; (\Gamma, \Box\varphi \Rightarrow \Sigma); (\quad \Rightarrow \varphi)}}{G; (\Gamma \Rightarrow \Sigma); (\quad \Rightarrow \varphi)}$$

$\square$

Lemma 2.11. *The rule*

$$L\Box_1 \frac{G; (\Gamma, \Box\varphi \Rightarrow \Sigma); (\Delta \Rightarrow \Lambda); H}{G; (\Gamma \Rightarrow \Sigma); (\Delta, \Box\varphi \Rightarrow \Lambda); H}$$

is derivable.

Proof. This lemma is proved by the following deduction.

$$\mathrm{Cut}\,\dfrac{\mathrm{TR}\,\dfrac{G; (\Gamma, \Box\varphi \Rightarrow \Sigma); (\Delta \Rightarrow \Lambda); H}{G; (\Gamma, \Box\varphi \Rightarrow \Sigma); (\Delta, \Box\varphi \Rightarrow \Lambda); H} \qquad \mathrm{EE}\,\dfrac{\mathrm{R}\Box\,\dfrac{\mathrm{EE}\,\dfrac{\mathrm{L}\Box\,\dfrac{\mathrm{TL/TR}\,\dfrac{\mathrm{EW}\,\dfrac{\mathrm{Id}\,\dfrac{}{(\varphi \Rightarrow \varphi)}}{(\varphi \Rightarrow \varphi); (\ \Rightarrow\ ); \ldots; (\ \Rightarrow\ )}}{(\varphi \Rightarrow \varphi); (\Delta \Rightarrow \Lambda); (\Gamma \Rightarrow \Sigma); G; H}}{(\ \Rightarrow \varphi); (\Delta, \Box\varphi \Rightarrow \Lambda); (\Gamma \Rightarrow \Sigma); G; H}}{(\ \Rightarrow \varphi); (\Gamma \Rightarrow \Sigma); G; (\Delta, \Box\varphi \Rightarrow \Lambda); H}}{(\Gamma \Rightarrow \Box\varphi, \Sigma); G; (\Delta, \Box\varphi \Rightarrow \Lambda); H}}{G; (\Gamma \Rightarrow \Box\varphi, \Sigma); (\Delta, \Box\varphi \Rightarrow \Lambda); H}}{G; (\Gamma \Rightarrow \Sigma); (\Delta, \Box\varphi \Rightarrow \Lambda); H}$$

$\square$

Lemma 2.12. *If $\vdash_{hs5} G$ then there is no model $M_{\mathscr{F}}$ such that $\mathscr{F} \in \mathbb{L}$ and $M_{\mathscr{F}}$ is a counter-example to G.*

Proof. Let $\delta \vdash G$. This lemma is proved by induction on the length of δ. As above all but the case of EE has already been considered. For the last case let $\nvdash_{hs5}$ $(S_1); \ldots; (S_i); (S_{i+1}); \ldots; (S_n)$. There is a counter-example $M_{\mathscr{F}}$ with world, $w_1, \ldots, w_n \in W_{\mathscr{F}}$ such that $w_j R w_{j-1}$ and $M_{\mathscr{F}}$ is a counter-model to S_j at w_j. Since for any worlds w_j and w_k $w_j R_{\mathscr{F}} w_k$, $w_i R_{\mathscr{F}} w_{i+1}$. So $M_{\mathscr{F}}$ is a counter-model to $(S_1); \ldots; (S_{i+1}); (S_i); \ldots; (S_n)$. $\square$

Theorem 2.3.6. $\vdash_{s5} \Gamma \Rightarrow \Sigma$ *iff* $\vdash_{hs5} \Gamma \Rightarrow \Sigma$.

Proof. For the left to right direction let $\delta \vdash_{s5} \Gamma \Rightarrow \Sigma$. It is shown by induction on the length of δ that $\vdash_{hs5} (\Gamma \Rightarrow \Sigma)$. The only case that has not been treated above is the case where the last inference of δ is an instance of 5. In that case δ has the following form.

$$\begin{array}{c} \delta \\ \vdots \\ 5\ \dfrac{\Box\Gamma \Rightarrow \varphi, \Box\Sigma}{\Box\Gamma \Rightarrow \Box\varphi, \Box\Sigma} \end{array}$$

By IH there is a deduction $\hat{\delta}$ such that $\hat{\delta} \vdash_{hs5} (\Box\Gamma \Rightarrow \varphi, \Box\Sigma)$. Let Σ be $\{\sigma_1, \ldots, \sigma_n\}$ and the cardinality of Γ be m. The following deduction establishes the left to right direction of this lemma.

$$\begin{array}{rc}
 & \hat{\delta} \\
 & \vdots \\
\text{GE}_1 \times n & \dfrac{(\Box\Gamma \Rightarrow \varphi, \Box\Sigma)}{(\Box\Gamma \Rightarrow \varphi); (\ \Rightarrow \sigma_1); \ldots; (\ \Rightarrow \sigma_n)} \\
\text{EW} & \dfrac{}{(\Box\Gamma \Rightarrow \varphi); (\ \Rightarrow\ ); (\ \Rightarrow \sigma_1); \ldots; (\ \Rightarrow \sigma_n)} \\
\text{L}\Box_1 & \dfrac{}{(\ \Rightarrow \varphi); (\Box\Gamma \Rightarrow\ ); (\ \Rightarrow \sigma_1); \ldots; (\ \Rightarrow \sigma_n)} \\
\text{R}\Box & \dfrac{}{(\Box\Gamma \Rightarrow \Box\varphi); (\ \Rightarrow \sigma_1); \ldots; (\ \Rightarrow \sigma_n)} \\
\text{EE} & \dfrac{}{(\ \Rightarrow \sigma_1); (\Box\Gamma \Rightarrow \Box\varphi); (\ \Rightarrow \sigma_2); \ldots; (\ \Rightarrow \sigma_n)} \\
\text{R}\Box & \dfrac{}{(\Box\Gamma \Rightarrow \Box\varphi, \Box\sigma_1); (\ \Rightarrow \sigma_2); \ldots; (\ \Rightarrow \sigma_n)} \\
\text{EE} + \text{R}\Box \times n-1 & \dfrac{}{(\Box\Gamma \Rightarrow \Box\varphi, \Box\Sigma)}
\end{array}$$

For the left to right direction let $\nvdash_{s5} \Gamma \Rightarrow \Sigma$. Because the class of models with frames in $\mathscr{L}$ are adequate for the sequent calculus $S5$ there is a model $M_{\mathscr{F}}$ with a world $w \in W_{\mathscr{F}}$ such that $M_{\mathscr{F}}$ is a counter-model to $\Gamma \Rightarrow \Sigma$ at w. It follows that $M_{\mathscr{F}}$ is a counter-example to $(\Gamma \Rightarrow \Sigma)$. By lemma 2.12 $\nvdash_{hs5} (\Gamma \Rightarrow \Sigma)$.

□

2.4 Technical Results

This section explores some features of the systems described above.

2.4.1 Interdefinability of $\Box$ and $\Diamond$

The use of $\Box$ as the primitive modal connective makes no difference to the results above. $\Diamond$ could be introduced using the rules L$\Diamond$ and R$\Diamond$.

$$\mathrm{L}\Diamond\,\frac{(\varphi \Rightarrow \quad);(\Gamma \Rightarrow \Sigma);G}{(\Gamma, \Diamond\varphi \Rightarrow \Sigma);G} \qquad\qquad \mathrm{R}\Diamond\,\frac{G;(\Gamma \Rightarrow \varphi,\Sigma);(\Delta \Rightarrow \Lambda);H}{G;(\Gamma \Rightarrow \Sigma);(\Delta \Rightarrow \Diamond\varphi,\Lambda);H}$$

If $\Diamond\varphi$ is defined as $\neg\Box\neg\varphi$, $\Diamond\varphi$ obeys L$\Diamond$ and R$\Diamond$. This is proved by the following two deductions.

$$\begin{array}{l}\mathrm{R}\neg\,\dfrac{(\varphi \Rightarrow \quad);(\Gamma \Rightarrow \Sigma);G}{(\quad \Rightarrow \neg\varphi);(\Gamma \Rightarrow \Sigma);G}\\ \mathrm{R}\Box\,\dfrac{}{(\Gamma \Rightarrow \Box\neg\varphi,\Sigma);G}\\ \mathrm{L}\neg\,\dfrac{}{(\Gamma,\neg\Box\neg\varphi \Rightarrow \Sigma);G}\end{array} \qquad\qquad \begin{array}{l}\mathrm{L}\neg\,\dfrac{G;(\Gamma \Rightarrow \varphi,\Sigma);(\Delta \Rightarrow \Lambda);H}{G;(\Gamma,\neg\varphi \Rightarrow \Sigma);(\Delta \Rightarrow \Lambda);H}\\ \mathrm{L}\Box\,\dfrac{}{G;(\Gamma \Rightarrow \Sigma);(\Delta,\Box\neg\varphi \Rightarrow \Lambda);H}\\ \mathrm{R}\neg\,\dfrac{}{G;(\Gamma \Rightarrow \Sigma);(\Delta \Rightarrow \neg\Box\neg\varphi,\Lambda);H}\end{array}$$

Conversely, taking $\Diamond$ as primitive $\Box\varphi$ is adequately definable as $\neg\Diamond\neg\varphi$. Let $K^{\Diamond}$ be the calculus that results from the calculus of fig. 2.3 by replacing L$\Box$ and R$\Box$ by L$\Diamond$ and R$\Diamond$. The following two deductions establish that $\Box\varphi$ is definable by $\neg\Diamond\neg\varphi$ in the calculus $K^{\Diamond}$. Since this result holds in $K^{\Diamond}$ it holds in any extension of this logic by the addition of external structural rules.

$$\begin{array}{l}\mathrm{R}\neg\,\dfrac{G;(\Gamma,\varphi \Rightarrow \Sigma);(\Delta \Rightarrow \Lambda);H}{G;(\Gamma \Rightarrow \neg\varphi,\Sigma);(\Delta \Rightarrow \Lambda);H}\\ \mathrm{R}\Diamond\,\dfrac{}{G;(\Gamma \Rightarrow \Sigma);(\Delta \Rightarrow \Diamond\neg\varphi,\Lambda);H}\\ \mathrm{L}\neg\,\dfrac{}{G;(\Gamma \Rightarrow \Sigma);(\Delta,\neg\Diamond\neg\varphi, \Rightarrow \Lambda);H}\end{array} \qquad\qquad \begin{array}{l}\mathrm{L}\neg\,\dfrac{(\quad \Rightarrow \varphi);(\Gamma \Rightarrow \Sigma);H}{(\neg\varphi \Rightarrow \quad);(\Gamma \Rightarrow \Sigma);H}\\ \mathrm{L}\Diamond\,\dfrac{}{(\Gamma,\Diamond\neg\varphi \Rightarrow \Sigma);H}\\ \mathrm{R}\neg\,\dfrac{}{(\Gamma \Rightarrow \neg\Diamond\neg\varphi,\Sigma);H}\end{array}$$

2.4.2 Uniqueness

As mentioned in section 2.2 if a calculus is to be used to underwrite an inferentialist theory of meaning it is desirable that that calculus uniquely determines the expressions it introduces. The sequent calculi discussed in section 2.2 do not uniquely determine the expression $\Box$. On the other hand, each of the hypersequent calculi do uniquely determine the expression $\Box$.[5]

Let $K^{\boxdot}$ be the calculus that results from adding the following rules to the rules of fig. 2.3.

$$\text{L}\boxdot\,\frac{G; (\Gamma, \varphi \Rightarrow \Sigma); (\Delta \Rightarrow \Lambda); H}{G; (\Gamma \Rightarrow \Sigma); (\Delta, \boxdot\varphi \Rightarrow \Lambda); H} \qquad\qquad \text{R}\boxdot\,\frac{(\quad \Rightarrow \varphi); (\Gamma \Rightarrow \Sigma); H}{(\Gamma \Rightarrow \boxdot\varphi, \Sigma); H}$$

Lemma 2.13. *$\vdash_{K^{\boxdot}} G; (\Gamma, \Box\varphi \Rightarrow \Sigma); H$ iff $\vdash_{K^{\boxdot}} G; (\Gamma, \boxdot\varphi \Rightarrow \Sigma); H$.*

Proof. For the left to right direction let $\delta \vdash_{K^{\boxdot}} \Gamma, \Box\varphi \Rightarrow \Sigma$. It is proved by induction on the length of δ that $\vdash_{K^{\boxdot}} \Gamma, \boxdot\varphi \Rightarrow \Sigma$. Let I be the last inference of δ. If I is Id then neither $\boxdot$ nor $\Box$ can occur in the conclusion hypersequent of δ. There are ten other cases to consider. Only several of these cases are considered explicitly.

Case 1 (I is TL). Without loss of generality let $\Box\varphi \in \Gamma$. In this case δ has the form

$$\begin{array}{c} \delta \\ \vdots \\ \text{TL}\,\dfrac{G; (\Gamma, \Box\varphi \Rightarrow \Sigma); H}{G; (\Gamma, \Box\varphi, \psi \Rightarrow \Sigma); H} \end{array}$$

By IH there is a deduction $\hat{\delta}$ of $G; (\Gamma, \boxdot\varphi \Rightarrow \Sigma); H$. An application of TL to $\hat{\delta}$ yields a deduction of $G; (\Gamma, \boxdot\varphi, \psi \Rightarrow \Sigma); H$. If $\Box\varphi$ is the main formula in TL then the

[5] They also uniquely characterize the extensional connectives.

deduction with $\boxdot\varphi$ is the main formula of TL yields a deduction of $G;(\Gamma,\boxdot\varphi,\psi \Rightarrow \Sigma);H$.

Case 2 (I is L$\rightarrow$). Without loss of generality let $\Box\varphi$ occur in Γ. In this case δ is a deduction with the following form.

$$\text{Cut}\ \dfrac{\begin{matrix}\delta_1\\ \vdots\\ G;(\Gamma,\Box\varphi\Rightarrow\psi,\Sigma);H\end{matrix}\qquad\qquad \begin{matrix}\delta_2\\ \vdots\\ G;(\Gamma,\Box\varphi,\theta\Rightarrow\Sigma);H\end{matrix}}{G;(\Gamma,\Box\varphi,\psi\rightarrow\theta\Rightarrow\Sigma);H}$$

By IH there are deductions $\hat{\delta_1}$ and $\hat{\delta_2}$ of $G;(\Gamma,\boxdot\varphi \Rightarrow \psi,\Sigma);H$ and $G;(\Gamma,\boxdot\varphi,\theta \Rightarrow \Sigma);H$ respectively. An application of L$\rightarrow$ to these deductions yields a deduction of $G;(\Gamma,\boxdot\varphi,\psi\rightarrow\theta \Rightarrow \Sigma);H$.

Case 3 (I is L$\Box$). If $\Box\varphi$ is not the main formula of I the result follows in a way similar to the above cases. If, on the other hand, $\Box\varphi$ is the main formula of I then δ has the form

$$\text{L}\Box\ \dfrac{\begin{matrix}\delta\\ \vdots\\ G;(\Delta,\varphi\Rightarrow\Lambda);(\Gamma\Rightarrow\Sigma);H\end{matrix}}{G;(\Delta\Rightarrow\Lambda);(\Gamma,\Box\varphi\Rightarrow\Sigma);H}$$

In this case, replacing I with an instance of L$\boxdot$ yields a deduction of $G;(\Delta \Rightarrow \Lambda);(\Gamma,\boxdot\varphi \Rightarrow \Sigma);H$.

The right to left direction of this lemma is analogous to the above proof.

$\square$

Lemma 2.14. $\vdash_{K\boxdot} (\Gamma \Rightarrow \Box\varphi,\Sigma)$ *iff* $\vdash_{K\boxdot} (\Gamma \Rightarrow \boxdot\varphi,\Sigma)$.

Proof. The proof of this is analogous to the proof of lemma 2.13. For the left to right direction let δ be a deduction of $G; (\Gamma \Rightarrow \Sigma); H$ whose last inference is I. In the case that I is R$\Box$ and $\Box\varphi$ is the formula introduced by R$\Box$ δ has the form

$$\dfrac{\begin{array}{c}\delta\\ \vdots\\ (\quad \Rightarrow \varphi); (\Gamma \Rightarrow \Sigma); H\end{array}}{(\Gamma \Rightarrow \Box\varphi, \Sigma); H}\,\mathrm{R}\Box$$

Replacing the occurrence of R$\Box$ by R$\boxdot$ as the last inference of δ establishes the result.

The right to left direction is analogous. $\square$

The proof that the other hypersequent calculi uniquely characterize $\Box$ requires showing that the induction over deductions done in lemmas 2.13 and 2.14 holds when external structural rules are added to the calculi. These proofs follow the general pattern of applying the inductive hypothesis to generate a deduction to which the appropriate inference for the case can be applied. These proofs are left out in the interest of space.

2.4.3 Cut Elimination

Two of the hypersequent calculi presented in section 2.3 are known to be cut admissible. I prove that the calculus given in section 2.3.2 is cut eliminable in chapter 3. It is proved below that the calculus given in section 2.3.6 is cut admissible. Restall [51] shows that an equivalent system is cut eliminable and also there proves the soundness and completeness results for that system. In "A Cut-Free Sequent System for Two-Dimensional Modal Logic and Why it Matters" ([55]), he proves the same

results for a calculus of which the hypersequent calculus for S5 is a part. The proof is rehearsed here for the sake of completeness. It is an open question whether or not the other hypersequent calculi are cut eliminable.

Lemma 2.15. *The following rule*

$$L\Box_U \frac{(\Gamma_1, \varphi \Rightarrow \Sigma_1); \ldots ; (\Gamma_i, \varphi \Rightarrow \Sigma_i); \ldots ; (\Gamma_n, \varphi \Rightarrow \Sigma_n)}{(\Gamma_1 \Rightarrow \Sigma_1); \ldots ; (\Gamma_i, \Box\varphi \Rightarrow \Sigma_i); \ldots ; (\Gamma_n \Rightarrow \Sigma_n)}$$

is derivable in the cut free fragment of the hypersequent calculus for S5.

Proof. This lemma is established by the following deduction.

$$\text{EC}\,\dfrac{\text{L}\Box\,\dfrac{\text{EW + TL+ TR}\,\dfrac{\text{EE + L}\Box\,\dfrac{\text{L}\Box\,\dfrac{\text{EE}\,\dfrac{(\Gamma_1, \varphi \Rightarrow \Sigma_1); \ldots ; (\Gamma_i, \varphi \Rightarrow \Sigma_i); \ldots ; (\Gamma_n, \varphi \Rightarrow \Sigma_n)}{(\Gamma_1, \varphi \Rightarrow \Sigma_1); (\Gamma_i, \varphi \Rightarrow \Sigma_i); \ldots ; (\Gamma_n, \varphi \Rightarrow \Sigma)}}{(\Gamma_1 \Rightarrow \Sigma_1); (\Gamma_i, \Box\varphi, \varphi \Rightarrow \Sigma_i); \ldots ; (\Gamma_n, \varphi \Rightarrow \Sigma)}}{(\Gamma_1 \Rightarrow \Sigma_1); \ldots ; (\Gamma_i, \Box\varphi, \varphi \Rightarrow \Sigma_i); \ldots ; (\Gamma_n \Rightarrow \Sigma_n)}}{(\Gamma_1 \Rightarrow \Sigma_1); \ldots ; (\Gamma_i, \Box\varphi, \varphi \Rightarrow \Sigma_i); (\Gamma_i \Rightarrow \Sigma_i); \ldots ; (\Gamma_n \Rightarrow \Sigma_n)}}{(\Gamma_1 \Rightarrow \Sigma_1); \ldots ; (\Gamma_i, \Box\varphi \Rightarrow \Sigma_i); (\Gamma_i, \Box\varphi \Rightarrow \Sigma_i); \ldots ; (\Gamma_n \Rightarrow \Sigma_n)}}{(\Gamma_1 \Rightarrow \Sigma_1); \ldots ; (\Gamma_i, \Box\varphi \Rightarrow \Sigma_i); \ldots ; (\Gamma_n \Rightarrow \Sigma_n)}$$

□

Let G be a hypersequent. A tree with G at the root built according to the following procedure is called $\pi(G)$. $\pi(G)$ is constructed in stages. A leaf G' of the tree being constructed is *closed* iff there is a sequent $\Gamma \Rightarrow \Sigma \in G'$ such that $\Gamma \cap \Sigma \neq \varnothing$. A leaf is *open* iff it is not closed. At each stage consider each open leaf $(\Gamma_1 \Rightarrow \Sigma_1); \ldots ; (\Gamma_n \rightarrow \Sigma_n)$ and sequent $\Gamma_i \Rightarrow \Sigma_i$ occurring in $(\Gamma_1 \Rightarrow \Sigma_1); \ldots ; (\Gamma_n \Rightarrow \Sigma_n)$. For each sentence $\varphi \in \Gamma_i \cup \Sigma_i$ do the following.

1. If φ is p_i do nothing.

2. If φ is $\neg\psi$ then

(a) if $\varphi \in \Gamma_i$ then expand that branch by

$$\frac{(\Gamma_1 \Rightarrow \Sigma_1); \dots (\Gamma_i, \varphi \Rightarrow \psi, \Sigma_i); \dots; (\Gamma_n \Rightarrow \Sigma_n)}{(\Gamma_1 \Rightarrow \Sigma_1); \dots (\Gamma_i, \varphi \Rightarrow \Sigma_i); \dots; (\Gamma_n \Rightarrow \Sigma_n)}$$

(b) if $\varphi \in \Sigma_i$ then expand that branch by

$$\frac{(\Gamma_1 \Rightarrow \Sigma_1); \dots (\Gamma_i, \psi \Rightarrow \varphi, \Sigma_i); \dots; (\Gamma_n \Rightarrow \Sigma_n)}{(\Gamma_1 \Rightarrow \Sigma_1); \dots (\Gamma_i \Rightarrow \varphi, \Sigma_i); \dots; (\Gamma_n \Rightarrow \Sigma_n)}$$

3. if φ is $\psi \to \theta$ then

(a) if $\varphi \in \Gamma_i$ then expand that branch by

$$\frac{(\Gamma_1 \Rightarrow \Sigma_1); \dots (\Gamma_i, \varphi \Rightarrow \psi, \Sigma_i); \dots; (\Gamma_n \Rightarrow \Sigma_n) \qquad (\Gamma_1 \Rightarrow \Sigma_1); \dots (\Gamma_i, \varphi, \theta \Rightarrow \Sigma_i); \dots; (\Gamma_n \Rightarrow \Sigma_n)}{(\Gamma_1 \Rightarrow \Sigma_1); \dots (\Gamma_i, \varphi \Rightarrow \Sigma_i); \dots; (\Gamma_n \Rightarrow \Sigma_n)}$$

(b) if $\varphi \in \Sigma_i$ then expand that branch by

$$\frac{(\Gamma_1 \Rightarrow \Sigma_1); \dots (\Gamma_i, \psi \Rightarrow \theta, \varphi, \Sigma_i); \dots; (\Gamma_n \Rightarrow \Sigma_n)}{(\Gamma_1 \Rightarrow \Sigma_1); \dots (\Gamma_i \Rightarrow \varphi, \Sigma_i); \dots; (\Gamma_n \Rightarrow \Sigma_n)}$$

4. if φ is $\Box\psi$ then

(a) if $\varphi \in \Gamma_i$ then expand that branch by

$$\frac{(\Gamma_1, \psi \Rightarrow \Sigma_1); \dots; (\Gamma_i, \psi, \varphi \Rightarrow \Sigma_i); \dots; (\Gamma_n, \psi \Rightarrow \Sigma_n)}{(\Gamma_1 \Rightarrow \Sigma_1); \dots; (\Gamma_i, \varphi \Rightarrow \Sigma_i); \dots; (\Gamma_n \Rightarrow \Sigma_n)}$$

(b) if $\varphi \in \Sigma_i$ and there is no $\Gamma_j \Rightarrow \Sigma_j$ such that $\psi \in \Sigma_j$ then expand that branch by

$$\frac{(\quad \Rightarrow \psi); (\Gamma_1 \Rightarrow \Sigma_1); \dots; (\Gamma_i \Rightarrow \varphi, \Sigma_i); \dots; (\Gamma_n \Rightarrow \Sigma_n)}{(\Gamma_1 \Rightarrow \Sigma_1); \dots; (\Gamma_i \Rightarrow \varphi, \Sigma_i); \dots; (\Gamma_n \Rightarrow \Sigma_n)}$$

Since each of the steps in building $\pi(G)$ is a derivable rule if every branch of $\pi(G)$ closes, $\pi(G) \vdash^{cf}_{hs5} G$. It follows that if $\nvdash^{cf}_{hs5} G$ then there is an open, possibly infinite, branch of $\pi(G)$.

Let β be a branch in $\pi(G_1)$ of the form $G_1, G_2, \ldots$. Consider two hypersequents G_i and G_{i+1} such that $\Gamma \Rightarrow \Sigma \in G_i$ and $\Delta \Rightarrow \Lambda \in G_{i+1}$. $\Delta \Rightarrow \Lambda$ is the *successor* of $\Gamma \Rightarrow \Sigma$ if either $\Gamma \Rightarrow \Sigma$ is $\Delta \Rightarrow \Lambda$ or $\Gamma \Rightarrow \Sigma$ is $\Delta, \varphi \Rightarrow \Lambda$ and φ is the result of the step taken at stage i. The relation of successor is a function. If $\Delta \Rightarrow \Lambda$ is the successor of $\Gamma \Rightarrow \Sigma$ this is indicated by $\#(\Gamma \Rightarrow \Sigma) = \Delta \Rightarrow \Lambda$. Let $\Gamma \Rightarrow \Sigma$ be a sequent featuring in G_i. $Ch(\Gamma \Rightarrow \Sigma)$ is a set defined inductively by

- $\Gamma \Rightarrow \Sigma \in Ch(\Gamma \Rightarrow \Sigma)$,
- $\#(\Gamma \Rightarrow \Sigma) \in Ch(\Gamma \Rightarrow \Sigma)$,
- and if $\Delta \Rightarrow \Lambda \in Ch(\Gamma \Rightarrow \Sigma)$ then $\#(\Delta \Rightarrow \Lambda) \in Ch(\Gamma \Rightarrow \Sigma)$,
- Nothing else is in $Ch(\Gamma \Rightarrow \Sigma)$.

Given a set $Ch(\Gamma_1 \Rightarrow \Sigma_1) = \{\Gamma_1 \Rightarrow \Sigma_1, \Gamma_2 \Rightarrow \Sigma_2, \ldots\}$ the sequent $SCh(\Gamma_1 \Rightarrow \Sigma_1)$ is defined as

$$SCh(\Gamma_i \Rightarrow \Sigma_i) := \bigcup_i \Gamma_i \Rightarrow \bigcup_i \Sigma_i$$

Let β be an open branch in the tree $\pi(G)$. The hypersequent $\mathfrak{M}(\beta)$ is the set[6]

$$\mathfrak{M}(\beta) := \{SCh(\Gamma \Rightarrow \Sigma) : \exists G(G \in \beta \& \Gamma \Rightarrow \Sigma \in G)\}$$

[6]As mentioned above given the rules EC and EE hypersequents can be treated as sets when considering the hypersequent calculus for S5.

Lemma 2.16. *Let G be such that $\nvdash^{cf}_{hs5} G$ and β be an open branch in $\pi(G)$. Let $\Gamma \Rightarrow \Sigma \in \mathfrak{M}(\beta)$. $\mathfrak{M}(\beta)$ has the following properties:*

1. *If $\neg\varphi \in \Gamma$ then $\varphi \in \Sigma$.*

2. *If $\neg\varphi \in \Sigma$ then $\varphi \in \Gamma$.*

3. *If $\varphi \to \psi \in \Gamma$ then either $\psi \in \Gamma$ or $\varphi \in \Sigma$.*

4. *If $\varphi \to \psi \in \Sigma$ then $\varphi \in \Gamma$ and $\psi \in \Sigma$.*

5. *If $\Box\varphi \in \Gamma$ then for any $\Delta \Rightarrow \Lambda \in \mathfrak{M}(\beta)$, $\varphi \in \Delta$.*

6. *If $\Box\varphi \in \Sigma$ then there is a $\Delta \Rightarrow \Lambda \in \mathfrak{M}(\beta)$ such that $\varphi \in \Lambda$.*

Proof. The proofs of *(1) – (4)* follow straightforwardly from the construction of $\pi(G)$. To prove *(5)* let $\Gamma \Rightarrow \Sigma \in \mathfrak{M}(\beta)$ and $\Box\varphi \in \Gamma$. Suppose that there is a $\Delta \Rightarrow \Lambda \in \mathfrak{M}(\beta)$ such that $\varphi \notin \Delta$. Let $SCh(\hat{\Gamma} \Rightarrow \hat{\Sigma}) = \Gamma \Rightarrow \Sigma$ and $Sch(\hat{\Delta} \Rightarrow \hat{\Lambda}) = \Delta \Rightarrow \Lambda$. There is a stage with hypersequent G_j such that $\hat{\Gamma}' \Rightarrow \hat{\Sigma}' \in Ch(\hat{\Gamma} \Rightarrow \hat{\Sigma})$, $\Box\varphi \in \hat{\Gamma}'$, $\hat{\Gamma}' \Rightarrow \hat{\Sigma}'$ occurs in G_j, $\hat{\Delta}' \Rightarrow \hat{\Lambda}' \in Ch(\hat{\Delta} \Rightarrow \hat{\Lambda})$, $\hat{\Delta}' \Rightarrow \hat{\Lambda}'$ occurs in G_j, and $\Box\varphi$ is under consideration. However at the following stage G_{j+1} φ is added to the left of every sequent in G_j by (4a) in the construction of $\pi(G)$. Let $\hat{\Delta}'' \Rightarrow \hat{\Lambda}'' = \#(\hat{\Delta}' \Rightarrow \hat{\Lambda}')$. It follows that $\varphi \in \hat{\Delta}''$ and so $\varphi \in \Delta$ contradicting the assumption.

To prove *(6)* let $\Box\varphi \in \Sigma$. Let $SCh(\hat{\Gamma} \Rightarrow \hat{\Sigma}) = \Gamma \Rightarrow \Sigma$. There is a sequent $\hat{\Gamma}' \Rightarrow \hat{\Sigma}' \in Ch(\hat{\Gamma} \Rightarrow \hat{\Sigma})$ occurring in a hypersequent $G_i \in \beta$ such that $\Box\varphi \in \hat{\Sigma}'$ with the sentence $\Box\varphi$ under consideration. At that stage either there is a sequent $\Delta \Rightarrow \Lambda \in G_i$ such that $\varphi \in \Lambda$ or not. If there is, then $\varphi \in SCh(\Delta \Rightarrow \Lambda)$. If not then

by (4b) in the construction of $\pi(G)$ G_{i+1} is $G_i; (\quad \Rightarrow \varphi)$. Let $SCh(\quad \Rightarrow \varphi) = \Pi \Rightarrow \Theta$. It follows that $SCh(\quad \Rightarrow \varphi) \in \mathfrak{M}(\beta)$ and $\varphi \in \Theta$. $\square$

Lemma 2.17. *If $\nvdash^{cf}_{hs5} G$ then there is a model $M_{\mathscr{F}}$ that is a counter-example to G.*

Proof. Let $\nvdash^{cf}_{hs5} G$. As noted above there is a open branch β in $\pi(G)$. Let $\mathfrak{M}(\beta) = (\Gamma_1 \Rightarrow \Sigma_1); \ldots; (\Gamma_n \Rightarrow \Sigma_n); \ldots$. Let $\mathscr{F}$ be a frame such that for each $\Gamma_i \Rightarrow \Sigma_i \in \mathfrak{M}(\beta)$ there is a world $w_i \in W_{\mathscr{F}}$ and no other worlds in $W_{\mathscr{F}}$. Let $R_{\mathscr{F}}$ be universal. It follows $\mathscr{F} \in \mathbb{L}$. Let $M_{\mathscr{F}}$ be the ordered pair $\langle I_{M_{\mathscr{F}}}, \mathscr{F}\rangle$. $I_{M_{\mathscr{F}}}(w_i, p) = 1$ iff $p \in \Gamma_i$. It is proved by induction on the rank of φ that for any φ if $\varphi \in \Gamma_j$ then $I_{M_{\mathscr{F}}}(w_j, \varphi) = 1$ and if $\varphi \in \Sigma_j$ then $I_{M_{\mathscr{F}}}(w_j, \varphi) = 0$. The atomic case is given above. This leaves the three other inductive cases.

Case 1 (φ is $\neg\psi$). Let $\varphi \in \Gamma_j$. By lemma 2.16 *(1)*, $\psi \in \Sigma_j$. By IH $I_{M_{\mathscr{F}}}(w_j, \psi) = 0$, so $I_{M_{\mathscr{F}}}(w_j, \neg\psi) = 1$. The case where $\varphi \in \Sigma_j$ is similar.

Case 2 (φ is $\psi \to \theta$). This case is similar the above case.

Case 3 (φ is $\Box\psi$). Let $\varphi \in \Gamma_j$. It follows from lemma 2.16 *(5)* that for every $\Delta \Rightarrow \Lambda \in \mathfrak{M}(\beta)$, $\psi \in \Delta$. Let $\Gamma_i \Rightarrow \Sigma_i \in \mathfrak{M}(\beta)$. So $\psi \in \Gamma_i$. By IH, $I_{M_{\mathscr{F}}}(w_i, \psi) = 1$. Since $\Gamma_i \Rightarrow \Sigma_i$ is arbitrary it holds for all such w_i that $I_{M_{\mathscr{F}}}(w_i, \psi) = 1$. So $I_{M_{\mathscr{F}}}(w_j, \Box\psi) = 1$.

Let $\varphi \in \Sigma_j$. It follows from lemma 2.16 *(6)* that there is a $\Gamma_i \Rightarrow \Sigma_i \in \mathfrak{M}(\beta)$ such that $\psi \in \Sigma_i$. By IH $I_{M_{\mathscr{F}}}(w_i, \psi) = 0$. So $I_{M_{\mathscr{F}}}(w_j, \Box\psi) = 0$.

It follows that there is a branch of worlds $w_1, \ldots, w_n, \ldots \in W_{\mathscr{F}}$ such that $M_{\mathscr{F}}$ is a counter-model to $\Gamma_i \Rightarrow \Sigma_i$ at w_i. So $M_{\mathscr{F}}$ is a counter-example to $\mathfrak{M}(\beta)$.

For each sequent $\Delta \Rightarrow \Lambda$ occurring in G there is a sequent $\Gamma_i \Rightarrow \Sigma_i \in \mathfrak{M}(\beta)$ such that $\Delta \subseteq \Gamma_i$ and $\Lambda \subseteq \Sigma_i$. Since $M_{\mathscr{F}}$ is a counter-example to $\mathfrak{M}(\beta)$ $M_{\mathscr{F}}$ is also a

counter-example to G. $\square$

Theorem 2.4.1. *If* $\vdash_{hs5} G$ *then* $\vdash_{hs5}^{cf} G$.

Proof. Let $\vdash_{hs5} G$. By lemma 2.12 there is no counter-example to G. By lemma 2.17, if there is no counter-example to G then $\vdash_{hs5}^{cf} G$. So $\vdash_{hs5}^{cf} G$. $\square$

2.5 Discussion

The hypersequent calculi presented in section 2.3 have several desirable properties. The rules governing modal expressions do not change from system to system. They also have all of the important properties discussed in section 2.2.

1. The rules have only one occurrence of $\Box$ in the conclusion.
2. Only $\Box$ appears essentially in the conclusion of L$\Box$ and R$\Box$.
3. Every rule governing $\Box$ either introduces $\Box$ on the left of a sequent or on the right and there is a rule to do each.
4. No rule features $\Box$ essentially in its premises.

Thus the rules of the hypersequent calculi are explicit, separated, symmetrical, and non-circular. In addition to this the results of section 7.3.2 show that they uniquely characterize the expression $\Box$. Finally it is known that the hypersequent calculi for the modal logics D and S5 are cut admissible. The calculus for the modal logic D is in addition to this cut eliminable. It follows that both hypersequent calculi have the sub-formula property.

Restall [51] concludes "Proofnets for S5: Sequents and Circuits for Modal Logic" by saying

> The aim, of course, is an account of proof in which the rules for the modal operators are untouched, and the structural rules (in this case, the behaviour of nearness and the relations of ancestor/descendant) play the role of determining which modal logic is found.

This chapter offers such an account of proof. The logics developed in section 2.3 vary only in their external structural rules. All of the logics have the advantage over the standard sequent account of modal logic presented in section 2.2 of uniquely characterizing the expression $\Box$. This suggests that this hypersequent approach to modal logic is well worth continued exploration. It is a valuable tool for exploring the proof theoretic properties of modal logics and may be of value to philosophers in developing an inferentialist theory of meaning for modal expressions.

Chapter 3

Hypersequent System D

Abstract. If inferentialism is to be a viable theory of meaning, then it must be able to offer an account of the meaning of modal expressions. This chapter lays the groundwork for such an account by proposing constraints on a theory of meaning. In particular it is required that the Rule of Cut be admissible and that each expression of the language whose meaning is being given be characterized uniquely. Many proof-theoretic accounts of modality fail to meet this second constraint. The achievement of this chapter is to offer a hypersequent calculus that does meet those constraints for System D. System D is selected as an example to show that the theory of meaning proposed is viable and can be extended to modal notions with inferential patterns weaker than S5. Cut elimination (Theorem 3.6.1) and the unique characterization of 'it is obligatory that...' are proved (Theorem 3.7.1) for the proposed hypersequent calculus.

Keywords. Modal Logic, Inferentialism, System D, Hypersequents

3.1 Inferentialism and Meaning

The main aim of this chapter is to introduce a viable example, and so a starting point, for an inferentialist account of modality. If inferentialism is to be a viable theory of meaning it must offer a plausible account of modality. The first requirement of such a theory is that it be adequate given the other commitments of inferentialism. The second is that it be able to offer a variety of modal systems to mirror the variety of modalities that appear to be a part of meaningful discourse. This chapter lays the groundwork for addressing these concerns. It is the first of the above concerns that motivates use of the hypersequent calculus. The bulk of the chapter introduces this formalism and shows that it meets the formal constraints required by an inferentialist account of meaning. The second of the above concerns prompts an exploration of Lemmon and Scott's [30] System D. Much of the literature on inferentialist accounts of modality has centered on the modal operators 'It is necessary that ...' and 'it is possible that ...' (see Brandom [10], Došen [12], Avron [2], and Restall [55]). This chapter illustrates that an inferentialist account of modality can capture a wider range of modal operators.

Before exploring the specific inferentialist theory that will be offered, some general points are worth noting. Inferentialism is at least the claim that the use of an expression determines its meaning. The salient use here is generally given by a set of rules. This is a minimal account of the commitments of inferentialism. Most inferentialists will subscribe to some stronger claim about the relation between the meaning of an expression and its use. What follows from this characterization of inferentialism thus has wide applicability.

Since a set of rules is said to *determine* the meaning of an expression it must be that there is only one consistent understanding of that expression. Suppose that two expressions were introduced using the same set of rules, and let there be a place where the two expressions, so introduced, were not intersubstitutable. In this case there would be an ambiguity as to what meaning the rules conferred, they could not confer the same meaning to both expressions. Thus, the set of rules did not uniquely determine a meaning. For any set of rules to confer a meaning on an expression, it must be provable that a second expressions governed by the same rules would be everywhere intersubstitutable with the first. This was first proposed by Belnap [4] in response to Prior [44]. More recently, this constraint has been put forward in Humberstone [21] as the *Uniqueness Criterion.* According to it, a meaning conferring set of rules must determine a single meaning for an expression. This is of particular concern in here because standard sequent presentations of modal logic fail to meet this constraint.

Similarly, a meaning conferring set of must be complete in the sense that they rely on no other rules to confer the meaning of an expression. It must be that when an expression can be shown to have some status in a proof system, e.g. proved, incoherent to assert, refutable, etc. there must be a deduction in the proof system that establishes that status using only meaning conferring (and structural) rules for that expression.[1] If this were to fail, then the original set of rules could not be said to be properly meaning conferring. They would not be sufficient to fix the meaning

[1] For simplicity let operational rules only add formulas to a sequent, i.e. if any formula appears in a premise sequent then either it appears as in the conclusion sequent or it is a sub-sentence of some sentence of the conclusion sequent.

of that expression in every context.[2] It seems plausible that "It is obligatory that ...", negation, and the conditional have their meanings determined by the set of rules governing each expression respectively and no others, e.g. the meaning that negation contributes to sentences in which it occurs is governed solely by L$\neg$ and R$\neg$ of Figure 3.2. In this circumstance this constraint amounts to the requirement that deductions be normalizable in the sense of Prawitz [42] or Dummett [13] or in the case of sequents and hypersequents that the calculus is cut admissible.

It should be noted that if the rule of cut is not valid, as suggested by Ripley [**?**], then this constraint is trivially met. If it is valid, then showing that it is admissible shows that when a sentence featuring that expression is established, then there is a deduction that establishes that sentence featuring a rule explicitly mentioning that expression. This constraint has stronger force in other inferentialist theories of meaning. For instance, Brandom's [9] account of logical vocabulary as expressive requires that cut be admissible. If there were a deduction involving logical vocabulary that established a sentence not establishable without the use of logical vocabulary, the logical vocabulary would have made a substantial contribution to the meaning of other sentences. In such a case, the logical vocabulary could not be purely expressive in Brandom's [9] sense.

The uniqueness constraint and cut admissibility are discussed in detail below. Most of this chapter is devoted to showing that the calculus of Figure 3.2 meets these two requirements of a theory of meaning. The former is established by Theorem 3.7.1, the latter by Theorem 3.6.1.

[2]It should be noted that this does not entail a molecularism about meaning. It is compatible with the above that rules governing the conditional are meaning conferring for negation.

Fix a set of atomic sentences, $p_1, p_2, \ldots$. The set of sentences of the language under consideration is specified by

$$\varphi := |\neg\varphi|(\varphi \to \psi)| \Box \psi$$

The specific theory of meaning that this chapter pursues is bilateralist in the sense of Rumfitt [59]. There are two independent uses of sentences, they can either be asserted or denied. Following Restall [50, 53, 55], a *position* is a pair of sets of sentences, one set of which represents assertions the other of which represents denials. If Γ and Σ are sets of sentences, then $\Gamma \Rightarrow \Sigma$ is the position one would take up if one asserted all of Γ and denied all of Σ. Positions can be coherent or incoherent. For instance it is incoherent to assert and deny the same sentence.[3] It is incoherent to assert "The sky is blue" while also denying that sentence.

Positions provide an adequate theory of meaning for classical propositional logic. The meaning of a sentence, 'P and Q', whose main operator is a conjunction is given by the following set of rules:

- If it is coherent to assert 'P and Q' in a position then it is coherent to assert 'P' in that position and it is coherent to assert 'Q' in that position.
- If it is coherent to deny 'P and Q' in a position then it is either coherent to deny 'P' in that position and it is coherent to deny 'Q' in that position.

These rules are given a formal treatment in by the following three rules of the sequent calculus. Let Γ and Σ be sets of sentences.

[3] This may not hold for all circumstances, for instances indexicals present a possible counter-example, but these cases are left out of consideration for present purposes.

$$\text{Land}\,\frac{\Gamma, P \Rightarrow \Sigma}{\Gamma, PandQ \Rightarrow \Sigma} \qquad \text{Land}\,\frac{\Gamma, Q \Rightarrow \Sigma}{\Gamma, PandQ \Rightarrow \Sigma} \qquad \text{Rand}\,\frac{\Gamma \Rightarrow P, \Sigma \qquad \Gamma \Rightarrow Q, \Sigma}{\Gamma \Rightarrow PandQ, \Sigma}$$

Let $\vdash \Gamma \Rightarrow \Sigma$ indicate that there is a deduction of the sequent $\Gamma \Rightarrow \Sigma$. Let 'und' be an expression that is introduced into the language by the meaning conferring rules:

$$\text{Lund}\,\frac{\Gamma, P \Rightarrow \Sigma}{\Gamma, PundQ \Rightarrow \Sigma} \qquad \text{Lund}\,\frac{\Gamma, Q \Rightarrow \Sigma}{\Gamma, PundQ \Rightarrow \Sigma} \qquad \text{Rund}\,\frac{\Gamma \Rightarrow P, \Sigma \qquad \Gamma \Rightarrow Q, \Sigma}{\Gamma \Rightarrow PundQ, \Sigma}$$

The rules governing 'und' are the same as those governing 'and'. The uniqueness constraint can be formally stated as the requirement that the following two facts hold.

- $\vdash \Gamma, PandQ \Rightarrow \Sigma$ iff $\vdash \Gamma, PundQ \Rightarrow \Sigma$.
- $\vdash \Gamma \Rightarrow PandQ \Rightarrow \Sigma$ iff $\vdash \Gamma \Rightarrow PundQ, \Sigma$.

3.2 Sequent System D

The above discussion of positions suggests that the most natural approach to offering a theory of meaning for a modal logic is to simply add modal operators to a sequent system that has already been proposed. The standard account of System D is given in Figure 3.1. Similar presentations can be found in Wansing [73], Poggiolesi [39] and Valentini [71].

The calculus of Figure 3.1 is unable to underwrite a theory of meaning for the modal operator $\Box$. It fails to the uniqueness constraint. Let the language be expanded by $\boxdot$, and System D$^+$ be the calculus of Figure 3.1 expanded by the rules:

Figure 3.1: Sequent System D

STRUCTURAL RULES

$$\text{Axiom}\ \frac{}{\varphi \Rightarrow \varphi}$$

$$\text{WL}\ \frac{\Gamma \Rightarrow \Sigma}{\Gamma, \varphi \Rightarrow \Sigma} \qquad \text{WL}\ \frac{\Gamma \Rightarrow \Sigma}{\Gamma \Rightarrow \varphi, \Sigma}$$

OPERATIONAL RULES

$$\text{L}\neg\ \frac{\Gamma \Rightarrow \varphi, \Sigma}{\Gamma, \neg\varphi \Rightarrow \Sigma} \qquad \text{R}\neg\ \frac{\Gamma, \varphi \Rightarrow \Sigma}{\Gamma \Rightarrow \neg\varphi, \Sigma}$$

$$\text{L}\rightarrow\ \frac{\Gamma \Rightarrow \varphi, \Sigma \qquad \Gamma, \psi \Rightarrow \Sigma}{\Gamma, \varphi \rightarrow \psi \Rightarrow \Sigma} \qquad \text{R}\rightarrow\ \frac{\Gamma, \varphi \Rightarrow \psi, \Sigma}{\Gamma \Rightarrow \varphi \rightarrow \psi, \Sigma}$$

$$\text{k}\ \frac{\Gamma \Rightarrow \varphi}{\Box\Gamma \Rightarrow \Box\varphi} \qquad \text{d}\ \frac{\Gamma \Rightarrow}{\Box\Gamma \Rightarrow}$$

$$\Box\Gamma = \{\Box\varphi : \varphi \in \Gamma\}$$

$$k^+\ \frac{\Gamma \Rightarrow \varphi}{\boxdot\Gamma \Rightarrow \boxdot\varphi} \qquad \text{d}^+\ \frac{\Gamma \Rightarrow}{\boxdot\Gamma \Rightarrow}$$

In system $\text{D}^+ \vdash \Box\varphi \Rightarrow \Box\varphi$, but $\nvdash \boxdot\varphi \Rightarrow \Box\varphi$. While cut is admissible for this system (see Valentini [71]) it fails the uniqueness constraint. The rules k and d thus do not uniquely determine the meaning of the operator $\Box$. This failure of uniqueness is not particular to the sequent account of system D. It fails for K, T, B, $S4$, and $S5$ as they are characterized in Wansing [73].

This suggests that sequents are not the appropriate formal tools to offer an inferentialist treatment of modality.

3.3 Hypersequent System D

A hypersequent is a finite sequence of sequents. So if for each $1 \leqslant i \leqslant n$, $\Gamma_i \Rightarrow \Sigma_i$ is a sequent, then $(\Gamma_1 \Rightarrow \Sigma_1); \ldots ; (\Gamma_n \Rightarrow \Sigma_n)$ is a hypersequent. Upper-case Roman Letters towards the beginning of the alphabet, e.g. G, H, J ... are used as hypersequent variables. When convenient parentheses around sequents are dropped. When considering a sentence, φ appearing in a hypersequent, it is convenient to discuss the specific sequent in which φ appears. As a shorthand, '$\varphi \in \Gamma \cup \Sigma$' is used to indicate that $\varphi \in \Gamma \cup \Sigma$. '$\varphi \in \Delta \Rightarrow \Lambda \in H$' is used to indicate that $\varphi \in \Delta \cup \Lambda$ and $\Delta \Rightarrow \Lambda \in H$.

Hypersequents were first introduced by Pottinger [40] to give a proof system for modal logic. They have since been used by Avron [1], Lahav [23], and Lellmann [29] for that same purpose. The approach offered by this chapter differs considerably from these approaches in two important ways. The first is that this approach makes crucial use of the external structural rules of the calculus. The hypersequents of this chapter are lists, not sets or multisets. This approach allows for many different systems of modal logic to be obtained simply by manipulation of structural rules of the calculus. The second difference is that the rules governing the modal operators are specified so that they satisfy the uniqueness constraint discussed above.

The goal of this chapter is to introduce a viable example, and so a starting point, for an inferentialist account of modality. Sequents have been rejected as a plausible structure to account for all the features of a theory of meaning that are required. In order for hypersequents to be satisfactory structure, then in addition to meeting the formal constraints discussed in Section 3.1 there must be a semantic interpretation

of hypersequents. For different systems of modal logic, the semantic interpretation is different. A hypersequent consisting of just one sequent is given the same reading as above. If that hypersequent, $\Gamma \Rightarrow \Sigma$, is deducible, then it is incoherent to assert all of Γ and deny all of Σ. Let $(\Gamma_1 \Rightarrow \Sigma_1); \ldots; (\Gamma_n \Rightarrow \Sigma_n)$ be a hypersequent. Corresponding to it is a hyper-position. Restall [55] gives an account of how to understand both the expressions. On that account, to be in a hyper-position is to be addressing the relationship of other positions to one's own position. This strategy is followed: modality is considered to ultimately be a way of considering positions other than the one that one has actually taken up. For deontic modality, in the case where $n = 2$, if $(\Gamma_1 \Rightarrow \Sigma_1); (\Gamma_2 \Rightarrow \Sigma_2)$ is deducible then it is incoherent to take up the position $\Gamma_2 \Rightarrow \Sigma_2$ while holding $\Gamma_1 \Rightarrow \Sigma_1$ to be morally ideal relative to $\Gamma_2 \Rightarrow \Sigma_2$. For example, the hypersequent, $(\Rightarrow \varphi); (\Box\varphi \Rightarrow)$, is deducible in the hypersequent calculus given by Figure 3.2. This corresponds to the incoherence of taking up a position asserting that it ought to be that φ, but denying φ in a position morally ideal relative to the one taken up. Simplified, it is incoherent to assert that φ ought to be the case, but deny φ is morally ideal.

For the case where $n > 2$, the relation '...is morally ideal relative to ...' is iterated. To say a hyperposition, $(\Gamma_1 \Rightarrow \Sigma_1); \ldots; (\Gamma_n \Rightarrow \Sigma_n)$, is incoherent is to say that it is incoherent to take up $\Gamma_n \Rightarrow \Sigma_n$ and hold that for each $1 \leqslant i, i+1 \leqslant n$, $\Gamma_i \Rightarrow \Sigma_i$ is morally ideal relative to $\Gamma_{i+1} \Rightarrow \Sigma_{i+1}$. This is a only a sketch of the semantic interpretation. The aim of this chapter is to introduce the formal apparatus and show that it meets the constraints suggested in Section 3.1.

Figure 3.2 gives a hypersequent calculus for system D. $\vdash_h G$ indicates that G

is deducible in this system. Theorem 3.8.2 and Theorem 3.8.1 establish that this system is equivalent to sequent system D in that it proves a hypersequent of the form $\Gamma \Rightarrow \Sigma$ iff that sequent is provable according to the rules of Figure 3.1. Theorem 3.6.1 establishes that the Cut rule is eliminable, and Theorem 3.7.1 establishes that the uniqueness constraint is met. It is worth noting that there may be other hypersequent calculi that have these features. The advantage of this approach is that other modal logics, for instance K, can be obtained by manipulating external structural rules. System K is given by the calculus that retains all the rules of Figure 3.2 except for Drop. It can be shown that cut is eliminable for that calculus by a proof similar to the one given here.

Section 3.4 offers theorems that are useful for acquaintance with the system. The theorems are useful in Section 3.7. Section 3.5 offers lemmas that allow a deduction featuring a sentence φ, to be transformed into a deduction featuring a sentence or sentences of rank less than φ. These lemmas are instrumental in the proof of Theorem 3.6.1. Section 3.6 proves that Cut is eliminable. This is proved by introducing a Cut*-rule of which all instances of the Cut rule are instances. It is proved that Cut* is eliminable by showing that it can be eliminated from proofs having only one occurrence of Cut*. This is shown by an induction over the rank of the formula that is main in the Cut* rule. The lemmas of Section 3.5 are instrumental in the base case and two inductive cases. For the inductive case where the main formula of the Cut* rule is a formula of the form $\Box\varphi$, a sub-induction over a newly defined measure, *disarray* (see Definition 19) is required. It is shown that Cuts* can be pushed up deductions while preserving disarray and that when there is no

disarray the rank of the Cut* formula can be reduced.

Section 3.8 shows that the calculus of Figure 3.2 deduces all and only the singleton hypersequents that are deducible in Figure 3.1. Section 3.7 establishes that the rules of Figure 3.2 uniquely characterize the modal operator.

3.4 Preliminaries

Theorem 3.4.1 shows that the rule Ax holds of all sentences, not only atomics. Many of the proofs in the subsequent sections require this result. Theorem 3.4.1 also plays a key role in the proof of Theorem 3.7.1.

Definition 7 (Rank of a sentence)**.** The rank of φ, $rk(\varphi)$ is defined inductively as follows:

- If φ is atomic, then $rk(\varphi) = 0$.
- If φ is $\Box\psi$ or $\neg\psi$, then $rk(\varphi) = rk(\psi) + 1$.
- If φ is $\gamma \to \delta$, then $rk(\varphi) = rk(\gamma) + rk(\delta) + 1$.

Theorem 3.4.1. *$\vdash_h \varphi \Rightarrow \varphi$, for any φ.*

Proof. This proof proceeds by induction on the rank of sentences. The base case if given by Ax. There are then three inductive cases: *Case 1* (Negation).

$$\dfrac{\dfrac{\dfrac{\text{IH}}{\varphi \Rightarrow \varphi}}{\neg\varphi, \varphi \Rightarrow}\ \text{L}\neg}{\neg\varphi \Rightarrow \neg\varphi}\ \text{R}\neg$$

Case 2 (Conditional).

$$\mathrm{R}{\to}\dfrac{\mathrm{L}{\to}\dfrac{\dfrac{\text{IH}}{\varphi \Rightarrow \varphi} \qquad \dfrac{\text{IH}}{\psi \Rightarrow \psi}}{\varphi, \varphi \to \psi \Rightarrow \psi}}{\varphi \to \psi \Rightarrow \varphi \to \psi}$$

Case 3 (Necessity).

$$\mathrm{R}\Box\dfrac{\mathrm{L}\Box\dfrac{\mathrm{W}\dfrac{(\varphi \Rightarrow \varphi)}{(\varphi \Rightarrow \varphi); (\Rightarrow)}}{(\Rightarrow \varphi); (\Box\varphi \Rightarrow)}}{(\Box\varphi \Rightarrow \Box\varphi)}$$

□

Corollary 1. $\vdash_h \Gamma, \varphi \Rightarrow \varphi, \Sigma$

Proof. This follows from Theorem 3.4.1, TL, and TR. □

3.5 Important Lemmas

3.5.1 Reduction Lemmas

This subsection offers the standard reduction lemmas of a cut-elimination proof. These lemmas show that given a deduction of a hypersequent featuring a sentence, $\neg\varphi$ or $\varphi \to \psi$, it is possible to get deductions featuring their subsentences.

The rules in Figure 3.2 are schemata. In what follows it is sometimes important to draw attention to different aspects of each of these schemata. Definition 8 and Definition 9 make referring to these aspects of schema easy. They are also crucial for definition the measure Definition 10 that is used in the base case of Lemma 3.13.

Definition 8 (Main sentence)**.** φ is the main sentence of TL, TR, L$\neg$, R$\neg$, L$\rightarrow$, R$\rightarrow$, L$\neg$, or R$\neg$, if φ appears in the conclusion of one of those but not the premise(s). In the case of Cut, the main sentence is the sentence not appearing in the conclusion hypersequent. In the case of Ax, the main sentence is the only sentence present.

Definition 9 (Main Sequent)**.** The main sequent of a rule schema, I, is either the sequent containing the main sentence of I, or is given by the following list:

- Drop has no main sequent.
- The main sequent of W is ($\Rightarrow$).
- For L$\Box$ and R$\Box$, the *right-main* sequent is the one containing the main sentence. The *left-main* sequent is either the one immediately preceding the *right-main* sentence or none at all.

Definition 10 (Depth)**.** Let δ be a deduction. The depth of a sentence, φ, in that deduction is defined relative to an occurrence of a sequent in a hypersequent in that deduction. If φ does not appear in the sequent in question in that deduction, then its depth is undefined.

- The depth of the main sentence of an instance of Ax is 1, $d(p, \delta, p \Rightarrow p) = 1$.
- Let the last inference of δ be I, and the deduction(s) preceding I to be $\delta_1(\delta_2)$. Let $\Delta \Rightarrow \Lambda$ be the sequent in δ_1 or δ_2, such that $\Gamma \Rightarrow \Sigma$ results from it after an application of I, e.g. if I is TL, and $\Gamma \Rightarrow \Sigma$ is main in I, with main sentence, ψ, then $\Delta \Rightarrow \Lambda$ is $\Gamma/\{\psi\} \Rightarrow \Sigma$, if $\Gamma \Rightarrow \Sigma$ is not main, then $\Delta \Rightarrow \Lambda$ is $\Gamma \Rightarrow \Sigma$.
- If I is Drop, and $\varphi \in \Gamma \Rightarrow \Sigma$, then $d(\varphi, \delta, \Gamma \Rightarrow \Sigma) = d(\varphi, \delta_1, \Delta \Rightarrow \Lambda) + 1$

- If I is W, and $\varphi \in \Gamma \Rightarrow \Sigma$, then $d(\varphi, \delta, \Gamma \Rightarrow \Sigma) = d(\varphi, \delta_1, \Delta \Rightarrow \Lambda) + 1$.

- If I is TL, φ is main in I, the main sequent of I is $\Gamma, \varphi \Rightarrow \Sigma$, and $\varphi \notin \Gamma$, then $d(\varphi, \delta, \Gamma, \varphi \Rightarrow \Sigma) = 1$. Otherwise, $d(\varphi, \delta, \Gamma, \varphi \Rightarrow \Sigma) = d(\varphi, \delta_1, \Delta \Rightarrow \Lambda) + 1$.

- If I is TR, φ is main in I, the main sequent of I is $\Gamma \Rightarrow \varphi, \Sigma$, and $\varphi \notin \Sigma$, then $d(\varphi, \delta, \Gamma \Rightarrow \varphi, \Sigma) = 1$. Otherwise, $d(\varphi, \delta, \Gamma \Rightarrow \varphi, \Sigma) = d(\varphi, \delta_1, \Delta \Rightarrow \Lambda) + 1$.

- If I is Cut, $d(\varphi, \delta, \Gamma \Rightarrow \Sigma) = d(\varphi, \delta_1, \Delta \Rightarrow \Lambda) + d(\varphi, \delta_2, \Delta \Rightarrow \Lambda) + 1$, where $\varphi \in \Gamma \cup \Sigma$.

- If I is L$\neg$, φ is main in I, the main sequent of I is $\Gamma, \varphi \Rightarrow \Sigma$, and $\varphi \notin \Gamma$, then $d(\varphi, \delta, \Gamma, \varphi \Rightarrow \Sigma) = 1$. Otherwise, $d(\varphi, \delta, \Gamma, \varphi \Rightarrow \Sigma) = d(\varphi, \delta_1, \Delta \Rightarrow \Lambda) + 1$.

- If I is R$\neg$, φ is main in I, the main sequent of I is $\Gamma \Rightarrow \varphi, \Sigma$, and $\varphi \notin \Sigma$, then $d(\varphi, \delta, \Gamma \Rightarrow \varphi, \Sigma) = 1$. Otherwise, $d(\varphi, \delta, \Gamma \Rightarrow \varphi, \Sigma) = d(\varphi, \delta_1, \Delta \Rightarrow \Lambda) + 1$.

- If I is L$\rightarrow$, φ is main in I, the main sequent of I is $\Gamma, \varphi \Rightarrow \Sigma$, and $\varphi \notin \Gamma$, then $d(\varphi, \delta, \Gamma, \varphi \Rightarrow \Sigma) = 1$. Otherwise, $d(\varphi, \delta, \Gamma, \varphi \Rightarrow \Sigma) = d(\varphi, \delta_1, \Delta \Rightarrow \Lambda) + d(\varphi, \delta_2, \Delta \Rightarrow \Lambda) + 1$.

- If I is R$\rightarrow$, φ is main in I, the main sequent of I is $\Gamma \Rightarrow \Sigma$, and $\varphi \notin \Sigma$, then $d(\varphi, \delta, \Gamma \Rightarrow \varphi, \Sigma) = 1$. Otherwise, $d(\varphi, \delta, \Gamma \Rightarrow \varphi, \Sigma) = d(\varphi, \delta_1, \Delta \Rightarrow \Lambda) + 1$.

- If I is L$\Box$, φ is main in I, the right main sequent of I is $\Gamma \Rightarrow \Sigma$, and $\varphi \notin \Gamma$, then $d(\varphi, \delta, \Gamma, \varphi \Rightarrow \Sigma) = 1$. Otherwise, $d(\varphi, \delta, \Gamma, \varphi \Rightarrow \Sigma) = d(\varphi, \delta_1, \Delta \Rightarrow \Lambda) + 1$.

- If I is R$\Box$, φ is main in I, the right main sequent of I is $\Gamma \Rightarrow \Sigma$, and $\varphi \notin \Sigma$, then $d(\varphi, \delta, \Gamma \Rightarrow \varphi, \Sigma) = 1$. Otherwise, $d(\varphi, \delta, \Gamma \Rightarrow \varphi, \Sigma) = d(\varphi, \delta, \Delta \Rightarrow \Lambda) + 1$

Let δ be a deduction, its length, $l(\delta)$, is defined as usual.

Lemma 3.1. *If $\delta \vdash_h G; \Gamma, \neg\varphi \Rightarrow \Sigma; H$, then there is a δ', such that $\delta' \vdash_h G; \Gamma \Rightarrow \varphi, \Sigma; H$ where $l(\delta') \leqslant l(\delta)$ and for any atomic $p \in \Delta \Rightarrow \Lambda \in G; \Gamma, \neg\varphi \Rightarrow \Sigma; H$, $d(p, \delta', \Delta \Rightarrow \Lambda) \leqslant d(p, \delta, \Delta \Rightarrow \Lambda)$.*

Proof. The proof proceeds by induction on the length of deductions. The smallest deductions are the instances of Ax, which do meet the antecedent of the lemma. It follows that the lemma holds for the base case. Let δ be given, and its last inference be I.

Case 1(I is Cut). If $\Gamma, \neg\varphi \Rightarrow \Sigma$ is not the main sequent in I, then the result follows from two applications IH followed by I. If, on the other hand $\Gamma, \neg\varphi \Rightarrow \Sigma$ is the main sequent in I, then the following deduction is given:

$$\text{Cut}\ \dfrac{\dfrac{\vdots}{G; \Gamma, \psi, \neg\varphi \Rightarrow \Sigma; H} \qquad \dfrac{\vdots}{G; \Gamma, \neg\varphi \Rightarrow \psi, \Sigma; H}}{G; \Gamma, \neg\varphi \Rightarrow \Sigma; H}$$

In this case IH allows the following deduction, δ', to be constructed:

$$\text{Cut}\ \dfrac{\dfrac{\text{IH}}{G; \Gamma, \psi \Rightarrow \varphi, \Sigma; H} \qquad \dfrac{\text{IH}}{G; \Gamma \Rightarrow \varphi, \psi, \Sigma; H}}{G; \Gamma \Rightarrow \varphi, \Sigma; H}$$

For each branch of δ, δ_1 and δ_2, $l(\delta_1), l(\delta_2) < l(\delta)$. By IH the each branches of δ', δ'_1, δ'_2 are such that $l(\delta'_1) \leqslant l(\delta_1)$, and $l(\delta'_2) \leqslant l(\delta_2)$. So $l(\delta') \leqslant l(\delta)$. Similarly, the depth of p in δ in $\Delta \Rightarrow \Lambda$ is preserved by IH.

Case 2(I is Drop or W) In either of these cases an application of IH followed by I suffices. Since IH preserves the length of deductions, an application of I to

a deduction whose length is at most one less than δ will be at most as long as δ. Similarly for the depth of any atomic in δ.

Case 3(I is TL or TR). If $\neg\varphi$ is not the main sentence, this follows from an application of IH to the deduction preceding I. The length of the resulting deduction is less than or equal to the length of the deduction to which it was applied. Applying I to that deduction gives δ', where $l(\delta') \leqslant l(\delta)$. If it is, δ' is the deduction immediately preceding I, whose length is less than δ. The depth of p in δ in $\Delta \Rightarrow \Lambda$ is preserved by IH.

Case 4(I is L$\neg$). As above, if $\neg\varphi$ is not the main sentence, this case follows from IH and I. If it is the main sentence, then δ' is the deduction preceding I, or an application of IH. The depth of p in δ in $\Delta \Rightarrow \Lambda$ is preserved by IH.

Case 5(I is R$\neg$, L$\rightarrow$, or R$\rightarrow$). These cases also follow form IH and I.

Case 6(I is L$\Box$). There are three sub-cases here: *Case 6.1*($\Gamma, \neg\varphi \Rightarrow \Sigma$ is not main in I). This sub-case follows from IH and I.

Case 6.2($\Gamma, \neg\varphi \Rightarrow \Sigma$ is left-main in I). In this case, the following deduction is given:

$$\text{L}\Box\,\dfrac{\dfrac{\vdots}{G; \Gamma, \psi, \neg\varphi \Rightarrow \Sigma; \Delta \Rightarrow \Lambda; H}}{G; \Gamma, \neg\varphi \Rightarrow \Sigma; \Delta, \Box\psi \Rightarrow \Lambda; H}$$

IH allows for the construction of the following deduction:

$$\text{L}\Box\,\dfrac{\dfrac{\text{IH}}{G; \Gamma, \psi \Rightarrow \varphi, \Sigma; \Delta \Rightarrow \Lambda; H}}{G; \Gamma \Rightarrow \varphi, \Sigma; \Delta, \Box\psi \Rightarrow \Lambda; H}$$

Let δ_1 be the deduction preceding I. In this case applying IH gives δ_1', such that $l(\delta_1') \leqslant l(\delta_1)$. So applying L$\Box$ gives a deduction of length less than or equal to δ.

Case 6.3($\Gamma, \neg\varphi \Rightarrow \Sigma$ is right-main in I). This is similar to the above case.

Case 7(I is R$\Box$). This case has two sub-cases: *Case 7.1*($\Gamma, \neg\varphi \Rightarrow \Sigma$ is not main in I). In this case the δ has the following form:

$$\mathrm{R}\Box\,\frac{\dfrac{\text{IH}}{\Rightarrow \psi; \Delta \Rightarrow \Lambda; G; \Gamma, \neg\varphi \Rightarrow \Sigma; H}}{\Delta \Rightarrow \Box\psi, \Lambda; G; \Gamma, \neg\varphi \Rightarrow \Sigma; H}$$

IH allows the following deduction to be constructed:

$$\mathrm{R}\Box\,\frac{\dfrac{\text{IH}}{\Rightarrow \psi; \Delta \Rightarrow \Lambda; G; \Gamma \Rightarrow \varphi, \Sigma; H}}{\Delta \Rightarrow \Box\psi, \Lambda; G; \Gamma \Rightarrow \varphi, \Sigma; H}$$

Applying IH preserves the length of the deduction immediately preceding I. Applying R$\Box$ to this gives δ' with $l(\delta') \leqslant l(\delta)$.

Case 7.2($\Gamma, \neg\varphi \Rightarrow \Sigma$ is right-main in I). In this case δ has the following form:

$$\mathrm{R}\Box\,\frac{\dfrac{\vdots}{\Rightarrow \psi; \Gamma, \neg\varphi \Rightarrow \Sigma; H}}{\Gamma, \neg\varphi, \Box\psi \Rightarrow \Sigma; H}$$

IH can be used to construct the following deduction:

$$\mathrm{R}\Box\,\frac{\dfrac{\text{IH}}{\Rightarrow \psi; \Gamma \Rightarrow \varphi, \Sigma; H}}{\Gamma \Rightarrow \Box\psi, \varphi, \Sigma; H}$$

Applying IH preserves the length of the deduction immediately preceding I. Applying R$\Box$ to this gives δ' with $l(\delta') \leqslant l(\delta)$. $\square$

Lemma 3.2. *If $\delta \vdash_h G; \Gamma \Rightarrow \neg\varphi, \Sigma; H$, then there is a δ' such that $\delta' \vdash_h \Gamma, \varphi \Rightarrow \Sigma; H$, where $l(\delta') \leqslant l(\delta)$ and for any atomic $p \in \Delta \Rightarrow \Lambda \in G; \Gamma, \neg\varphi \Rightarrow \Sigma; H$, $d(p, \delta', \Delta \Rightarrow \Lambda) \leqslant d(p, \delta, \Delta \Rightarrow \Lambda)$.*

Proof. The proof of this is similar to Lemma 3.1. $\square$

Lemma 3.3. *If $\delta \vdash_h G; \Gamma, \varphi \to \psi \Rightarrow \Sigma; H$, then there is a δ' such that $\delta' \vdash_h G; \Gamma \Rightarrow \varphi, \Sigma; H$ and $\delta'' \vdash_h G; \Gamma, \psi \Rightarrow \Sigma; H$, where $l(\delta'), l(\delta'') \leqslant l(\delta)$ and for any atomic $p \in \Delta \Rightarrow \Lambda \in G; \Gamma, \varphi \to \psi \Rightarrow \Sigma; H$, $d(p, \delta', \Delta \Rightarrow \Lambda), d(p, \delta'', \Delta \Rightarrow \Lambda) \leqslant d(p, \delta, \Delta \Rightarrow \Lambda)$.*

Proof. The base case for this is given by the fact that the antecedent of the lemma is vacuous in that case. Let I be the last inference of δ.

Case 1 (I is Cut). There are two sub-cases:

Case 1.1 ($\Gamma, \varphi \to \psi \Rightarrow \Sigma$ is not main). In this case, δ has the following form:

$$\text{Cut}\ \frac{\dfrac{\vdots}{\Delta, \gamma \Rightarrow \Lambda; G; \Gamma, \varphi \to \psi \Rightarrow \Sigma; H} \qquad \dfrac{\vdots}{\Delta \Rightarrow \gamma, \Lambda; G; \Gamma, \varphi \to \psi \Rightarrow \Sigma; H}}{\Delta \Rightarrow \Lambda; G; \Gamma, \varphi \to \psi \Rightarrow \Sigma; H}$$

In this case IH gives deductions of the following four hypersequents:

δ_1' $\Delta, \gamma \Rightarrow \Lambda; G; \Gamma \Rightarrow \varphi, \Sigma; H$, with $l(\delta_1') < l(\delta)$

δ_2' $\Delta \Rightarrow \gamma, \Lambda; G; \Gamma \Rightarrow \varphi, \Sigma; H$, with $l(\delta_2') < l(\delta)$

δ_3' $\Delta, \gamma \Rightarrow \Lambda; G; \Gamma, \psi \Rightarrow \Sigma; H$, with $l(\delta_3') < l(\delta)$

δ_4' $\Delta \Rightarrow \gamma, \Lambda; G; \Gamma, \psi \Rightarrow \Sigma; H$, with $l(\delta_4') < l(\delta)$

Applying Cut to (δ_1') and (δ_2') gives a deduction, δ' of $\Delta \Rightarrow \Lambda; G; \Gamma \Rightarrow \varphi, \Sigma; H$, with $l(\delta') \leqslant l(\delta)$. Applying Cut to ($\delta_3$) and($\delta_4$) gives a deduction, δ'', of $\Delta \Rightarrow \Lambda; \Gamma, \psi \Rightarrow \Sigma; H$, where $l(\delta'') \leqslant l(\delta)$.

Case 1.2 ($\Gamma, \varphi \to \psi \Rightarrow \Sigma$ is main). This is similar to the above sub-case.

Case 2 (I is W or Drop). In both of these cases IH and an application of I gives the result.

Case 3 (I is TL or TR). If $\varphi \to \psi$ is not the main sentence, these follow from an application of IH followed by I. If $\varphi \to \psi$ is the main sentence in TL, then TL can be used to weaken in the desired sentence.

Case 4 (I is L$\neg$, R$\neg$, or R$\to$). In these cases the result follows from IH followed by I.

Case 5 (I is L$\to$). In this case if $\varphi \to \psi$ is not the main sentence, the result follows from IH and an application of I. If $\varphi \to \psi$ is the main sentence the result is either the deductions immediately preceding I, or follows from them by IH.

Case 6 (I is L$\Box$ or R$\Box$). This is similar to the cases in Lemma 3.1.

$\square$

Lemma 3.4. *If $\delta \vdash_h G; \Gamma \Rightarrow \varphi \to \psi, \Sigma; H$ then there is a δ' such that $\delta' \vdash_h G; \Gamma, \varphi \Rightarrow \psi, \Sigma; H$, where $l(\delta') \leqslant l(\delta)$ and for any atomic $p \in \Delta \Rightarrow \Lambda \in G; \Gamma \Rightarrow \varphi \to \psi, \Sigma; H$, $d(p, \delta', \Delta \Rightarrow \Lambda) \leqslant d(p, \delta, \Delta \Rightarrow \Lambda)$.*

Proof. This proof is similar to the above. $\square$

Lemma 3.5. *If $\delta \vdash_h G; \Delta \Rightarrow \Lambda; \Gamma, \Box\varphi \Rightarrow \Sigma; H$, then there is a δ' such that $\delta' \vdash_h G; \Delta, \varphi \Rightarrow \Lambda; \Gamma \Rightarrow \Sigma; H$, where $l(\delta') \leqslant l(\delta)$ and for any atomic $p \in \Delta \Rightarrow \Lambda \in G; \Gamma, \Box\varphi \Rightarrow \Sigma; H$, $d(p, \delta', \Delta \Rightarrow \Lambda) \leqslant d(p, \delta, \Delta \Rightarrow \Lambda)$.*

Proof. This proceeds by induction on the length of deductions. Again the base case does not meet the antecedent of the lemma. Let I be the last inference of δ

Case 1 (I is Cut.) If $\Gamma, \Box\varphi \Rightarrow \Sigma$ is neither right nor left-main in I, then the result follows from two applications of IH and I. If it is right-main in I, then δ has the following form:

$$\text{Cut}\ \dfrac{\dfrac{\vdots}{G; \Delta \Rightarrow \Lambda; \Gamma, \Box\varphi, \psi \Rightarrow \Sigma; H} \quad \dfrac{\vdots}{G; \Delta \Rightarrow \Lambda; \Gamma, \Box\varphi \Rightarrow \psi, \Sigma; H}}{G; \Delta \Rightarrow \Lambda; \Gamma, \Box\varphi \Rightarrow \psi, \Sigma; H}$$

IH makes the following deduction constructible:

$$\dfrac{\dfrac{\text{IH}}{G; \Delta, \varphi \Rightarrow \Lambda; \Gamma, \psi \Rightarrow \Sigma; H} \quad \dfrac{\text{IH}}{G; \Delta, \varphi \Rightarrow \Lambda; \Gamma \Rightarrow \psi, \Sigma; H}}{G; \Delta, \varphi \Rightarrow \Lambda; \Gamma \Rightarrow \psi, \Sigma; H}$$

As above, the application of IH preserves the length of the sub-deductions of δ, and so the constructed deduction is of length at most δ.

Case 2 (I is W or Drop). In either case, the result follows from IH and I.

Case 3 (I is TL or TR). If $\Box\varphi$ is not the main sentence then δ' is the result of IH and I. If $\Box\varphi$ is the main sentence then weakening in φ to $\Delta \Rightarrow \Lambda$ as opposed to $\Box\varphi$ to $\Gamma \Rightarrow \Sigma$ gives the result.

Case 4 (I is L$\neg$, R$\neg$, L$\rightarrow$, or R$\rightarrow$). All of these cases follow from IH and I.

Case 5 (I is L$\Box$). If $\Box\varphi$ is not main, then this follows from IH and I. If $\Box\varphi$ is main, then either δ' is the deduction preceding I, or is given by IH.

Case 6 (I is R$\Box$). In this case the result follows from IH and I.

□

3.5.2 Modal Lemmas

This subsection gives the conditions under which something like the above lemmas can be applied to sentences of the form $\Box\varphi$.

Definition 11 (Modal Rank)**.** The modal rank of φ, $mr(\varphi)$ is defined inductively as follows:[4]

- If φ is atomic then $mr(\varphi) = 0$.
- If φ is $\neg\psi$, and $mr(\psi) = 0$ then $mr(\varphi) = 0$.
- If φ is $\psi \to \theta$ where $mr(\psi) = 0, mr(\theta) = 0$, then $mr(\varphi) = 0$.
- If φ is $\Box\psi$, then $mr(\varphi) = 1$
- If φ is $\neg\psi$ and $mr(\psi) = n \geqslant 1$, then $mr(\varphi) = n + 1$.
- If φ is $\psi \to \theta$, where $mr(\psi) = j$ and $mr(\theta) = k$, and $j + k \geqslant 1$, then $mr(\varphi) = j + k + 1$

If $\Gamma \Rightarrow \Sigma$ is a sequent, then the modal rank of that sequent is the ordered pair of the maximum of the modal ranks of sentences in $\Gamma \cup \Sigma$, and the number of sentences of that modal rank, $mr(\Gamma \Rightarrow \Sigma) = \langle max\{mr(\varphi) : \varphi \in \Gamma \cup \Sigma\}, |\{\psi : mr(\psi) = max\{mr(\varphi) : \varphi \in \Gamma \cup \Sigma\}\}|\rangle$, where $|S|$ is the cardinality of the set S.[5] The modal ranks of sequents are ordered lexicographically. The modal rank of a hypersequent G, $mr(G)$, is the ordered pair of the maximum of the modal ranks of sequents in G, and the number of sequents of that modal rank in G, $mr(G) = \langle max\{mr(\Gamma \Rightarrow \Sigma) : \Gamma \Rightarrow \Sigma \in G\}, |\{\Delta \Rightarrow \Lambda : mr(\Delta \Rightarrow \Lambda) = max\{mr(\Gamma \Rightarrow \Sigma) : \Gamma \Rightarrow \Sigma \in G\}\}|\rangle$. These too are ordered lexicographically.

[4] This measure is similar to one defined by Lellmann [29]. It is a measure of how deeply a sentence whose main expression is $\Box$ is nested in a sentence.

[5] The function max returns the maximum number in a set based on whatever ordering is given. If the given set is empty it returns the lowest value of in the ordering in question.

Remark 1. If $mr(\Gamma \Rightarrow \Sigma) = \langle 1, n \rangle$, then $\Gamma \Rightarrow \Sigma$ can be partitioned into a set where $\Box$ does not occur, and a set where the main operator of each sentence is $\Box$. This warrants rewriting this sequent as $\Delta \Box \Delta' \Rightarrow \Lambda \Box \Lambda'$ where for any $\varphi \in \Delta \cup \Lambda$, $mr(\varphi) = 0$.

Lemma 3.6. *If* $\delta \vdash_h G; \Gamma \Rightarrow \Sigma; \Delta\Delta' \Rightarrow \Lambda\Lambda'; H$ *and* $\sum_{\varphi \in \Delta \cup \Lambda} mr(\varphi) = 0$*, then* $\delta' \vdash_h G; \Gamma \Rightarrow \Sigma; \Delta' \Rightarrow \Lambda'; H$*, where* $l(\delta) \leqslant l(\delta')$ *and for any atomic* $p \in \Pi \Rightarrow \Theta \in G; \Gamma \Rightarrow \Sigma; \Delta\Delta' \Rightarrow \Lambda\Lambda'; H$*,* $d(p, \delta', \Pi \Rightarrow \Theta) \leqslant d(p, \delta, \Pi \Rightarrow \Theta)$*.*

Proof. Ax does not meet the antecedent of the lemma. Let I be the last inference of δ

Case 1(I is Cut). If $\Gamma \Rightarrow \Sigma$ is not the main sequent in the cut, the result follows from IH and I or merely IH. For the sake of clarity the case where $\Gamma \Rightarrow \Sigma$ is the main sequent in the cut is rehearsed. In this case δ has the following form:

$$\text{Cut}\,\frac{\dfrac{\vdots}{G; \Gamma, \varphi \Rightarrow \Sigma; \Delta\Delta' \Rightarrow \Lambda\Lambda'; H} \qquad \dfrac{\vdots}{G; \Gamma \Rightarrow \varphi, \Sigma; \Delta\Delta' \Rightarrow \Lambda\Lambda'; H}}{G; \Gamma \Rightarrow \Sigma; \Delta\Delta' \Rightarrow \Lambda\Lambda'; H}$$

The following deduction is can be constructed:

$$\text{Cut}\,\frac{\dfrac{\text{IH}}{G; \Gamma, \varphi \Rightarrow \Sigma; \Delta' \Rightarrow \Lambda'; H} \qquad \dfrac{\text{IH}}{G; \Gamma \Rightarrow \varphi, \Sigma; \Delta' \Rightarrow \Lambda'; H}}{G; \Gamma \Rightarrow \Sigma; \Delta' \Rightarrow \Lambda'; H}$$

Case 2(I is Drop or W). In this case IH and I give the result.

Case 3(I is TL or TR). This case is also given by IH followed by I.

Case 4(I is L$\neg$, R$\neg$, L$\rightarrow$, or R$\rightarrow$). Again, these cases are given by IH followed by I.

Case 5(I is L$\Box$). *Case 5.1*($\Delta\Delta' \Rightarrow \Lambda\Lambda'$ is right-main in I). In this case δ is

$$\mathrm{L}\Box\,\dfrac{\dfrac{\vdots}{G; \Gamma, \varphi \Rightarrow \Sigma; \Delta\Delta' \Rightarrow \Lambda\Lambda'; H}}{G; \Gamma \Rightarrow \Sigma; \Delta\Delta', \Box\varphi \Rightarrow \Lambda\Lambda'; H}$$

The following deduction can be constructed using IH

$$\mathrm{L}\Box\,\dfrac{\dfrac{\mathrm{IH}}{G; \Gamma, \varphi \Rightarrow \Sigma; \Delta' \Rightarrow \Lambda'; H}}{G; \Gamma \Rightarrow \Sigma; \Delta', \Box\varphi \Rightarrow \Lambda'; H}$$

Case 5.2($\Delta\Delta' \Rightarrow \Lambda\Lambda'$ is left-main in I). In this case δ has the following form:

$$\mathrm{L}\Box\,\dfrac{\dfrac{\vdots}{G; \Gamma \Rightarrow \Sigma; \Delta\Delta', \varphi \Rightarrow \Lambda\Lambda'; \Pi \Rightarrow \Theta; H}}{G; \Gamma \Rightarrow \Sigma; \Delta\Delta' \Rightarrow \Lambda\Lambda'; \Pi, \Box\varphi \Rightarrow \Theta; H}$$

In this case the following deduction is given:

$$\mathrm{L}\Box\,\dfrac{\dfrac{\mathrm{IH}}{G; \Gamma \Rightarrow \Sigma; \Delta', \varphi \Rightarrow \Lambda'; \Pi \Rightarrow \Theta; H}}{G; \Gamma \Rightarrow \Sigma; \Rightarrow; \Pi, \Box\varphi \Rightarrow \Theta; H}$$

Case 5.3(Either $\Gamma \Rightarrow \Sigma$ is right-main in I or neither $\Gamma \Rightarrow \Sigma$ nor $\Delta\Delta' \Rightarrow \Lambda\Lambda'$ is main in I). In this case the result is given by an application of IH followed by I.

Case 6(I is R$\Box$). This case is given by an application of IH followed by I.

$\square$

Lemma 3.7. *If* $\delta \vdash_h G; \Gamma \Rightarrow \Sigma, \Delta \Rightarrow \Box\varphi, \Lambda; H$, *then* $\delta' \vdash_h G; \Gamma \Rightarrow \Sigma, \Delta \Rightarrow \Lambda; H$, *where* $l(\delta') \leqslant l(\delta)$ *and* $p \in \Pi \Rightarrow \Theta \in G; \Gamma \Rightarrow \Sigma; \Delta\Delta' \Rightarrow \Lambda\Lambda'; H$, $d(p, \delta', \Pi \Rightarrow \Theta) \leqslant d(p, \delta, \Pi \Rightarrow \Theta)$.

Proof. The proof proceeds by induction on the length of δ. The base case is trivial. For the inductive cases let I be the last inference of δ.

Case 1([I is Drop or W]). In either of these cases the result follows from IH and I.

Case 2([I is TL or TR]). If I is TL the result follows from IH and I. If I is TR, then either it follows from IH and I, or in the case that $\Box\varphi$ is the main sentence of I, δ' is the δ without I.

Case 3([I is Cut]). This case follows from two applications of IH followed by I.

Case 4([I is L$\neg$, R$\neg$, L$\rightarrow$, or R$\rightarrow$]). In any of these cases the result follows from IH and I.

Case 5([I is L$\Box$]). This follows from IH and I.

Case 6([I is R$\Box$]). If $\Gamma \Rightarrow \Sigma$ is not right-main in I, then the result follows from IH and I since $\Box\varphi$ cannot be the main formula in I. The only possibly worrying case is if it is. In this case δ is

$$\text{R}\Box\,\frac{\dfrac{\vdots}{\Rightarrow \psi; \Gamma \Rightarrow \Sigma/\{\Box\psi\}; \Delta \Rightarrow \Box\varphi, \Lambda; H}}{\Gamma \Rightarrow \Sigma; \Delta \Rightarrow \Box\varphi, \Lambda; H}$$

IH allows the following deduction to be constructed.

$$\text{R}\Box\,\frac{\dfrac{\text{IH}}{\Rightarrow \psi; \Gamma \Rightarrow \Sigma/\{\Box\psi\}; \Delta \Rightarrow \Lambda; H}}{\Gamma \Rightarrow \Sigma; \Delta \Rightarrow \Lambda; H}$$

$\square$

Lemma 3.8. *If $mr(\Gamma \Box \Gamma' \Rightarrow \Sigma \Box \Sigma'; H) = \langle\langle 1, n\rangle, 1\rangle$ and $\delta \vdash_h \Gamma \Box \Gamma' \Rightarrow \Sigma \Box \Sigma'; H$, then either there is a δ' such that $\delta' \vdash_h \Gamma \Rightarrow \Sigma; H$ or there is a δ'' such that $\delta'' \vdash_h \Box\Gamma' \Rightarrow \Box\Sigma'; H$.*

Proof. The base case is given by the fact that Ax is not of the right form to meet the antecedent of the lemma. Let I be the last inference of δ. For the inductive cases, it is important to note that because $mr(\Gamma \Box \Gamma' \Rightarrow \Sigma \Box \Sigma'; H) = \langle\langle 1, n\rangle, 1\rangle$,

for any $\Delta \Rightarrow \Lambda \in \Gamma \,\square\, \Gamma' \Rightarrow \Sigma \,\square\, \Sigma'; H$, if $(\Delta \Rightarrow \Lambda) \neq (\Gamma \,\square\, \Gamma \,\square\, \Gamma' \Rightarrow \Sigma \,\square\, \Sigma')$ then $mr(\Delta \Rightarrow \Lambda) = \langle 0, m \rangle$, for some m.

Case 1(I is TL or TR). These follow from IH, and possibly an application of TL or TR.

Case 2(I is Cut).

Case 2.1(The Main Sequent is the left-most). In this case the given δ is

$$\text{Cut}\ \dfrac{\dfrac{\vdots}{\Gamma \,\square\, \Gamma', \varphi \Rightarrow \Sigma \,\square\, \Sigma'; H} \qquad \dfrac{\vdots}{\Gamma \,\square\, \Gamma' \Rightarrow \varphi, \Sigma \,\square\, \Sigma'; H}}{\Gamma \,\square\, \Gamma' \Rightarrow \Sigma \,\square\, \Sigma'; H}$$

In this case an sub-induction on the modal rank of φ is required. There are several sub-cases that need to be considered.

Case 2.1.1($mr(\varphi) = 0$). By IH either $\vdash_h \Gamma, \varphi \Rightarrow \Sigma; H$ or $\vdash_h \square\Gamma' \Rightarrow \square\Sigma'; H$. In the second case, the result is given by IH alone. In the first case, it is necessary to consider what is given by IH when applied to the deduction of $\Gamma \,\square\, \Gamma' \Rightarrow \varphi, \Sigma \,\square\, \Sigma'; H$. If, on the one hand $\vdash_h \square\Gamma' \Rightarrow \square\Sigma'; H$, then IH gives the result. If, on the other, $\vdash_h \Gamma \Rightarrow \varphi, \Sigma; H$. The following deduction gives the result:

$$\text{Cut}\ \dfrac{\dfrac{\text{IH}}{\Gamma, \varphi \Rightarrow \Sigma; H} \qquad \dfrac{\text{IH}}{\Gamma \Rightarrow \varphi, \Sigma; H}}{\Gamma \Rightarrow \Sigma; H}$$

Case 2.1.2($mr(\varphi) = 1$). This case is analogous to the above case where $mr(\varphi) =$ 0

Case 2.1.3($mr(\varphi) = n$). In this case, it will be shown that cuts can be pushed back up δ. This allows the outer inductive hypothesis to be applied at a stage where

the main sequents of the cut have a modal rank of at most $\langle 1, m\rangle$, for some m. There are two sub-sub-cases to consider here.

2.1.3.1 (φ is $\neg\gamma$). δ then has the form:

$$\text{Cut}\,\dfrac{\dfrac{\text{IH}}{\Gamma, \neg\gamma \Rightarrow \Sigma; H} \qquad \dfrac{\text{IH}}{\Gamma \Rightarrow \neg\gamma, \Sigma; H}}{\Gamma \Rightarrow \Sigma; H}$$

Applying Lemma 3.1 to the deduction of $\Gamma, \neg\gamma \Rightarrow \Sigma; H$ and Lemma 3.2 to the deduction of $\Gamma \Rightarrow \neg\gamma, \Sigma; H$ and cutting gives the result.

2.1.3.2 (φ is $\gamma \to \delta$). Again in this case applying Lemma 3.3 and Lemma 3.4 and cutting will give the result.

Case 2.2 (The main sequent is other than the left-most). A similar induction over the modal rank of the cut sentence is required. In this case δ is

$$\text{Cut}\,\dfrac{\dfrac{\vdots}{\Gamma \sqcap \Gamma' \Rightarrow \Sigma \sqcap \Sigma'; G'; \Delta, \varphi \Rightarrow \Lambda; H} \qquad \dfrac{\vdots}{\Gamma \sqcap \Gamma' \Rightarrow \Sigma \sqcap \Sigma'; G'; \Delta \Rightarrow \varphi, \Lambda; H}}{\Gamma \sqcap \Gamma' \Rightarrow \Sigma \sqcap \Sigma'; G'; \Delta \Rightarrow \Lambda; H}$$

Case 2.2.1 ($mr(\varphi) = 0$). In this case an application of Lemma 3.6 to one of the deductions followed by TL and TR will give the result.

Case 2.2.2 ($mr(\varphi) = 1$). In this case, an application of Lemma 3.7 to one of the branches of δ will give the result.

Case 2.2.3 ($mr(\varphi) = n$). There are two sub-cases to be considered.

Case 2.2.3.1 (φ is $\neg\psi$). In this case, δ is

$$\text{Cut}\ \frac{\dfrac{\vdots}{\Gamma \mathbin{\Box} \Gamma' \Rightarrow \Sigma \mathbin{\Box} \Sigma'; G'; \Delta, \neg\psi \Rightarrow \Lambda; H} \quad \dfrac{\vdots}{\Gamma \mathbin{\Box} \Gamma' \Rightarrow \Sigma \mathbin{\Box} \Sigma'; G'; \Delta \Rightarrow \neg\psi, \Lambda; H}}{\Gamma \mathbin{\Box} \Gamma' \Rightarrow \Sigma \mathbin{\Box} \Sigma'; G'; \Delta \Rightarrow \Lambda; H}$$

An application of Lemma 3.1 to the right branch of δ, and application of Lemma 3.2 to the left branch, and a cut give the result.

Case 2.2.3.2(φ is $\psi \to \theta$). In this case δ is

$$\text{Cut}\ \frac{\dfrac{\vdots}{\Gamma \mathbin{\Box} \Gamma' \Rightarrow \Sigma \mathbin{\Box} \Sigma'; G'; \Delta, \psi \to \theta \Rightarrow \Lambda; H} \quad \dfrac{\vdots}{\Gamma \mathbin{\Box} \Gamma' \Rightarrow \Sigma \mathbin{\Box} \Sigma'; G'; \Delta \Rightarrow \psi \to \theta, \Lambda; H}}{\Gamma \mathbin{\Box} \Gamma' \Rightarrow \Sigma \mathbin{\Box} \Sigma'; G'; \Delta \Rightarrow \Lambda; H}$$

An application of Lemma 3.3, an application of Lemma 3.4, and two cuts give the result.

Case 3(I is L$\neg$, R$\neg$, or R$\to$). These cases are all given by an application of IH, and possibly I.

Case 4(I is L$\to$). In this case the following deduction is given:

$$\text{L}\!\to\ \frac{\dfrac{\vdots}{\Gamma \mathbin{\Box} \Gamma' \Rightarrow \Sigma, \varphi, \Box\Sigma'; H} \quad \dfrac{\vdots}{\Gamma, \psi, \Box\Gamma' \Rightarrow \Sigma \mathbin{\Box} \Sigma'; H}}{\Gamma, \varphi \to \psi, \Box\Gamma' \Rightarrow \Sigma \mathbin{\Box} \Sigma'; H}$$

In this case $mr(\varphi \to \psi) = 0$. So if IH in either case delivers $\Box\Delta' \Rightarrow \Box\Lambda'; H$, we are done. On the other hand, the following deduction can be constructed:

$$\text{L}\!\to\ \frac{\dfrac{\text{IH}}{\Gamma, \Rightarrow \varphi, \Sigma; H} \quad \dfrac{\text{IH}}{\Gamma, \psi \Rightarrow \Sigma; H}}{\Gamma, \varphi \to \psi \Rightarrow \Sigma; H}$$

If $\varphi \to \psi$ is not in the left-most sequent in then an application of Lemma 3.6 to one of the branches, and TL give the result.

Case 5(I is L$\Box$). Since the only sequent in $\Gamma \Box \Gamma' \Rightarrow \Sigma \Box \Sigma'; H$, is $\Gamma \Box \Gamma' \Rightarrow \Sigma \Box \Sigma'$, L$\Box$ cannot be the last rule in δ.

Case 6(I is R$\Box$). In this case δ is

$$I \dfrac{\dfrac{\vdots}{\Rightarrow \varphi; \Gamma \Box \Gamma' \Rightarrow \Sigma \Box \Sigma'; H}}{\Gamma \Box \Gamma' \Rightarrow \Sigma \Box \Sigma', \Box\varphi; H}$$

Since $mr(\Gamma \Box \Gamma' \Rightarrow \Sigma \Box \Sigma'; H) = \langle\langle 1, n\rangle, n\rangle$, it follows that $\sum_{\varphi \in \Gamma \cup \Sigma} mr(\varphi) = 0$. Lemma 3.6 can be applied to the deduction preceding I yielding a deduction of $\Rightarrow \varphi; \Box\Gamma' \Rightarrow \Box\Sigma'; H$. An application of R$\Box$ to this gives a deduction of $\Box\Gamma' \Rightarrow \Box\varphi, \Box\Sigma'; H$

□

Remark 2. In each of the above Lemmas, the only cases that relied on the use of the cut rule were cases where the inference being considered was an instance of Cut. Therefore, if a deduction that meeting the antecedent of any of the lemmas does not make use of Cut, then an application of one of the lemmas will preserve this feature.

3.6 Cut Elimination

3.6.1 Important Lemmas for Cut Elimination

This section proves that Cut is eliminable. It first introduces a new rule, Cut*, of which all Cuts are instances. It proves that Cut* is eliminable by induction over the rank of the main formula of a Cut* rule. There are two sub-inductions in this proof. The first, in the base case, is an induction over the depth of the main formula of the Cut* rule. The second sub-induction is in the case where the main formula of the

Cut* rule is of the form $\Box\varphi$. In this case a new measure, disarray, on hypersequents is defined. The sub-induction is over the amount of disarray in the hypersequents under consideration.

Definition 12 (Cut-Free Deduction)**.** A deduction δ making no use of the cut-rule is called *cut-free*. $\delta \vdash_h^{cf} G$ indicates that δ is a cut-free deduction of G, and $\vdash_h^{cf} G$ indicates that there is some cut-free deduction of G.

Definition 13 (Fine Modal Rank)**.** The *fine modal rank* of φ, $fmr(\varphi)$ is defined inductively as follows:

- If φ is atomic then $fmr(\varphi) = \langle 0, 0\rangle$.
- If φ is $\neg\gamma$, and $fmr(\gamma) = \langle 0, n\rangle$ then $fmr(\varphi) = \langle 0, n+1\rangle$.
- If φ is $\gamma \to \delta$ where $fmr(\gamma) = \langle 0, n\rangle, fmr(\delta) = \langle 0, m\rangle$, then $fmr(\varphi) = \langle 0, m + n + 1\rangle$.
- If φ is $\Box\psi$, where $fmr(\psi) = \langle m, n\rangle$ then $fmr(\varphi) = \langle m+1, 0\rangle$.
- If φ is $\neg\gamma$ and $fmr(\gamma) = \langle n, m\rangle$, then $fmr(\varphi) = \langle n, m+1\rangle$.
- If φ is $\gamma \to \delta$, where $fmr(\gamma) = \langle j, k\rangle$ and $fmr(\delta) = \langle m, n\rangle$, $j + k \geqslant 1$, then $fmr(\varphi) = \langle j+m, k+n\rangle$

Fine modal ranks are ordered lexicographically. If $\Gamma \Rightarrow \Sigma$ is a sequent, then the fine modal rank of that sequent is the ordered pair of the maximum of the fine modal ranks of sentences in $\Gamma \cup \Sigma$, and the number of sentences of that fine modal rank, $fmr(\Gamma \Rightarrow \Sigma) = \langle max\{fmr(\varphi) : \varphi \in \Gamma \cup \Sigma\}, |\{\psi : fmr(\psi) = max\{fmr(\varphi) : \varphi \in$

$\Gamma \cup \Sigma\}\}|\rangle$. The fine modal ranks of sequents are ordered lexicographically. The fine modal rank of a hypersequent G, $fmr(G)$, is the ordered pair of the maximum of the fine modal ranks of sequents in G, and the number of sequents of that fine modal rank in G, $fmr(G) = \langle max\{fmr(\Gamma \Rightarrow \Sigma) : \Gamma \Rightarrow \Sigma \in G\}, |\{\Delta \Rightarrow \Lambda : fmr(\Delta \Rightarrow \Lambda) = max\{fmr(\Gamma \Rightarrow \Sigma) : \Gamma \Rightarrow \Sigma \in G\}\}|\rangle$. These too are ordered lexicographically.

Remark 3. The restriction on many lemmas that $mr(\Gamma \Box \Gamma' \Rightarrow \Sigma \Box \Sigma'; H) = \langle\langle 1, n\rangle, 1\rangle$, for some n, could now be written as the restriction that $fmr(\Gamma, \Box\Gamma' \Rightarrow \Sigma \Box \Sigma'; H) = \langle\langle\langle m, 0\rangle, n\rangle, 1\rangle$, for some n and m. Use of the modal rank as opposed to the fine modal rank where possible allows simpler and more general proofs. this is particularly true of Section 3.8. Because the definition of fine modal rank has more clauses than that of modal rank cases that would have been repetitive can often be glossed over when using modal rank as the inductive measure.

Lemma 3.9. *If* $\delta \vdash_h^{cf} \Gamma, \Box\varphi \Rightarrow \Sigma; H$, *where* $mr(\Gamma, \Box\varphi \Rightarrow \Sigma) = \langle 1, m\rangle$ *for some* m, *and* $mr(H) = \langle\langle 0, n\rangle, o\rangle$ *for some* n, o, *and for any* $\varphi \in \Sigma, mr(\varphi) = 0$ *then either there is a* δ' *such that* $\delta' \vdash_h^{cf} \Gamma \Rightarrow \Sigma; H$, *or there is a* δ'' *such that* $\delta'' \vdash_h^{cf} \varphi \Rightarrow; \Gamma \Rightarrow \Sigma; H$

Proof. The proof of this lemma proceeds by induction on the length of δ. The base case is trivial. For the inductive case let I be the last inference in δ.

Case 1(I is W or TR) These cases follow from IH and I.

Case 2(I is TL). If $\Box\varphi$ is not main, then the result follows from IH and I. If it is, then the result is given by the deduction preceding I.

Case 3(I is Drop). In this case an application of Lemma 3.5 to the deduction preceding I gives the result.

Case 4 (I is L$\neg$, R$\neg$, R$\rightarrow$). These cases follow from IH and I.

Case 5 (I is L$\rightarrow$). There are two sub-cases. Either the main sentence of I, $\psi \rightarrow \theta$, is in the left-most sequent or not.

Case 5.1 ($\psi \rightarrow \theta$ is in the left-most sequent). In this case δ is

$$I\ \dfrac{\dfrac{\vdots}{\Gamma, \Box\varphi, \theta \Rightarrow \Sigma; H} \qquad \dfrac{\vdots}{\Gamma, \Box\varphi \Rightarrow \psi, \Sigma; H}}{\Gamma, \Box\varphi, \psi \rightarrow \theta, \Sigma; H}$$

Applying IH to the left branch either gives a deduction of $\Gamma, \theta \Rightarrow \Sigma; H$ or a deduction of $\varphi \Rightarrow; \Gamma, \theta \Rightarrow \Sigma; H$. In the second case an application of Lemma 3.6, followed by TL gives a deduction of $\varphi \Rightarrow; \Gamma, \psi \rightarrow \theta \Rightarrow \Sigma; H$. Applying IH to the right branch either gives a deduction of $\Gamma \Rightarrow \psi, \Sigma; H$ or a deduction of $\varphi \Rightarrow; \Gamma \Rightarrow \psi, \Sigma; H$. In the latter case an application of Lemma 3.6 and TL gives a deduction of $\varphi \Rightarrow; \Gamma, \psi \rightarrow \theta \Rightarrow \Sigma; H$. This leaves the case where there are deductions of $\Gamma, \theta \Rightarrow \Sigma; H$ and $\Gamma \Rightarrow \psi, \Sigma; H$, in this case application of L$\rightarrow$ gives a deduction of $\Gamma, \psi \rightarrow \theta \Rightarrow \Sigma; H$.

Case 5.2 ($\psi \rightarrow \theta$ is not in the left-most sequent). In this case after an application of IH, an application of Lemma 3.6 and TL gives the result.

Case 6 (I is L$\Box$ or R$\Box$). I cannot be L$\Box$ or R$\Box$.

$\square$

Lemma 3.10. *If $\delta \vdash^{cf}_{h} \Box\Gamma \Rightarrow \Box\Sigma; H$, where $\Box\Sigma \neq \varnothing$ and $mr(H) = \langle\langle 0, n\rangle, m\rangle$ for some m and n, then there is a δ' and $\varphi \in \Sigma$, such that $\delta' \vdash^{cf}_{h} \Rightarrow \varphi; \Box\Gamma \Rightarrow; H$*

Proof. This is proved by induction on the length of δ. The base case is trivial. Let the last inference of δ be I.

Case 1(I is Drop). In this case an application of Lemma 3.7 and TR gives the result.

Case 2(I is W, TL, or TR). These cases follow from IH and I.

Case 3(I is L$\neg$, R$\neg$, or R$\rightarrow$). These cases follow from IH and I.

Case 4(I is L$\rightarrow$). After an application of IH to one of the branches, an application of Lemma 3.6 followed by TL gives the result.

Case 5(I is L$\Box$). I cannot be L$\Box$

Case 6(I is R$\Box$). In this case apply Lemma 3.7 deduction preceding I, perhaps several times, to get a deduction of $\varphi \Rightarrow; \Box\Gamma \Rightarrow; H$.

$\square$

Lemma 3.11. *If $\delta \vdash_h^{cf} \Gamma \Rightarrow \Sigma; G; \Rightarrow$ then there is a δ' such that $\delta' \vdash_h^{cf} \Gamma \Rightarrow \Sigma; G$ where $l(\delta') \leqslant l(\delta)$, and for any atomic $p \in \Delta \Rightarrow \Lambda \in \Gamma \Rightarrow \Sigma; H$, $d(p, \delta') \leqslant d(p, \delta)$.*

Proof. This lemma is proved by induction on the length of deductions. In the base case the Lemma holds trivially. For the inductive cases let δ be given and let I be the last inference in δ. The only case where the result does not follow immediately from an application of IH followed by I is where I is W. But in this case δ' is the deduction immediately preceding I.

$\square$

3.6.2 Cut Elimination

The proof of cut-elimination proceeds much the same way as a Gentzen-style deduction. The main deduction is over the rank of the cut-formula. The lemmas from

Section 3.5 do much of the work that the sub-induction over the depth of the cut-formula would do. The greater complexity comes in the modal case, where it must be shown that hypersequents involving cuts can be transformed into hypersequents to which Lemma 3.5 and Lemma 3.10 can be applied.

Definition 14 (Product)**.** If G is the hypersequent $\Gamma_1 \Rightarrow \Sigma_1; \ldots; \Gamma_n \Rightarrow \Sigma_n$ and G' is the hypersequent $\Gamma'_1 \Rightarrow \Sigma'_1; \ldots; \Gamma'_n \Rightarrow \Sigma'_n$, both containing n-many sequents, then the *even product* of G and G', $[GG']$, is $\Gamma_1\Gamma'_1 \Rightarrow \Sigma_1\Sigma'_1; \ldots; \Gamma_n\Gamma'_n \Rightarrow \Sigma_n\Sigma'_n$. If G is the hypersequent $\Gamma_1 \Rightarrow \Sigma_1; \ldots; \Gamma_n \Rightarrow \Sigma_n$ and G' is the hypersequent $\Gamma'_1 \Rightarrow \Sigma'_1; \ldots; \Gamma'_m \Rightarrow \Sigma'_m$, where $n \leqslant m$, then the *uneven product* of G and G', $G \otimes G'$, is the hypersequent $\Gamma_1\Gamma'_1 \Rightarrow \Sigma_1\Sigma'_1; \ldots; \Gamma_n\Gamma'_n \Rightarrow \Sigma_n\Sigma'_n; \Gamma'_{n+1} \Rightarrow \Sigma'_{n+1}; \ldots; \Gamma'_m \Rightarrow \Sigma'_m$.

Importantly, $G \otimes G'$ is the same as $G' \otimes G$.

Definition 15 (Cut*)**.** The *Cut** rule is the following:

$$\text{Cut*}\ \frac{G; \Gamma, \varphi \Rightarrow \Sigma; H \qquad G'; \Gamma' \Rightarrow \varphi, \Sigma'; H'}{[GG']; \Gamma\Gamma' \Rightarrow \Sigma\Sigma'; H \otimes H'}$$

The Cut* rule requires that the sequents in which the main sentence of the rule occurs, must be the same distance from the left of the hypersequent. What follows them is the uneven product of the hypersequents following the main sequent.

Let $h*$ be the system characterized by replacing Cut with Cut* in Figure 3.2.

Definition 16 (Cut-Proof)**.** A *cut-proof* is a cut-free deduction whose only application of the Cut* rule is the last inference. Call the main sentence of this inference the *cut*-sentence*

Definition 17 (In Position). A sentence, φ, in a hypersequent H, is *in position* when either it is of modal rank 0, or it is both of modal rank 1 and in the left-most sequent of that hypersequent.

Definition 18 (Complete). A hypersequent, H, is *complete* when all of its sentences are in position.

Definition 19 (Disarray). Let H be a hypersequent. The disarray, $dis(H)$, of H is given by $dis(H) = \langle max\{fmr(\varphi) : \varphi \in H \text{ and is not in position}\}, |\{\psi \in H : fmr(\psi) = max\{fmr(\varphi) : \varphi \in H \text{ and is not in position}\} \text{ and } \psi \text{ is not in position}\}|\rangle$.

Note 1. A hypersequent has disarray of $\langle\langle 0, 0\rangle, 0\rangle$ iff that hypersequent is complete.

Lemma 3.12. *If Cut* is eliminable from Cut-proofs then Cut* is eliminable h∗, i.e. if there is a deduction of G in h∗, then there is a cut*-free deduction of G in h∗.*

Proof. Let Cut* be eliminable from cut-proofs. Let δ be a deduction involving cuts*. Consider the top-most and left-most cut*. The sub-deduction of δ ending in this cut* is a cut*-proof. By assumption, there is a cut*-free deduction of the same hypersequent. Replace the cut*-proof with this deduction. Again consider the top-most and left-most cut* in the new δ'. In the same way, this can be eliminated. Repeat this process until all the cuts* are eliminated. This will be a cut*-free deduction of end-sequent of δ. □

Lemma 3.13. *Cut* is eliminable from Cut-proofs.*

Proof. The proof of this proceeds by induction on the rank of the cut*-sentence. There are, furthermore, two sub-inductions over the depth of the sentence in the

atomic case, and the modal rank of the main sequent in the case where the main operator of the cut*-sentence is $\Box$. These sub-inductions occur in case 1, and case 2.3. Let δ be the deduction in question, and I be its last inference. Let the left branch of δ be δ_1 and the right branch be δ_2.

Case 1 (Base Case: $rk(\varphi) = 0$). In this case δ has the following form:

$$\text{Cut*}\ \frac{\dfrac{\vdots}{G; \Gamma \Rightarrow p, \Sigma; H} \qquad \dfrac{\vdots}{G'; \Gamma', p \Rightarrow \Sigma'; H'}}{[GG']; \Gamma\Gamma' \Rightarrow \Sigma\Sigma'; H \otimes H'}$$

As mentioned above, a induction on the depth of p in δ is required.

Case 1.1 ($d(p, \delta_1, \Gamma \Rightarrow \Sigma = d(p, \delta_2, \Gamma' \Rightarrow \Sigma') = 1$). In this case both δ_1 and δ_2 introduce the cut*-sentence, which is atomic. Either p was introduced into δ_1 by means of Ax or TL. In the latter case, the result is given by the deduction preceding TL and a combination of TL and TR. In the former case, either p was introduced into δ_2 by Ax or TR. In the latter case the result is given by the deduction immediately preceding TR and a combination of TL and TR. In the former case, δ has the following form:

$$\text{Cut*}\ \frac{\overline{p \Rightarrow p} \qquad \overline{p \Rightarrow p}}{p \Rightarrow p}$$

Here the result is given by an instance of Ax.

For the inductive case let the maximum of $d(p, \delta_1, \Gamma \Rightarrow \Sigma), d(p, \delta_2, \Gamma' \Rightarrow \Sigma')$ be n. It is necessary to reduce the depth of p in whichever of the two is higher, if any. For the sake of brevity let p be of greater depth in δ_1, the case where it is in δ_2 is similar. This will proceed by checking the last inference of δ_1, and showing that the depth of

p can be reduced. If $\Gamma, p \Rightarrow \Sigma$ is not the left-most sequent in the hypersequent, then the result is given directly by Lemma 3.6, TL, and TR. Therefore, it is only necessary to consider cases where $\Gamma, p \Rightarrow \Sigma$ is the left-most sequent in the hypersequent.

Case 1.2(I is W). In this case δ has the following form:

$$\cfrac{\text{W}\ \cfrac{\cfrac{\vdots}{\Gamma, p \Rightarrow \Sigma; H}}{\Gamma, p \Rightarrow \Sigma; H; \Rightarrow} \qquad \cfrac{\vdots}{\Gamma' \Rightarrow p, \Sigma'; H'}}{\Gamma\Gamma' \Rightarrow \Sigma\Sigma'; H' \otimes (H; \Rightarrow)}\ \text{Cut*}$$

There are two possibilities that require consideration. *Case 1.2.1* $H; \Rightarrow$ is longer than H', in this case cutting* before W and applying W will suffice.

Case 1.2.2 $H; \Rightarrow$ is less than or the same size as H'. In this case, the fact that the product of $\Rightarrow$ and $\Gamma \Rightarrow \Sigma$ is $\Gamma \Rightarrow \Sigma$, means that Cut* can be applied above W to give the desired deduction.

Case 1.3(I is TL or TR). Only the case of TL will be considered since the case of TR is similar. There are two possibilities

Case 1.3.1(p is the Main sentence in I). In this case the result is given by using TL and TR to get the desired hypersequent.

Case 1.3.2(p is not the Main sentence in I). In this case the following deduction is given:

$$\cfrac{\text{TL}\ \cfrac{\cfrac{\vdots}{\Gamma, p \Rightarrow \Sigma; H}}{\Gamma, p, \varphi \Rightarrow \Sigma; H} \qquad \cfrac{\vdots}{\Gamma' \Rightarrow p, \Sigma'; H'}}{\Gamma\Gamma', \varphi \Rightarrow \Sigma\Sigma'; H \otimes H'}\ \text{Cut*}$$

The following deduction only uses Cut* on a deduction, δ', where $d(p, \delta', \Gamma \Rightarrow \Sigma) \leqslant d(p, \delta, \Gamma \Rightarrow \Sigma)$.

$$\text{TL}\,\dfrac{\text{Cut*}\,\dfrac{\dfrac{\vdots}{\Gamma, p \Rightarrow \Sigma; H} \qquad \dfrac{\vdots}{\Gamma' \Rightarrow p, \Sigma'; H'}}{\Gamma\Gamma' \Rightarrow \Sigma\Sigma'; H \otimes H'}}{\Gamma\Gamma', \varphi \Rightarrow \Sigma\Sigma'; H \otimes H'}$$

Case 1.4 (I is L¬, R¬, L→ or R→). Each of these cases follows from cutting* above I, then applying I. The case of L→ is given as an example. Let the main sequent in I not be $\Gamma, p \Rightarrow \Sigma$, the case where it is is analogous. In this case δ has the form:

$$\text{Cut*}\,\dfrac{\text{L}\!\rightarrow\dfrac{\dfrac{\vdots}{\Gamma, p \Rightarrow \Sigma; G; \Delta \Rightarrow \varphi, \Lambda; H} \qquad \dfrac{\vdots}{\Gamma, p \Rightarrow \Sigma; G; \Delta, \psi \Rightarrow \Lambda; H}}{\Gamma, p \Rightarrow \Sigma; G; \Delta, \varphi \to \psi \Rightarrow \Lambda; H} \qquad \dfrac{\vdots}{\Gamma' \Rightarrow p, \Sigma'; H'}}{\Gamma\Gamma' \Rightarrow \Sigma\Sigma'; H' \otimes (G; \Delta, \varphi \to \psi \Rightarrow \Lambda; H)}$$

Since the any product, even or uneven, will preserve the distance of a sequent from the left, it is possible to assume, without loss of generality, that H' is longer than $G; \Delta, \varphi \to \psi \Rightarrow \Lambda; H$. In this case H' can be rewritten as $G'; \Delta' \Rightarrow \Lambda'; J'$, where G is the same length as G'. In this case let δ' be

$$\text{Cut*}\,\dfrac{\dfrac{\vdots}{\Gamma, p \Rightarrow \Sigma; G; \Delta \Rightarrow \varphi, \Lambda; H} \qquad \dfrac{\vdots}{\Gamma' \Rightarrow p, \Sigma'; G'; \Delta' \Rightarrow \Lambda'; J}}{\Gamma\Gamma' \Rightarrow \Sigma\Sigma'; [GG']; \Delta\Delta' \Rightarrow \varphi, \Lambda\Lambda'; H \otimes J}$$

and δ'' be

$$\text{Cut*}\,\dfrac{\dfrac{\vdots}{\Gamma, p \Rightarrow \Sigma; G; \Delta, \psi \Rightarrow \Lambda; H} \qquad \dfrac{\vdots}{\Gamma' \Rightarrow p, \Sigma'; G'; \Delta' \Rightarrow \Lambda'; J}}{\Gamma\Gamma' \Rightarrow \Sigma\Sigma'; [GG']; \Delta\Delta', \psi \Rightarrow \Lambda\Lambda'; H \otimes J}$$

The following deduction has cuts* only when the depth of p is less than its depth in δ.

$$\text{L}\!\rightarrow \frac{\delta' \qquad \delta''}{\Gamma\Gamma' \Rightarrow \Sigma\Sigma'; [GG']; \Delta\Delta', \varphi \rightarrow \psi \Rightarrow \Lambda\Lambda'; H \otimes J}$$

Case 1.5(I is L$\Box$). As above, this case is given by switching the Cut* with L$\Box$.

Case 1.6(I is R$\Box$). This case follows from an application of Lemma 3.6 to the deduction preceding I followed by I.

Case 2($rk(\varphi) = n > 0$). *Case 2.1*(φ is $\neg\psi$). In this case δ has the form:

$$\text{Cut*}\ \frac{\dfrac{\vdots}{G; \Gamma, \neg\psi \Rightarrow \Sigma; H} \qquad \dfrac{\vdots}{G'; \Gamma' \Rightarrow \neg\psi, \Sigma'; H'}}{[GG']; \Gamma\Gamma' \Rightarrow \Sigma\Sigma'; H \otimes H'}$$

Applying Lemma 3.1 to $G; \Gamma, \neg\psi \Rightarrow \Sigma; H$ gives a deduction of $G; \Gamma \Rightarrow \psi, \Sigma; H$. Similarly applying Lemma 3.2 to $G'; \Gamma \Rightarrow \neg\psi, \Sigma'; H'$ gives a deduction of $G; \Gamma, \psi \Rightarrow \Sigma; H'$. Applying Cut* to these two deductions gives a deduction of $[GG']; \Gamma\Gamma' \Rightarrow \Sigma\Sigma'; H \otimes H'$.

Case 2.2(φ is $\gamma \rightarrow \delta$). This case is similar to the above with the exception that Lemma 3.3 and Lemma 3.4 are used, and two cuts* are required as opposed to one.

Case 2.3(φ is $\Box\psi$). If the main sentences of the cut* are not in the left-most sequent of the end hypersequents of δ_1 and δ_2, then the result follows from an application of Lemma 3.7, TL, TR, and possibly W will give the result.

This only leaves the case where the cut* sentences are in the left-most sequent in both end hypersequents. This case requires an induction over the disarray of the final hypersequent of the deduction. It is shown that the rank of the formula that

is cut* can be reduced when that hypersequent is complete, and that cuts* can be transformed into cuts* over hypersequents of less disarray.

The base case is that the end hypersequents of δ_1 and δ_2 are of the form: $\Gamma, \Box\Gamma' \Rightarrow \Sigma \Box \Sigma'; H$, where $mr(H) = \langle\langle\langle 0, m\rangle, n\rangle$ for some m and n, and $mr(\Gamma \Box \Gamma' \Rightarrow \Sigma \Box \Sigma') = \langle 1, n\rangle$, i.e. both end hypersequents of δ_1 and δ_2 are *complete*.

The inductive case is on the sum of the disarrays of the two end hypersequents, call this *dis*. The sum of disarrays of two hypersequents is the ordered pair of the maximum of fmr of sentences not in position in either hypersequent, and number of formulas in either hypersequent that are of that fine modal rank and not in position, i.e. if G and H are hypersequents where $dis(G) = \langle\langle j, k\rangle, l\rangle$ and $dis(H) = \langle\langle m, n\rangle, o\rangle$, then the sum of their disarrays is $\langle max(\langle j, k\rangle, \langle m, n\rangle), |\{\psi : \psi \in G \cup H, \psi$ is not in position, and $fmr(\psi) = max(\langle j, k\rangle, \langle m, n\rangle)\}$.

Case 2.3.1 (Both end hypersequents are complete). If they are, then δ is

$$\text{Cut*}\ \frac{\dfrac{\vdots}{\Gamma \Box \Gamma', \Box\varphi \Rightarrow \Sigma \Box \Sigma'; H} \qquad \dfrac{\vdots}{\Delta \Box \Delta' \Rightarrow \Lambda, \Box\Lambda', \Box\varphi; H'}}{\Gamma\Delta \Box \Gamma' \Box \Delta' \Rightarrow \Sigma\Lambda \Box \Sigma' \Box \Lambda'; H \otimes H'}$$

Importantly, for any $\psi \in \Gamma \cup \Sigma \cup \Delta \cup \Lambda$, $mr(\psi) = 0$. Similarly, $mr(H) = \langle\langle 0, j\rangle, k\rangle$ and $mr(H') = \langle\langle 0, l\rangle, m\rangle$. So an application of Lemma 3.8 to $\Gamma \Box \Gamma', \Box\varphi \Rightarrow \Sigma \Box \Sigma'; H$ either gives a deduction of $\Gamma \Rightarrow \Sigma; H$, in which case the result is given by TL, TR, and possibly W, or a deduction of $\Box\Gamma', \Box\varphi \Rightarrow \Box\Sigma'; H$. Similarly, an application of Lemma 3.8 to $\Delta \Box \Delta' \Rightarrow \Lambda, \Box\Lambda', \Box\varphi; H'$ either gives a deduction of $\Delta \Rightarrow \Lambda; H'$, in which case the result is given by TL, TR, and possibly W, or a deduction of $\Box\Delta' \Rightarrow \Box\Lambda', \Box\varphi; H'$. This leaves a case where there are deductions of $\Box\Gamma', \Box\varphi \Rightarrow \Box\Sigma'; H$ and $\Box\Delta' \Rightarrow \Box\Lambda', \Box\varphi; H'$.

Applying to Lemma 3.10 to $\Box\Delta' \Rightarrow \Box\Lambda', \Box\varphi; H'$ gives a deduction of $\Rightarrow \lambda; \Box\Delta' \Rightarrow ; H'$ for some $\lambda \in \Lambda'$. If λ is not φ, then the result is given by R$\Box$, TL, TR, and possibly W. Assume that it is, i.e. there is a deduction of $\Rightarrow \varphi; \Box\Delta' \Rightarrow; H'$.

If $\Box\Sigma' \neq \varnothing$, then an application of Lemma 3.10 to $\Box\Gamma', \Box\varphi \Rightarrow \Box\Sigma'; H$ gives a deduction of $\Rightarrow \sigma; \Box\Gamma', \Box\varphi \Rightarrow; H$. Applying Lemma 3.5 to this gives a deduction of $\varphi \Rightarrow \sigma; \Box\Gamma' \Rightarrow; H$. This can be cut* with $\Rightarrow \varphi; \Box\Delta'; H'$ to get $\Rightarrow \sigma; \Box\Gamma' \Box\Delta' \Rightarrow; H \otimes H'$. Applying R$\Box$, TL and TR gives the result.

If $\Box\Sigma' = \varnothing$, then Lemma 3.9 can be applied to $\Box\Gamma', \Box\varphi \Rightarrow \Box\Sigma'; H$ to get either a deduction of $\varphi \Rightarrow; \Box\Gamma' \Rightarrow \Box\Sigma'; H$ or a deduction of $\Box\Gamma' \Rightarrow \Box\Sigma'; H$. In the latter case the result is given by TL, TR, and possibly W. In the former case, $\varphi \Rightarrow; \Box\Gamma' \Rightarrow \Box\Sigma'; H$ can be cut* with $\Rightarrow \varphi; \Box\Delta' \Rightarrow; H'$ to get a deduction of $\Rightarrow; \Box\Gamma' \Box \Delta' \Rightarrow \Box\Sigma'; H \otimes; H'$. An application of Drop, TL and TR, gives the result.

There is no case where $dis = \langle\langle 0, 0\rangle, 1\rangle$.

Case 2.3.2$(dis = \langle\langle 1, 0\rangle, 1\rangle)$. In this case there is only one sentence that is not in position, and it is of fine modal rank $\langle 1, 0\rangle$. If it is in the succedent of the sequent in which it occurs Lemma 3.7 will reduce the case to one where both hypersequents are complete. In this case applying the above will give a cut*-free deduction, an application of TR will give the result. If it is on the left suppose it is in the end hypersequent of δ_1, the other case is analogous. Let δ be

$$\text{Cut*}\;\frac{\dfrac{\vdots}{\Gamma \Box \Gamma' \Box \varphi \Rightarrow \Sigma \Box \Sigma'; G; \Pi, \Box\gamma \Rightarrow \Theta; G'} \qquad \dfrac{\vdots}{\Delta \Box \Delta' \Rightarrow \Lambda, \Box\Lambda', \Box\varphi; H'}}{\Gamma\Delta \Box \Gamma' \Box \Delta' \Rightarrow \Sigma\Lambda \Box \Sigma' \Box \Lambda'; (G; \Pi, \Box\gamma \Rightarrow \Theta; G') \otimes H'}$$

where $H = G; \Pi, \Box\gamma \Rightarrow \Theta; G'$. Lemma 3.5 can be applied to $\Gamma\Box\Gamma'\Box\psi \Rightarrow \Sigma\Box\Sigma'; G; \Pi \Rightarrow$

$\Theta; \Pi', \Box\gamma \Rightarrow \Theta'; G'$ to get a deduction of $\Gamma \Box \Gamma' \Box \psi \Rightarrow \Sigma \Box \Sigma'; G; \Pi \Rightarrow \Theta; G'$. Where $G \neq \varnothing$, γ will appear in the sequent preceeding $\Pi \Rightarrow \Theta$, where $G = \varnothing$, γ will appear in Γ. Since in either case, the hypersequent under consideration is complete, the base case will give that there is a cut*-free deduction of $\Gamma\Delta \Box \Gamma' \Box \Delta' \Rightarrow \Sigma\Lambda \Box \Sigma' \Box \Lambda'; (G; \Pi, \gamma \Rightarrow \Theta; \Pi' \Rightarrow \Theta'; G') \otimes H'$. Applying L$\Box$ gives the result.

Case 2.3.3 ($dis = \langle\langle 1, 0\rangle, j\rangle$, $dis = \langle\langle k, 0\rangle, 1\rangle$ or $dis = \langle\langle n, 0\rangle, j\rangle$). If any sentences of the form $\Box\gamma$ are in the succedent of a sequent, then an application of Lemma 3.7 followed by cut* will give a deduction with a cut* only over hypersequents of less disarray. Applying TR to the result will suffice. Otherwise, an application of Lemma 3.5 either reduce the maximum fine modal rank of a formula not in position, or reduce the number of such formulas. Cutting* over the result, and applying L$\Box$ will give the required hypersequent.

Case 2.3.4 ($dis = \langle\langle 1, m\rangle, 1\rangle$, $dis = \langle\langle 1, m\rangle, j\rangle$, or $dis = lr\langle k, m\rangle, j\rangle$). In this case either there is only one sentence of fine modal rank $\langle 1, n\rangle$ or there are multiple, let one such formula be θ. There are four cases to consider:

Case 2.3.4.1 (θ is $\neg\gamma$ on the left). In this case the end-hypersequent of δ_1 is $\Gamma, \Box\psi \Rightarrow \Sigma; G; \Delta, \neg\gamma \Rightarrow \Lambda; G'$, where $G; \Delta, \neg\gamma \Rightarrow \Lambda; G' = H$. An application of Lemma 3.1 yields a deduction of $\Gamma, \Box\psi \Rightarrow \Sigma; G; \Delta \Rightarrow \gamma, \Lambda; G'$. Cutting* yields a deduction of $\Gamma\Gamma' \Rightarrow \Sigma\Sigma'; (G; \Delta \Rightarrow \gamma, \Lambda; G') \otimes H'$. Since the cut* was over hypersequents the sum of whose fine modal ranks is less than dis, by the inner-inductive hypothesis, there is a cut*-free deduction of $\Gamma\Gamma' \Rightarrow \Sigma\Sigma'; (G; \Delta \Rightarrow \gamma, \Lambda; G') \otimes H'$. Applying L$\neg$ yields the required hypersequent.

Case 2.3.4.2 (θ is $\neg\gamma$ on the right). This case is similar to the above with the

exception that it uses Lemma 3.2 and R$\neg$.

Case 2.3.4.3 (θ is $\gamma \to \delta$ on the left). In this case applying Lemma 3.3 to the end hypersequent of δ_1 yields deductions of Lemma 3.3 yields deductions of $\Gamma \Rightarrow \Sigma; G; \Delta \Rightarrow \gamma, \Lambda; G'$ and $\Gamma \Rightarrow \Sigma; G; \Delta, \delta \Rightarrow \Lambda; G'$. Two applications of cut* to the end hypersequent of δ_2 yields deductions of $\Gamma\Gamma' \Rightarrow \Sigma\Sigma'; (G; \Delta \Rightarrow \gamma, \Lambda; G') \otimes H'$ and $\Gamma\Gamma' \Rightarrow \Sigma\Sigma'; (G; \Delta, \delta \Rightarrow \Lambda; G') \otimes H'$. Applying L$\to$ to this yields the result.

Case 2.3.4.4 (θ is $\gamma \to \delta$ on the right). This case is analogous to the above cases.

In the case where $dis = \langle\langle 1, m\rangle, 1\rangle$, the above cases lower dis by lowering the maximum fine modal rank of a formula not in position. In the case where there are multiple such formulas, this will lower the number of formulas at that fine modal rank.

□

Lemma 3.14. *Cut* is eliminable from* $h*$.

Proof. This follows immediately from Lemma 3.13 and Lemma 3.12. □

Theorem 3.6.1. *Cut is eliminable from Figure 3.2.*

Proof. Let δ be a deduction of G using only the rules of Figure 3.2. Since every application of Cut is also an application of Cut*, and all the other rules of Figure 3.2 are rules of $h*$, this is also a deduction in $h*$. By Lemma 3.14 there is a cut*-free deduction of G. Since this deduction does not use Cut* it is also a deduction using only rules of Figure 3.2 and not using the Cut rule. □

Definition 20 (Subformula Property)**.** A deductive system has the *subformula property* if when δ is a proof of G, then any formula in any sequent in a hypersequent in

δ is a subformula of a formula in a sequent of G

Lemma 3.15. *Cut-Free Hypersequent System D has the Subformula Property.*

Proof. This is proved by induction on the length of deductions. The base case is an instance of Ax in which only two atomic sentences appear. So every formula appearing in δ is a subformula of the instance of Ax in question. For the inductive case let δ be given, and I be the last inference in δ. For brevity only select cases will be considered.

Case 1 (I is W). In this case δ is

$$\mathrm{I}\,\frac{\dfrac{\vdots}{G}}{G; \Rightarrow}$$

By IH any formula in any sequent of $\delta' = \delta/I$ is a subformula of a formula in a sequent of G. Since I does not remove any formulas, this property holds of $G; \Rightarrow$

Case 2 (I is Drop). In this case δ is

$$\mathrm{I}\,\frac{\dfrac{\vdots}{\Rightarrow; G}}{G}$$

As above, the deduction preceding I has the property in questions. Since no formulas occur in $\Rightarrow$, this property holds of G and δ.

Case 3 (I is TL, TR, L¬, R¬, L→, R→, L□, or R□). In these cases formulas are only added, either not appearing in the sequent before, as in TL, or TR, or whose subformulas appear in the previous sequent. □

Corollary 2. *For any deduction $\delta \vdash_h G$, there is a deduction $\delta' \vdash_h G$, such that any formula in any sequent in a hypersequent in δ is subformula of a formula in a sequent of G.*

Proof. This follows from Theorem 3.6.1 and Lemma 3.15. □

3.7 Uniqueness

The previous section establishes that the calculus of Figure 3.2 meets the requirement on a theory of meaning that the Cut rule be eliminable. This section establishes the result that that calculus meets the uniqueness constraint also. These two results show that the calculus of Figure 3.2 can support a viable inferentialist account of the meaning of deontic vocabulary.

Let Hypersequent System D′ be Hypersequent System D extended by the rules

$$\text{L}\boxdot\ \frac{G; (\Delta, \varphi \Rightarrow \Lambda); (\Gamma, \varphi \Rightarrow \psi, \Sigma); H}{G; (\Delta \Rightarrow \Lambda); (\Gamma, \boxdot\varphi \Rightarrow \Sigma); H} \qquad\qquad \text{R}\boxdot\ \frac{(\Rightarrow \varphi); (\Delta \Rightarrow \Lambda); H}{(\Delta \Rightarrow \boxdot\varphi, \Lambda); H}$$

Let $\vdash_{h'} G$ indicate that the hypersequent G is provable in Hypersequent System D′.

Theorem 3.7.1. *The following two facts hold:*

1. $\vdash_{h'} G; \Gamma, \Box\varphi \Rightarrow \Sigma; H$ *iff* $\vdash_{h'} G; \Gamma, \boxdot\varphi \Rightarrow \Sigma; H$

2. $\vdash_{h'} G; \Gamma \Rightarrow \Box\varphi, \Sigma; H$ *iff* $\vdash_{h'} G; \Gamma \Rightarrow \boxdot\varphi, \Sigma; H$

Proof. Case 1(1). Suppose that $\delta \vdash_{h'} G; \Gamma \Box \varphi \Rightarrow \Sigma; H$. Either $G = \varnothing$ or not. If $G = \varnothing$, as a consequence of Theorem 3.4.1 $\vdash_{h'} \Box\varphi \Rightarrow \boxdot\varphi$. Several applications of W followed by several of TL or TR yields that $\vdash_{h'} \Gamma, \Box\varphi \Rightarrow \boxdot\varphi, \Sigma; H$. Cutting this hypersequent with the end-hypersequent of δ yields that $\vdash_{h'} \Gamma \Rightarrow \boxdot, \Sigma; H$. If, on the

other hand, $G \neq \varnothing$ the result follows from Lemma 3.7 and TR. The lemmas used here hold trivially for Hypersequent System D′.

The left to right direction is analogous.

Case 2(2). Suppose that $\delta \vdash_{h'} G; \Gamma \Box \varphi \Rightarrow \Sigma; H$. Again, either $G = \varnothing$ or $G \neq \varnothing$. In the first case, it follows from Theorem 3.4.1, W, TL, and TR that $\vdash_{h'} \Gamma, \boxdot\varphi \Rightarrow \Box\varphi, \Sigma; H$. Cutting this with the end hypersequent of δ yields the result. If $G \neq \varnothing$, let $G = D; \Delta \Rightarrow \Lambda$. By Lemma 3.5, $\vdash_{h'} D; \Delta, \varphi \Rightarrow \Lambda; \Gamma \Rightarrow \Sigma; H$. By L$\boxdot$, $\vdash_{h'} G; \Gamma, \boxdot\varphi \Rightarrow \Sigma; H$.

The proof of the other direction is analogous. □

3.8 Equivalence To Sequent System D

Theorem 3.8.1. *If* $\vdash_s \Gamma \Rightarrow \Sigma$ *then* $\vdash_h \Gamma \Rightarrow \Sigma$*.*

Proof. This is proved by induction on the length of deductions. The base case is simply an instance of Axiom. But that all the instances of Axiom are derivable is given by Corollary 1. For the inductive cases let δ be a derivation whose last inference is I.

Case 1(I is WL or WR). In this case IH and WL or WR give the result.

Case 2(I is L¬, R¬, L→, or R→). As above these cases will follow from IH and an application of the corresponding rule.

Case 3(I is k). In this case δ has the following form:

$$\mathrm{k}\,\dfrac{\dfrac{\vdots}{\Gamma \Rightarrow \varphi}}{\Box\Gamma \Rightarrow \Box\varphi}$$

Let the cardinality of Γ be n. With the help of IH, the following deduction can be constructed:

$$\dfrac{\dfrac{\dfrac{\dfrac{\text{IH}}{\Gamma \Rightarrow \varphi}}{\Gamma \Rightarrow \varphi; \Rightarrow}\ \text{W}}{\Rightarrow \varphi; \Box\Gamma \Rightarrow}\ \text{L}\Box \times n}{\Box\Gamma \Rightarrow \Box\varphi}\ \text{R}\Box$$

Case 4 (I is d). In this case δ has the following form:

$$\dfrac{\dfrac{\vdots}{\Gamma \Rightarrow}}{\Box\Gamma \Rightarrow}\ \text{d}$$

Again, for convenience let the cardinality of Γ be n.

$$\dfrac{\dfrac{\dfrac{\dfrac{\text{IH}}{\Gamma \Rightarrow}}{\Gamma \Rightarrow; \Rightarrow}\ \text{W}}{\Rightarrow; \Box\Gamma \Rightarrow}\ \text{L}\Box \times n}{\Box\Gamma \Rightarrow}\ \text{Drop}$$

□

Lemma 3.16. *If* $\vdash_h^{cf} \Gamma \Rightarrow \Sigma$ *then* $\vdash_s \Gamma \Rightarrow \Sigma$

Proof. This proof proceeds by induction on the length of deductions in the hypersequent system D. The base case is a deduction consisting solely of Ax, but any instance of Ax is also an instance of Axiom from the sequent system D. For the inductive case let δ be given and I be the last inference in δ.

Case 1 (I is Drop). In this case δ has the following form:

$$\dfrac{\dfrac{\vdots}{\Rightarrow; \Gamma \Rightarrow \Sigma}}{\Gamma \Rightarrow \Sigma}\ \text{Drop}$$

Let the deduction immediately preceding I be δ'. This case requires a sub-induction over the modal rank of $\Gamma \Rightarrow \Sigma$.

Case 1.1 (Base Case). For the base case let $mr(\Gamma \Rightarrow \Sigma) = \langle 0, 0\rangle$. It follows from Lemma 3.6 that $\vdash_h \Rightarrow ; \Rightarrow$, but by Corollary 2 this is impossible.

Case 1.2 ($mr(\Gamma \Rightarrow \Sigma) = \langle 1, 1\rangle$). In this case there is one formula of modal rank 1, call it $\Box\varphi$. If $\Box\varphi \in \Sigma$, by Lemma 3.7 and Lemma 3.6, there is a deduction of $\Rightarrow ; \Rightarrow$, but this is impossible. If, on the other hand $\Box\varphi \in \Gamma$ then there by Lemma 3.5, there is a deduction $\delta' \vdash_h^{cf} \varphi \Rightarrow ; \Gamma/\{\varphi\} \Rightarrow \Sigma$, where $l(\delta') < l(\delta)$. By Lemma 3.6, there is a deduction, $\delta'' \vdash_h^{cf} \varphi \Rightarrow ; \Rightarrow$, where $l(\delta'') \leqslant l(\delta')$. Finally, by Lemma 3.11, there is a deduction $\delta''' \vdash_h^{cf} \varphi \Rightarrow$, where $l(\delta''') \leqslant l(\delta'')$. IH can be applied to this deduction to get a deduction, $\delta_1 \vdash_s \varphi \Rightarrow$. An application of d, WL, and WR, then give the result.

The inner inductive hypothesis, IH_1 is that if $\delta_a \vdash_h^{cf} \Rightarrow ; \Gamma \Rightarrow \Sigma$ then there is a δ_b such that $\delta_b \vdash_s \Gamma \Rightarrow \Sigma$, where $l(\delta_b) \leqslant l(\delta_a)$.

Case 1.3 ($mr(\Gamma \Rightarrow \Sigma) = \langle 1, n\rangle$).

In this case there are multiple sentences of modal rank 1 in $\Gamma \Rightarrow \Sigma$. If any are on the right, then an application of Lemma 3.7 yields a deduction $\delta' \vdash_h^{cf} \Rightarrow ; \Gamma \Rightarrow \Sigma/\{\Box\varphi\}$. Since this is of modal rank less than $\Gamma \Rightarrow \Sigma$, an application of IH_1 yields that $\vdash_s \Gamma \Rightarrow \Sigma/\{\Box\varphi\}$. Applying WR yields the result.

If for each formula of modal rank 1 is in Γ, then applying Lemma 3.5 n-many times yields a deduction, δ' such that $\delta_h'^{cf} \Gamma' \Rightarrow ; \Gamma/\Box\Gamma' \Rightarrow \Sigma$, where $l(\delta') < l(\delta)$. An application of Lemma 3.6 yields that there is a δ'' such that $\delta'' \vdash_h^{cf} \Gamma' \Rightarrow ; \Rightarrow$. Finally, an applicatoin of Lemma 3.11 yields a deduction, δ''', such that $\delta''' \vdash_h^{cf} \Gamma' \Rightarrow$, where $l(\delta''') < l(\delta)$. Applying IH yields that $\vdash_s \Gamma' \Rightarrow$. Application of d, WL, and WR

yields the required sequent.

Case 1.4 ($mr(\Gamma \Rightarrow \Sigma) = \langle n, 1\rangle$). In this case there is a sentence, φ, in $\Gamma \cup \Sigma$, such that $mr(\varphi) = n$. There are four cases to consider

Case 1.4.1 (φ is $\psi \to \theta \in \Gamma$). In this case an application of Lemma 3.3 gives deduction δ' and δ'' such that $\delta' \vdash_h^{cf} \Rightarrow; \Gamma, \theta \Rightarrow \Sigma$, and $\delta'' \vdash_h^{cf} \Gamma \Rightarrow \psi, \Sigma$, where $l(\delta'), l(\delta'') < l(\delta)$. By IH_1, there are deductions δ'_a and δ'_b such that $\delta'_a \vdash_s \Gamma, \theta \Rightarrow \Sigma$, and $\delta''_a \Gamma \Rightarrow \psi, \Sigma$. An application of L$\to$ to these yields the required sequent.

Case 1.4.2 (φ is $\psi \to \theta \in \Sigma$). In this case an application of Lemma 3.4, gives a deduction to which IH_1 can be applied. Applying R$\to$ to the result, gives that $\vdash_s \Gamma \Rightarrow \Sigma$.

Case 1.4.3 (φ is $\neg\psi \in \Gamma$). An application of Lemma 3.1 gives a hypersequent to which IH_1 can be applied. Applying L$\neg$ to that yields the required sequent.

Case 1.4.4 (φ is $\neg\psi \in \Sigma$). Applying Lemma 3.2 to the hypersequent preceeding Drop gives a hypersequent to which IH_1 can be applied. Applying R$\neg$ to that yields the required sequent.

Case 1.5 ($mr(\Gamma \Rightarrow \Sigma) = \langle m, n\rangle$). This case is similar to the above case. Select one of the formulas of modal rank m. Apply the same technique as cases 1.4.1–1.4.4 depending on the formula and its location in $\Gamma \Rightarrow \Sigma$. This will reduce n. Applying IH_1 and the relevant rule of Figure 3.1 will yield the required sequent.

Case 2 (I is TL or TR] This case follows from IH and WL or WR respectively.

Case 3 (I is L$\neg$, R$\neg$, L$\to$, R$\to$] These cases also follow from IH and the corresponding inference in the sequent presentation of D.

Case 4 (I is L$\Box$] I cannot be L$\Box$ in this situation.

Case 5(I is R$\Box$] In this case δ has the following form:

$$\mathrm{R}\Box\,\dfrac{\dfrac{\vdots}{\Rightarrow \varphi; \Gamma \Rightarrow \Sigma}}{\Gamma \Rightarrow \Box\varphi; \Sigma}$$

This case again requires a sub-induction on the modal rank of $\Gamma \Rightarrow \Sigma$. Let δ be the deduction that precedes I.

Case 5.1($mr(\Gamma \Rightarrow \Sigma) = \langle 0, n\rangle$). In this case it follows from Lemma 3.6 that for some δ', $\delta' \vdash_h^{cf} \Rightarrow \varphi; \Rightarrow$ where $l(\delta') < l(\delta)$. From Lemma 3.11 it follows that there is a δ'' that $\delta'' \vdash_h^{cf} \Rightarrow \varphi$, where $l(\delta'') < l(\delta')$. From IH it follows that $\vdash_s \Rightarrow \varphi$. Applying k to this gives $\vdash_s \Rightarrow \Box\varphi$. The result then follows from WL and WR.

Case 5.2($mr(\Gamma \Rightarrow \Sigma) = \langle 1, 1\rangle$). In this case there is a formula $\Box\psi \in \Gamma \cup \Sigma$.

If $\Box\psi \in \Sigma$, an application of Lemma 3.7 gives a deduction δ' of $\Rightarrow \varphi; \Gamma \Rightarrow \Sigma/\{\Box\psi\}$, where $l(\delta') < l(\delta)$. Applying Lemma 3.6 and Lemma 3.11 gives a deduction δ'' of $\Rightarrow \varphi$, where $l(\delta'') < l(\delta)$. Applying IH, k, WL, and WR yields the required sequent.

If $\Box\psi \in \Gamma$, then an application of Lemma 3.5 gives a deduction δ' of $\psi \Rightarrow \varphi; \Gamma/\{\Box\psi\} \Rightarrow \Sigma$, where $l(\delta') < l(\delta)$. An application of Lemma 3.6 and Lemma 3.11 gives a deduction δ'' of $\psi \Rightarrow \varphi$, where $l(\delta'') \leqslant l(\delta)$. An application of IH followed by k, WL, and WR yields the required sequent.

Case 5.3($mr(\Gamma \Rightarrow \Sigma) = \langle 1, n\rangle$). In this case applications of Lemma 3.7 and Lemma 3.5, will give a deduction, δ', of $\Gamma' \Rightarrow \varphi; \Gamma/\Box\Gamma' \Rightarrow \Sigma/\{\Box\varphi : \Box\varphi \in \Sigma\}$, where $l(\delta') < l(\delta)$. Applying Lemma 3.6 and IH give that $\vdash_s \Gamma' \Rightarrow \varphi$. An application of k followed by WL and WR give the result.

The inner inductive hypothesis, IH_1 is that if $\Rightarrow \varphi; \Gamma \Rightarrow \Sigma$, where $mr(\Gamma \Rightarrow \Sigma)$ is reduced, then $\vdash_s \Gamma \Rightarrow \Box\varphi, \Sigma)$.

Case 5.4 ($mr(\Gamma \Rightarrow \Sigma) = \langle n, 1\rangle$). In this case there is some one formula, γ, of modal rank n. There are four sub-cases to be considered.

Case 5.4.1 (γ is $\psi \to \theta \in \Gamma$). In this case an application of Lemma 3.3, gives deductions δ' and δ'' of $\Rightarrow \varphi; \Gamma, \theta \Rightarrow \Sigma$ and $\Rightarrow \varphi; \Gamma, \psi \Rightarrow \Sigma$. Applying IH_1 to both gives deductions in sequent system D of $\Gamma, \theta \Rightarrow \Box\varphi, \Sigma$ and $\Gamma \Rightarrow \psi, \Box\varphi, \Sigma$. An application of L$\to$ gives the result.

Case 5.4.2 (γ is $\psi \to \theta \in \Sigma$). In this case an application of Lemma 3.4 yields a deduction to which IH_1 can be applied. Applying R$\to$ to the sequent that results finishes this case.

Case 5.4.3 (γ is $\neg\psi \in \Gamma$). In this case an application of Lemma 3.1 gives a deduction to which IH_1 can be applied. Applying L$\neg$ to that deduction gives the result.

Case 5.4.4 (γ is $\neg\psi \in \Sigma$). In this case an application of Lemma 3.2 yields a deduction to which IH_1 can be applied. Applying R$\neg$ to that deduction gives the result.

Case 5.4.5 ($mr(\Gamma \Rightarrow \Sigma) = \langle n, m\rangle$). In this case a formula whose modal rank is n is either $\neg\gamma$ or $\gamma \to \delta$, and is either in Γ or Σ. Applying the same process as described in 5.4.1–5.4.4 depending on which formula is selected will give the result.

□

Theorem 3.8.2. *If* $\vdash_h \Gamma \Rightarrow \Sigma$ *then* $\vdash_s \Gamma \Rightarrow \Sigma$

Proof. This follows immediately from Theorem 3.6.1 and Lemma 3.16. □

Theorem 3.8.1 and Theorem 3.8.2 establish that the calculus of Figure 3.2 captures System D. Theorem 3.6.1 and Theorem 3.7.1 establish that that calculus meets

the uniqueness constraint and is Cut eliminable. It can, therefore, be used to underwrite a theory of the meaning of an operator that behaves as the $\Box$ of System D does. Given the requirements on an inferentialist theory of meaning given in Section 3.1, Figure 3.2 can offer an inferentialist treatment of at least one modal concept.

Figure 3.2: Hypersequent System D

STRUCTURAL RULES

$$\text{Ax}\,\frac{}{p \Rightarrow p}$$

$$\text{Drop}\,\frac{(\Rightarrow); G}{G} \qquad \text{W}\,\frac{G}{G; (\Rightarrow)}$$

$$\text{TL}\,\frac{G; (\Gamma \Rightarrow \Sigma); H}{G; (\Gamma, \varphi \Rightarrow \Sigma); H} \qquad \text{TR}\,\frac{G; (\Gamma \Rightarrow \Sigma); H}{G; (\Gamma \Rightarrow \varphi, \Sigma); H}$$

$$\text{Cut}\,\frac{G; (\Gamma, \varphi \Rightarrow \Sigma); H \qquad G; (\Gamma \Rightarrow \varphi, \Sigma); H}{G; (\Gamma \Rightarrow \Sigma); H}$$

OPERATIONAL RULES

$$\text{L}\neg\,\frac{G; (\Gamma \Rightarrow \varphi, \Sigma); H}{G; (\Gamma, \neg\varphi \Rightarrow \Sigma); H} \qquad \text{R}\neg\,\frac{G; (\Gamma, \varphi \Rightarrow \Sigma); H}{G; (\Gamma \Rightarrow \neg\varphi, \Sigma); H}$$

$$\text{L}\!\to\,\frac{G; (\Gamma \Rightarrow \varphi, \Sigma); H \qquad G; (\Gamma, \psi \Rightarrow \Sigma); H}{G; (\Gamma, \varphi \to \psi \Rightarrow \Sigma); H} \qquad \text{R}\!\to\,\frac{G; (\Gamma, \varphi \Rightarrow \psi, \Sigma); H}{G; (\Gamma \Rightarrow \varphi \to \psi, \Sigma); H}$$

$$\text{L}\Box\,\frac{G; (\Gamma, \varphi \Rightarrow \Sigma); (\Delta \Rightarrow \Lambda); H}{G; (\Gamma \Rightarrow \Sigma); (\Delta, \Box\varphi \Rightarrow \Lambda); H} \qquad \text{R}\Box\,\frac{(\Rightarrow \varphi); (\Delta \Rightarrow \Lambda); H}{(\Delta \Rightarrow \Box\varphi, \Lambda); H}$$

Chapter 4

Sellars, Second-Order Quantification, and Ontological Commitment

Abstract. Sellars [63, 67] argues that the truth of a second-order sentence, e.g. $\exists f fa$, does not incur commitment to there being any sort of abstract entity. This chapter begins by exploring the arguments that Sellars offers for the above claim. It then develops those arguments by pointing out places where Sellars has been unclear or ought to have said more. In particular, Sellars's arguments rely on there being a means by which language users could come to understand sentences of a second-order language wherein the truth of sentences of the form $\exists f fa$ do not require there to be abstract entities. In addition to this, as Sellars [67] notes, a formal account of quantification is required that does not make use of the apparatus of sequences.

Both a translation of $\exists f fa$ and a formal account of quantification are provided by this chapter.

Keywords. Ontological Commitment, Wilfrid Sellars, Second-Order Quantification, W. V. Quine

In “On What There Is”, Quine [47] argues that the ontological commitments of a theory are laid bare by first formalizing that theory and then examining what entities must be in the range of that theory’s bound variables. This account of ontological commitment places special emphasis on quantifiers and the variables bound by them. Quine famously stated his view as “To be assumed as an entity is, purely and simply, to be reckoned as the value of a variable”. This account of ontological commitment entails that any language with second-order quantifiers will require there to be entities which can be the values of the second-order variables of quantification. Twelve years after the publication of “On What There Is”, Sellars published “Grammar and Existence: A Preface to Ontology”. In that paper he argues that a theory making use of second-order quantifiers need not be committed to any entities which are the values of its second-order variables. In “Naturalism and Ontology”, he elaborates on this position by offering further arguments in support of it. This chapter explores Sellars’s arguments from those two works and develops novel arguments for the same conclusions.

The first half of this chapter explores Sellars’s arguments that second-order quantification does not, of itself, incur ontological commitments to abstract entities. There are two main arguments to this effect. One argument attempts to cast doubt on the claim that the only natural language translation of the formal sentence, “$\exists f fa$”, is one that clearly incurs commitment to some abstract entity or other. The second argument examines the standard model-theoretic definition of the consequence relation —which Sellars clearly ascribes to Quine —and argues that if it is adopted second-order variables must be assigned entities as values. This offers a vindication of a

Quinean position that also adopts the standard model-theoretic account of the consequence relation. The second half of the chapter addresses the weaknesses in Sellars's arguments. It offers an alternative translation of the formal sentence, "$\exists f f a$", into English and provides a proof-theoretic alternative to the standard model-theoretic account of the consequence relation. It is argued that Sellars [63, 67] could and may have endorsed the alternative proof-theoretic account of the consequence relation. The chapter concludes by showing that the proof-theoretic alternative does not require entities to be assigned as the values of second-order variables of quantification.

4.1 A Quinean Argument

This chapter is concerned with arguments offered by Sellars. It is clear in "Naturalism and Ontology" that he takes his disputant to be Quine. This does not entail that Sellars has correctly interpreted Quine's position. In what follows, Sellars arguments are said to be leveled against a "Quinean" view. This Quinean view may not have been Quine's own in detail, but it is the view that Sellars argues against in "Grammar and Existence" and "Naturalism and Ontology" and it is faithful to Quine's views in broad strokes.

Sellars [67] makes use of the term 'reference'. This is unfortunate because that term has been used variously by philosophers and it is unclear how Sellars's use of it there fits with his other work. 'Reference' is also unfortunately used by Quine [47] in the expression 'range of reference', while Sellars [67] talks of 'determinate reference' and 'indeterminate reference'. Whether there is a unified account of the meaning

of that term is set aside. To make the argument under consideration clearer, new terminology is introduced. The term ‘supposition’ describes the relation that an expression must have to the world when that expression occurs in a true sentence.[1] For instance, if the sentence ‘Helen is happy’ is true, then ‘Helen’ must supposit for Helen. Similarly, if the sentence ‘Some donkey is running’ is true, then the expression ‘Some’ must supposit for some objects.[2]

Call the following Quinean[3] argument the Argument from Abstracta:

1. Commitment to the truth of a sentence containing an expressions that supposits for an entity entails an ontological commitment to that entity.—supposition entails ontological commitment.

2. For any quantifier, if it is the main operator in a true sentence, it supposits for something.

3. Second-order quantifiers could only supposit for properties, attributes, classes, etc.

4. Properties, attributes, classes, etc. do not exist.

(C) Therefore, introducing second-order quantification into a language incurs commitment to entities that do not exist.

[1]The medieval term is used both because it is long enough out of use that it can be re-purposed for this chapter, and it may offer clarity to anyone familiar with the term to grasp the concept aimed at by its use.

[2]Medieval philosophers also made a distinction between determinate and indeterminate supposition. It is crucial to note that these notions of ‘determinate’ and ‘indeterminate’ have nothing to do with the notions discussed below. ‘Supposition’ is a term used only to indicate a relation between expressions in true sentences and entities in the world.

[3]It is emphasized that this argument may not be the one actually given by Quine, but it is one to which Sellars responds.

Suppose that second-order quantifiers were introduced into a language. In such a language '$\exists f fa$' is a logical truth, for any term 'a'. By premise (2), the quantifier $\exists f$ supposits in that sentence. By premise (3) that entity is a property, attribute, class, or other sort of abstract entity. Since any theory is committed to the logical truths, by premise (1), every theory is committed to there being some property, attribute, class or other abstract entity. By premise (4), those do not exist. So any theory in a second-order language is committed to there being entities that do not exist.

Premise one is designed to be true. Its truth is built into the notion of supposition that was introduced above. Since the sort of entity that is assigned to first-order variables is the sort of entity that names stand for, the sort of entity that is assigned to a second-order variable is the sort of entity that predicates stand for. Sellars is committed to premise (4) by his nominalism.[4] [5]

Premise (3) is worth an aside. Boolos [8] argued that second-order quantifiers do supposit but do so multiply. They supposit for the same objects as first-order quantifiers, but do so plurally. As he says in "To Be is to Be the Value of a Variable (or some Values of Some Variables)", "There are, rather, two (at least) different ways of referring to the same things" [8, pg. 449]. The purpose of this chapter is not to

[4]There are nuances of Sellars's view of abstract objects that are set to the side for the purposes of this chapter. Sellars [64, 67] tries to preserve the grammar of sentences involving apparent reference to abstract entities so that it turns out that it is true that 'There are properties'. But he also acknowledges that there *really* are no abstract entities, as he says in "Naturalism and Ontology" "I shall, however, as you might expect, go on to argue that although there are attributes, there *really* are no attributes."[emphasis in original] [67, pg. 41]

[5]The conclusion of this argument locates the problem of second-order quantification. There are interpretations of Quine [48] according to which second-order quantifiers are illegitimate for use in theorizing because they mask the true set-theoretic nature of second-order quantification. Quine [48] calls second-order logic "set theory in sheep's clothing". Whether this argument can be sustained is a contentious issue. For arguments that it cannot, see Boolos [6],and Shapiro [69]. This argument is set aside for the purposes of this chapter

refute Boolos [8], but to offer an alternative Sellarsian justification of second-order logic. Premise (3) is taken for granted in what follows.

This leaves only premise (2) to be denied. Premise (2) follows from the Quinean claim introduced above that variables of quantification are assigned entities when assessing the ontological commitments of a theory. Call the set of entities so assigned to a variable the range of that variable. Second-order quantifiers bind second-order variables and so if a second-order quantifier is the main operator of a sentence, there must be some entity assigned to the variable it binds. If premise (2) is false, then the Quinean idea that the ontological commitments of a theory are laid bare by examining the entities in the ranges of variables of that theory must also be false.

It is easier to make explicit what it would take for a sentence of a formal language to be true than to make explicit what it would take for a sentence of English to be true. This chapter examines two ways of doing this, a model-theoretic method and a proof-theoretic method. If formal methods are to offer clarification in metaphysical or other discussion, there must be a way of translating discourse in those areas into the formal language and vice versa. Since the truth conditions of a set of sentences of a formal language are explicit, the translation from the formal language into natural language provides truth conditions for the translated sentences of natural language. Similarly, if S is a sentence of natural language that does not obviously carry ontological commitment when accepted, and S is an adequate translation of the sentence S' of a formal language, then this provides reason to doubt that S' carries ontological commitment when accepted. The process of coordinating the two languages will ultimately lay bare the ontological commitments of both languages.

The goal of section 4.2.1 is to suggest that there is a natural language translation of the formal sentence '$\exists f f a$' that does not incur ontological commitment to there being some abstract entity, except possibly what is named by 'a'. Section 4.4 provides a translation of '$\exists f f a$' and offers an explanation of how other sentences with second-order quantifiers may come to be understood. Even if this is successful, if there is no available account of the commitments of the formal sentence '$\exists f f a$' that rules out that the second-order quantifier supposits then the above translation cannot be correct. Section 4.2.2 shows that if the model-theoretic account of the consequence relation is accepted, commitment to the truth of '$\exists f f a$' results in commitment to an entity in to be the value of the varable 'f'. Section 4.3 provides an alternative account of the consequence relation and shows that that account does not require that second-order quantifiers supposit. The conclusion of this chapter is that there is a coherent account of second-order quantification on which philosophers need not be hesitant to make use of them.

Quine's ([47, 48]) official position is that variables are the expressions of a first-order quantified sentence that supposit. Sellars [67], on the other hand, holds that quantifiers are the expressions of a first-order quantified sentence that supposit. When there is no loss of clarity this distinction between the two philosophers is ignored. Sellars [67] discusses the difference between himself and Quine as being over how the truth conditions of quantified sentences of a formal language are to be determined.

Sellars [63] attempts to cast doubt on Premise (2) of the Argument from Abstracta by fixing what sentential forms of natural language carry ontological commitment

when accepted. He then suggests that the proper translation of '$\exists f fa$' is neither of one of those forms nor commits one to a sentence of one of those forms. If a translation of '$\exists f fa$' can be found which is neither of an ontologically committing form nor entails commitment to a sentence with such a form, then the sentence '$\exists f fa$' does not carry ontological commitment in virtue of the quantifier '$\exists f$'. Under these circumstances, by Premise (1) of the Argument from Abstracta, the quantifier '$\exists f$' does not supposit in '$\exists f fa$'. From this it follows that premise (2) of the Argument from Abstracta is false.

Sellars points out that the paradigm case of an ontologically committing sentence is 'There is a K' and the equivalent sentence 'There are Ks'. This is because they are the most straightforward answer to the question 'What is there?', the central question of ontology. Commitment to the truth of such sentences results in commitment to there being Ks. Any sentence of this form, or that entails a sentence of this form, is ontologically committing. These forms are taken from [63, pg. 235]

> . . . among the forms by the use of which one most clearly and explicitly asserts the existence of objects of a certain sort . . . are the forms "There is an N", "Something is an N", and "There are N's."

4.2 Not All Quantifiers Supposit

4.2.1 The First Argument

For convenience call the sentences that have one of the above forms ontologically committing. The question under discussion then is whether the formal sentence

'$\exists f f a$', when translated into natural language, is or entails an ontologically committing sentence. In order to answer, in the first case, that the sentence '$\exists f f a$' is not itself of an ontologically committing form, Sellars reasons in analogy with the accepted translation of first-order sentences.

Consider the pair of sentences

(A) $\exists x x$ is warm.

(B) $\exists f$ S is f.

A standard translation of sentence (A) is 'There is an x such that x is warm'. This, however, cannot be an adequate translation. The first occurrence of 'x' is a common noun, as 'donkey' in 'Fred is a donkey' is. It must be a common noun because it is preceded by the indefinite article 'a'. But the second occurrence of 'x' is a singular term, the sort of grammatical item that names an entity, as 'Fred' is in 'Fred is a frog'. No expression used univocally can serve both purposes. Grammatically, that translation of (A) has the same form as either 'There is an apple such that apple is warm' or 'There is a Fred such that Fred is warm'. Neither sentence is grammatically correct. A similar issue arises if the same technique is used to translate (B). The analogous translation is 'There is an f such that S is f'. The first occurrence of 'f' is again a common noun such as 'apple', but the second is a predicate adjective such as 'red', 'cold', or 'loud'. That 'f' is not being treated univocally is shown by the sentences substituting a common noun and predicate adjective for 'f': 'There is an apple such that S is apple', and 'There is a red such that S is red'. Since neither of these sentences is grammatical, the original translation of (B) is not grammatical.

The most immediate remedy is to introduce a Carnapian ([11]) universal word, or what Sellars calls a category word, such as 'thing'. (A) could then be translated as 'There is a thing, x, such that x is warm'. 'x' is being treated univocally in both positions of that sentence. It is a singular term in both. The analogous translation of (B) would then be 'There is a property, f, such that S is f'. Importantly, the first occurrence of 'f' renames the property that is supposited for by the quantifier —Sellars has granted that 'There is a property' is of an ontologically committing form —whereas the second occurrence of 'f' is an adjective. If this sentence were grammatical the sentence, 'There is a property, loud, such that S is loud' would have to be. The proper names of properties are generally words with '-ness', '-ity', and '-hood' suffixes. The properly grammatical sentence is 'There is a property, loudness, such that S is loud'.

If Quine [48] thought that the correct translation of (A) was 'There is a thing, x, such that x is warm', and by analogy concluded that the correct translation of any existentially quantified sentence required the introduction of a Carnapian universal word, followed by a name for an entity of that category, then it is easy to make sense of arguments he gives in *Philosophy of Logic* ([48]).

> Consider first some ordinary quantifications: '$\exists x$ (x walks)', '$\forall x$ (x walks)', '$\exists x$ (x is prime)'. The open sentence after the quantifiers shows 'x' in a position where a name could stand; ... To put the predicate letter 'F' in a quantifier, then, is to treat predicate positions suddenly as name positions, and hence to treat predicates as names of entities of some sort.

The second sentence of this passage is an indication that Quine takes the correct

translation of 'x is warm' to require 'x' to be a name. When 'x' occurs "in a quantifier", it must also be treated as a name to preserve grammaticality. Since $\exists x$ and $\exists F$ are both quantifiers, 'F' must have the same grammatical category as 'x'. The only grammatical treatment of '$\exists F$' is to treat 'F' as a name, and so the only translation of '$\exists f f a$' is "There is a property, f-ness, such that S is f".

Boolos [6] criticizes Quine's argument here on the grounds that he does not consider the analogous argument:

> Consider some extraordinary quantification: '$(\exists F)$(Aristotle F)' ... the open sentence after the quantifier shows 'F' in a position where a predicate could stand ... to put the variable 'x' in a quantifier, then is to treat name positions suddenly as predicate positions, and hence to treat names as predicates with extensions of some sort.

If Quine's reasons for making the above argument were that he could only see how to translate the sentence '$\exists f$ S is f' as 'There is a property, f-ness, such that S is f', then Boolos's argument is off the mark. The open sentence 'Aristotle is F' does have F in a place where a predicate could stand, but the quantifier '$\exists F$' , and for similar reasons '$\forall F$', cannot. It must have F in a place where a *name* could go. If the above translation is the only one available, that phrase must be translated as 'There is a property F-ness ... '

If, contrary to the above assumption, another translation of '$\exists f$ S is f' is available then it may be that second-order quantifiers do not suddenly treat predicate positions as name positions. Sellars attempts to provide such a translation of (A), one that is not *itself* of an ontologically committing form, though in the first-order case it

entails a sentence of such a form. Sellars offers ‘Something is warm’ as a translation of (A). This sentence is not among the forms listed above that were ontologically committing. However, it entails a sentence which *is* of an ontologically committing form, ‘There are warm things’. This explains the potentially perplexing remark he makes in “Grammar and Existence”

> ...not even quantification over singular term variables of type 0 makes, *as such*, an existence commitment involving an ontological category, *i.e. says* ‘There are particulars’[emphasis in original]

‘Something is warm’ does not differ ‘only graphologically’ from ‘There is a warm thing’. They are two importantly distinct sentences of English. The two sentences, though of different form, are logically equivalent. Commitment to the difference being merely the way the sentences are written or said may tempt one to think that all uses of the word ‘something’ can be exchanged for the ontologically committing ‘There is a ...thing’. This deprives formal language from the resources to merely generalize without a corresponding ontological commitment. It sets the role of quantifiers as expressions which can supposit in the forefront. This chapter suggests that the generalizing role of quantification should be emphasized. The role of a quantifier as an expression for suppositing is a secondary feature of those quantifiers that generalize over grammatical categories that are themselves used for suppositing. This is contrary to the received opinion that the role of quantifiers is primarily for suppositing and only incidentally that they are used to generalize. There are other philosophers who deny Quine’s Dictum. For instance, Azzouni [3] also denies that quantification is the main vehicle of ontological commitment. The view in this chap-

ter is importantly different from that account. Sellars, and the view espoused here, take it that true sentences of the form "There are K's" are ontologically committing. There is nothing on the view under consideration that wishes to "free ontology from its linguistic straitjacket." Wright [76] has also argued against Quine's dictum. As is suggested at the end of this chapter, the view of quantification offered here complements Wright's views nicely. Sellars [67] offers an outline of how an alternative to Quine's Dictum might be articulated. The goal of this chapter is to elaborate on and defend a Sellarsian account of quantification and ontological commitment.

The analogous way of translating (B) that Sellars first recommends is 'S is something'. If the translation of (A) as 'Something is warm' is not itself of an ontologically committing form, but only entails a sentence of that form, then there is, at first glance, reason to deny that commitment to the truth of a second-order sentence incurs commitment to there being properties, attributes, sets, etc.

A possible concern is that the sentence 'S is something' is not grammatical. The above argument relies heavily enough on finding a grammatical translation of '$\exists f fa$' that if the translation Sellars recommends is not grammatical, one might accuse Sellars of not being even-handed in his treatment of the issue. This criticism as it is currently stated is too strong. One reply is that the ungrammaticality of 'S is something' is of a different sort from the ungrammaticality of 'There is an f such that S is f'. If 'S is something' is ungrammatical this is not because it can only be understood to be treating expressions of different grammatical categories as if they were the same expression. Sellars may argue that since natural language has some flexibility to it, all that is required is a way of using natural language to teach

others the meaning of the newly introduced sentence ‘S is something’ and so to understand the meaning of ‘$\exists f f a$’. This would explain why Sellars says, “Now it is easy enough, if I may be permitted a paradox, to invent an ‘ordinary language’ equivalent of [(B)]”[63, pg. 502]. It is, however, preferable to avoid such paradoxes. A grammatical translation of ‘$\exists f f a$’ is presented below. (See section 4.4.)

A second concern is that just as ‘Something is warm’ entails a sentence of an ontologically committing form, so too might ‘S is something’. Sellars addresses this concern at length in both “Grammar and Existence” and “Naturalism and Ontology”.

The sentence ‘Something is warm’, as noted above, entails the sentence ‘There is a warm thing’. So commitment to the truth of (A) translated as ‘Something is warm’ would commit one to there being warm things. Note, however, commitment to the sentence ‘Something is warm’ would not commit one to there being warms in the way that commitment to the sentence ‘Something is a cat’ commits one to there being cats. If commitment to ‘Something is warm’ did not entail a sentence of ontologically committing form, then by premise one of the Argument from Abstracta ‘Something’ in ‘Something is warm’ would not supposit for anything.

An argument that second-order quantifiers supposit can be generated by analogy with the case of first-order quantifiers. Suppose that ‘$\exists f$ S is f’ is true and consider the following series of inferences:

1. ‘$\exists f$ S is f’ is translated as ‘S is something’.

2. Thus, there is something that S is.

3. Thus, there is a property which S is.

(C) Therefore, there are properties.

Sellars denies that the second sentence entails the third sentence in this case. While ‘There is something. . . ’ appears to be an ontologically committing form, it is not. ‘Something’ does not appear in that sentence as a kind term, or common noun. In that sentence, ‘something’ takes the place of a predicate adjective. Sentences of an ontologically committing form require a kind term to follow the ‘There is’, as in ‘There is a K’. The inference from the second sentence to the third substitutes ‘a property’ for ‘something’ but ‘property’ is a common noun. This substitution makes a grammatical mistake akin to the grammatical mistakes discussed above. In order for the substitution to be valid, it must be valid to transform ‘something’ into a common noun. The above argument requires this inference in the second-order case:

$$\frac{\text{S is something}}{\text{S is some thing}}$$

‘There is a property which S is’ does follow from ‘S is some thing’. But in order to go from ‘S is something’ to ‘There is a property which S is’ relies on the above inference. The inference might be made more explicit with the use of formal notation. It could then be represented as

$$\frac{\exists x \text{ S is } f}{\exists f(f \text{ is a thing } \wedge \text{ S is } f)}$$

Since in the corresponding first-order inference the instances of the quantified sentence will replace the variable with names of *things* the inference is innocent. But the

above inference relies on holding that predicates stand for things. If the reasoning above with respect to grammaticality is correct, then this conclusion of the inference may not even be grammatical. The second instance of 'f' is a name, while the third is an adjective. The sentence should be '$\exists f$(f-ness is a thing $\wedge$ S is f)'.

If translating '$\exists f$ S is f' as 'S is something' avoids grammaticality worries and does not entail 'There is a property which S is', then Sellars has taken the first steps in divorcing quantification from supposition. If correct, this shows that to put a predicate-variable after a quantifier is not to treat predicates as names for objects.

This section offered an understanding of formal sentences in terms of natural language in such a way that premise two of the Argument from Abstracta is false. It relied on the claim that the only sentences of natural language that are of an ontologically committing form are 'Something is a K, 'There is a K', and 'There are Ks'. Sellars offers no real argument for this claim. As was suggested above these forms take center stage because they are the most perspicuous answers to the central question of ontology, "What is there?". There is, however, an argument that the best understanding of the *formal* language requires a particular interpretation of our natural language sentences such that premise two of the Argument from Abstracta is true.

4.2.2 The Second Argument

Another argument for premise two of the Argument from Abstracta is as follows. The best account of quantifiers in a formal language is the familiar Tarskian model-theoretic account. This theory requires first-order quantifiers that are the main

operators of true sentences to supposit. There is no reason to suppose that the account should be any different from that of any higher-order quantifier. Therefore, the best theory of quantification requires that all quantifiers supposit.

Sellars ([67, Ch. 1]) considers some form of this argument. There he considers two different accounts of the existential quantifier, call them (a) and (b). The first account, (a), again bifurcates, into (a1) and (a2). The second account, (b), is the standard Tarskian model-theoretic account. On (a1) a sentence of the form '$\exists x Fx$' is true iff either 'Fa_1' is true or 'Fa_2' is true and so on for each expression 'a_i'. This is standardly called substitutional quantification. On (a2) a sentence of the from '$\exists x Fx$' is true iff some statement of the form 'Fc' is true. It is crucial that the right hand side of the biconditional is not to be interpreted as referring to a list as in the strategy (a1). The crucial feature of strategy (a2) is that

> "...a language not only consists of more than the grammatical strings which are actaully deployed at any one time ... It also includes, in a sense difficult to define, the resources by which the language could be enriched through being extended in specific ways."[67, pg.8]

Sellars ultimately opts for strategy (a2), but does not give an explicit formal account of this theory of quantification. A formal account is proposed in section 4.3 below.

The Tarskian model-theoretic account of quantification, (b) above, gives the truth conditions for existential sentences by defining their truth in a model conditions. Fix a language. A model for that language is a pair of a domain of objects and an interpretation function. The interpretation function assigns each n-ary predicate in the language a subset of the n^{th}-Cartesian product of the domain of the model and

each term an element of the domain of the model. Along with each model comes a set of sequences. A sequence is a function that assigns each variable of the language an object of the domain. The set of sequences for a model is the set of all such functions. For convenience consider only the sentence '$\exists x Fx$'. This sentence is said to be satisfied by a sequence, s, in a model, M, if some sequence, s', exactly like s except possibly for what it assigns to x, satisfies Fx. A sequence satisfies Fx iff the object assigned to x by that sequence is in the set that the interpretation function assigns to F. A sentence is true iff it is satisfied by every sequence.

Sequences, therefore, do the main work in making quantified sentences true. It is the variables of quantified formulas that are assigned objects in the domain of a model. There is no need for a quantified sentence to get its truth conditions via a sentence involving a name. Sellars [67] says that this is not a helpful model of natural language. But the details of his concerns would take this chapter off topic. The goal of this chapter is merely to show that there is a viable alternative to the Tarskian model-theoretic account of quantification. It is not here argued that the Tarskian model-theoretic account of quantification will not serve the purposes of a philosophical logician in trying to model natural language.

The Tarskian model-theoretic account of quantification, as stated above, entails that if a person is committed to the truth of '$\exists x Fx$', then that person is committed to there being an entity assigned to 'x' by some sequence, and that entity is F. Similarly, if the account of quantification is to be generalized, it will hold that a quantified sentence of the form '$\exists f f a$' is true if and only if there is an entity assigned to 'f' by some sequence such that the entity denoted by 'a' is or has whatever entity is

denoted by 'f'. It follows that the Tarskian model-theoretic account of quantification entails that all quantifiers, if they are the main operators of a sentence, supposit.

This means that if the response to Argument from Abstracta given in section 4.2 is to succeed, an alternative to the Tarskian model-theoretic account of quantification is required.

4.3 An Alternative Account of Quantification

The account of quantification presented here is embedded in a framework first developed by Restall [53].[6] Restall offers an explication of the notion of a position used in the question 'what is Y's position on topic T?' or the statement 'My position on T is X'. A position is an ordered pair of sets of sentences. Positions can be coherent or incoherent depending on the sentences that they contain. The rules governing the assertion and denial conditions of sentences featuring logical vocabulary determine the coherent and incoherent positions. Those positions in which a sentence can be (in)coherently asserted or denied are the primary semantic machinery of the framework. The rules governing the structure of positions and the conditions of assertion and denial for a sentence determine whether or not it is coherent to assert or deny that sentence in a position. The semantic contribution that a logical expression, such as $\neg$ or $\exists$, makes to a sentence is given by the rules governing coherent assertion and denial of that sentence. This inferentialist theory of the meaning of sentences follows the spirit of Sellars [66, 67].

It is important to note that the truth of this framework is not argued for here. It

[6]This is similar to work done by Koslow [22]

is presented only as an alternative to the Tarskian model-theoretic picture presented above. The goal of this chapter is only to show that there is a prima facie viable picture of quantification and ontological commitment that can germinate from the kernel planted by Sellars [63, 67]. In what follows a formal account both of positions and of ontological commitment are developed. It is then shown that on this view not all quantifiers which are the main operators in true sentence must supposit. More precisely, if an expression in a true atomic sentence does not supposit, then the quantified sentence that generalizes over that expression does not supposit in any way that the original atomic sentence does not.

Fix a formal language. Let there be an infinite set of names $c_1, \ldots, c_n, \ldots$, an infinite set of variables, $x_1, \ldots, x_n, \ldots$, and for each arity, m, an inifinite set of predicates, $R^m_1, \ldots, R^m_n, \ldots$ Call the union of the set of names and the set of variables the set of terms. The set of logical vocabulary is $\{\neg, \wedge, \exists\}$. If φ is a formula, then $\varphi[t_i/t_j]$ is the result of replacing every occurrence of t_j in φ by t_i. The set of formulas of the language is defined recursively by

- If R^m_n is an m-ary predicate, and $t_1, \ldots, t_m$ are m-many terms, then $R^m_n t_1, \ldots, t_n$ is a formula.
- If φ and ψ are formulas then so are $\neg(\varphi)$, $(\varphi \wedge \psi)$, and $\exists x_i(\varphi)$[7]
- Nothing else is a formula.

A sentence is a formula with no unbound variables.

[7] Parenthesis are dropped when convenient.

Definition 21 (Position)**.** Let Γ and Σ be sets of sentences. $\Gamma \Rightarrow \Sigma$ is the position one takes up by asserting all of Γ and denying all of Σ.

$\Gamma \Rightarrow \Sigma$ in the formal language represents the position a person would be in if they were to assert all of Γ and deny all of Σ.

Figure 4.1: First-Order Logic

STRUCTURAL RULES

$$\text{Id}\ \frac{}{\varphi \Rightarrow \varphi} \qquad \text{Cut}\ \frac{\Gamma \Rightarrow \varphi, \Sigma \qquad \Gamma, \varphi \Rightarrow \Sigma}{\Gamma \Rightarrow \Sigma}$$

$$\text{WL}\ \frac{\Gamma \Rightarrow \Sigma}{\Gamma, \varphi \Rightarrow \Sigma} \qquad \text{WR}\ \frac{\Gamma \Rightarrow \Sigma}{\Gamma \Rightarrow \varphi, \Sigma}$$

OPERATIONAL RULES

$$\text{L}\neg\ \frac{\Gamma \Rightarrow \varphi, \Sigma}{\Gamma, \neg\varphi \Rightarrow \Sigma} \qquad \text{R}\neg\ \frac{\Gamma, \varphi \Rightarrow \Sigma}{\Gamma \Rightarrow \neg\varphi, \Sigma}$$

$$\text{L}\wedge\ \frac{\Gamma, \varphi, \psi \Rightarrow \Sigma}{\Gamma, \varphi \wedge \psi \Rightarrow \Sigma} \qquad \text{R}\wedge\ \frac{\Gamma \Rightarrow \varphi, \Sigma \qquad \Gamma \Rightarrow \psi, \Sigma}{\Gamma \Rightarrow \varphi \wedge \psi, \Sigma}$$

$$\text{L}\exists^{*}\ \frac{\Gamma, \varphi[t/x] \Rightarrow \Sigma}{\Gamma, \exists x\varphi \Rightarrow \Sigma} \qquad \text{R}\exists\ \frac{\Gamma \Rightarrow \varphi[t/x], \Sigma}{\Gamma \Rightarrow \exists x\varphi, \Sigma}$$

* t does not occur in the conclusion of L$\exists$

A deduction is also defined inductively

- All instances of Id are deductions.
- If $\delta_1, \ldots, \delta_n$ are deductions, R an n-premise rule that has the last positions of $\delta_1, \ldots, \delta_n$ as premises and $\Gamma \Rightarrow \Sigma$ as a conclusion, then

$$R\,\frac{\begin{matrix}\delta_1 & & \delta_n\\ \vdots & \dots & \vdots\end{matrix}}{\Gamma \Rightarrow \Sigma}$$

is a deduction.

A position, $\Gamma \Rightarrow \Sigma$ is incoherent iff there is a deduction of $\Gamma \Rightarrow \Sigma$, written $\vdash \Gamma \Rightarrow \Sigma$. This generates the reading a rule of fig. 4.1, e.g. R

$$\mathrm{R}\,\frac{\Gamma \Rightarrow \Sigma}{\Delta \Rightarrow \Lambda}$$

as indicating that if it is incoherent to assert all of Γ and deny all of Σ, then it is incoherent to assert all of Δ and deny all of Λ. On this understanding, Id corresponds to the fact that assertion and denial are exclusive speech acts. It is incoherent to assert and deny the same sentence. WL and WR indicate that if a position is incoherent, then asserting or denying more sentences will not change that.

R read contrapositively indicates that if it is coherent to assert all of Δ and deny all of Λ, then it is coherent to assert all of Γ and deny all of Σ. Cut, on this reading, indicates that if it is coherent to assert all of Γ and deny all of Σ, then for any sentence φ, either it is coherent to assert all of Γ and assert φ and deny all of Σ, or it is coherent to assert all of Γ and deny φ and all of Δ. Cut thus entails that any coherent position can be expanded to a coherent position that either asserts or denies each sentence of the language.

Similar to the Cut rule, the L$\exists$ rule is a rule for the expansion of a position. The L$\exists$ rule read from bottom to top indicates that if $\Gamma, \exists x\varphi \Rightarrow \Sigma$ is coherent, then it is coherent to add a new term, t, to one's language and assert $\varphi[t/x]$. On this account of quantification, the contribution that a quantifier makes to the sense of

a sentence is to mark what is coherent or incoherent in expansions of the language. The general role of quantifiers in a language, on the view under consideration, is to account for the ways in which that language can be expanded. L$\exists$ indicates that if it is coherent to assert $\exists x\varphi$, then it is coherent to expand the resources of the language by a new name, c, and assert $\varphi[c/x]$. This is embodied in the familiar pattern of reasoning that names what is existentially quantified over, e.g. moving from "There is a set containing such and such numbers" to "Call it 'A'." and asserting "A contains such and such numbers". R$\exists$ indicates that if it is coherent to deny $\exists x\varphi$, then for any term, t —even terms in an expansion of the language —it is coherent to deny $\varphi[t/x]$. This account of quantifiers thus offers a way of understanding one sense in which a language that includes quantification has the resources to put restrictions on expansions of the language. In the first-order case, quantifiers mark what is coherent and incoherent when the language is expanded by new names. Second-order quantifiers mark what is coherent and incoherent given expansions of the language by predicates. Sentential quantifiers cover the case of expansions of the language by sentences.

It is more appropriate, on this account of quantification, to assign quantifiers the role of generalizing rather than of marking the number of entities that fall under a kind, property, attribute, etc. This particular role of first-order quantifiers is due to the fact that they are generalizing over names. As is discussed below, that quantifiers have such ontological significance is not essential to quantifiers as such but is a feature of those quantifiers that generalize over grammatical expressions that have ontological significance.

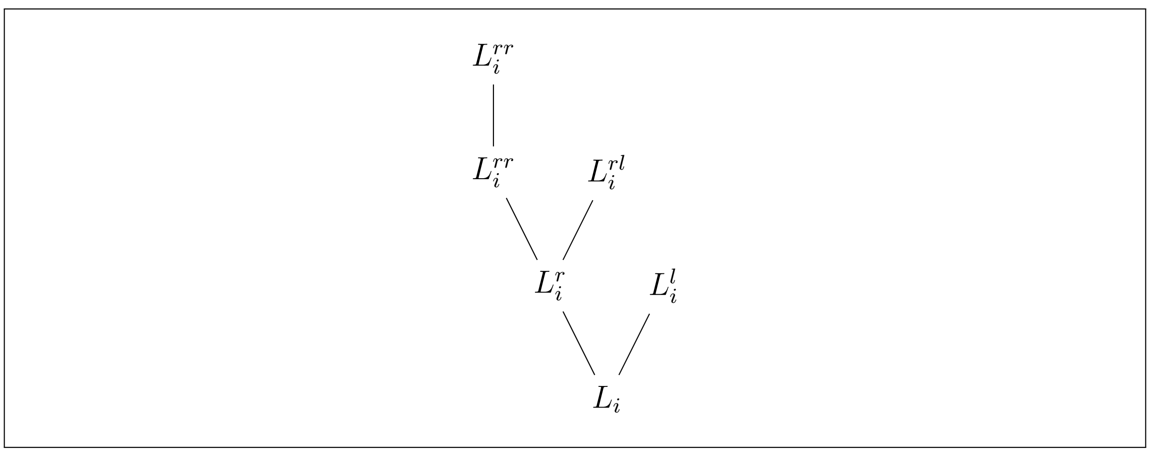
Ontological commitment is a species of commitment. This chapter agrees with the standard picture that the most straightforward way to be committed to Fs is to be committed to the sentence $\exists xFx$. A position, $\Gamma \Rightarrow \Sigma$ is said to be committed to a sentence, φ iff $\vdash \Gamma \Rightarrow \varphi, \Sigma$, i.e. a position is committed to φ when it is incoherent to take up that position and deny φ.

To come to an explication of the notion of an ontological commitment it is necessary to consider all the ways that a coherent position can be expanded. $\Delta \Rightarrow \Lambda$ is an *expansion* of $\Gamma \Rightarrow \Sigma$ iff $\Gamma \subseteq \Delta$ and $\Sigma \subseteq \Lambda$. A position is *maximal* iff for each sentence, φ, it either asserts φ or it denies φ. A maximal coherent expansion of a position is a way of completing that position's account of everything. A maximal position takes a stand on every sentence of the language. An expansion of a position asserts and denies all the sentences that the original position asserts and denies, and possibly others. A coherent expansion of a position is thus a coherent way of continuing the description that the original position makes. The maximal coherent expansions of some position thus correspond to complete characterizations of ways things could be given that the original position is adopted.

Let $\Gamma \Rightarrow \Sigma$ be a coherent position. The set of maximal coherent expansions of $\Gamma \Rightarrow \Sigma$, $ME(\Gamma \Rightarrow \Sigma)$ is defined by a construction that makes use of both the Cut rule and the L$\exists$ rule. Both rules guarantee ways of coherently expanding a position. The L$\exists$ rule offers a way of expanding a position by a new term, and the Cut rule guarantees that given a coherent position, there is a maximal coherent expansion of that position. Using these two rules a tree is constructed. The set of maximal coherent expansions of a position are constructed using this tree.

In order to construct the tree, let $\varphi_1, \ldots, \varphi_n, \ldots$ be an ordering on the sentences of $\mathscr{L}$. Let W be a denumerably infinite set of witnesses, and let $w_1, \ldots, w_n, \ldots$ be an ordering on W. W is the set of expressions that account for the dynamic nature of language. They are the terms that are not in the language of a coherent position, but are added to witness the truth of existential sentences. Let L_W and $L_\exists$ be two lists that at the beginning of the procedure are empty. Each branch will have its own $L_\exists$ list. This list will be identified by a superscript of a sequence of 'l's and 'r's. This index corresponds to the branch of the tree to which that list belongs. For instance, the list $L_\exists^{rr}$ is thus associated with the right branch of the right branch of the tree. An example is given in fig. 4.2.

Figure 4.2: A Sample Tree With Labeled Lists



Sequences of 'l's and 'r's are abbreviated by 'd's. To determine the set $ME(\Gamma \Rightarrow \Sigma)$, begin a tree whose root is $\Gamma \Rightarrow \Sigma$. The leaves of the tree are those positions that do not have any positions above them. A leaf of the tree, $\Delta \Rightarrow \Lambda$, is open

iff $\not\models \Delta \Rightarrow \Lambda$, otherwise it is closed. At stage n consider sentence φ_n and do the following for each open leaf $\Delta \Rightarrow \Lambda$ of the tree:

- If for any sentence of the form $\exists x\varphi$, $\exists x\varphi \in \Delta$ and $\exists x\varphi \notin L^d_\exists$, then take the least element of $W \notin L_W$, w_n, expand the branch by

$$\frac{\Delta, \varphi[w_n/x] \Rightarrow \Lambda}{\Delta \Rightarrow \Lambda}$$

  Add w_n to L_W, φ to $L^d_\exists$, and add the set of sentences formed using w_n to the end of the list of sentences.

- If $\not\models \Delta, \varphi_n \Rightarrow \Lambda$ and $\not\models \Delta \Rightarrow \varphi_n, \Lambda$, the tree branches and becomes

$$\frac{\Delta, \varphi_n \Rightarrow \Lambda \qquad \Delta \Rightarrow \varphi_n, \Lambda}{\Delta \Rightarrow \Lambda}$$

  If $L^d_\exists$ was associated with that branch, then $L^{dl}_\exists$ is associated with the left branch, and $L^{dr}_\exists$ is associated with the right branch.

- If exactly one of $\not\models \Delta, \varphi_n \Rightarrow \Lambda$ or $\not\models \Delta \Rightarrow \varphi_n, \Lambda$ holds extend the branch by that sequent.

Cycle through the sentences of the language in this way until no new sentences are added to any open leaves of the tree.

Let $\Gamma \Rightarrow \Sigma, \Delta_1 \Rightarrow \Lambda_1, \ldots, \Delta_n \Rightarrow \Lambda_n, \ldots$ be an open branch on this tree. Call $\bigcup_i \Delta_i \Rightarrow \bigcup_i \Lambda_i$ the maximal leaf of this branch.

Definition 22 (Maximal Coherent Extensions)**.** The set of *maximal coherent extensions* of a position $\Gamma \Rightarrow \Sigma$, $ME(\Gamma \Rightarrow \Sigma)$ is the set of maximal leaves after the process described above is completed.

As discussed above, the most natural answer to the question "What is there?" is "There are K's", or some variant. The central question of ontology is thus best answered by indicating what *non-empty kinds* there are. An account of ontological commitment requires an account of what non-empty kinds a position is committed to. If φ is a sentence in which a term t occurs, let $\varphi[\xi/t]$ be a kind term. Intuitively, the kind-term '$F\xi$' corresponds to being an F.

The maximal coherent extensions of a position are all the ways that things could be given the truth of that position, i.e. that was it asserts is true and what it denies is false. A position $\Gamma \Rightarrow \Sigma$ is said to be committed to a sentence φ iff for each $\Delta \Rightarrow \Lambda \in ME(\Gamma \Rightarrow \Sigma)$, $\varphi \in \Delta$. A position is thus committed to a sentence if on any way of coherently filling it in, φ is asserted. It is worth remarking that $\vdash \Gamma \Rightarrow \varphi, \Sigma$ iff for any $\Delta \Rightarrow \Lambda \in ME(\Gamma \Rightarrow \Sigma)$, $\varphi \in \Delta$. From this it follows that a position is committed to φ iff it is incoherent, given that position, to deny φ. A position $\Gamma \Rightarrow \Sigma$ is committed to there being entities of kind K iff for each $\Delta \Rightarrow \Lambda \in ME(\Gamma \Rightarrow \Sigma)$, $K[t/\xi] \in \Delta$ for some term or witness t. Given the construction of $ME(\Gamma \Rightarrow \Sigma)$ it follows from this definition that $\Gamma \Rightarrow \Sigma$ is committed to there being K's iff it is committed to $\exists x K[x/\xi]$ iff $\vdash \Gamma \Rightarrow \exists x K[x/\xi], \Sigma$.[8] Thus a position $\Gamma \Rightarrow \Sigma$ is committed to there being K's iff $\vdash \Gamma \Rightarrow \exists x K[x/\xi], \Sigma$. The first order existential sentences to which a position is committed are those closely tied to the ontological commitments of a position.

The ontological commitments of a position $\Gamma \Rightarrow \Sigma$ on this account, depend on what sentences are in the set of maximal coherent extensions of $\Gamma \Rightarrow \Sigma$, and what

[8] A proof of these is offered in the section 4.5.

parts of unquantified sentences are taken to be ontologically significant. On the above definition kinds were taken to be of primary ontological significance. But it is consistent with this view that whatever is named by a predicate has ontological significance as well. Whether or not commitment to the sentence '$\exists f fa$' carries commitment to there being something which the entity named by 'a' is or has, depends on whether commitment to sentences of the form 'Ha' carry such a commitment. That will depend on what the right account of the supposition of atomic sentences is. What determines the ontological significance of a quantified sentence depends on what the ontological significance of its instances in an expanded language are.

Sellars [65, 68] goes to great lengths to provide a theory of predication whereby predicates occurring in true sentences do not supposit. If both Sellars's account of predication and the present treatment of quantification are correct, then there is no reason to think that second-order quantifiers, when occurring as the main operator of a true sentence, supposit. Thus, premise two of the Argument from Abstracta on this account is false. The sentence '$\exists f fa$' does not mask commitment to properties, attributes, sets, classes, etc. Coherently asserting '$\exists f fa$', does not that there be some entity which a is or has. A coherent assertion of '$\exists f fa$' only requires that there is a coherent expansion of the language by a newly defined predicate H, such that it is coherent to assert 'Ha'. This account of quantification and ontological commitment serve Sellars's purposes. It is worth noting that it can also be seen as an account that can serve the purposes of Wright [76], who claims that the ontological commitments of a quantified sentence are reducible to the ontological commitments of its instance. He calls this position 'Neutralism'.

4.4 A Grammatical Interpretation of '$\exists f fa$'

Sellars argues against several translations of '$\exists f fa$' into English on the grounds that they are not grammatical. He offers various alternatives as translations, such as 'a is something', 'a is somequale', and 'a is somehow'.[9] It may be that none of these sentences are grammatical either. Their ungrammaticality, however, does not consist in translating 'f' equivocally, sometimes as an adjective, sometimes as a common noun, and sometimes as a name. Sellars's suggested translations do not commit *that* grammatical error. Instead, they are not obviously sentences of English. This may be acceptable given that natural language is flexible, and they could be introduced as sentences of English with little trouble.

New sentences such as 'a is somehow' can be introduced into English and given a coherent meaning. The introduction of such sentences would provide correlates for the sentence '$\exists f fa$'. These sentences could then be used to assess whether the second-order quantifier, when the main operator of a true sentence, supposits. If 'a is somehow' could be introduced so that it is clearly a translation of '$\exists f fa$' and does not clearly entail a sentence like 'There is some how or way that a is', this could defuse worries that the sentence '$\exists f fa$' had problematic ontological commitments.

Even with the introduction of new sentences, it is helpful to point out that such resources are already available in English. It may be difficult to make clear what one might express by asserting the sentence 'Homer is somequale' or 'Fred is somehow'. The latter is worrying because it looks like an incomplete sentence of English to

[9] The latter of these was first suggested by Prior [45], and later used by Rayo [49] as a translation of second-order sentences.

which an interlocutor might respond 'Fred is somehow what?'[10] These sentences are indicated in question and answer dialogs of the sort that Sellars considers at the end of "Grammar and Existence". Consider the following:

A : Bart is doing something that you won't like.

B : What is Bart doing?

A : He's being mean to Lisa.

The first sentence of the dialogue is the one containing the translation of the second-order quantifier. For a small set of cases it is appropriate to translate '$\exists f f a$' as 'a is doing something'. Appropriate specifications of the first sentence are phrases such as 'being mean', 'being kind', etc. The expression 'is doing something' is as much a part of English as the expressions, 'Lisa is doing something about the issue', or 'Gil is doing something new'. The above dialogue suggests that it is not inappropriate to use the expression 'doing something' to generalize over ways that persons might be. 'a is doing something' is not itself of an ontologically committing form as defined above. It also does not entail the sentence 'There is some thing that a is doing' which is of an ontologically committing form. The latter is equivalent to the sentence 'a is doing some thing' the former is not.

If 'a is doing something' is a legitimate translation of '$\exists f f a$', this would dispel the worry introduced at the end of section 4.2.1 that there is no translation of '$\exists f f a$' into English that is not of an ontologically committing form.

[10] One such interlocutor posed exactly this question to Sellars in the discussion period following Sellars's first Dewey lecture. (See https://www.youtube.com/watch?v=6UiV-vMOueY). Sellars's response is that 'somehow' is being used in a non-standard way.

4.5 Proof of Claims

Lemma 4.1. *If* $\Delta \Rightarrow \Lambda \in ME(\Gamma \Rightarrow \Sigma)$, *then* $\Gamma \subseteq \Delta$ *and* $\Sigma \subseteq \Lambda$

Proof. There is no step in the construction of $ME(\Gamma \Rightarrow \Sigma)$ that removes sentences from $\Gamma \Rightarrow \Sigma$. So every element of $ME(\Gamma \Rightarrow \Sigma)$ is an expansion of it. □

Theorem 4.5.1. $\vdash \Gamma \Rightarrow \varphi, \Sigma$ *iff for every* $\Delta \Rightarrow \Lambda \in ME(\Gamma \Rightarrow \Sigma)$, $\varphi \in \Delta$, *for a sentence* φ *containing no witnesses.*

Proof. For the left to right direction, suppose that $\vdash \Gamma \Rightarrow \varphi, \Sigma$. Suppose that there is a $\Delta \Rightarrow \Lambda \in ME(\Gamma \Rightarrow \Sigma)$ such that $\varphi \in \Lambda$. By lemma 4.1, $\Gamma \subseteq \Delta$ and $\Sigma \cup \{\varphi\} \subseteq \Lambda$. Since $\vdash \Gamma \Rightarrow \varphi, \Sigma$ by weakening $\vdash \Delta \Rightarrow \Lambda$. But then $\Delta \Rightarrow \Lambda \notin ME(\Gamma \Rightarrow \Sigma)$. So there is no $\Delta \Rightarrow \Lambda \in ME(\Gamma \Rightarrow \Sigma)$ such that $\varphi \in \Lambda$. Let $\Delta \Rightarrow \Lambda \in ME(\Gamma \Rightarrow \Sigma)$. Since $\varphi \notin \Lambda$ and $\Delta \Rightarrow \Lambda$ is maximal, $\varphi \in \Delta$.

For the right to left direction suppose that for every $\Delta \Rightarrow \Lambda \in ME(\Gamma \Rightarrow \Sigma)$, $\varphi \in \Delta$. But also suppose that $\nvdash \Gamma \Rightarrow \varphi, \Sigma$. The only way for this to be the case is that at the stage, call it n, where φ was considered, there was no open leaf, $\Pi \Rightarrow \Theta$ such that $\nvdash \Pi \Rightarrow \varphi, \Theta$. Since $\nvdash \Gamma \Rightarrow \varphi, \Sigma$, there must be a stage earlier than n, m, such that it was still the case that $\nvdash \Pi' \Rightarrow \varphi, \Theta'$, but that at stage $m + 1 \vdash \Pi'' \Rightarrow \varphi, \Theta''$. Let each time a witness is added for an existential sentence count as a sub-stage. The sub-stages of a stage i are $i.1, i.2, \ldots$. Let the sentence in consideration at stage $m + 1$ be ψ. There are four cases to consider.

Case 1. Suppose that this happens at stage $m.i$. Since witnesses are added there is a sentence, $\exists x\psi \in \Pi'$, such that $\nvdash \Pi', \exists x\psi \Rightarrow \varphi, \Theta$ but $\vdash \Pi', \psi[w_i/x] \Rightarrow \varphi, \Theta$. However,

since, w_i does not occur in φ, it is guaranteed by L$\exists$ that $\nvdash \Pi', \psi[w/x] \Rightarrow \varphi, \Theta$. So this cannot be where the each leaf became committed to φ.

Case 2. Suppose that $\nvdash \Pi' \Rightarrow \varphi, \Theta'$, that both $\nvdash \Pi', \psi \Rightarrow \Theta'$ and $\nvdash \Pi' \Rightarrow \psi, \Theta'$. From the assumption it follows that both $\vdash \Pi', \psi \Rightarrow \varphi, \Theta'$ and $\vdash \Pi' \Rightarrow \psi, \varphi, \Theta'$. But then by Cut $\vdash \Pi' \Rightarrow \varphi, \Theta$, contradicting the assumption.

Case 3. Suppose that $\nvdash \Pi' \Rightarrow \varphi, \Theta'$, and that both $\nvdash \Pi', \psi \Rightarrow \Theta'$ and $\vdash \Pi' \Rightarrow \psi, \Theta'$. By assumption $\vdash \Pi', \psi \Rightarrow \varphi, \Theta'$. But then by Cut $\vdash \Pi' \Rightarrow \varphi, \Theta$, a contradiction.

Case 4. Suppose that $\nvdash \Pi' \Rightarrow \varphi, \Theta'$, and that both $\vdash \Pi', \psi \Rightarrow \Theta'$ and $\nvdash \Pi' \Rightarrow \psi, \Theta'$. By assumption $\vdash \Pi' \Rightarrow \psi, \varphi, \Theta'$. But then by Cut $\vdash \Pi' \Rightarrow \varphi, \Theta$, a contradiction.

□

Theorem 4.5.2. *If for each $\Delta \Rightarrow \Lambda \in ME(\Gamma \Rightarrow \Sigma)$, $K[t/\xi] \in \Delta$ for some t, then $\vdash \Gamma \Rightarrow \exists x K[x/\xi], \Sigma$.*

Proof. Suppose that for each $\Delta \Rightarrow \Lambda \in ME(\Gamma \Rightarrow \Sigma)$, $K[t/\xi] \in \Delta$ for some t. Suppose that there is a $\Delta \Rightarrow \Lambda \in ME(\Gamma \Rightarrow \Sigma)$ such that $\exists x K[x/\xi] \notin \Delta$. Since $\Delta \Rightarrow \Lambda$ is maximal, $\exists x K[x/\xi] \in \Lambda$. Thus, there is some finite stage on that branch with position, $\Pi \Rightarrow \Theta$, where $K[t/\xi] \in \Pi$ for some t and $\exists x K[x/\xi] \in \Theta$. The following deduction establishes that that branch is closed

$$\text{WL/WR}\,\dfrac{\text{R}\exists\,\dfrac{\text{Id}\,\dfrac{}{K[t/\xi] \Rightarrow K[t/\xi]}}{K[t/\xi] \Rightarrow \exists x K[x/\xi]}}{\Pi, K[t/\xi] \Rightarrow \exists x K[x/\xi], \Theta}$$

This contradiction the assumption that $\Delta \Rightarrow \Lambda \in ME(\Gamma \Rightarrow \Sigma)$. Thus, there is no such $\Delta \Rightarrow \Lambda$. So for each $\Delta \Rightarrow \Lambda \in ME(\Gamma \Rightarrow \Sigma)$, $\exists x K[x/\xi] \in \Delta$. By theorem 4.5.1, $\vdash \Gamma \Rightarrow \exists x K[x/\xi], \Sigma$. □

Chapter 5

Atomic Ontology

Abstract. The aim of this chapter is to offer a method for determining the ontological commitments of a formalized theory. The second section shows that determining the consequence relation of a language model-theoretically entails that the ontology of a theory is tied very closely to the variables that feature in that theory. The third section develops an alternative way of determining the ontological commitments of a theory given a proof-theoretic account of the consequence relation for the language that theory is in. It is shown that the proof-theoretic account of ontological commitment does not entail that the ontological commitments of a theory depend on the variables of that theory. The last section of the chapter discusses how this account of ontological commitment can be used in other philosophical projects such as Wright's abstractionism. The chapter concludes with a discussion of the upshots of adopting the proof-theoretic account of ontological commitment for ontology generally.

Keywords. Ontological Commitment, Quine's Dictum, Quantification

5.1 Introduction

The primary question of ontology is "What is there?" After the linguistic turn this question has been answered by a two step divide and conquer strategy. In the first step a general account is given of what entities a person is committed to by the statements that they accept. This is the formal answer to the primary question of ontology. The second step is to figure out which theory ought to be accepted. This is the material answer to that question. The task of offering a material answer belongs to the various areas of – broadly construed – scientific inquiry. The answer to the primary question of ontology is thus given in a roundabout way. It is answered materially by the physicists, ethicists, mathematicians, etc. Offering a formal answer, on the other hand, is the task of the philosopher of language and the philosophical logician.

Providing a formal answer to the primary question of ontology is providing a method for determining how a person commits themselves to there being entities of such and such a sort. Theories themselves do not have commitments. A person who endorses a particular theory may incur certain commitments because of the content of that theory. For instance, a person endorsing the statement that it is raining and it is Tuesday is committed to the statement that it is raining. This is because the statement that it is raining follows from the statement that it is raining and it is Tuesday. In what follows the phrase "ontological commitments of a theory" should

be understood in light of the above. The ontological commitments of a theory are what a person would be committed to there being if they were to endorse that theory.

A formal answer to the primary question of ontology was famously given by Quine [46, 47]. According to Quine the ontological commitments of a theory are determined by examining what entities are in the ranges of that theory's variables. Quine sums this up as, "To be assumed as an entity is, purely and simply, to be reckoned as the value of a variable" [47]. Call this Quine's Dictum. Quine's Dictum is the dominant account of how to determine the ontological commitments of a theory. The goal of this chapter is to offer an alternative to Quine's Dictum.

As [72] points out, things are not so simple as the preceding paragraphs may suggest. In fact two steps are required for determining the ontological commitments of a theory. First the theory must be formalized and then, according to Quine's Dictum, the ontological commitments of that theory are determined by examining the ranges of the variables occurring in that theory. One way to think of the process of formalizing a theory is as a translation from natural language into a formal language. A concern with this first step is that there is no apparent reason to think that there is a unique translation into a formal language or that there is a particular formal language that must be the object of the translation. That there is no unique translation would only be a problem if what was to be determined was the ontology of the theory as stated unreflectively by the theorist. But the aim of someone offering a formal answer to the primary question of ontology is to discover the commitments of the theorist who has carefully translated their theory. As Van Inwagen points out the formalization of a theory that is ultimately adopted may arise from some charitable

back and forth between theorist and formalizer. One attempted formalization and analysis may be met by the theorist with "I did not mean to commit myself to that" after which some refinement could take place.

In order to avoid this difficulty it is assumed either that this has already taken place or that the theorist is fluent in the formalism. In either case what formalization is settled on is assumed as given and that the theorist offering the theory will accept what commitments are determined to come along with that theory. The determination of what formal sentences a theory accepts is taken for granted in what follows.

Below (section 5.2) it is argued that the standard model theoretic account of consequence entails Quine's Dictum. It follows that in order to offer an alternative to Quine's Dictum some other account of consequence must be offered. Section 5.3 presents an alternative account of consequence and ontological commitment. Section 5.4 discusses upshots of and objections to the view presented in section 5.3.

The only formal language that Quine admits for determining the ontological commitments of a theory is the language of first-order logic with identity. The points made in this chapter can be made without invoking identity. For the sake of simplicity it is left to the side. Call the first-order set of sentences under consideration $\mathscr{L}$. Let $N = \{c_1, \ldots, c_n, \ldots\}$ be a denumerably infinite set of names, $V = \{x_1, \ldots, x_n, \ldots\}$ be a denumerably infinite set of variables,[1] and $P_m = \{F^m_1, \ldots, F^m_n, \ldots\}$ be a denumerably infinite set of predicates of arity m for each natural number m. Let $T = N \bigcup V$ be the set of terms of $\mathscr{L}$ and $P = \bigcup_{i \in \mathbb{N}} P_i$ be the set of predicates of $\mathscr{L}$. The set $\mathscr{L}$

[1] Officially variables are 'x's decorated with sub-scripts, but for convenience 'y' and 'z' are be used.

is defined recursively by

- If $F^n \in P$ and $t_1, \ldots, t_n \in T$ then $F^n t_1, \ldots, t_n \in \mathscr{L}$.
- If $\varphi \in \mathscr{L}$ then $\neg\varphi \in \mathscr{L}$.
- If φ and ψ are both in $\mathscr{L}$ then $(\varphi \wedge \psi) \in \mathscr{L}$.
- If $\varphi \in \mathscr{L}$ and $x \in V$ then $(\exists x \varphi) \in \mathscr{L}$.

If $\varphi \in \mathscr{L}$ then $\varphi[t_1/t_2]$ is the sentence that results from replacing every occurrence t_2 by t_1 in φ. A sentence is called closed iff there are no variables occurring in that sentence that are not bound by a quantifier. A *consequence* relation on a language is a relation that holds between sets of sentences and sets of sentences.[2]

The most natural form for an answer to the question "What is there?" to take is "There are Ks". It is this form of the answer to the question that is considered in detail below.[3] Commitment to there being K's is commitment to there being an entity that is a K. A kind-term is an expression in the metalanguage of $\mathscr{L}$. A kind-term is formed by replacing some term in a closed sentence of $\mathscr{L}$ by the place holder ξ. If φ is a closed sentence of $\mathscr{L}$ then $\varphi[\![\xi/t]\!]$ is the kind-term that results from replacing t everywhere by ξ. For instance if Ftt is a closed sentence of $\mathscr{L}$ then $F\xi t$, $Ft\xi$ and $F\xi\xi$ are all kind-terms. If K is a kind-term with place holder ξ and t is a term then $K[\![t/\xi]\!]$ is the sentence that results from replacing t for every instance of ξ in K. The ontological commitments of a theory are given by the set of kinds-terms

[2]The non-standard set-set consequence relation is adopted for ease in dealing with the proof-theory presented in section 5.3.

[3]One upshot of the view offered here is that this is not necessarily the only intelligible question that can be asked.

such that according to the theory there are entities of which those kind terms are true. For instance, a theory is committed to there being lions if according to that theory there are entities of which the kind-term "ξ is a lion" is true.

5.2 Model-Theory and Quine's Dictum

An account of quantification is fixed by a way of determining the consequence relation for a language. This section argues that if the consequence relation of a language is determined in the standard model-theoretic way then Quine's Dictum is correct.

A model M is an ordered pair $\langle D_M, I_M \rangle$ of a non-empty domain of objects D_M and an interpretation function I_M. For each name c, $I_M(c) \in D_M$ and for each n-ary predicate P^n, $I_M(P^n) \subseteq D_M^n$. Let Σ_M be the set of functions from V into D_M. An individual such function σ is called a variable assignment. Two variable assignments σ and σ' are v-variants of one another, written $\sigma \sim_v \sigma'$, iff for any variable v' that is not v, $\sigma(v') = \sigma'(v')$. A denotation function δ is defined relative to variable assignments. Let t be a term of $\mathscr{L}$. The denotation of t relative to a variable assignment σ is given by

$$\delta_\sigma(t) = \begin{cases} I_M(t), & \text{if } t \in N \\ \sigma(t), & \text{if } t \in V \end{cases}$$

A model M relative to a variable assignment σ satisfies a sentence φ, $M, \sigma \vDash \varphi$, when

- if φ is $P^n t_1, \ldots, t_n$ then $M, \sigma \vDash \varphi$ iff $\langle \delta_\sigma(t_1), \ldots, \delta_\sigma(t_n) \rangle \in I_M(P^n)$;

- if φ is $\neg\psi$ then $M, \sigma \vDash \varphi$ iff $M, \sigma \nvDash \psi$;
- if φ is $\psi \wedge \theta$ then $M, \sigma \vDash \varphi$ iff $M, \sigma \vDash \psi$ and $M, \sigma \vDash \theta$; and
- if φ is $\exists x \psi$ then $M, \sigma \vDash \varphi$ iff there is a $\sigma' \sim_x \sigma$ such that $M, \sigma' \vDash \psi$

A sentence φ is *true-in-a-model* iff it for any variable assignment σ, $M, \sigma \vDash \varphi$. A model M is a model of a set of sentences Γ, $M \vDash \Gamma$ iff for each $\gamma \in \Gamma$, γ is true-in-M. M is an anti-model of a set of sentences Γ, $M \Dashv \Gamma$, iff for each $\gamma \in \Gamma$ $M \nvDash \gamma$. A set of sentences Σ is a *model-theoretic consequence* of another set Γ, $\Gamma \vDash \Sigma$ iff there is no model M such that $M \vDash \Gamma$ and $M \Dashv \Sigma$.

5.2.1 Ontological Commitment

As discussed above the question of what the ontological commitments of a theory are in this instance is a question about what kinds of entities that theory is committed to there being. A theory is a set of sentences. A theory Γ is committed to there being Ks when there is no model that makes Γ true but has no entities of which the kind term K is true.

A variable v is free for a kind-term K when v does not occur in K. All the sentences of the language are finitely long and there are an infinite number of variables so for any kind-term K there is a variable that is free for K.

Definition 23 (Empty in a Model)**.** A kind-term K is *empty in a model M* iff there is no variable assignment $\sigma \in \Sigma_M$ such that $K[\![x/\xi]\!]$ is satisfied by σ, where x is free for K.

A theory Γ is committed to there being entities of which K is true iff there is no model M such that $M \models \Gamma$ and K is empty in M. The use of variable assignments in definition 23 is crucial. There may be a model M such that there is an entity $a \in D_M$ but no name c such that $I_M(c) = a$. On the other hand for any variable v there is a variable assignment σ such that $\sigma(v) = a$. On this account of ontological commitment then variables play an indispensible role. It is variables via variable assignments that have guaranteed contact with every element in the domain of a model.

Theorem 5.2.1. *Let Γ be a theory. Γ is committed to there being entities of which K is true iff for any model M if $M \models \Gamma$ then $M, \sigma \models K[\![x/\xi]\!]$ for some $\sigma \in \Sigma_M$ and x that is free for K.*[4]

Theorem 5.2.1 shows that the model-theoretic account of quantification entails Quine's Dictum in the first-order case. A theory is committed to there being entities of a certain sort if and only if there is an entity of that sort assigned to some variable on every model making that theory true. On this account of consequence one could know whether there are entities of such and such a kind by examining the ranges of the variables of that theory.

Quine's dictum requires that any variable that can be bound by a quantifier have some ontological significance. Nothing about theorem 5.2.1 relied on the fact that the language under consideration was first-order. If a second-order language were given the same model-theoretic treatment Quine's Dictum would hold for second-order variables as well. On the model-theoretic account of quantification which

[4]Each theorem mentioned is proved in the appendix to this chapter.

requires variable assignments and entities in a domain to be the values of variables, all quantification is ontologically significant. A theory with quantifiers binding variables of a particular syntactic expression requires that variables of that syntactic expression have values in a domain.

Quine's Dictum leads to limiting results when considering higher-order languages. Since quantifiers require variables of quantification and variables of quantification require entities in domains, quantification requires entities in domains. As Boolos [7] has pointed out, second-order quantification can express things not expressible in first order logic. On the model-theoretic account of quantification this expressive power must be bought with the coin of ontology. Introducing second-order quantifiers to express what cannot be expressed in a first-order language requires introducing second-order variables. The values of those variables are most naturally kinds, properties, attributes, etc. Any of which are not acceptable to a philosopher with nominalist leanings. One upshot of the alternative account of ontological commitment proposed below is that the expressive power of second-order quantification can be bought without paying the model-theoretic cost.

5.3 Proof Theory and Ontological Commitment

The alternative account of consequence is presented in a framework developed by Restall [50, 52, 53]. That framework takes as primitive two speech acts, assertion and denial. A position is an ordered pair of assertions and denials. If Γ and Δ are sets of sentences then $\Gamma \Rightarrow \Sigma$ is the position a person takes up by asserting all of

Γ and denying all of Σ. Positions can either be coherent or incoherent. These two statuses of positions are used to give the assertion and denial conditions of sentences of $\mathscr{L}$.

A horizontal bar separating positions indicates that if the positions above the bar are incoherent then the position below the bar is also incoherent. For instance

$$\frac{\Delta \Rightarrow \Lambda \qquad \Pi \Rightarrow \Theta}{\Gamma \Rightarrow \Sigma}$$

indicates that if both of $\Delta \Rightarrow \Lambda$ and $\Pi \Rightarrow \Theta$ are incoherent then so is $\Gamma \Rightarrow \Sigma$. Contrapositively, if $\Gamma \Rightarrow \Sigma$ is coherent then either $\Delta \Rightarrow \Lambda$ is coherent or $\Pi \Rightarrow \Theta$ is coherent.

Assertion and denial are exclusive speech acts. Asserting a sentence precludes denying it and denying a sentence precludes asserting it. It is incoherent to assert and deny the same sentence. If a person were to assert and deny the same sentence they would either be doing something wrong or may be attempting to express two propositions using homonymous sentences. There is no coherent interpretation of a person univocally asserting and denying the same sentence. This is the sole notion of incoherence that is at work in Restall's account. That assertion and denial are exclusive is given by Id in fig. 5.1.

If a position $\Gamma \Rightarrow \Sigma$ is incoherent then asserting (denying) more sentences will not change that. Contrapositively, if a position $\Gamma \Rightarrow \Sigma$ is coherent then taking back some assertions (denials) will preserve coherence. This corresponds to the rules of TL and RL of fig. 5.1

A derivation δ is a tree whose root is a position $\Gamma \Rightarrow \Sigma$, each of whose nodes is an application of some rule of fig. 5.1, and whose leaves are all instances of Id. In that

Figure 5.1: First-Order Logic

STRUCTURAL RULES

$$\text{Id}\ \frac{}{Rc_1,\ldots,c_n \Rightarrow Rc_1,\ldots,c_n}\ c_i \in N \qquad \text{Cut}\ \frac{\Gamma \Rightarrow \varphi,\Sigma \qquad \Gamma,\varphi \Rightarrow \Sigma}{\Gamma \Rightarrow \Sigma}$$

$$\text{TL}\ \frac{\Gamma \Rightarrow \Sigma}{\Gamma,\varphi \Rightarrow \Sigma} \qquad \text{TR}\ \frac{\Gamma \Rightarrow \Sigma}{\Gamma \Rightarrow \varphi,\Sigma}$$

OPERATIONAL RULES

$$\text{L}\neg\ \frac{\Gamma \Rightarrow \varphi,\Sigma}{\Gamma,\neg\varphi \Rightarrow \Sigma} \qquad \text{R}\neg\ \frac{\Gamma,\varphi \Rightarrow \Sigma}{\Gamma \Rightarrow \neg\varphi,\Sigma}$$

$$\text{L}\wedge\ \frac{\Gamma,\varphi,\psi \Rightarrow \Sigma}{\Gamma,\varphi\wedge\psi \Rightarrow \Sigma} \qquad \text{R}\wedge\ \frac{\Gamma \Rightarrow \varphi,\Sigma \qquad \Gamma \Rightarrow \psi,\Sigma}{\Gamma \Rightarrow \varphi\wedge\psi,\Sigma}$$

$$\text{L}\exists^{1}\ \frac{\Gamma,\varphi[t/x] \Rightarrow \Sigma}{\Gamma,\exists x\varphi \Rightarrow \Sigma} \qquad \text{R}\exists\ \frac{\Gamma \Rightarrow \varphi[t/x],\Sigma}{\Gamma \Rightarrow \exists x\varphi,\Sigma}$$

1. t does not appear in the conclusion of this inference.

case δ is said to be a derivation of $\Gamma \Rightarrow \Sigma$. Since all instances of Id are incoherent and the rules of fig. 5.1 preserve incoherence, a derivation of $\Gamma \Rightarrow \Sigma$ entails that $\Gamma \Rightarrow \Sigma$ is incoherent. This is indicated by prefixing a derivable position with a turnstile, e.g. $\vdash \Gamma \Rightarrow \Sigma$. Furthermore, in this setting the only way for a position to be incoherent is for there to be a derivation of it.

A rule, for instance R$\wedge$, can be read as the claim that if $\Gamma \Rightarrow \varphi, \Sigma$ and $\Gamma \Rightarrow \psi, \Sigma$ are both incoherent then so is $\Gamma \Rightarrow \varphi \wedge \psi, \Sigma$. The contrapositive of this is that if $\Gamma \Rightarrow \varphi \wedge \psi, \Sigma$ is coherent then either $\Gamma \Rightarrow \varphi, \Sigma$ is coherent or $\Gamma \Rightarrow \psi, \Sigma$ is coherent. In other words, if it is coherent to deny $\varphi \wedge \psi$ while asserting all of Γ and denying all of Σ then either it is coherent to deny φ while asserting all of Γ and denying all of Σ or it is coherent to deny ψ while asserting all of Γ and denying all of Σ. This bears a resemblance to the model-theoretic account given in section 5.2. On that account if $\varphi \wedge \psi$ is false, then either φ is false or ψ is false. Conversely if $\varphi \wedge \psi$ is true then φ is true and ψ is true. This mirrors L$\wedge$ which amounts to the claim that if it is coherent to assert $\varphi \wedge \psi$ then it is coherent to assert both φ and ψ.

The rules for quantification offer a reading that suggests a divergence from the model-theoretic account. R$\exists$ can be read as the claim that if it is coherent to deny $\exists x\varphi$ while asserting all of Γ and denying all of Σ then for any term t that might be introduced it is coherent to deny $\varphi[t/x]$ while asserting all of Γ and denying all of Σ. L$\exists$ amounts to the claim that if it is coherent to assert $\exists x\varphi$ while asserting all of Γ and denying all of Σ then it is coherent to introduce a new term t for which it is coherent to assert $\varphi[t/x]$ while asserting all of Γ and denying all of Σ.

This account of quantification is broadly substitutional. It is not substitutional

in the sense that $\exists x F x$ is equivalent to the disjunction of all the instances of Ft for a term t in the language. Such an account of quantification is called simple substitutional quantification. Any view according to which an existential quantifier is logically equivalent to a disjunction of its instances, i.e. any simple substitutional account, is not recursively definable. This result was proved by Belnap [14]. In that article Dunn and Belnap also proposed an improvement on simple substitutional quantification. On the improved account of quantification a quantified sentence $\exists x \varphi$ of a language $\mathscr{L}$ is true iff there is an extension of that language $\mathscr{L}'$ by a term t such that $\varphi[t/x]$ is true in $\mathscr{L}'$. Other formal accounts of a similar quantifier have been given by Geach [17] ,Bonevac [5], and [27]. An account of quantification along these lines is referred to as expanding substitutional quantification.

Theorem 5.3.1. *The proof-theoretic (fig. 5.1) and model-theoretic (section 5.2) accounts of quantification agree, i.e.* $\vdash \Gamma \Rightarrow \Sigma$ *iff* $\Gamma \models \Sigma$.

Lavine [27] shows that the other expanding substitutional accounts of quantification are also sound and complete with respect to model-theoretic account of consequence given in section 5.2. Formally, all of these structures define the same consequence relation on $\mathscr{L}$. The view endorsed in this chapter takes proof-theory to be primitive in determining the consequence relation. This chapter therefore can offer an inferentialist explanation of the role of quantification in ontology and the groundwork for an inferentialist picture of ontological commitment in general. [5]

[5]Lance [26] also proposes an inferentialist account of ontological commitment. Lance's methodology is to hold Quine's dictum fixed and to offer an inferentialist explanation of the content of existential sentences. This chapter offers an alternative to Quine's dictum along inferentialist lines and makes no commitment as to the particular content of existential sentences other than what can be determined from their assertion and denial conditions.

Among the results of this chapter is a formal explanation of why a person endorsing the proof-theoretic view of the consequence relation need not endorse Quine's dictum.[6] This, as is shown below, does not entail that quantification has no ontological significance at all. Quantifiers that bind ontologically significant syntactic positions are themselves ontologically significant. The ontological significance of, for instance, first-order quantifiers does not derive from the fact that they are quantifiers but from the fact that they quantify the syntactic position where a name could appear. The fact that names are ontologically significant then explains the fact that first-order quantifiers are ontologically significant.

5.3.1 Proof-Theory and Ontological Commitment

The Cut rule read from bottom to top says that for any sentence φ if $\Gamma \Rightarrow \Sigma$ is a coherent position then one of $\Gamma, \varphi \Rightarrow \Sigma$ and $\Gamma \Rightarrow \varphi, \Sigma$ is coherent. Cut entails the claim that any coherent position can be filled in to a coherent position that either

[6] Bonevac [5] suggests that this account of quantification is a formal treatment of a view endorsed by Sellars [62]. This is suggested by the following passage from that (1948) article.

> It has not always, however, been realized that this train of thought leads directly to the conclusion that our language claims somehow to contain a designation for every element in every state of affairs, past, present, and future; that, in other words, it claims to mirror the world by a complete and systematic one-to-one correspondence of designations with individuals. If it is obvious that our language does not explicitly contain such designations (and it would hardly be illuminating to say that it contains them implicitly), it is equally clear that our language behaves as though it contained them. [62, pg. 603]

More evidence that Sellars would have endorsed such a view can be found in Sellars [67]. There he argues that an adequate account of quantification should reduce the "indefinite reference" of quantified sentences to the "definite reference" of unquantified ones. In a footnote, he suggests that this is best accomplished by considering expansions of a language. Sellars [63, 67] spends considerable efforts arguing against Quine's dictum. Similar remarks to those made by Sellars [63] are made by Prior [45]. The results of this chapter can be seen as a formal explanation of the insights of those philosophers.

asserts or denies any sentence of $\mathscr{L}$. The L$\exists$ rule read from bottom to top says that if $\Gamma, \exists x\varphi \Rightarrow \Sigma$ is coherent then it is coherent to introduce into $\mathscr{L}$ a term t to serve as a witness for that existential, i.e. the position $\Gamma, \exists x\varphi, \varphi[t/x] \Rightarrow \Sigma$ is coherent where t does not occur in $\Gamma \bigcup \{\exists x\varphi\} \bigcup \Sigma$.

A position $\Gamma \Rightarrow \Sigma$ is *maximal* iff for any sentence φ it either asserts φ or it denies φ, i.e. $\varphi \in \Gamma$ or $\varphi \in \Sigma$. One position $\Gamma \Rightarrow \Sigma$ is a *superposition* of another $\Delta \Rightarrow \Lambda$ iff $\Gamma \Rightarrow \Sigma$ asserts all the sentences that $\Delta \Rightarrow \Lambda$ asserts and denies all the sentences that $\Delta \Rightarrow \Lambda$ denies, i.e. $\Delta \subseteq \Gamma$ and $\Lambda \subseteq \Sigma$. The set of maximal coherent superpositions of a position $\Gamma \Rightarrow \Sigma$, written $\triangledown(\Gamma \Rightarrow \Sigma)$, are used to determine the ontological commitments of $\Gamma \Rightarrow \Sigma$.

Above it was noted that the rule of Cut entails that if a position $\Gamma \Rightarrow \Sigma$ is coherent then there is a coherent position $\Delta \Rightarrow \Lambda$ such that $\Delta \Rightarrow \Lambda$ is a maximal superposition of $\Gamma \Rightarrow \Sigma$. The Cut rule entails the stronger claim that if $\Gamma \Rightarrow \Sigma$ is a coherent position of $\mathscr{L}$ then if $\mathscr{L}'$ is an expansion of $\mathscr{L}$ there is a maximal coherent superposition of $\Gamma \Rightarrow \Sigma$ in the language of $\mathscr{L}'$. All the rules of fig. 5.1 are valid in any expansion of the language $\mathscr{L}$ by additional expressive resources.

In order to generate $\triangledown(\Gamma \Rightarrow \Sigma)$ a tree is constructed with $\Gamma \Rightarrow \Sigma$ at its root. Let W be a denumerably infinite set of witness terms. The witness terms are those expressions which are not in $\mathscr{L}$ but are in expansions of $\mathscr{L}$. The elements of W will extend the language $\mathscr{L}$ in the course of the construction. They serve, as in standard completeness proofs, as the witnesses to the truth of existentially quantified sentences. Let $c_1, \ldots, c_n, \ldots$ be an ordering on the set N and $\varphi_1, \ldots, \varphi_n, \ldots$ be an ordering on the set of sentences of $\mathscr{L}$. A leaf $\Delta \Rightarrow \Lambda$ of the tree under construction

is *closed* iff $\vdash \Delta \Rightarrow \Lambda$. A leaf is *open* iff it is not closed.

The construction of the tree proceeds in stages. Each stage consists of sub-stages which themselves consist of sub-sub-stages. Let φ_i be the i^{th} sentence in the ordering. At stage i consider the left-most open leaf of the construction that has not been considered at stage i. Let $\Delta \Rightarrow \Lambda$ be the n^{th} such open leaf, call this sub-stage $i.n$. At this stage extend the tree according to the following rules.

(a) If φ_j is of the form $\exists x\psi$, $\varphi_j \in \Delta$, and there is no sentence of the form $\psi[w/x]$ in Δ, then take the least element of W that does not appear in $\Delta \bigcup \Lambda$, w_k, and extend the branch by

$$\frac{\Delta, \varphi[w_k/x] \Rightarrow \Lambda}{\Delta \Rightarrow \Lambda}$$

Call this stage $i.n.j$.

(b) If $\nvdash \Delta, \varphi_i \Rightarrow \Sigma$ and $\nvdash \Delta \Rightarrow \varphi_i, \Sigma$ then extend the branch by

$$\frac{\Delta, \varphi_i \Rightarrow \Lambda \qquad \Delta \Rightarrow \varphi_i, \Lambda}{\Delta \Rightarrow \Lambda}$$

Call this stage $i.n.0$.

(c) If $\vdash \Delta, \varphi_i \Rightarrow \Lambda$ then extend the branch by

$$\frac{\Delta \Rightarrow \varphi_i, \Lambda}{\Delta \Rightarrow \Lambda}$$

Call this stage $i.n.0$

(d) If $\vdash \Delta \Rightarrow \varphi_i, \Lambda$ then extend the branch by

$$\frac{\Delta, \varphi_i \Rightarrow \Lambda}{\Delta \Rightarrow \Lambda}$$

Call this stage $i.n.0$

Let $\tau(\Gamma \Rightarrow \Sigma)$ be a tree constructed in the above way with $\Gamma \Rightarrow \Sigma$ as its root. Let $\Gamma \Rightarrow \Sigma, \Delta_1 \Rightarrow \Lambda_1, \ldots, \Delta_n \Rightarrow \Lambda_n, \ldots$ be an open branch in this tree. Call $\bigcup_i \Delta_i \Rightarrow \bigcup_i \Lambda_i$ the maximal leaf of this branch.

Definition 24 (Maximal Coherent Superposition)**.** If $\Gamma \Rightarrow \Sigma$ is a position then the set of maximal coherent superpositions of that position $\nabla(\Gamma \Rightarrow \Sigma)$ is the set of maximal leaves of open branches in $\tau(\Gamma \Rightarrow \Sigma)$.

A position is a reckoning of how things are. A maximal position, if coherent, is an account of how everything is. The maximal coherent positions leave nothing unsaid. They are complete stories of how the world could be. The set $\nabla(\Gamma \Rightarrow \Sigma)$ is the set of complete stories of the world that the position $\Gamma \Rightarrow \Sigma$ could tell. The complete stories that a position could tell reveal the ontological commitments of that position. If on every complete way of filling in a position a sentence of the form Fa is asserted, then that position is committed to asserting that there are F's.

Theorem 5.3.2. *For any sentence φ that does not contain elements of W, it holds that $\vdash \Gamma \Rightarrow \varphi, \Sigma$ iff for every $\Delta \Rightarrow \Lambda \in \nabla(\Gamma \Rightarrow \Sigma)$, $\varphi \in \Delta$.*

Theorem 5.3.2 says that for a position $\Gamma \Rightarrow \Sigma$ it is incoherent to deny a sentence φ of $\mathscr{L}$ while asserting all of Γ and denying all of Σ iff φ is asserted in every maximal coherent superposition of $\Gamma \Rightarrow \Sigma$. The sentences that a position rules out denying are the sentences that each of its maximal coherent superpositions asserts. Restall [52]

suggests that a position $\Gamma \Rightarrow \Sigma$ is committed to a sentence φ when it is incoherent to maintain that position and deny φ, i.e. $\vdash \Gamma \Rightarrow \varphi, \Sigma$. For instance, any position asserting $\varphi \wedge \psi$ is committed to the sentence φ because it is incoherent to assert $\varphi \wedge \psi$ and deny φ, $\vdash \varphi \wedge \psi \Rightarrow \varphi$. This account of commitment avoids it being the case that a person must know, assert, or explicitly believe what they are committed to. A person is committed to what, given their assertions and denials, it is incoherent for them to deny.

One of the benefits of this account of commitment is that it makes clear the commitments of a position that asserts a disjunction. A position that asserts $\varphi \vee \psi$ is not committed to either φ or to ψ, though it is incoherent to deny both at the same time. It is natural to move from commitment to a sentence to commitment to a set of sentences. Any position that asserts $\varphi \vee \psi$ is set-committed to $\{\varphi, \psi\}$. The disadvantage of this approach is made clear by the consideration of quantified sentences. In both of the above examples the commitments of a sentence could be expressed in terms of their sub-sentences. A position that asserts $\exists x\varphi$ is not committed to any sentence of the form $\varphi[t/x]$ and not even set-committed to any set of the form $\{\varphi[t_0/x], \varphi[t_1/x], \ldots\}$. The simple substitutional approach to quantification enforces the latter set-commitment of an existential sentence. Theorem 5.3.2 shows that Restall's account of commitment can be captured by considering the set of maximal coherent superpositions of a position. For instance, it follows from theorem 5.3.2 that if a position $\Gamma \Rightarrow \Sigma$ asserts $\varphi \wedge \psi$ then φ appears in every maximal coherent superposition of $\Gamma \Rightarrow \Sigma$. The commitments of a position which asserts an existential sentence can be described in terms of the instances of that existential sentence

appearing in maximal coherent superpositions of the original position. The ability to adequately capture the commitments incurred by asserting a sentence in terms of its sub-sentences is what makes it possible to generate an account of the ontological commitments of a position.

Definition 25 (Ontological Commitment)**.** A position $\Gamma \Rightarrow \Sigma$ is ontologically committed to there being entities of which K is true iff for each $\Delta \Rightarrow \Lambda \in \triangledown(\Gamma \Rightarrow \Sigma)$ there is a name or witness t such that $K[\![t/\xi]\!] \in \Delta$.

On this account of ontological commitment a position is committed to there being an entity of which K is true iff on every maximally coherent way of filling in that position a sentence of the form $K[\![t/\xi]\!]$ is true, i.e. K is true of something. Ontological commitment to a kind is commitment to naming something of which K is true if one were to tell the complete story of the world. According to this view to be assumed as an entity is to be reckoned nameable in all maximal coherent superpositions. This account of ontological commitment does not require that variables be given values or even that variables have "ranges". In order to learn the ontological commitments of a position, one does not need to explore the ways that variables are matched to objects in a domain, but to explore what objects are named in the maximal coherent superpositions of that position.[7]

Expanding substitutional quantifiers when paired with this account of ontological

[7]This account of ontological commitment shows formally a way of accomplishing what Sellars [67] was hoping to accomplish. He argues that the ontological commitments of a quantified sentence (one that he says "refers indeterminately") ultimately must rest on the non-quantified instances of that sentence (sentences that he says "refer determinately"). In Sellars's terminology this account of ontological commitment is an explanation of how 'indirect reference' can be explained in terms of 'direct reference'.

commitment treat variables as little more than bookkeeping devices. There is no special relation that variables have to domains in a model or to ontological commitment generally. Variables, and the quantifiers that bind them, mark which sentences may be asserted or denied in complete stories of the world. The above construction makes plain that the ontological commitments of a theory whose quantifiers are interpreted in this way need not bear a special relation to a domain of quantification.

Call the account of ontological commitment described in this section proof-theoretic ontological commitment and call the account of the previous section (section 5.2) model-theoretic ontological commitment. Before considering extensions of first-order logic the proof theoretic and the Quinean account of ontological commitment agree with one another.

Theorem 5.3.3. *A set of sentences* Γ *has a model-theoretic ontological commitment to* K *iff the position* $\Gamma \Rightarrow \quad$ *has a proof-theoretic commitment to* K*.*

5.4 Discussion

5.4.1 Second-Order Quantification

Theorem 5.3.3 shows that these two accounts of ontological commitment, at least in the first-order case, completely agree. Their disagreement will arise when certain other forms of quantification are introduced. An important philosophical difference between the two accounts is that if a quantifier binds variables whose syntactic category is in general non-denoting then the model-theoretic account of quantification requires that entities be introduced to be the values of those variables. A propo-

nent of the model-theoretic account of ontological commitment therefore cannot in good conscience allow quantifiers to bind variables whose syntactic category is non-denoting. For instance, if predicates do not denote then second-order quantifiers cannot bind variables in predicate position. This would require there to be entities to be the values of those variables.

Things go differently for the proof-theoretic account of ontological commitment. Let $\mathscr{L}$ be expanded to $\mathscr{L}_2$ by the addition of second-order quantifiers and variables. The rules governing the second-order quantifiers are

$$\mathrm{L}\exists_2 \frac{\Gamma, \varphi[F/X] \Rightarrow \Sigma}{\Gamma, \exists X\varphi \Rightarrow \Sigma} \qquad\qquad \mathrm{R}\exists_2 \frac{\Gamma \Rightarrow \varphi[F/X], \Sigma}{\Gamma \Rightarrow \exists X\varphi, \Sigma}$$

where F does not occur in the conclusion of $\mathrm{L}\exists_2$.[8] According to the specification of the consequence relation offered in section 5.3 $\mathrm{L}\exists_2$ says that if it is coherent to assert $\exists X\varphi$ while asserting Γ and denying Σ, then there is an expansion of the language by a new predicate F such that it is coherent to assert $\varphi[F/X]$. $\mathrm{R}\exists_2$ says that if it is coherent to deny $\exists X\varphi$ while asserting Γ and denying Σ then on any expansion of the language and for any predicate F it is coherent to deny $\varphi[F/X]$.

Let $\Gamma \Rightarrow \Sigma$ be a position all of whose sentences are elements of $\mathscr{L}_2$. In order to generate $\triangledown(\Gamma \Rightarrow \Sigma)$ add the set of predicates to the set of names and let W_2 be a denumerable set of second-order witnesses. Add the following step to the process that generated $\tau(\Gamma \Rightarrow \Sigma)$ in the first-order case. Let the stage in question be $i.n$ and the open leaf being considered be $\Delta \Rightarrow \Lambda$.

[8]For convenience only atomic predication is considered. A discussion of comprehension principles and quantification introduces unnecessary complication.

(a*) If φ_j is of the form $\exists X\psi$, $\varphi_j \in \Delta$, and there is no sentence of the form $\psi[W/x]$ in Δ, then take the least element of W_2 that does not appear in $\Delta \bigcup \Lambda$, W_k, and extend the branch by

$$\frac{\Delta, \varphi[W_k/X] \Rightarrow \Lambda}{\Delta \Rightarrow \Lambda}$$

Call this stage $i.n.j$.

The ontological commitments of a position are defined in the same way as the first-order case. There is nothing in this definition that requires second-order variables to be matched via variable assignments to some element of a model. Whether or not second-order sentences incur commitment to properties, attributes, sets, classes, etc. is a matter of whether predicates themselves incur such commitment.

Without anything like an existence predicate defined independently of quantification, a difference in ontology makes itself apparent in the object language only in the case that the language in question is not recursively specifiable. A standard example of such a language is second-order logic as characterized by the standard models (as opposed to Henkin models). For any language that is recursively specifiable the proof-theoretic and the model-theoretic characterizations of the language will agree. There will be no object language sentence of, for instance, second-order logic as characterized by the Henkin models over which the two accounts of ontological commitment disagree. Any sentence of one language has the same consequences and is a consequence of the same sentences in the other language. The difference between the proof-theoretic and the model-theoretic account of ontological commitment is only made apparent by considerations of how the consequence relation of

the language is itself specified and how the ontological commitments of a theory or position are to be determined. The ontological differences between the two ways of specifying the consequence relation are only made apparent when the language itself is analyzed.

Of particular interest for this account of ontological commitment is Wright's ([75]) project called abstractionism. The abstractionist program attempts to derive various branches of mathematics by the addition of abstraction principles to logic. Because the only resources used by abstractionists are logic and abstraction principles it is claimed that abstractionism has the potential to offer a plausible epistemology of mathematical truths. It is essential to the abstractionist project that any resource used that is not an abstraction principle is purely logical.

The program was initiated when it was shown that the Peano-Dedekind axioms for the natural numbers follow from an abstraction principle, Hume's Principle, in second-order logic. In order for the abstractionist program to be successful it must be the case that second-order quantification is purely logical. This claim was denied by Quine [48]. It is commonly thought that a necessary condition on a piece of vocabulary being logical vocabulary is that introducing it to a language does not require the introduction of a new category of entity. If the model-theoretic account of ontological commitment is correct second-order quantification along with a commitment to nominalism violates this constraint.

Wright [76] proposes that the ontological commitments of a second-order sentence are nothing more than the ontological commitments of any of its instances. He calls this view neutralism. The proof-theoretic account of ontological commitment

presented in section 5.3 is neutralist in this sense and so meets that need of the abstractionist project. A theory Γ containing the sentence $\exists FFa$ will be committed to there being an entity a such that in every complete story of the world that is an extension of $\Gamma \Rightarrow$ there is a sentence of the form Wa. If, however, the sentence Wa incurs no commitment to properties, attributes, classes, etc. then neither does $\exists FFa$. The sentence $\exists FFa$ indicates that a sentence of the form Wa occurs in the complete story of the world for any position in which it is asserted. It does not, thereby, say that there are properties, attributes, classes, etc. The proof-theoretic account of ontological commitment thus offers a response to an objection that second-order logic is not purely logical.[9]

5.4.2 Expanding Substitutional Quantification

This section deals with objections to the proposed method of determining the ontological commitments of a theory.

Empty Names

The first objection is plausibly what motivated Quine [47] in "On What There Is" to put forward his dictum that "to be assumed as an entity is, purely and simply, to be reckoned as the value of a variable". Let φ be a proper translation of the sentence 'Pegasus does not exist'. On the proposed account of ontological commitment the position $\varphi \Rightarrow$ is committed to the kind term "ξ does not exist". Most people are willing to endorse the position that asserts that Pegasus does not exist but are not

[9]It should be noted that it only shows that second-order logic meets a necessary condition on a consequence relation being logical consequence.

thereby committed to there being non-existent entities. The issue is how a position could, as Quine puts it, "deny Pegasus" without being committed to there being entities that do not exist.

The objection as put requires serious attention. In the construction of the maximally coherent superpositions of $\varphi \Rightarrow \quad$ no sentences are removed from the original position. φ is asserted in every member of $\nabla(\varphi \Rightarrow \quad)$. Given the proposed account of ontological commitment this position is committed to the kind-term "ξ does not exist" being non-empty. There are two avenues of response to this objection.

The first strategy follows Quine. It holds that at the stage where the theory in question is translated into the language of first-order logic names are replaced with predicates. A proper translation of "Pegasus does not exist" is the sentence $\neg\exists x Px$ where $P\xi$ is the kind-term "ξ is a pegasizer". On this strategy every position in $\nabla(\neg\exists x Px \Rightarrow \quad)$ includes an assertion of $\neg Pt$ for every term or witness, t. That is to be expected. Since Pegasus does not exist, everything fails to be a pegasizer.

A second strategy points out that the above view assumes that all names denote. An alternative to the Quinean use of predicates like "pegasizer" was pioneered by Leonard [31] and later Lambert [24].[10] Free logics allow non-denoting terms in a language. If Γ and Σ are sets of sentences and A and B sets of names then $A : \Gamma \Rightarrow \Sigma : B$ is a *name-position.*[11] $A : \Gamma \Rightarrow \Sigma : B$ is the name-position that one takes up by accepting all of A, asserting all of Γ denying all of Σ, and rejecting all of B.

According to this strategy the ability to "deny Pegasus" is taken as primitive. A term is taken not to denote by a position if that position rejects that term. The

[10]For a helpful overview of free logics see Lehmann [28].

[11]The idea of a name-position was introduced by Restall [56, 57].

rules governing quantification are modified along the lines of Restall [57] or Gratzl [19] by using the following two instead of L∃ and R∃:

$$\mathrm{L}\exists_f \frac{A, t : \Gamma, \varphi[t/x] \Rightarrow \Sigma : B}{A : \Gamma, \exists x\varphi \Rightarrow \Sigma : B} \qquad \mathrm{R}\exists_f \frac{A : \Gamma \Rightarrow \varphi[t/x], \Sigma : B \qquad A : \Gamma \Rightarrow \Sigma : t, B}{A : \Gamma \Rightarrow \exists x\varphi, \Sigma : B}$$

where t does not appear in the conclusion of $\mathrm{L}\exists_f$. $\mathrm{L}\exists_f$ says that if it is coherent to assert $\exists x\varphi$ while accepting all of A, asserting all of Γ, denying all of Σ, and rejecting all of B then there is an expansion of the language by a term t, such that it is coherent to accept t and assert $\varphi[t/x]$ while accepting all of A, asserting all of Γ, denying all of Σ, and rejecting all of B. The construction of $\tau(A : \Gamma \Rightarrow \Sigma : B)$ is only modified slightly to account for this difference. (a) is modified so that the witness w_k is added to the accepted names of the position under consideration. On this account a position $A : \Gamma \Rightarrow \Sigma : B$ is committed to a kind-term K iff for any position $C : \Delta \Rightarrow \Lambda : D \in \nabla(A : \Gamma \Rightarrow \Sigma : B)$ there is a name or witness $t \in C$ such that $K[t/\xi] \in \Delta$.

Let the sentence "Pegasus does not exist again" be φ. On the account of quantification being proposed the position $: \varphi \Rightarrow \ :$ is not committed to there being a non-existent entity. There is a position in $\nabla(: \varphi \Rightarrow \ :)$ that rejects every term. So it fails to meet the first condition for a kind-term to be empty in a maximal coherent superposition.

Unnameable Entities

Another objection to this account of ontological commitment is that it assumes that everything can be named. So stated the objection requires clarification. One way of

making this objection more precise is to rephrase it as the claim that on this account of ontological commitment there is no way for a person to commit themselves to the existence of an entity that cannot be named. This objection is only worth considering in this chapter if it goes on to add that on the model-theoretic account of ontological commitment it is possible to commit oneself to there being entities which cannot be named. But by theorem 5.3.3, this is not the case. Any ontological commitment of a first-order theory determined proof-theoretically is an ontological commitment of that theory determined model-theoretically, and vice-versa. If there is a sentence φ that says the same thing as "There are unnamable entities" then asserting it on the model-theoretic and on the proof-theoretic account commits one to the same entities.

The objection might continue by pointing out that the rule L$\exists$ *assumes* that every entity can be named and it is this fact that explains the coincidence of the two accounts of ontological commitment, i.e. the truth of theorem 5.3.3. To address this agreement let Ƹ be a quantifier *defined* by the following model-theoretic clause

* $M \models \text{Ƹ}x\varphi$ iff there is an expansion $\mathscr{L}$ of $\mathscr{L}'$ by a term t such that $M \models \varphi[t/x]$.

$\text{Ƹ}x\varphi$ is true in a model, by definition, iff there is an expansion of $\mathscr{L}$ by a term t that makes $\varphi[t/x]$ true that model. This definition does not assume that every entity can be named, but $\text{Ƹ}x\varphi$ will be true only of an entity if it can be named. Given this definition the following two proof rules are sound for the model-theoretic condition *

$$\text{L}\text{Ƹ}\,\frac{\Gamma\varphi[w/x] \Rightarrow \Sigma}{\Gamma, \text{Ƹ}\varphi \Rightarrow \Sigma} \qquad\qquad \frac{\Gamma \Rightarrow \varphi[t/x], \Sigma}{\Gamma \Rightarrow \text{Ƹ}\varphi, \Sigma}$$

where w does not occur in the conclusion in LƸ. The objection that the proof-

theoretic account assumes that there are no unnamed entities (while the model-theoretic account makes no such assumption) can now be restated as the point that there are models in which the sentence "$\exists xx$ cannot be named" are true but "$\reflectbox{$\Sigma$}xx$ cannot be named" are false. That the objection so stated is false is established by the following argument. The rules L$\exists$ and R$\exists$ are sound and complete for the model-theoretic account of quantification. The rules L$\reflectbox{$\Sigma$}$ and R$\reflectbox{$\Sigma$}$ are at least sound for the model-theoretic condition *. Let w not appear in a sentence φ of the language that results from adding $\reflectbox{$\Sigma$}$ to $\mathscr{L}$. The following is a valid deduction in that language.

$$\dfrac{\dfrac{\varphi[w/x] \Rightarrow \varphi[w/x]}{\varphi[w/x] \Rightarrow \reflectbox{Σ}x\varphi[w/x]}\,\mathrm{R}\reflectbox{Σ}}{\exists x\varphi \Rightarrow \reflectbox{Σ}x\varphi}\,\mathrm{L}\exists$$

Since the rules L$\reflectbox{$\Sigma$}$ and R$\reflectbox{$\Sigma$}$ are at least sound for the rules presented above there is no counter-example to the position $\exists x\varphi \Rightarrow \reflectbox{Σ}x\varphi$ for any φ. Thus, there is no model M that makes $\exists x\varphi$ true and $\reflectbox{$\Sigma$}x\varphi$ false. The objection so formulated cannot be sustained.

Another way of putting the objection may be as a cardinality concern. Perhaps there are more entities than there are expansions of a language. Let there be $\beth$-many entities in the domain of a model M. The objection continues that there is no way on the proof-theoretic account of ontological commitment to be committed to all those entities. The objection, if it is aimed at the proof-theoretic account of ontological commitment overshoots. The Löwenheim-Skolem theorem says that for any theory T that has M as a model there is another model with $\aleph_0$-many entities in its domain. On the model-theoretic account of ontological commitment given above, T is not *committed* to there being $\beth$-many entities. In fact, there is no sentence

of a recursively specifiable language that generates such a commitment. So long as languages are thought to be recursively specifiable, this objection is not an objection to the proof-theoretic account of ontological commitment.

A final way of stating the objection is specifically against the expanding substitutional account of quantification. The expanding substitutional account of quantification draws a tight connection between quantification and names. In certain circumstances, in particular when no one object is specifiable, quantified sentences do work that names could not. There are two responses to this objection in the literature. Lance [26] argues that we can be entitled to $\exists x F x$ even though we are not entitled to name an object. We can introduce into our language names that do not have definite denotations but whose meaning is given by the inferential connections they have to other names and predicates. For instance the name w can be introduced as a name of one of the o's without it being specified which o it denotes. It can then be used as the witness for the existential $\exists x F x$. The response denies the assumption made in the objection that every name has a specifiable denotation.

Lavine [27] has proposed an alternative answer to the objection. He argues that a language user's inability to specify the expansion of their language does not mean that there is no such expansion. The L$\exists$ rule read properly says that if it is coherent to assert $\exists x\varphi$ then there is an expansion of the language by a witness w such that it is coherent to assert $\varphi[w/x]$. This response denies that if there is an expansion of the language then there is a specifiable expansion of the language.

Because the aim of this chapter is to offer a proof-theoretic account of ontological commitment the question of which of these is correct does not need to be settled

here. It is enough for the purposes of this chapter that there are responses to the objection so stated, even if the author is unable to specify which response is the right one.

From Formal Language to Natural Language

Another concern for the account of ontological commitment proposed comes by way of considering how sentences of a language like $\mathscr{L}$ are translated into sentences of English. The sentence $\exists x(x \text{ walks})$ is commonly translated into English as 'There is a thing that walks'. Reasoning by analogy one may be tempted to translate $\exists F(F \text{ Aristotle})$ as 'There is a thing that Aristotle is (or has)'. This sentence, however, talks about *things* which other things are or have. While it makes sense to, in the same sentence name the walker as in, 'There is a thing, Zoe, that walks', the sentence 'There is a thing, person, that Aristotle is' makes no sense. This suggests that if the right translation of any existentially quantified sentence of the form $\exists \varepsilon \varphi$ is 'There is a thing such that φ' then it only really makes sense for quantifiers to bind variables in name position.

One response to this objection requires a translation to be provided for any sentence of the form $\exists \varepsilon \varphi$ into English. The translation must not have the same obvious ontological significance as the above translation of $\forall F(F \text{ Aristotle})$. This has been attempted by Boolos [8] and others following him (see e.g. Hewitt [20]) who translate the sentence $\exists F Fa$ as 'There are some things such that a is one of them'. Others such as Prior [45], Sellars [67], and Rayo and Yablo [49] offer the translation 'a is somehow'.

Another response to this objection is to attempt to show that no translation is required. The formal language $\mathscr{L}$ is a language like any other. It can be learned in the ways that other languages are learned —though perhaps not in all the ways other languages are learned. This response would maintain that learning a language need not be purely translational. One could learn under what circumstances assertions, denials, etc. are appropriate even if one does not already have the resources to translate the new language. An approach along these lines has recently been proposed by Rossberg [58]. Rossberg cites the inability of the above solution to translate relational quantifiers as in $\exists X(Xab)$ as a concern.

A middle way points out that there are devices for generalizing over predicates in English. These devices are not clearly tied to ontology and so can help orient someone bothered by quantifiers that merely generalize. For example, one can say "Aristotle is doing something"[12] if Aristotle is fixing his car or going for a walk. These sorts of sentences can serve to point a person to the meaning of sentences such as '$\exists F(F$ Aristotle)'. Once one has a grip on those sentences, then the rules for using other second-order sentences such as "$\exists X(X$ Aristotle, Plato)" can convey the meaning of those sentences

5.4.3 Atomic Ontology

The upshot of determining the ontological commitments of a position $\Gamma \Rightarrow \Sigma$ in the way discussed in section 5.3 is that they can be completely determined by an examination of the atomic sentences appearing in $\nabla(\Gamma \Rightarrow \Sigma)$. The primary locus of

[12] This sentence is importantly distinct from the sentence "Aristotle is doing some thing".

ontological commitment is thus not a variable but an atomic sentence. This frees up the apparatus of quantification from ontological significance. Quantification over a syntactic category incurs commitments to there being entities denoted by expressions of that syntactic category iff expressions of that syntactic category denote entities in atomic sentences.

This feature of quantifiers explains why first-order quantifiers are often taken to have the ontological significance they do. First-order quantifiers bind variables in name-position and names are taken to be the paradigmatic case of expressions that denote entities in atomic sentences. Given the assumption that names are denoting expressions it is clear why first-order quantifiers incur ontological commitment.

This also brings to light an assumption that when questioned leads to further avenues of research. It has been assumed throughout that the primary ontologically interesting expressions are kind-terms. Perhaps there are other sorts of ontological commitment that are of interest. If predicates denote entities then commitment to a kind-term may incur not only commitment to entities of which that kind-term is true but also to universals denoted by that kind-term.

These questions would be answered by offering a complete account of the ontological commitments of atomic sentences. A theory of how atomic sentences come to have contact with the world would, on the account of ontological commitment presented above, complete the formal answer to the primary question of ontology. If, as has been assumed, the only expressions in atomic sentences that denote are names, the account presented above is complete. But whether this assumption is justified remains to be shown.

Appendix

Lemma 5.1. *If* $\Delta \Rightarrow \Lambda \in \nabla(\Gamma \Rightarrow \Sigma)$*, then* $\Gamma \subseteq \Delta$ *and* $\Sigma \subseteq \Lambda$*.*

Proof. Let $\Gamma \Rightarrow \Sigma, \Delta_1 \Rightarrow \Lambda_1, \ldots$ be an open branch in $\tau(\Gamma \Rightarrow \Sigma)$. Since the root of this branch is $\Gamma \Rightarrow \Sigma$, $\Gamma \subseteq \Gamma \cup \bigcup_i \Delta_i$ and $\Sigma \subseteq \Sigma \cup \bigcup_i \Lambda_i$. Since that branch was arbitrary this holds for every open branch in $\tau(\Gamma \Rightarrow \Sigma)$ and thus for every position in $\nabla(\Gamma \Rightarrow \Sigma)$. □

Theorem 1. *Let* Γ *be a theory.* Γ *is committed to there being entities of which* K *is true iff for any model* M *if* $M \models \Gamma$ *then* $M, \sigma \models K[\![x/\xi]\!]$ *for some* $\sigma \in \Sigma_M$ *and* x *that is free for* K*.*

Proof. For the left to right direction let Γ be committed to there being entities of which K is true. It follows that there is no model M such that $M \models \Gamma$ and K is empty in M. Let M be a model of Γ, i.e. $M \models \Gamma$. Since K is not empty in M there is variable assignment σ and variable x such that x is free for K and $M, \sigma \models K[\![x/\xi]\!]$.

For the right to left direction let it be that for any model M if $M \models \Gamma$ then $M, \sigma \models K[\![x/\xi]\!]$ for some $\sigma \in \Sigma_M$ and X that is free for K. Suppose that Γ is not committed to there being entities of which K is true. It follows that there is a model M such $M \models \Gamma$ and K is empty in M. Since K is empty in M there is no variable assignment $\sigma \in \Sigma_M$ and variable x that is free for K such that $M, \sigma \models K[\![x/\xi]\!]$. But by the assumption there is such a σ and variable. □

Theorem 2. *The proof-theoretic and model-theoretic accounts of quantification agree, i.e.* $\Gamma \models \Sigma$ *iff* $\vdash \Gamma \Rightarrow \Sigma$*.*

Proof. This is a familiar result for first-order classical logic. For a proof of this see Ebbinghaus, Flum, and Thomas[15].

□

Theorem 3. *For any sentence φ that does not contain elements of W, it holds that $\vdash \Gamma \Rightarrow \varphi, \Sigma$ iff for every $\Delta \Rightarrow \Lambda \in \nabla(\Gamma \Rightarrow \Sigma)$, $\varphi \in \Delta$.*

Proof. For the left to right direction suppose that $\vdash \Gamma \Rightarrow \varphi, \Sigma$. Let $\Delta \Rightarrow \Lambda \in \nabla(\Gamma \Rightarrow \Sigma)$ be such that $\varphi \in \Lambda$ and β be the open branch of $\tau(\Gamma \Rightarrow \Sigma)$ from which $\Delta \Rightarrow \Lambda$ is generated. By lemma 5.1 $\Gamma \subseteq \Delta$ and $\Sigma \bigcup \{\varphi\} \subseteq \Lambda$. Since $\varphi \in \Lambda$ there is an n such that $\Delta' \Rightarrow \Lambda'$ in β and $\varphi \in \Lambda'$. Let $\delta \vdash \Gamma \Rightarrow \varphi, \Sigma$. The following deduction establishes that $\vdash \Delta' \Rightarrow \Lambda'$ and thus that β is not an open branch in $\tau(\Gamma \Rightarrow \Sigma)$ contradicting the assumption that it was.

$$\begin{array}{c} \delta \\ \vdots \\ \text{TL/ TR}\,\dfrac{\Gamma \Rightarrow \varphi, \Lambda}{\Delta' \Rightarrow \Lambda'} \end{array}$$

For the right to left direction suppose that $\varphi \in \Delta$ for every $\Delta \Rightarrow \Lambda \in \nabla(\Gamma \Rightarrow \Sigma)$ and for reductio that

(H) $\nvdash \Gamma \Rightarrow \varphi, \Sigma$

Let φ be the n^{th} sentence. Suppose that at stage n of the construction there is a branch $\beta \in \tau(\Gamma \Rightarrow \Sigma)$ with an open leaf $\Delta' \Rightarrow \Lambda'$ such that $\nvdash \Delta' \Rightarrow \varphi, \Lambda'$. At this stage β is at least extended to β' with open leaf $\Delta' \Rightarrow \varphi, \Lambda'$. But in this case there is a branch β'' that passes through β' when the construction of $\tau(\Gamma \Rightarrow \Sigma)$ is completed such that the position $\Delta \Rightarrow \Lambda$ is the maximal leaf of β'' and $\varphi \in \Lambda$. Since $\vdash \Delta' \Rightarrow \varphi, \Lambda'$ $\varphi \notin \Delta$, contradicting the assumption. This establishes

(B) There is no open leaf $\Delta' \Rightarrow \Lambda'$ at stage n such that $\nvDash \Delta' \Rightarrow \varphi, \Lambda'$.

Claim 2. *There is no stage i such that there is an open leaf $\Pi \Rightarrow \Theta$ in the construction of $\tau(\Gamma \Rightarrow \Sigma)$ where $\nvDash \Pi \Rightarrow \varphi, \Theta$ but at $i+1$ there is no open leaf $\Pi' \Rightarrow \Theta'$ such that $\nvDash \Pi' \Rightarrow \varphi, \Theta'$.*

Suppose that claim 2 is true. It follows that for any stage i if there is an open leaf $\Pi \Rightarrow \Theta$ at a stage i such that $\nvDash \Pi \Rightarrow \varphi, \Theta$ then there is an open leaf $\Pi' \Rightarrow \Theta'$at $i+1$ such that $\nvDash \Pi' \Rightarrow \varphi, \Theta'$. By (H) at stage 0, there is an open leaf $\Gamma \Rightarrow \Sigma$ such that $\nvDash \Gamma \Rightarrow \varphi, \Sigma$. This contradicts (B).

This leaves only the proof of claim 2. It is proved by reductio. Let i be a stage such that there is an open leaf $\Pi \Rightarrow \Theta$ in the construction of $\tau(\Gamma \Rightarrow \Sigma)$ where $\nvDash \Pi \Rightarrow \varphi, \Theta$ but at $i+1$ there is no open leaf $\Pi' \Rightarrow \Theta'$ such that $\nvDash \Pi' \Rightarrow \varphi, \Theta'$. To be specific let $\Pi \Rightarrow \Theta$ be the position under consideration at $i.j.k$. Either there is a step $i.j.k+l$ with position $\Pi' \Rightarrow \Theta'$ such that $\vdash \Pi' \Rightarrow \varphi, \Theta'$ or not.

Case 1 (There is a step $i.j.k+l$ with position $\Pi' \Rightarrow \Theta'$ such that $\vdash \Pi' \Rightarrow \varphi, \Theta'$). In this case $\Pi' \Rightarrow \Theta'$ is $\Pi, \psi[w/x] \Rightarrow \Theta$. It follows that $\exists x\psi_j \in \Pi$, w does not appear in $\Pi \bigcup \Theta$, $\nvDash \Pi, \exists x\psi_j \Rightarrow \Theta$. It was assumed that no witnesses occur in φ so w does not appear in φ. By assumption $\vdash \Pi, \exists x\psi_j, \psi[w/x] \Rightarrow \varphi, \Theta$ and $\nvDash \Pi, \exists x\psi_j \Rightarrow \varphi, \Theta$. However, L$\exists$ read from bottom to top guarantees that if $\nvDash \Pi, \exists x\psi_j \Rightarrow \varphi, \Theta$ and w does not occur in $\Pi \bigcup \Theta \bigcup \{\varphi\}$ then $\nvDash \Pi, \exists x\psi_j, \psi[w/x] \Rightarrow \varphi, \Theta$ contradicting the assumption.

Case 2 (There is no step $i.j.k+l$ with position $\Pi' \Rightarrow \Theta'$ such that $\vdash \Pi' \Rightarrow \varphi, \Theta'$). Without loss of generality let there be no existential steps between i and $i+1$. If

there are then there is a point at which case 1 applies. There are then three sub-cases for how i is extended.

Case 1 (The branch with leaf $\Pi \Rightarrow \Theta$ is expanded according (b) in the construction of $\tau(\Gamma \Rightarrow \Sigma)$). In this case $\nvdash \Pi, \psi_i \Rightarrow \Theta$ and $\nvdash \Pi \Rightarrow \psi_i, \Theta$. By assumption there are no open leaves $\Pi' \Rightarrow \Theta'$ at this stage such that $\vdash \Pi' \Rightarrow \varphi, \Theta'$, so there are deduction δ and δ' such that $\delta \vdash \Pi, \psi_i \Rightarrow \varphi, \Theta$ and $\delta' \vdash \Pi \Rightarrow \psi_i, \varphi, \Theta$. It was also assumed that $\nvdash \Pi \Rightarrow \varphi, \Theta$. The following deduction contradicts this assumption.

$$\dfrac{\begin{matrix}\delta \\ \vdots \\ \Pi, \psi_i \Rightarrow \varphi, \Theta\end{matrix} \qquad \begin{matrix}\delta' \\ \vdots \\ \Pi \Rightarrow \psi_i, \varphi, \Theta\end{matrix}}{\Pi \Rightarrow \varphi, \Theta}\,\text{Cut}$$

Case 2 (The branch with leaf $\Pi \Rightarrow \Theta$ is expanded according to (c) in the construction of $\tau(\Gamma \Rightarrow \Sigma)$). In this case there is a δ such that $\delta \vdash \Pi, \psi_i \Rightarrow \Theta$ but $\nvdash \Pi \Rightarrow \psi_i, \Theta$. By assumption there is no open leaf $\Pi' \Rightarrow \Theta'$ at this stage such that $\nvdash \Pi' \Rightarrow \varphi, \Theta'$. It follows that there is a δ' such that $\delta' \vdash \Pi \Rightarrow \psi_i, \varphi, \Theta$. The following deduction contradicts the assumption that $\nvdash \Pi \Rightarrow \varphi, \Theta$.

$$\dfrac{\dfrac{\begin{matrix}\delta \\ \vdots \\ \Pi, \psi_i \Rightarrow \Theta\end{matrix}}{\Pi, \psi_i \Rightarrow \varphi, \Theta}\,\text{TR} \qquad \begin{matrix}\delta' \\ \vdots \\ \Pi \Rightarrow \psi_i, \varphi, \Theta\end{matrix}}{\Pi \Rightarrow \varphi, \Theta}\,\text{Cut}$$

Case 3 (The branch with leaf $\Pi \Rightarrow \Theta$ is expanded according to (d) in the construction of $\tau(\Gamma \Rightarrow \Sigma)$). In this case there is a δ such that $\delta \vdash \Pi \Rightarrow \psi_i, \Theta$ but $\nvdash \Pi, \psi_i \Rightarrow \Theta$. By assumption there is no open leaf $\Pi' \Rightarrow \Theta'$ at this stage such that $\nvdash \Pi' \Rightarrow \varphi, \Theta'$.

It follows that there is a δ' such that $\delta' \vdash \Pi, \psi_i \Rightarrow \varphi \Rightarrow \Theta$. The following deduction contradicts the assumption that $\nvdash \Pi \Rightarrow \varphi, \Theta$.

$$\dfrac{\mathrm{TR}\ \dfrac{\begin{matrix}\delta\\ \vdots\\ \Pi \Rightarrow \psi_i, \Theta\end{matrix}}{\Pi \Rightarrow \psi_i, \varphi, \Theta} \qquad \begin{matrix}\delta'\\ \vdots\\ \Pi, \psi_i \Rightarrow \varphi, \Theta\end{matrix}}{\Pi \Rightarrow \varphi, \Theta}\ \mathrm{Cut}$$

□

Theorem 4. *A set of sentences Γ has a model-theoretic ontological commitment to K iff the position $\Gamma \Rightarrow$ has a proof-theoretic commitment to K.*

Proof. For the left to right direction suppose that Γ has a model-theoretic commitment to K. By theorem 5.2.1 for any model M such that $M \vDash \Gamma$ there is a $\sigma \in \Sigma_M$ and variable x that is free for K such that $M, \sigma \vDash K[\![x/\xi]\!]$. It follows that $M \vDash \exists x K[\![x/\xi]\!]$. Since this holds for any model M it follows that $\Gamma \vDash \exists x K[\![x/\xi]\!]$. By theorem 5.3.1 $\vdash \Gamma \Rightarrow \exists x K[\![x/\xi]\!]$. It follows from theorem 5.3.2 that for any $\Delta \Rightarrow \Lambda \in \nabla(\Gamma \Rightarrow\ )$, $\exists x K[\![x/\xi]\!] \in \Delta$. Let $\Delta \Rightarrow \Lambda \in \nabla(\Gamma \Rightarrow\ )$ that is the maximal leaf of a branch β. Since $\exists x K[\![x/\xi]\!] \in \Delta$ there is a node $\Pi \Rightarrow \Theta$ in β such that $\exists x K[\![x/\xi]\!] \in \Pi$ was considered at stage $n.j$ of the construction of $\tau(\Gamma \Rightarrow\ )$. Let $\exists x K[\![x/\xi]\!]$ be the k^{th} sentence. At stage $n.j.k$ a sentence the node directly above $\Pi \Rightarrow \Theta$ is $\Pi, K[\![w/\xi]\!]$ for a witness w. By lemma 5.1 $K[\![w/\xi]\!] \in \Delta$. Since $\Delta \Rightarrow \Lambda$ was selected arbitrarily it holds that for any $\Delta \Rightarrow \Lambda \in \nabla(\Gamma \Rightarrow\ )$ there is a sentence of the form $K[\![w/\xi]\!] \in \Delta$. It follows that $\Gamma \Rightarrow$ has a proof-theoretic commitment to K.

For the right to left direction suppose that $\Gamma \Rightarrow \quad$ has a proof-theoretic commitment to K. It follows that for any $\Delta \Rightarrow \Lambda \in \nabla(\Gamma \Rightarrow \quad)$ there is a sentence of the form $K[\![w/\xi]\!] \in \Delta$. The sentence $\exists x K[\![x/\xi]\!]$ is closed and contains no witnesses. Since $K[\![w/\xi]\!] \in \Delta$ $\exists x K[\![x/\xi]\!] \notin \Lambda$. If it were then $\Delta \Rightarrow \Lambda$ would not be coherent. Since $\Delta \Rightarrow \Lambda$ is maximal $\exists x K[\![x/\xi]\!] \in \Delta$. By theorem 5.3.2 $\vdash \Gamma \Rightarrow \exists x K[\![x/\xi]\!]$. By theorem 5.3.1 $\Gamma \models \exists x K[\![x/\xi]\!]$. Let M be a model of Γ. It follows that $M \models \exists x K[\![x/\xi]\!]$ and that there is a $\sigma \in \Sigma_M$ such that $M, \sigma \models K[\![x/\xi]\!]$. So K is not empty in M. Since M was arbitrary it holds that there are no models of Γ for which K is empty. Γ has a model-theoretic commitment to K.

$\square$

Chapter 6

Theories of Meaning and Existence

Abstract. This paper motivates an account of first-order quantification that addresses concerns with non-denoting names. It begins with a traditional puzzle about non-denoting terms and offers an explanation of how the proposed account of quantification can be motivated and resolve the problem.

Keywords. Non-denoting terms, Free-logic, Ontological commitment

6.1 An Argument

The following argument has been put forth for consideration in various guises throughout the history of philosophy. The conclusion of the argument is that every meaningful genuine proper name must denote. Consider, for example, the sentence ‘Pegasus does not exist’. This sentence is true. But if it is to be true, it must be true of something. The only candidate is Pegasus. But if the sentence is about Pegasus, then

Pegasus must in some sense *be*, otherwise there would be no way for a sentence to be about Pegasus. So perhaps it is true that Pegasus does not exist, but nonetheless Pegasus has some sort of being. Denying that Pegasus exists requires that there be something that does not exist. But this can be generalized: for any singular term, either what it denotes exists or not.

This argument is formally underwritten by classical logic. As is standard practice, let the claim 'Pegasus exists' be formalized as '$\exists x(x = p)$', where 'p' is another name for Pegasus. The following deduction derives a contradiction from the claim that Pegasus does not exist

$$\dfrac{\dfrac{\dfrac{\neg\exists x(x = p)}{\forall x(x \neq p)}}{p \neq p} \qquad \dfrac{}{p = p}}{p = p \wedge p \neq p}$$

Solutions to this puzzle have become part of the standard curriculum for students of analytic philosophy. Famously Russell [60] offers a solution wherein he claims that 'Pegasus' is not a logically proper name, but a disguised definite description. On this account to deny that Pegasus exists is to deny that there are any flying horses. But that is no contradiction at all. Quine [47] takes this solution a step further proposing that in an ideally regimented language there would be no logically proper names.

Plantinga [38] calls the argument of the first paragraph the *Classical Argument*. The *Classical Argument* can be made explicit in the following way:

1. Pegasus does not exist.

2. For any true sentence if it features a name, then that name denotes.

3. Premise (1) is true and features the name 'Pegasus'.

C 'Pegasus' denotes.

The conclusion of the argument is that the singular term 'Pegasus' denotes. This suggests that it is possible to name those entities that do not exist.

Philosophers such as Meinong and Parsons [37] suggest that this argument is sound, and posit a difference between entities that exist and those that are. Call such an account of being a Dense Ontology. The conclusion of this paper is that such a distinction is under-motivated and can be avoided while maintaining most of the intuitions that underwrite the classical argument. The conclusion of this paper is that the Classical Argument does not underwrite a Dense Ontology.

The methodology of the paper is to apply a theory of meaning that has been developed by Rumfitt [59] and Restall [50, 53], and apply it to the issue at hand. This will result in an argument from the theory of meaning to a metaphysical conclusion, and will provide a formalism to underwrite the conclusions drawn by Plantinga [38]. In the section 6.7 a model theory is shown to be sound and complete for the theory of meaning that has been offered.

6.2 A Theory of Meaning

The theory of meaning endorsed in this paper is inferentialist in the following sense: it is the use of a sentence that determines the meaning of that sentence. A declarative sentence has two primary uses, it can he asserted or denied. The meaning of a sentence is determined by the rules governing the coherent and incoherent assertion

and denial of that sentence. [1] Following Restall [53], a *position* is an ordered pair of assertions and denials. Positions can be either coherent or incoherent.

The incoherence in question is the sort of incoherence that is engendered by someone who asserts and denies the same thing. If a person asserts 'It is snowing' and denies 'It is snowing' while refusing to retract either the assertion or the denial, then one can reasonably conclude that either the person was not asserting and deny the same sentence, but some homophones, or that they had done something rationally wrong. There is no uniform way of understanding what position that person intends to be adopting, assuming they intend to remain understandable. Throughout, this is the sort of incoherence that is referred to.

The meanings of the sentences in a position determine whether or not that position is incoherent. A formalization of the language thus makes explicit and precise the rules governing the use of certain sentences. In particular, they make precise the rules governing those sentences featuring the portion of the language for which the theory of meaning is being given. It also allows for strict criteria of adequacy to be stated and shown to hold of the theory. Without such constraints, the concern that nothing has been clearly enough stated to be of value has greater pull, especially so in the case of a theory of meaning.

[1] Coherence of assertion or denial is not a feature of the sentences themselves, but a feature of the contexts in which they are asserted or denied. some sentences may be incoherent to assert in any context, and that may underwrite the tendency to call such sentences incoherent. For instance, there is a tendency to call the sentence 'It is raining and not.' incoherent. This cannot mean that the sentence has no meaning; it is composed in a standard way from the meanings of 'it is raining', 'and', and 'not'. Calling the sentence 'It is raining and not' incoherent amounts to claiming that it is incoherent, in any context, to assert that sentence. Coherence is a feature of the context in which a sentence is asserted or denied, not a feature of the sentence itself. The relevant notion of context is a position.

6.2.1 An Example Theory of Meaning

Let $\{p_1, p_2, \ldots\}$ be a denumerable set of atomic sentences. These are the set of sentences whose meaning assumed to be given. The sequent calculus given in fig. 1.1 gives the rules determining the meanings of complex sentences using atomic sentences as a base. The complete set of sentences of the language is given by the recursive definition

- If $\varphi \in \{p_1, p_2, \ldots\}$, then φ is a sentence.
- If φ is a sentence then $\neg\varphi$ is a sentence.
- If φ and ψ are sentences then $(\varphi \to \psi)$ is a sentence.
- Nothing else is a sentence.

A sequent is a pair of sets of sentence. If Γ and Σ are sets of sentences, then $\Gamma \Rightarrow \Sigma$ is a sequent. The sequent $\Gamma \Rightarrow \Sigma$ corresponds to the position that asserts all of Γ and denies all of Σ. A calculus such as fig. 1.1, determines the set of coherent and incoherent sequents. A derivation is a tree whose root is a sequent, whose leaves are axioms, and is such that for each node in the tree, the node(s) above are the premises of a rule of which the original node is the conclusion. If a sequent is derivable, then the position to which it corresponds is incoherent. If a sequent S is derivable, this is write, $\vdash S$. $\vdash S$, therefore, indicates that the position corresponding to S is incoherent. At a minimum, if two sentences have the same meaning, then the positions in which it is incoherent to assert or deny that sentence must be the same. No commitment here is made that the converse must hold. It is,

Figure 6.1: Propositional Logic

Structural Rules

$$\text{Id}\ \frac{}{p \Rightarrow p} \qquad \text{Cut}\ \frac{\Gamma \Rightarrow \varphi, \Sigma \qquad \Gamma, \varphi \Rightarrow \Sigma}{\Gamma \Rightarrow \Sigma}$$

$$\text{WL}\ \frac{\Gamma \Rightarrow \Sigma}{\Gamma, \varphi \Rightarrow \Sigma} \qquad \text{WR}\ \frac{\Gamma \Rightarrow \Sigma}{\Gamma \Rightarrow \varphi, \Sigma}$$

Operational Rules

$$\text{L}\neg\ \frac{\Gamma \Rightarrow \varphi, \Sigma}{\Gamma, \neg\varphi \Rightarrow \Sigma} \qquad \text{R}\neg\ \frac{\Gamma, \varphi \Rightarrow \Sigma}{\Gamma \Rightarrow \neg\varphi, \Sigma}$$

$$\text{L}\!\rightarrow\ \frac{\Gamma, \psi \Rightarrow \Sigma \qquad \Gamma \Rightarrow \varphi, \Sigma}{\Gamma, \varphi \rightarrow \psi \Rightarrow \Sigma} \qquad \text{R}\!\rightarrow\ \frac{\Gamma, \varphi \Rightarrow \psi, \Sigma}{\Gamma \Rightarrow \varphi \rightarrow \psi, \Sigma}$$

without further argument, plausible to hold that $\neg(p \wedge q)$ and $\neg p \vee \neg q$ do not have the same meaning, though they are assertible or deniable in all the same positions.

The rules of a calculus thus have an intuitive reading in terms of coherence and incoherence. The structural rules correspond to the the rules that govern assertion and denial itself. Id corresponds to the fact, indicated above, that assertion and denial are exclusive uses of sentences. It is incoherent to assert and deny the same sentence. WL and WR correspond to the fact that if a position is incoherent, more assertions and denial will not make it coherent. Contrapositively, if a position is coherent, taking back assertions and denials will preserve its coherence. The rule of Cut says that if it is incoherent to assert φ in a position, and it is incoherent to deny φ in that position, then that position is itself incoherent. Reading Cut

contrapositively, it amounts to the claim that if a position is coherent, then for any sentence φ, it must be either coherent to assert φ or coherent to deny φ. For any coherent position, there is a coherent way of expanding it to one such that for any sentence of the language, it is either asserted or denied.

The operational rules of the language give the use of sentences that is particular to the main connective of the sentence. The rules of negation establish that if it is incoherent to assert a sentence in a position, then it is incoherent to deny the negation of that sentence in that position; and if it is incoherent to deny a sentence in a position then it is incoherent to assert the negation of that sentence in that position. The meaning of 'It is not raining', is thus given by the rules saying that it is incoherent to assert 'It is not raining' when it is incoherent to deny 'It is raining', and vice-versa for denial and assertion. The L$\rightarrow$ rule says that if it is incoherent to assert ψ in a position and incoherent to deny φ in that position, then it is incoherent to assert $\varphi \rightarrow \psi$, in that position. This rule may be more illuminating when considered contrapositively as the claim that if it is coherent to assert the sentence $\varphi \rightarrow \psi$ in a position then it must either be coherent to assert ψ in that position or coherent to deny φ in that position. The R$\rightarrow$ rule claims that if it is coherent to deny $\varphi \rightarrow \psi$, in a position, then it is coherent to assert φ and deny ψ in that position.

This formalization makes the following correlates of intuitive notions definable.

Definition 26 (Commitment)**.** A position $\Gamma \Rightarrow \Sigma$ is *committed* to a sentence φ iff $\vdash \Gamma \Rightarrow \varphi, \Sigma$.

A position is committed to a sentence, when it is incoherent to deny that sentence, in that position.

Definition 27 (Good Inference)**.** There is a *good inference* from a set of sentences Γ to a sentence φ iff $\vdash \Gamma \Rightarrow \varphi$.

If adopting a set of sentences, Γ commits one to a sentence, φ, then there is a good inference from Γ to φ. The validity of the Cut rule ensures that if it is coherent to assert all of Γ, then whenever there is a good inference from Γ to φ then it is coherent to assert φ.[2]

Definition 26 and definition 27 illustrate the link between the theory of meaning for the logical connectives and logic as it is normally conceived as the theory of good inference. Fixing in which positions a sentence is coherent to assert or deny, thus fixes which inferences featuring the vocabulary in question are good.

In what follows the theory of meaning given by fig. 1.1 will be expanded to include quantification. In order to account for this, the language must be expanded to replace the set of atomic sentences above, with atomic sentences that make explicit the names and predicates featuring in them. As above, the meaning of names and predicates will be assumed to be given.[3]

6.2.2 Criteria for a Theory of Meaning

In what follows it will be important to have a concrete example to refer to. Let the following rule be that example

$$\Box\ \frac{\Gamma \Rightarrow \varphi}{\Box\Gamma \Rightarrow \Box\varphi}$$

[2] Let $\nvdash \Gamma \Rightarrow$, but $\vdash \Gamma \Rightarrow \varphi$. Cut guarantees that when $\nvdash \Gamma \Rightarrow$, either $\nvdash \Gamma, \varphi \Rightarrow$ or $\nvdash \Gamma \Rightarrow \varphi$. So $\nvdash \Gamma, \varphi \Rightarrow$.

[3] Even though the meaning of these expressions must be assumed to be given, something can still be learned about what the contribution is of those expressions to sentences in which they feature.

where $\Box\gamma = \{\Box\Gamma : \gamma \in \Gamma\}$. This rule gives the use of sentences featuring $\Box$. It generates the conditions under which it is coherent or incoherent to assert or deny a sentence featuring $\Box$. As with the rules above, it can be read in two ways. From top-to-bottom it says that if the position $\Gamma \Rightarrow \varphi$ is incoherent, then the position $\Box\Gamma \Rightarrow \Box\varphi$ is incoherent. From bottom-to-top the rule says that if the position $\Box\Gamma \Rightarrow \Box\varphi$ is coherent, then the position $\Gamma \Rightarrow \varphi$ is coherent.

Following Restall [53], the reading of the sequent rules from bottom-to-top suggests a way of expanding coherent sequents into ideally coherent sequents. Given a coherent sequent $\Gamma \Rightarrow \Sigma$, it is possible to apply one of the rules of which it is a conclusion to get other coherent sequents, $\Delta_1 \Rightarrow \Lambda_1, \ldots, \Delta_n \Rightarrow \Lambda_n$. This process can then be repeated until applying a rule would no longer yield a new coherent sequent. The result is a set of coherent sequents that contain all the sub-sentences of sentences in the original sequent. That set of coherent sequents can be thought of as generating a set of models, and so the notions of truth-in-a-model, denotation, etc. can be generated from the rules of the sequent calculus. This process is carried out in a standard completeness proof where a model is built from an underivable sequent. An example of this process can be found in section 6.7.

In this sense the truth-in-a-model conditions of a sentence in a position can be derived from the rules of the calculus. A coherent position carries a set of coherent expansions with it, that ideally considered give truth conditions for the sentences contained in the position. This is an example of how it is possible to derive what are generally considered to be representational aspects of language from their use. In the above case, the notion of truth conditions is derived from the notion of a coherent

position.

Since this account of language takes inference as explanatorily prior, the following criteria for a meaning conferring rule can be introduced. These are necessary, though not necessarily sufficient conditions for a set of rules to be meaning conferring. As mentioned above, inferentialism is the claim that the use of a sentence determines the meaning of that sentence. The rules of a sequent calculus correspond to those rules of use. It is necessary, then, to guarantee that the rules have determined a single meaning. If the rules that are given do not pin down a particular meaning, then it is hard to see how they could be rightly considered to be meaning *determining*. This corresponds to the requirement that Belnap [4] and Humberstone [21] calls *uniqueness*. If the rules given do, in fact, pin down the meaning of a sentence, then any connective introduced with those rules must be coherent and incoherent in all the same sequents.

For example, let the connective $\Box_0$ be governed by the rule[4]

$$\Box_0 \frac{\Gamma \Rightarrow \varphi}{\Box_0\Gamma \Rightarrow \Box_0\varphi}$$

A language containing only $\Box$ and $\Box_0$, cannot deduce the sequent $\Box p \Rightarrow \Box_0 p$. In other words, it is coherent, in this language to assert $\Box p$ and deny $\Box_0 p$. Though it is incoherent to assert $\Box p$ and deny $\Box p$. If a sentence is coherent to assert where another is incoherent to assert then those two sentences cannot have the same meaning. So in this language, $\Box p$ and $\Box_0 p$ have distinct meanings. But if the rules governing the use of a connective must pin down a single meaning, then this example shows that $\Box$ is insufficient to do this. Of a language containing only $\Box$, one could ask whether

[4]As above, $\Box_0\Gamma = \{\Box_0\gamma : \gamma \in \Gamma\}$.

the sentence $\Box p$ in the original language corresponded to $\Box p$ or $\Box_0 p$ in the expanded language. The rule $\Box$ does not determine uniquely a meaning for sentences of the form $\Box\varphi$.

The above example leads to the following criterion for a sequent calculus to provide an adequate account of meaning: For each n-ary connective $\odot$ it must be that the following two facts hold for any Γ, Σ, and $\odot_0$ introduced by the same rules as $\odot$

1. $\vdash \Gamma, \odot(\varphi_1, \ldots, \varphi_n) \Rightarrow \Sigma$ iff $\vdash \Gamma, \odot_0(\varphi_1, \ldots, \varphi_n) \Rightarrow \Sigma$.

2. $\vdash \Gamma \Rightarrow \odot(\varphi_1, \ldots, \varphi_n), \Sigma$ iff $\vdash \Gamma \Rightarrow \odot_0(\varphi_1, \ldots, \varphi_n), \Sigma$.

Call this requirement, following Humberstone [21], the *Uniqueness Criterion*.

The second criterion arises from considerations about the broad category of language that is being theorized about. Because conditionals and negations seem to be expressions that span various sub-species of a language, the next consideration will be in force. Before philosophical speculation, it appears as though biologists and bankers can communicate about issues not concerning their specialty with an adequate understanding of one another. Particularly, it appears as though they both fully grasp the meaning of conjunction. It is at least counter-intuitive to say that what the biologist means by 'I'm hungry and I'm tired' is no different from what the banker means by that same sentence. While there are portions of their dialect that diverge, there is a large swath of overlap between them. The theory of meaning that gives an account of such broadly linguistic notions as conjunction, conditional, negation, and quantification, should therefore do honor to this feature of our linguistic experience.

Suppose then that there were two such languages as the one spoken by the biologist, L_1, and the banker, L_2. Let L be the language that those two have in common. L contains a conjunction, for convenience it will be written as '$\wedge$'. Suppose that p and q are sentences completely in the language L, and that if no-one had every spoken L_1 or L_2, the position $p \wedge q \Rightarrow$ would be coherent, i.e. it is coherent in L, $L \nvdash p \wedge q \Rightarrow$ [5] Once L is expanded into L_1 and L_2, however, it turns out that $L_1 \nvdash p \wedge q \Rightarrow$ and $L_2 \vdash p \wedge q \Rightarrow$. This means that $p \wedge q$ cannot have the same meaning in L_1 and L_2. But as remarked above, it does not seem like knowledge of biology or knowledge of banking should change the meaning of a conjunction having to do with neither of those subjects.[6]

To put this matter more formally, let L_1 be an expansion of a language L by the addition of some new vocabulary. Let Γ and Σ be sets of sentences completely in the language L. The following must hold:

- $L \vdash \Gamma \Rightarrow \Sigma$ iff $L_1 \vdash \Gamma \Rightarrow \Sigma$.

In the current setting this fact is guaranteed by the admissibility of the Cut rule.

Definition 28 (Admissible Rule)**.** A rule R is *admissible* for a calculus, when for any derivation, δ, of $\Gamma \Rightarrow \Sigma$, in that calculus using R, there is a derivation δ', of $\Gamma \Rightarrow \Sigma$ that does not make use of R.

In the current setting the admissibility of the Cut rule guarantees that if there is a deduction of a sequent $\Gamma \Rightarrow \Sigma$, then there is a deduction wherein only sentences

[5] The language in which a position is coherent or not will be prefixed to $\vdash$. For example. $L^* \vdash \Gamma \Rightarrow \Sigma$ says that $\Gamma \Rightarrow \Sigma$ is incoherent in L^*.

[6] This requirement cannot hold for all language, it is plausible that there are some inferential relations among atomic sentences for which this will fail. The concern here, however, is with non-atomic language that appears to have a uniform meaning across a wide variety of discourse.

appearing in $\Gamma \cup \Sigma$ or their sub-sentences appear. If that is the case, then no sentence not in the language of $\Gamma \cup \Sigma$ can effect whether or not $\Gamma \Rightarrow \Sigma$ is derivable.

6.3 Adding Names

In the case of propositional logic, a sequent calculus provides an account of the contribution that a connective, say $\wedge$ makes to the sentences in which it occurs. This generates an account of the meaning of $\wedge$, and sentences such as $\varphi \wedge \psi$, given the meanings of φ and ψ.

Below several sequent calculi are presented that offer an account of the contribution that names make to atomic sentences in which they occur, and ultimately to quantified sentences. Just as there are two uses of atomic sentences, and sentences generally, there are two uses of names. Names can be accepted or rejected.[7]

Suppose it turns out that there was no individual answering to the name Homer. Under this supposition, the historical blind poet does not exist and never existed. The proper name 'Homer' would have no denotation. In the following dialog, speaker B rejects the name 'Homer'.

A: I was reading the Illiad a couple of days ago and decided to read up on its author, Homer.

B: No, no. As it turns out, there was no Homer.

[7] In 'On What There Is' Quine comes close to the terminology used here. For instance he says, "...what people are talking about when they deny Pegasus". If one reads that article as though Quine is looking for a way to deny Pegasus then the view expressed here responds to Quine by taking it as primitive that Pegasus can be denied.

There are two things worth noting about this dialog. The first is that *B* could have accomplished the same thing by simply saying 'No, no'. The second is that *B*'s utterance of 'No, no.' is not denying any *sentence* uttered by *A*.

If *B* had only said 'No, no.', they could have made the same move that they did in the first dialog. The fact that they go on to elaborate their position by uttering 'As it turns out, there was no person Homer', does not alter the force of what is *B* intended to say by means of 'No, no'. There may be other ways for a person in *B*'s position to go on after an utterance of 'No, no.' One could go on to say 'The life of Homer is really boring' or 'You don't have time to waste on Homer'. It suffices for the purposes at hand that 'No, no' can be used to express the attitude that 'Homer' does not denote.

That *B* is objecting to this presupposition is made clear by noting that there is no *sentence* uttered by *A*, to which *B* objects. It is coherent for *B* to accept everything that *A* has said. *A* can have read the Illiad, and have decided to read up on Homer. *B* can, accept both those claims and still coherently say 'No, no.' Without predicates such as 'exist' or quantification, there is no *sentence* that be could assert that would convey what was conveyed by their use of 'No, no', though without such vocabulary, *B* could still object to *A*'s endeavor on the grounds that they had rejected 'Homer'. The force of *B*'s 'No, no' was to reject the name 'Homer'.

Similarly, the following dialog is an instance of *B* accepting a name. As above, *B* is not accepting any *sentence* uttered by *A*, but is accepting the name 'Alex'.

A: I was talking to Alex yesterday.

B: Alex? Oh, yeah, Alex.

In this dialog B considers whether to accept the name 'Alex', then goes on to do so. Again, it would be coherent for B to go on to reject what A said. B could know that Alex was in another country yesterday whereas A was not. In such a case, B would be rejecting A's claim, while still accepting the name 'Alex'. In such a case, it would be wrong for A to criticize B for saying 'Oh, yeah, Alex'. B's saying that had nothing to do with their accepting or rejecting whether or not A did in fact meet Alex yesterday.[8]

The theory of meaning to be proposed must account for this aspect of the use of names. This, however, requires that the notion of a position be expanded to include accepted and rejected names. A name-position, then, is an ordered quadruple consisting of two sets of names and two sets of sentences. If A and B are sets of names, and Γ and Σ are sets of sentences, then the name-sequent $A : \Gamma \Rightarrow \Sigma : B$, corresponds to the position that accepts all of A, asserts all of Γ, denies all of Σ, and rejects all of B.

It is worth mentioning that a theory of language whereby names can be accepted or rejected rejects something that some have taken to be a central feature of a language: that the smallest unit of linguistic responsibility is a sentence. If names can be accepted and rejected then one can be responsible for the names that one accepts and rejects. On this view the smallest unit of responsibility is a name. This does not imply that all linguistic responsibility can be reduced to the responsibility that is incurred by the acceptance or the rejection of a name. Responsibility for

[8] While in *Reference and Generality*, Geach [17] suggests that there are uses of names independent of predication [Ch 2. ¶20], the uses he suggests appear to be different from the uses suggested in the previous two paragraphs.

assertions and denials, which is of a different sort altogether is still required to have a full blown linguistic practice. One could not reduce sentential responsibility to the acceptance and rejection of names.

As mentioned above, sequent calculi can be used to give the coherent and incoherent name-positions of a language. If a the name-sequent $A : \Gamma \Rightarrow \Sigma : B$ is derivable in a calculus, then the position corresponding to that sequent is incoherent. fig. 6.2 gives a sequent calculus for name-sequents from which the exploration of a theory of meaning can begin.

6.4 Formal Theory

Before presenting the formal theory itself, the language considered must be expanded. Let the language include a denumerably infinite set of names, $N = \{c_1, c_2, \ldots\}$, and for each n let $Pred_n = \{F_1^n, F_2^n, \ldots\}$ be a denumerably infinite set of n-ary relations. Let the set of predicates of the language, $Pred = \cup_n \in \mathbb{N} Pred_n$, be the union of all sets of n-ary predicates. Finally, let $Var = \{x_1, x_2, \ldots\}$ be a denumerably infinite set of variables. Given a sentence, φ, let $\varphi[x_j/c_i]$ be the result of replacing some occurrences of c_i in φ, by x_j. The set of sentences is given by the following recursive definition

- If $c_1, \ldots, c_n$ are n-many names and F_m^n is an n-ary predicate, then $Fc_{i_1}, \ldots, c_{i_n}$ is a sentence.[9]
- If φ is a sentence, then so is $\neg\varphi$.

[9] Following standard usage, the sentence formed at this stage are referred to as atomic sentences.

- If φ and ψ are sentences, then so is $(\varphi \to \psi)$.
- If φ is a sentence, then so is $\exists x \varphi[x/c]$.
- Nothing else is a sentence.

A quick terminological aside: Let $\exists x\varphi$ be called an existential sentence. If $\exists x\varphi$ has n-many occurrences of x bound by the outermost existential quantifier, then any sentence $\varphi[t/x]$, where r replaces each of the n-many occurrences of x, is called an *instance* of $\exists x\varphi$.

There are two crucial points to note about the first-pass at a theory in fig. 6.2. The first is that as opposed to having a single axiom, it has two. The first axiom, Id(s) amounts, as above, to the claim that it is incoherent to assert and deny the same sentence. Id(t) enforces that acceptance and rejection, like assertion and denial, are exclusive uses of a sentence. It is incoherent to accept and reject the same name. The other two structural rules governing names mimic those concerning sentences discussed above. If it were coherent to be in a name-position that accepted (rejected) a name, it would be coherent to be in a position that took no stance on that name. Similarly, the rule Cut(t) gives that it if is incoherent to accept a name in a position and to reject a name in a position, then that position is incoherent. For the purposes of this paper, the Cut(t) rule is trivially admissible.[10] For this reason is will be of little concern here.

The second important point is that the quantifiers of this language differ from

[10]The Cut(t) plays a crucial role in language if sub-nominal connectives are considered, such as the conjunction in 'Jack and Jill surrounded the building'. Whether or not it is admissible in such a language is a more complex matter.

the classical account of quantification. This requires, if not justification, explanation. The R$\exists$ rule requires that if it is coherent to deny $\exists x\varphi$, then for any term t, either it is coherent to deny $\varphi[t/x]$ or it is coherent to reject t. If a person denies that something

Figure 6.2: Positive Logic

STRUCTURAL RULES

$$\mathrm{Id(s)}\ \frac{}{:\varphi \Rightarrow \varphi :} \qquad \mathrm{Id(t)}\ \frac{}{c: \quad \Rightarrow \quad :c}$$

$$\mathrm{WL(s)}\ \frac{A:\Gamma \Rightarrow \Sigma : B}{A:\Gamma,\varphi \Rightarrow \Sigma : B} \qquad \mathrm{WR(s)}\ \frac{A:\Gamma \Rightarrow \Sigma : B}{A:\Gamma \Rightarrow \varphi,\Sigma : B}$$

$$\mathrm{WL(t)}\ \frac{A:\Gamma \Rightarrow \Sigma : B}{A,c:\Gamma \Rightarrow \Sigma : B} \qquad \mathrm{WR(t)}\ \frac{A:\Gamma \Rightarrow \Sigma : B}{A:\Gamma \Rightarrow \Sigma : B,c}$$

$$\mathrm{Cut(s)}\ \frac{A:\Gamma \Rightarrow \varphi,\Sigma : B \qquad A:\Gamma,\varphi \Rightarrow \Sigma : B}{A:\Gamma \Rightarrow \Sigma : B}$$

$$\mathrm{Cut(t)}\ \frac{A:\Gamma \Rightarrow \Sigma : B,c \qquad A,c:\Gamma \Rightarrow \Sigma : B}{A:\Gamma \Rightarrow \Sigma : B}$$

OPERATIONAL RULES

$$\mathrm{L}\neg\ \frac{\Gamma \Rightarrow \varphi,\Sigma}{\Gamma,\neg\varphi \Rightarrow \Sigma} \qquad \mathrm{R}\neg\ \frac{\Gamma,\varphi \Rightarrow \Sigma}{\Gamma \Rightarrow \neg\varphi,\Sigma}$$

$$\mathrm{L}\!\to\ \frac{\Gamma \Rightarrow \varphi,\Sigma \qquad \Gamma,\psi \Rightarrow \Sigma}{\Gamma,\varphi \to \psi \Rightarrow \Sigma} \qquad \mathrm{R}\!\to\ \frac{\Gamma,\varphi \Rightarrow \psi,\Sigma}{\Gamma \Rightarrow \varphi \to \psi,\Sigma}$$

$$^{1}\mathrm{L}\exists\ \frac{A,c:\Gamma,\varphi(c) \Rightarrow \Sigma : B}{A:\Gamma,\exists x\varphi(x) \Rightarrow \Sigma : B}$$

$$\mathrm{R}\exists\ \frac{A:\Gamma \Rightarrow \Sigma : B,c \qquad A:\Gamma \Rightarrow \varphi(c),\Sigma : B}{A:\Gamma \Rightarrow \exists x\varphi(x),\Sigma : B}$$

[1] c cannot occur in the conclusion.

is φ then they must be prepared for any term t to either reject t or deny that t is φ. Conversely, if one were to assert that something is φ then one must hold that there is an expansion of the language by a term w such that it is coherent to accept w and assert that w is φ. This account of quantification takes into consideration which terms are taken to denote and which are taken to fail to denote.

The alternative account of quantification does not take the acceptance and rejection of terms into consideration. The standard rules for quantification are

$$\mathrm{L}\reflectbox{Σ}\ \frac{A : \Gamma, \varphi[t/x] \Rightarrow \Sigma : B}{A : \Gamma, \reflectbox{Σ}x\varphi \Rightarrow \Sigma : B} \qquad\qquad \mathrm{R}\reflectbox{Σ}\ \frac{A : \Gamma \Rightarrow \varphi[t/x], \Sigma : B}{A : \Gamma \Rightarrow \reflectbox{Σ}x\varphi, \Sigma : B}$$

where t does not occur in the conclusion of $\mathrm{L}\reflectbox{$\Sigma$}$. Call the quantifiers characterized by these rules Meinongian, and the quantifiers characterized by the rules of fig. 6.2 Ontic. On the Meinongian account of quantification, denying an existential sentence requires that for any name, one be prepared to deny the corresponding instance of the existential sentence. The disagreement between these two theories will not be settled by any of the considerations for a theory of meaning mentioned above. But would have to be settled by other criteria. Both the Ontic and the Meinongian accounts of quantification meet the criterion specified for a set of rules to be meaning conferring.

One such criterion is that the Ontic account of quantification invalidates the argument given at the beginning of this paper. If it is true that Pegasus does not exist, it does not follow, according to the former account, that something does not exist. Similarly, if nothing is identical to Pegasus, it does not follow that Pegasus is not self-identical. The Meinongian account of quantification must admit that

Pegasus exists, or that something is identical to Pegasus. This is made clear by the following two derivations

$$\text{L}\exists\,\dfrac{\;:\quad\Rightarrow\quad : p \qquad \text{Ref=}\,\dfrac{}{\;:\quad\Rightarrow p = p :\;}}{:\quad\Rightarrow \exists x(x = p) :} \qquad\qquad \text{L}\reflectbox{$\exists$}\,\dfrac{\dfrac{}{\;:\quad\Rightarrow p = p :\;}}{:\quad\Rightarrow \reflectbox{$\exists$}x(x = p) :}$$

The Ontic derivation fails, so long as one is prepared to reject the name ‘Pegasus’. However, the Meinongian derivation establishes that it is incoherent to deny that something is Pegasus.

While this is merely grinding intuitions together, it is worth noting what is at stake. If there were a meaning theoretic criterion to adjudicate the situation, then what appears to be a metaphysical question could be settled by examining language.

The theory presented in fig. 6.2 corresponds to a positive free logic (see Lambert [25], or Lehmann [28]). A positive free logic is one which allows for atomic sentences featuring non-denoting names to be true. What it amounts to in this setting is that it is coherent to assert or deny an atomic sentence featuring a name that is either accepted or rejected. No such combination is itself incoherent. To the extent that there is an intuition that it is coherent to assert ‘Pegasus is a winged horse’, while rejecting ‘Pegasus’, this logic will capture that intuition.

6.5 Fully Positive, Negative, and Non-Denoting Terms

The above discussion concerns quantification. On the ontic account of quantification, it is coherent to assert that nothing is Pegasus, so long as one is prepared to reject

the name ‘Pegasus’. Another issue concerns the relation between atomic sentences and the acceptance or rejection of names that feature in those sentences. The rest of the paper assumes as given the logic of fig. 6.2. The remaining concerns are what further rules should be added to such a system. In particular, they are concerns about the rules relating the acceptance and rejection of terms and the assertion and denial of sentences.

The argument given above relied on the principle that in order for a sentence to be true, it must be true of something. Call this the Existence Principle. The Existence Principle, however, should not be accepted in full generality. Suppose that for any sentence, $\varphi(c)$, if it were coherent to assert $\varphi(c)$, then it would be coherent to accept c. The rule corresponding to this would be

$$\text{EP}\,\frac{A, c : \Gamma \Rightarrow \Sigma : B}{A : \Gamma, \varphi(c) \Rightarrow \Sigma : B}$$

This rule allows for the derivation that for any name, t, it is incoherent to reject t

$$\text{Cut(s)}\,\dfrac{\text{EP}\,\dfrac{\text{Id(t)}\,\dfrac{}{t :\ \Rightarrow\ : t}}{: Ft \Rightarrow\ : t} \qquad \text{Cut(s)}\,\dfrac{\dfrac{\text{Id(t)}\,\dfrac{}{t :\ \Rightarrow\ : t}}{: \neg Ft \Rightarrow\ : t} \quad \text{WR(t)}\,\dfrac{\text{R}\neg\,\dfrac{\text{Id(s)}\,\dfrac{}{: Ft \Rightarrow Ft :}}{:\ \Rightarrow Ft, \neg Ft :}}{:\ \Rightarrow Ft, \neg Ft : t}}{:\ \Rightarrow Ft : t}}{:\ \Rightarrow\ : t}$$

There are several points to be made against this strategy. The first is that the above deduction leads to what appears to be a false conclusion: that it is incoherent to reject any term. The second point is that this rule will enforce Meinongian quantifiers throughout. If it is incoherent to reject a term, then for any existential sentence, denying it will require denying any instance of the sentence. Finally the above derivation shows that this logic will not meet the criterion of cut admissibility. There is no cut-free derivation of $:\ \Rightarrow\ : t$.

While EP cannot underwrite a theory of meaning, there are plausible restrictions of this principle that can. As Plantinga [38] points out, restricting EP to atomic sentences appears to provide an intuitive account when non-denoting terms are involved. The restriction of EP to atomic sentences is

$$\text{AL}\,\frac{A, t : \Gamma \Rightarrow \Sigma : B}{A : \Gamma, Rt_1, \ldots, t, \ldots t_n \Rightarrow \Sigma}$$

in what follows $Rt_1, \ldots, t, \ldots, t_n$ is written as $R(t)$ for convenience. Read from bottom-to-top AL is the claim that if it is coherent to assert an atomic sentence featuring t, then it is coherent to accept t.

AL suggests three other rules concerning the use of atomic sentences featuring accepted or rejected names

$$\text{AR}\,\frac{A, t : \Gamma \Rightarrow \Sigma : B}{A : \Gamma \Rightarrow R(t), \Sigma : B}$$

$$\text{RL}\,\frac{A : \Gamma \Rightarrow \Sigma : B, t}{A : \Gamma, R(t) \Rightarrow \Sigma : B} \qquad \text{RR}\,\frac{A : \Gamma \Rightarrow \Sigma : B, t}{A : \Gamma \Rightarrow R(t), \Sigma : B}$$

Read from top-to-bottom, RR is the claim that if it is incoherent to reject a term then it is incoherent to assert an atomic sentence in which it features. There appears to be a simple counter-example to this. It is incoherent, in the position this paper adopts to reject the term 'RR', since it is the name of a rule. But it is not incoherent to assert 'RR is a rule', since it is. Similar considerations suggest that RL is not a valid rule.

AL and AR do not have such unintuitive features. According to the top-to-bottom reading of AL, it claims that if it is incoherent to accept a name, then it is incoherent to deny any atomic sentence featuring that name. Contrapositively, it

claims that if it is coherent to deny an atomic sentence featuring a term, t, then it is coherent to accept that term. If EP is restricted to in the way suggested above, it corresponds to AL, which enforces that if it is coherent to assert an atomic sentence featuring a term, t, then it is coherent to accept t.

This leaves four possible combinations of principles to cover the interaction between accepted names and atomic sentences

1. $\varnothing$
2. AL alone
3. AR alone
4. AL and AR

Following Lehmann [28] call the logic corresponding to (1) through (4), Positive Free Logic, Negative Free Logic, Fully Positive Free Logic, and Classical* Logic.

If the intuitions behind the classical argument are to be trusted, then both (1) and (3) can be ruled out as they deny the restricted form of EP, AL. This leaves Negative Free Logic and Classical* Logic.

If one held, as philosophers such as Frege [16] and Geach [18] did, that sentences featuring non-denoting names had no truth-value, then option (4) would appear best. When considered model-theoretically, it will follow on this logic that any sentence which features a non-denoting name is neither true nor false. Furthermore, there is a plausible argument in favor of such a view if one adopts the claim that sense determines reference.

Suppose that the term t is rejected, and so has no denotation though it does have a meaning. Let $R(t)$ be a sentence in which t features. The referent of that sentence, either the True or the False will be determined compositionally from the referents of $R()$ and t. But, as hypothesized t does not denote anything, so $R(t)$ cannot have a referent.

While this argument appears to be sound, it cannot be. The logic given by fig. 6.2, AL, and AR, is not cut-admissible. Thus it cannot underwrite a theory of meaning. This leaves only Negative Free Logic as offering a theory of meaning. If the Classical Argument makes a mistake, it is in taking the Meinongian rule of quantification either to be valid or to express existence. This motivated the theory of quantification given in fig. 6.2. Adding EP, the second premise of the Classical Argument, to fig. 6.2, was not cut admissible, and so a restricted version AL, was adopted. Finally, since adding AR to this, as is suggested by Frege and Geach resulted in the failure of cut admissibility, Negative Free Logic was settled as the logic which confers meaning.[11]

Before addressing the philosophical upshots of this, as well as offering a refinement on the notion of an atomic sentence, an explanation of the failure of Frege's argument is required.

The picture of the relationship between truth and falsity must thus be different than the one offered by Frege. The analogy that Frege suggests is of sentences referring to either the True or the False, but a more accurate analogy makes clear the distinction between the True and the False. An atomic sentence is true when the

[11] An account equivalent to the logic thus proposed as meaning conferring is given by Gratzl [19], and shown there to be cut eliminable. However, since the rules given there differ from those proposed here, a cut-admissibility result for the proposed system is required and offered in section 6.7.

object denoted by the name is as the predicate says it is. It is false if that is not the case. Frege's mistake was to suppose that there was only one way for it not to be the case that a sentence is true: when the object denoted by a name in an atomic sentence is not as the predicate says it is. The picture presented by the logic above holds that it is possible for an atomic sentence to be false also when there is no object denoted by the name. Falsity is thus not a polar opposite of truth. A sentence is false when it fails in some way. A better analogy is that of an arrow. The sentence aims at truth, if it fails to hit its target, then it is false. There are two ways an assertion of an atomic sentence can fail to be correct: either (1) it misapplies a predicate to a name, i.e. attempts to say of some existing thing that it has a property that it fails to have or (2) it applies a predicate to a name that does not denote. The second way for an atomic sentence to be false is not a sub-category of the first. The first way for an atomic sentence to be false requires that the name to which a predicate is being applied denotes.

It is surprising that Geach [18] holds a similar view of truth and falsity. He characterizes sentences as roads leading to a city. The true ones are those that will lead to the city, the false ones are those that lead away, though not to any particular place. The false sentence are those that are in some way defective. While the account of truth and falsity presented here follows those lines, Geach [17] maintained that sentences with non-denoting terms were neither true nor false.

The logic that has been argued for in this paper is given by fig. 6.3. Several expressions have been added to the logic. The first of note is the identity relation. Its addition does not disrupt the cut-admissibility result, and its use in much of the

paper prompts its addition.

The second addition is λ-expressions. The purpose of this is to make more plausible the restriction of EP. Once λ-expressions are added to the language the notion of an atomic sentence can be expanded. So that sentences such as $\lambda x.\neg\exists y \ \ y = x$[Pegasus] can be added to the language. This sentence says of Pegasus that it has the property of being identical to nothing. It is incoherent to assert this sentence given the rules of fig. 6.3

$$\dfrac{\dfrac{\dfrac{\dfrac{\text{L}\exists\ \dfrac{\text{WL(t)}\ \dfrac{\text{R=}\ \dfrac{}{: \ \Rightarrow p = p :}}{p : \ \Rightarrow p = p :} \qquad \text{Id(t)}\ \dfrac{}{p : \ \Rightarrow \ : p}}{p : \ \Rightarrow \exists y(y = p)}}{\text{L}\neg\ \ p : \neg\exists y(y = p) \Rightarrow \ :}}{\text{L}\lambda\ \ p : \lambda x(\exists y y = x)p \Rightarrow \ :}}{\text{AL}\ \ : \lambda x(\exists y y = x)p, \lambda x(\exists y y = x)p \Rightarrow \ :}}{: \lambda x(\exists y y = x)p \Rightarrow \ :}$$

λ-expressions can therefore be used to approximate the sentiments of EP.

In order to add λ-expressions to the language, the set of sentences must be redefined. Let N, $Pred$ and Var be as above. For any sentence, φ in the set of sentences let $\lambda x(\varphi[x/t])$ be in $Pred$. Thus, as the set of predicates expands, so does the set of atomic sentences, and so the set of sentences.

λ-expressions are used to form complex predicates. Consider the case of $\neg$. Say that $\neg$ features sententially in a sentence φ, when it does not occur within the scope of any λ-expression, and predicatively otherwise. Asserting a sentence $\neg\psi$, is thus the same as denying ψ, and denying $\neg\psi$ is thus the same as asserting ψ. But this is not the case when $\neg$ occurs predicatively. The sentence $\lambda x(\neg\varphi)a$ does not have this behavior. Asserting it is not only asserting $\neg\varphi[a/x]$, but also accepting a.

Similarly, denying it is not the same as denying $\neg\varphi[a/x]$, but may also have the force of rejecting a. For instance, suppose C takes it to be the case that 'Pegasus' does not exist. They can coherently deny that it is true of Pegasus that it is winged and that it is true of Pegasus that it is not winged, i.e. the position $\Rightarrow \lambda x(\neg Wx)p, \lambda x(Wx)p$ is coherent. C can do this only in circumstances where it is incoherent to accept Pegasus. C's position implicitly rejects Pegasus.

6.6 Conclusions

The Classical Argument of section 6.1 has the false statement that the term 'Pegasus' denotes as its conclusion. The problem with the Classical Argument is an ambiguity in its use of the word 'features'. If 'features' means 'appears in' then premise 2. is false. There are true sentences, e.g. 'It is not the case that Pegasus is winged', in which non-denoting names appear. If, on the other hand, 'features' means something like 'occurs as the logical subject' then premise 3. is false. In order for premise 1. (that Pegasus does not exist) to be true, 'Pegasus' cannot occur as its logical subject. This is established by the fact that it is incoherent to assert the predication of non-existence to any singular term. On either interpretation of 'features' the Classical Argument has a false premise.

It has been assumed that there are logically proper names, such as 'Pegasus' that do not denote. This assumption, however, does not rule out a classical account of quantification on the meaning theoretic grounds given in section 6.2. There is nothing about Meinongian quantifiers that rules them out on meaning theoretic

grounds. This suggests that the disagreement between philosophers who hold that a thing can be without existing and those that hold that being is existence is not over the legitimacy of an Ontic or Meinongian quantifier. A language that includes both Ontic and Meinongian quantifiers meets the meaning theoretic constraints discussed above. It is also sound and complete for exactly the same set of models for which the logic of fig. 6.3 is sound and complete.

Elsewhere (see Chapter 5 of this dissertation [36]) it is argued that not all quantification is ontologically significant. If that view is right then there is nothing that prevents a philosopher who identifies being and existence from adopting both Ontic and Meinongian quantifiers. The dispute between such a philosopher and one endorsing a Dense Ontology is not revealed by what sentences are asserted or denied but by what terms that position is committed to accepting or rejecting.

6.7 Cut Admissibility for Negative Free Logic

To prove that Cut(s) and Cut(t) are admissible, a model theory will be given and shown to be sound for negative free logic, and complete for the cut-free fragment of that system. It follows that for any deduction of a name-sequent, $A : \Gamma \Rightarrow \Sigma : B$, there is a cut-free deduction of that name-sequent.

6.7.1 Models

The set of models given here is a variation on the theme of those proposed by Geach [17] and Lavine [27]. Instead of making use of functions between variables and

the domain of quantification, it makes use of expansions of the language to handle quantification.

The syntax is as above, with the exception that a witness set, $\{w_1, w_2, \ldots\}$, is added to account for quantifiers.

A model, M, is a pair, $\langle D_M, I_M \rangle$, of a set of objects, partitioned into D_e and D_n, and an interpretation function defined by:

- For any term, t, $I_M(t) \in D$.
- For any n-ary atomic predicate, F, $I_M(F) \subseteq D_e$.
- For any complex predicate $\lambda x(\psi)$, $I_M(\lambda x(\psi) = \{y : y \in D_e \wedge$ for any term t such that $I_M(t) = y, I_M(\psi[t/x]) = 1\}$
- If φ is an atomic sentence $Ft_1, \ldots, t_n$, then $I_M(\varphi) = 1$ iff $\langle I_M(t_1), \ldots, I_M(t_n) \rangle \in I_M(F)$.
- If φ is $t_j = t_k$, then $I_M(\varphi) = 1$ iff $I_M(t_j)$ is $I_M(t_k)$.
- If φ is $\neg\psi$, then $I_M(\varphi) = 1$ iff $I_M(\psi) = 0$.
- If φ is $\psi \rightarrow \theta$, then $I_M(\varphi) = 1$ iff $I_M(\psi) = 0$ or $I_M(\theta) = 1$.
- If φ is $\exists x\psi$, then $I_M(\varphi) = 1$ iff there is an expansion of M, M', by a witness, w, such that $I_{M'}(\psi[w/x]) = 1$.

Definition 29 (Expansion)**.** M' is an expansion of M by an element of the witness set, w, when $I_M(w)$ is undefined, but $I_{M'}(w) \in D_e$, and $I_M = I_{M'}$ everywhere else.

Definition 30 (Counter-example)**.** A model, M, is a counter-example to a name-sequent, $A : \Gamma \Rightarrow \Sigma$, iff

1. for any term, $t \in A$, $I_M(t) \in D_e$;
2. for any sentence, $\gamma \in \Gamma$, $I_M(\gamma) = 1$;
3. for any sentence, $\sigma \in \Sigma$, $I_M(\sigma) = 0$;
4. and for any term, $t \in B$, $I_M(t) \in D_n$.

If there is no counter-example to a name-sequent, $A : \Gamma \Rightarrow \Sigma : B$, this is written as $\models A : \Gamma \Rightarrow \Sigma : B$.

6.7.2 Soundness

Definition 31 (Rank)**.** The rank of a sentence, $rk(\varphi)$, is defined inductively by

- If φ is atomic, then $rk(\varphi) = 0$.
- If φ is $\lambda x(\psi)t$, then $rk(\varphi) = rk(\psi) + 1$.
- If φ is $\neg\psi$, then $rk(\varphi) = rk(\psi) + 1$.
- If φ is $\psi \rightarrow \theta$, then $rk(\varphi) = rk(\psi) + rk(\theta) + 1$.
- If φ is $\exists x\psi$, then $rk(\varphi) = rk(\psi) + 1$.

Lemma 6.1. *Let M be a model with an extension M' making $\varphi[w/x]$ true or false, where w is a witness. Let $T = A \cup B$, be a finite set of terms, and $S = \Gamma \cup \Sigma$ finite sets of sentences. There is a model, N, such that $I_N(\varphi[t/x]) = I_{M'}(\varphi[w/x])$ and $I_N(t) \in D_e$ iff $I_{M'}(w) \in D_e$, for a term t not appearing in T or S.*

Proof. This is proved by induction on the rank of φ.

Case 1 ($rk(\varphi) = 0$). Let φ be $Ft_1, \ldots, w, \ldots, t_n$. We have that $I_{M'}(\varphi) = 1$. So $\langle I_{M'}(t_1), \ldots, I_{M'}(w), I_{M'}(t_n)\rangle \in I_M(F)$. It follows that $I_{M'}(w) \in D_e$. Let t be a term not occurring in T or in S. Such a term exists because the terms of the language is infinite while the sets T and S are only finite. N can be defined $D_N = D_M$, $I_M(t') = I_N(t')$ for any term that is not t, and $I_N(t) = I_{M'}(w)$. That $I_N(Ft_1, \ldots, t, \ldots, t_n) = 1$, is given by the fact that M, M', and N agree everywhere but possibly at t. But $I_N(t) = I_{M'}(t)$. The proof is similar where $I_{M'}$ makes φ false.

Case 2 (φ is $\lambda z(\psi)c$). Let $I_{M'}(\varphi[w/x]) = 1$. So $I_{M'}(c) \in \{y : y \in D_e \wedge$ for any term n such that $I_{M'}(n) = y, I_{M'}(\psi[n/z][w/x]) = 1\}$. So $I_{M'}(t) \in D_e$, and $I_{M'}(\psi[t/z]) = 1$. By IH there is a model, N such that $I_N(\psi[t/z])$ and $I_N(t) \in D_e$. It follows that $I_N(\varphi[t/x]) = 1$.

Let $I_{M'}(\varphi[w/x]) = 0$. So $I_{M'}(c) \notin \{y : y \in D_e \wedge$ for any term n such that $I_{M'}(n) = y, I_{M'}(\psi[n/z][w/x]) = 1\}$. This leaves two cases. Either $I_M(c) \notin D_e$ or $I_M(\varphi[w/x][c/z]) = 0$. In the first case, either $I_{M'}(\psi[w/x][c/z] = 1$ or $I_{M'}(\psi[w/x][c/z] = 0$. Either way, by IH there is an extension, N, such that $I_N(t) \notin D$, so $I_N(\varphi[t/x]) = 0$. In the second case, there is a model N, and term t, such that $I_N(\psi[t/x][c/z]) = 0$, so $I_M(\varphi[t/x]) = 0$.

Case 3 (φ is $\neg\psi$). Let $I_{M'}(\varphi[w/x]) = 1$. So $I_{M'}(\psi[w/x]) = 0$. By IH, there is a model, N, and term, t, such that $I_N(\psi[t/x]) = 0$. So $I_N(\varphi[t/x]) = 1$. The case where $I_{M'}(\varphi) = 0$ is similar.

Case 4 (φ is $\psi \to \theta$). Let $I_{M'}((\psi \to \theta)[w/x]) = 1$. Either $I_{M'}(\psi[w/x]) = 0$ or $I_{M'}(\theta[w/x]) = 1$. In the first case by IH there is a model, N, such that $I_N(\psi[t/x]) = 0$, so $I_N((\psi \to \theta)[t/x]) = 1$. In the second there is a model, N, such that $I_N(\theta[t/x]) =$

0, so $I_N((\psi \to \theta)[t/x]) = 1$. The case where $I_{M'}((\psi \to \theta)[w/x] = 0$ is similar.

Case 5 (φ is $\exists y\psi$). There are two sub-cases:

Case 4. ($I_{M'}(\exists y\psi[w/x]) = 1$). In this case there is an extension M'' agreeing everywhere with M', with the exception that $I_{M'}(w')$ is undefined but $I_{M''}(w') \in D_e$ and $I_M(\psi[w/x][w'/y]) = 1$. By IH, there is a model, N that agrees everywhere with M and M' except for w. But is such that $I_N(t) = I'_M(w)$. Furthermore, $I_N(\psi[t/x][w'/y]) = 1$. But then N is an extension of a model, N' by w'. So $I'_N(\exists y\psi[t/x]) = 1$.

Case 5. ($I_{M'}(\exists y\psi[w/x]) = 0$). In this case there is no extension, M'', of M' such that $I_{M''}(\psi[w/x][w'/y]) = 1$. Suppose that N' is an extension of N, where N agrees everywhere with M except on w, where $I_N(t) = I_{M'}(w)$. Let N' be an extension of N by w'. By IH, $I_{N'}(\psi[w/x][w'/y]) = 0$. But this was general so $I_N(\exists y\psi[t/x]) = 0$.

□

Lemma 6.2. *If N and M are models or extensions of models agreeing on their domain, and the interpretation of any expressions appearing in the name-sequent $A : \Gamma \Rightarrow \Sigma : B$, then for any sentence, $\varphi \in \Gamma \cup \Sigma$, $I_M(\varphi) = I_N(\varphi)$ and for any term t in $A \cup B$ or appearing in $\Gamma \cup \Sigma$, $I_N(t) = I_M(t)$.*

Proof. The proof of this lemma is standard. The only worrying cases are if φ has a λ-expression as its main operator or if φ is existential. First let φ be $\lambda x(\psi)a$.

Case 1 ($I_M(\varphi) = 1$). In this case $I_M(a) \in \{y : y \in D_e \wedge$ for any term n such that $I_M(n) = y, I_M(\psi[n/x]) = 1\}$. So $I_M(a) \in D_e$, and $I_M(\psi[a/x]) = 1$. By IH both $I_N(\psi[a/x])$ and $I_N(a) \in D_e$. But then $I_N(\lambda x(\psi)a) = 1$.

Case 2 ($I_M(\varphi) = 0$). In this case $I_M(a) \notin \{y : y \in D_e \wedge$ for any term n such that $I_M(n) = y, I_M(\psi[n/x]) = 1\}$. Either $I_M(a) \notin D_e$ or $I_M(\psi[a/x]) = 0$. In the first case, by IH, $I_N(a) \notin D_e$. It follows that $I_N(\lambda x(\psi)a) = 0$. In the second case $I_N(\psi[a/x]) = 0$, so $I_N(\varphi) = 0$.

Let φ be $\exists x\psi$. There are two cases:

Case 1 ($I_M(\varphi) = 1$). In this case there is an extension M' of M by w such that $I'_M(\psi[w/x]) = 1$. Let N' be the extension of N such that $I_{N'}(w) = I_{M'}(w)$. By IH, $I'_N(\psi[w/x]) = 1$. So $I_N(\varphi) = 1$.

Case 2 ($I_M(\varphi) = 0$). Let N' be an extension of N by w. Suppose that $I_{N'}(\psi[w/x]) = 1$. Similarly, let I_M be an extension of M by w such that $I_{M'}(w) = I_{N'}(w)$. But then by IH $I_{N'}(\psi[w/x]) = I_{M'}(\psi[w/x]) = 0$.

□

Lemma 6.3. *Let M and N be models or extensions. Let M and N have the same domain and agree on all the expressions in φ except for t, but $I_M(t) = I_N(t')$. $I_M(\varphi) = I_N(\varphi[t'/t]$.*

Proof. This is proved by induction on the rank of φ. Again the difficult case is if φ is $\exists x\psi$. Let $I_M(\varphi) = 1$. So there is an extension of M, M', by w, such that $I_M(\psi[w/x]) = 1$. Let N' be the extension of N by w such that $I_{N'}(w) = I_{M'}(w)$. By IH $I_{N'}(\psi[w/x][t'/t]) = 1$. So $I_N(\varphi[t'/t]) = 1$.

Let $I_M(\varphi) = 0$. So there is no extension of M, M', by any witness, w, such that $I_{M'}(\psi(w/x)] = 1$. Suppose that there is an extension N' of N by a witness, w, such that $I_{N'}(\psi[w/x][t'/t]) = 1$. Let M' be the extension of M by w such that

$I_{M'}(w) = I_{N'}(w)$. But then by IH, $I_{M'}(\psi[w/x]) = 1$, which is impossible. So there is no such extension of N, from which it follows that $I_N(\varphi[t'/t]) = 0$. $\square$

Theorem 6.7.1. *If* $\vdash A : \Gamma \Rightarrow \Sigma : B$*, then* $\vDash A : \Gamma \Rightarrow \Sigma : B$*.*

Proof. This is proved by induction on the length of the deduction of $A : \Gamma \Rightarrow \Sigma : B$. Most cases will be omitted for the sake of space. The base cases are Id(s) and Id(t). For these cases it is enough to note that no model assigns a sentence 1 and 0, and that $D_e \cup D_n = \varnothing$. For the inductive cases it suffices to show that given a counter-example to the conclusion name-sequent, there is a counter-example to one of the premise name-sequents. The cases of AL, Cut(s), R$\rightarrow$, L$\exists$, R$\exists$, L$=$, and R$=$ are considered below.

Case 1 (AL). Let M be a counter-example to $A : \Gamma, Ft_1, \ldots, t, \ldots, t_n \Rightarrow \Sigma : B$. So for any term, $t \in A$, $I_M(t) \in D_e$, for any sentence, $\gamma \in \Gamma \cup \{Ft_1, \ldots, t, \ldots, t_n\}$, $I_M(\gamma) = 1$, for any sentence, $\sigma \in \Sigma$, $I_M(\sigma) = 0$, and for any term, $t \in B$, $I_M(t) \in D_n$. In particular, $I_M(Ft_1, \ldots, t, \ldots, t_n) = 1$. So $\langle I_M(t_1), \ldots, I_M(t), \ldots, I_M(t_n)\rangle \in I_M(F)$. But $I_M(F) \subseteq D_e$. Thus, $I_M(t) \in D_e$. So M is a counter-example to $A, t : \Gamma \Rightarrow \Sigma : B$.

Case 2 (Cut(s)). Let M be a counter-example to $A : \Gamma \Rightarrow \Sigma : B$. It is shown by induction on the complexity of φ that there is a counter-example either to $A : \Gamma \Rightarrow \Sigma, \varphi : B$, or $A : \Gamma, \varphi \Rightarrow \Sigma : B$. It is important to note that elements of the witnessing set are not part of the language, but serve to as possible expansions of the language.[12] In the case that φ is $Ft_1, \ldots, t_n$, there are two sub-cases to consider:

[12]This is because, as Sellars [62] points out, the language under study is in some sense schematic: it does not itself contain a term for everything there is, but the quantifiers presume that such a language could exist.

Case 6 ($I_M(t_i) \in D_e$, $1 \leqslant i \leqslant n$). Either $\langle I_M(t_1), \ldots, I_M(t_n)\rangle \in I_M(F)$ or not. In the first case, M is counter-example to $A : \Gamma, \varphi \Rightarrow \Sigma : B$. In the second it is a counter-example to $A : \Gamma \Rightarrow \varphi, \Sigma : B$.

Case 7 ($I_M(t_i) \in D_n$ for some i). In this case $I_(\varphi) = 0$, because $I_M(F) \subseteq D_e$. So M is a counter-example to $A : \Gamma \Rightarrow \varphi, \Sigma : B$

If φ is not an atomic, the result follows from the sub-inductive hypothesis. In the case where φ is an existential sentence it follows from the fact that either there is an extension of M that makes an instance of φ with a witness true or not.

Case 3 (R$\rightarrow$). Let M be a counter-example to $A : \Gamma \Rightarrow \varphi \rightarrow \psi, \Sigma : B$. So $I_M(\varphi) = 1$ and $I_M(\psi) = 0$. But then M is a counter-example to $A : \Gamma, \varphi \Rightarrow \psi, \Sigma$.

Case 4 (Lλ). Let M be a counter-example to $A : \Gamma, \lambda x(\varphi)t \Rightarrow \Sigma : B$. In particular, $I_M(\lambda x(\varphi)t) = 1$. So $I_M(t) \in \{y : y \in D_e \wedge \text{ for any term } n \text{ such that } I_M(n) = y, I_M(\varphi[n/x]) = 1\}$. But then $I_M(t) \in D_e$ and $I_M(\varphi[t/x]) = 1$. So M is a counter-example to $A : \Gamma, \varphi[t/x] \Rightarrow \Sigma : B$.

Case 5 (Rλ). Let M be a counter-example to $A : \Gamma \Rightarrow \lambda x(\varphi)t, \Sigma : B$. So $I_M(\lambda x(\varphi)t) = 0$. It follows that either $I_M(t) \notin D_e$ or $I_M(\varphi[t/x]) = 0$. In the first case M is a counter-example to $A : \Gamma \Rightarrow \Sigma : B, t$. In the second, M is a counter-example to $A : \Gamma \Rightarrow \varphi[t/x], \Sigma : B$.

Case 6 (L$\exists$). Let M be a counter-example to $A : \Gamma, \exists x\varphi \Rightarrow \Sigma : B$. So there is an extension of M, M' by a witness w, such that $I_M(\varphi[w/x]) = 1$. By lemma 6.1, there is a model, N such that $I_N(\varphi[t/x]) = I_M(\varphi[w/x] = 1$, where t does not appear in any of A, B, Γ, or Σ, and N agrees everywhere else with M. So N is a counter-example

to $A, t : \Gamma \Rightarrow \Sigma : B$.

Case 7 (R$\exists$). Let M be a counter-example to $A : \Gamma \Rightarrow \exists x\varphi, \Sigma : B$. So for any extension M' of M by w, $I_{M'}(\varphi[w/x]) = 0$. Let t be a term in the language. There are two sub-cases to consider:

Case 8 ($I_M(t) \in D_e$). Suppose that $I_M(\varphi[t/x]) = 1$. Let M' be an extension of M by w, such that $I_{M'}(w) = I_M(t)$. By lemma 6.2, $I_M(\varphi[w/x]) = 1$. But this contradicts our hypothesis.

Case 9 ($I_M(t) \in D_n$). In this case M is a counter-example to $A : \Gamma \Rightarrow \Sigma : B, t$.

Case 8 (L=). Let M be a counter-example to $A : \Gamma, \varphi, t_i = t_j \Rightarrow \Sigma : B$. So $I_M(\varphi) = 1$, and $I_M(t_i) = I_M(t_j)$. That $I_M(\varphi[t_i/t_j]) = 1$ is proved by induction on the rank of φ, but from the more general fact that $I_M(\varphi) = I_M(\varphi[t_i/t_j])$ for any model or extension M, when $I_M(t_i) = I_M(t_j)$.

Case 10 (φ is atomic). Let φ be $Ft_1, \ldots, t_j, \ldots, t_n$. Since $I_M(\varphi) = 1$, $\langle t_1, \ldots, t_j, \ldots, t_n\rangle \in I_M(F)$. But $I_M(t_j) = I_M(t_i)$, so $\langle t_1, \ldots, t_i, \ldots, t_n\rangle \in I_M(F)$. The case where $I_M(\varphi) = 0$ is analogous.

Case 11 (φ is $\neg\psi$). Let $I_M(\varphi) = 1$. So $I_M(\psi) = 0$. But then by IH, $I_M(\psi[t_i/t_j]) = 0$. So $I_M(\varphi[t_i/t_j) = 1$. The case where $I_M(\varphi) = 0$ is similar.

Case 12 (φ is $\psi \to \theta$). Let $I_M(\varphi) = 1$. So either $I_M(\psi) = 0$ or $I_M(\theta) = 1$. In the first case, by IH $I_M(\psi[t_i/t_j]) = 0$, so $I_M(\varphi[t_i/t_j]) = 1$. In the second, by IH $I_M(\theta[t_i/t_j]) = 1$, so $I_M(\varphi[t_i/t_j]) = 1$.

Alternatively, let $I_M(\varphi) = 0$. So $I_M(\psi) = 1$ and $I_M(\theta) = 0$. By IH, $I_M(\psi[t_i/t_j]) = 1$ and $I_M(\theta[t_i/t_j]) = 0$. So $I_M(\varphi[t_i/t_j]) = 0$.

Case 13 (φ is $\exists x\psi$). Let $I_M(\varphi) = 1$. So there is an extension M', Such that $I_{M'}(\psi) = 1$. But I_M and $I_{M'}$ agree everywhere but W, so by lemma 6.2, $I_{M'}(t_i = t_j) = 1$ So by IH, $I_{M'}(\psi[t_i/t_j]) = 1$. But then $I_M(\exists x\psi[t_i/t_j]) = 1$.

Let $I_M(\varphi) = 0$. So there is no extension, M', of M, such that $I_{M'}(\psi) = 1$. Suppose that $I_M(\varphi[t_i/t_j]) = 1$. So there is an extension, M', of M such that $I_{M'}(\psi[t_i/t_j]) = 1$. But then by IH, $I_{M'}(\varphi) = 1$, which is impossible.

Case 9 (R=). Suppose that M was such that $I_M(t = t) = 0$. Then it would be that $I_M(t) \neq I_M(t)$ which is impossible.

$\square$

6.7.3 Completeness

Completeness is proved in two stages. In the first stage, a procedure for expanding an unprovable sequent is given. This relies on several lemmas, which are proved first. Once an unprovable sequent has been fully expanded, it is shown how this corresponds to a model. This model, finally, is shown to be a counter-example to the unprovable sequent.

Lemmas

Fact 1. The rule

$$\frac{A, t_1, \ldots, t_n : \Gamma \Rightarrow \Sigma : B}{A : \Gamma, Ft_1, \ldots, t_n \Rightarrow \Sigma : B}$$

is derivable.

Lemma 6.4. *The following rule is derivable*

$$\frac{t_1, \ldots, t_n : \Gamma \Rightarrow \Sigma, \varphi(t_1), \ldots, \varphi(t_n) : B}{t_1, \ldots, t_n : \Gamma \Rightarrow \Sigma, \exists x\varphi(x) : B}$$

Proof. This is proved by induction on the number of terms that the sequent accepts.

Let the base case be:

$$\text{L}\exists\,\frac{t_1, \ldots, t_n : \Gamma \Rightarrow \Sigma, \varphi(t_1), \ldots, \varphi(t_n) : B \qquad \text{TL/R(s/t)}\,\dfrac{t_1 : \quad \Rightarrow \quad : t_1}{t_1, \ldots, t_n : \Gamma \Rightarrow \Sigma, \varphi(t_2), \ldots, \varphi(t_n) : B, t_1}}{t_1, \ldots, t_n : \Gamma \Rightarrow \Sigma, \exists x\varphi(x), \varphi(t_2), \ldots, \varphi(t_n) : B}$$

For the inductive case

$$\text{TL/R(s/t)}\,\frac{t_i : \quad \Rightarrow \quad : t_i}{t_1, \ldots, t_n : \Gamma \Rightarrow \Sigma, \exists x\varphi(x), \varphi(t_{i+1}), \ldots, \varphi(t_n) : B}$$

$$\delta_1\vdots$$

$$\frac{t_1, \ldots, t_n : \Gamma \Rightarrow \Sigma, \exists x\varphi(x), \varphi(t_i), \ldots, \varphi(t_n) : B \qquad \delta_1\vdots}{t_1, \ldots, t_n : \Gamma \Rightarrow \Sigma : \exists x\varphi(x), \varphi(t_{i+1}), \ldots, \varphi(t_n) : B}$$

□

Lemma 6.5. *The following is derivable*

$$\frac{A, t : \Gamma, \varphi[t/x] \Rightarrow \Sigma : B}{A : \Gamma, \lambda x(\varphi)t \Rightarrow \Sigma : B}$$

Proof. □

Lemma 6.6. *The following is derivable*

$$\frac{A : \Gamma, a = c, a = b, b = c \Rightarrow \Sigma : B}{A : \Gamma, a = b, b = c \Rightarrow \Sigma : B}$$

Proof. This proof is made easier to see by making explicit the contraction steps that are involved.[13]

[13]Since sequents consist of *sets* of sentences, these steps are really no more than organizing notation.

$$\dfrac{\dfrac{\dfrac{A : \Gamma, a = c, a = b, b = c \Rightarrow \Sigma : B}{A : \Gamma, b = c, a = b, b = c, a = b \Rightarrow \Sigma : B}\text{L=}}{A : \Gamma, b = c, a = b, a = b \Rightarrow \Sigma : B}\text{Contraction}}{A : \Gamma, b = c, a = b \Rightarrow \Sigma : B}\text{Contraction}$$

Note, this did not depend on particular order of the identities, for any $t_i = t_j$, it could be replaced with $t_j = t_i$ above, and a proof of the same structure would work. □

Lemma 6.7. *The following is derivable*

$$\frac{A : \Gamma, a = b, b = a \Rightarrow \Sigma : B}{A : \Gamma, a = b \Rightarrow \Sigma : B}$$

Proof. Again, the contraction steps are made explicit and left until the end.

$$\dfrac{\dfrac{\dfrac{A : \Gamma, a = b, b = a \Rightarrow \Sigma : B}{A : \Gamma, a = b, b = b, a = b \Rightarrow \Sigma : B}\text{L=}}{A : \Gamma, a = b, a = b, a = b, a = b \Rightarrow \Sigma : B}\text{L=}}{A : \Gamma, a = b \Rightarrow \Sigma : B}\text{Contraction } \times 3$$

□

6.7.4 Constructing a Tree

The tree to be constructed will have as nodes sequents. Let $A : \Gamma \Rightarrow \Sigma : B$ be a sequent on the leaf of a tree. If either $A \cap B \neq \varnothing$,$\Gamma \cap \Sigma \neq \varnothing$, or $c = c \in \Sigma$ for some c, then the branch from the root of the tree to S is closed. Otherwise, the branch is open.

Let *Sent* be the set of sentences of the language. Let $C : \Delta \Rightarrow \Lambda : D$ be an unprovable sequent. The construction of the tree proceeds in stages. Let L be a list of sentences that grows with each stage. For each stage, S_i, let $\varphi_1, \varphi_2, \ldots$ be an enumeration of *Sent*. Let $A_1 : \Gamma_1 \Rightarrow \Sigma_1 : B_1, \ldots, \Gamma_n \Rightarrow \Sigma_n$ be the leaves of open branches of the tree. Let $\Gamma_i^n = \{t : t \text{ appears in a setence of } \Gamma\}$, then $A =$

$\bigcup_{1\leqslant i\leqslant n}(A_i \cup B_i \cup \Gamma_i^n \cup \Sigma_i^n$. For each element, $\langle \varphi_i, t_j \rangle$ of $Sent \times A$ —if A is empty let it be $\{t\}$ do the following:

1. If φ_i is atomic non-identity, then if t_j is an argument of φ, for any leaf of an open branch of the form $A_k : \Gamma_k, \varphi_i \Rightarrow \Sigma_k : B_k$, replace that sequent with the derivation

$$\frac{A_k : t_j : \Gamma_k \Rightarrow \varphi_i \Rightarrow \Sigma_k : B_k}{A_k : \Gamma_k, \varphi_i \Rightarrow \Sigma_k : B_k}$$

2. If φ_i is $t_n = t_m$, then for any leaf of an open branch of the form: $A_k : \Gamma_k, t_n = t_m \Rightarrow \Sigma$, replace that sequent with the derivation

$$\frac{A_k : \Gamma_k, t_n = t_m, t_m = t_n \Rightarrow \Sigma : B_k}{A_k : \Gamma_k, t_n = t_m, \Rightarrow \Sigma : B_k}$$

   For any leaf of an open branch of the form: $A_k : \Gamma_k, t_n = t_m, t_o = t_p \Rightarrow \Sigma_k : B_k$, where either $n \in \{o,p\}$ or $m \in \{o,p\}$ but not both, replace that sequent with the derivation

$$\frac{C_k : \Delta_k, t_n = t_m, t_o = t_p, t_q = t_r \Rightarrow \Lambda_k : D_k}{A_k : \Gamma_k, t_n = t_m, t_o = t_p \Rightarrow \Sigma_k : B_k}$$

   where t_q is one of t_n or t_m and t_r is one of t_o or t_p, and $C_k : \Delta_k \Rightarrow \Lambda_k : D_k$ is the result of making all possible substitutions of one of those terms for another of them in $A_k : \Gamma_k \Rightarrow \Sigma_k : B_k$.

3. If φ_i is $\lambda x(\psi)t$, then for any leaf of an open branch of the form $A_k, \Gamma_k, \lambda x(\psi)t \Rightarrow \Sigma_k : B_k$, replace that sequent with the derivation

$$\frac{A_k, t : \Gamma_k, \varphi[t/x] \Rightarrow \Sigma_k : B_k}{A_k : \Gamma_k, \lambda x(\varphi)t \Rightarrow \Sigma_k : B_k}$$

For any leaf of an open branch of the form $A_k : \Gamma_k \Rightarrow \lambda x(\varphi)t, \Sigma_k : B_k$ replace that sequent with the derivation

$$\frac{A_k : \Gamma_k \Rightarrow \varphi[t/x], \Sigma_k : B_k \qquad A_k : \Gamma_k \Rightarrow \Sigma_k : B_k, t}{A_k : \Gamma_k \Rightarrow \lambda x(\varphi)t, \Sigma_k : B_k}$$

4. If φ_i is $\neg\psi$, then for any leaf of an open branch of the form: $A_k, \Gamma_k, \neg\psi \Rightarrow \Sigma_k : B_k$, replace that sequent with the derivation

$$\frac{A_k : \Gamma_k, \neg\psi \Rightarrow \psi, \Sigma_k : B_k}{A_k, \Gamma_k, \neg\psi \Rightarrow \Sigma_k : B_k}$$

For any leaf of an open branch of the form $A_k : \Gamma_k \Rightarrow \neg\psi, \Sigma_k : B_k$, replace that sequent with the derivation

$$\frac{A_k : \Gamma_k, \psi \Rightarrow \neg\psi, \Sigma_k : B_k}{A_k : \Gamma_k, \neg\psi \Rightarrow \neg\psi, \Sigma_k : B_k}$$

5. If φ_i is $\psi \to \theta$, then for any leaf of an open branch of the form $A_k : \Gamma_k, \psi \to \theta \Rightarrow \Sigma_k : B_k$, replace that sequent with the derivation

$$\frac{A_k : \Gamma_k, \psi \to \theta \Rightarrow \psi, \Sigma_k : B_k \qquad A_k : \Gamma_k, \psi \to \theta, \theta \Rightarrow \Sigma_k : B_k}{A_k : \Gamma_k, \psi \to \theta \Rightarrow \Sigma_k : B_k}$$

For any leaf of an open branch of the form $A_k : \Gamma_k \Rightarrow \psi \to \theta, \Sigma_k : B_k$, replace that sequent with

$$\frac{A_k : \Gamma_k, \psi \Rightarrow \psi \to \theta, \theta, \Sigma_K : B_k}{A_k : \Gamma_k, \Rightarrow \psi \to \theta, \Sigma_k : B_k}$$

6. If φ_i is $\exists x\psi$, then for any leaf of an open branch of the form $A_k : \Gamma_k, \exists x\psi \Rightarrow \Sigma_k : B_k$ do the following. If $\varphi_i \in L$, then do nothing. If $\varphi_i \notin L$, then let $t_i \notin A$, and not appearing in $\Gamma_i \cup \Sigma_i$. Replace that leaf with

$$\frac{A_k, t_i : \Gamma_k, \exists x\psi, \psi[t_i/x] \Rightarrow \Sigma_k : B_k}{A_k : \Gamma_k, \exists x\psi \Rightarrow \Sigma_k : B_k}$$

and add φ_i to L.

If φ_i is $\exists x\psi$, then for any leaf of an open branch of the form $A_k : \Gamma_k \Rightarrow \exists x\psi, \Sigma_k : B_k$, replace that sequent with

$$\frac{A_k : \Gamma \Rightarrow \exists x\psi, \psi[t_j/x], \Sigma_k : B_k \qquad A_k : \Gamma \Rightarrow \exists x\psi, \Sigma_k : B_k, t_j}{A_k : \Gamma \Rightarrow \exists x\psi, \Sigma_k : B_k, t_j}$$

If no branch is open, then from the lemmas and fact of section 6.7.3, the tree is a deduction of $C : \Delta \Rightarrow \Lambda : D$, contradicting the assumption. Consider an open branch on the infinite tree that results from completing stage n for each $n \in \mathbb{N}$. Let node of the branch be enumerated from the root upwards, so that the name-sequent at node j has the form $A_j : \Gamma_j \Rightarrow \Sigma_j : B_k$. Define a sequent $P : \Pi \Rightarrow \Theta : Q$ as $P = \bigcup_{j \in \mathbb{N}} A_j$, $\Pi = \bigcup_{j \in \mathbb{N}} \Gamma_j$, $\Theta = \bigcup_{j \in \mathbb{N}} \Sigma_j$ and $Q = \bigcup_{j \in \mathbb{N}} B_j$.

Fact 2. For each $A_j : \Gamma_j \Rightarrow \Sigma_j : B_j$ and $A_n : \Gamma_n \Rightarrow \Sigma_n : B_n$ that are nodes in the infinite branch, if $j \leqslant n$, then $A_j \subseteq A_n$, $\Gamma_j \subseteq \Gamma_n$, $\Sigma_j \subseteq \Sigma_n$, and $B_j \subseteq B_n$.

Lemma 6.8. *If t appears in an atomic sentence in Π, then $t \in P$.*

Lemma 6.9. $\Pi \cap \Theta = \varnothing$, *$P \cap Q = \varnothing$ and for any term t, $t = t \notin \Theta$.*

Proof. Suppose there were a φ such that $\varphi \in \Pi$ and $\varphi \in \Theta$. Then there would have to be a stage , with $A_j : \Gamma_j \Rightarrow \Sigma_j : B_j$ and a stage m with $A_k : \Gamma_k \Rightarrow \Sigma_k : B_k$ such that $\varphi \in \Gamma_j$ and $\varphi \in \Sigma_k$. But then by fact 2, at stage $n+m$, there is a name-sequent $A_l : \Gamma_l \Rightarrow \Sigma_l : B_l$ such that $\varphi \in \Gamma_l \cap \Sigma_l$. But then this branch would have closed.

A similar argument establishes that $P \cap Q = \varnothing$.

Let $t = t \in \Theta$. Then there must be a stage with a name-sequent $A_j : \Gamma_j \Rightarrow \Sigma_j : B_j$ such that $t = t \in \Sigma_j$. But then the branch would have closed after that stage. $\square$

Proof. Let φ be an atomic sentence in Π that has t as an argument. There is a Γ_j such that $\varphi \in \Gamma_j$, appearing at stage n. At stage $n+1$, the pair, $\langle \varphi, t \rangle$ would have been considered, and t added to A_k. So by section 6.8, $t \in P$. $\square$

Lemma 6.10. *If $\lambda x(\varphi)t \in \Pi$, then $\varphi[t/x] \in \Pi$ and $t \in P$, and if $\lambda x(\varphi)t \in \Theta$ then either $t \in Q$ or $\varphi[t/x] \in \Theta$.*

Proof. Let $\lambda x(\varphi)t \in \Pi$. Then there must be some node $A_j : \Gamma_j \Rightarrow \Sigma_j : B_j$, where $\neg\varphi \in \Gamma_j$. This node would have been built at some stage n. Let k be some term in A at stage n. When considering the pair $\langle \lambda x(\varphi)t, k \rangle$, at stage $n+1$, both t would have been added to A_k, and $\varphi[t/x]$ added to Γ_k. By fact 2, these would have remained in P and Π.

Let $\lambda x(\varphi)t \in \Theta$. Then there is a node, $A_j : \Gamma_j \Rightarrow \Sigma_j : B_j$, where $\lambda x(\varphi)t \in \Sigma_j$. But at stage $n+1$, at any further stage in which that sentence was considered there is a branching such that t is added to the rejected terms on one branch, and $\varphi[t/x]$ to the denied sentences on the other. Since the branch in consideration must go through one of those, by fact 2 either $t \in Q$ or $\varphi[t/x] \in \Theta$. $\square$

Lemma 6.11. *If $\neg\varphi \in \Pi$ then $\varphi \in \Theta$, and if $\neg\varphi \in \Theta$ then $\varphi \in \Pi$.*

Proof. Let $\neg\varphi \in \Pi$. Then there must be some node $A_j : \Gamma_j \Rightarrow \Sigma_j : B_j$, where $\neg\varphi \in \Gamma_j$.This node would have been built at stage n. Let t be a term in A at stage n. At stage $n+1$, when considering $\langle \neg\varphi, t\rangle$, φ would have been added to Σ_m, $m \geqslant j$. But then $\varphi \in \Theta$. The second half of this proof is analogous. □

Lemma 6.12. *If $\varphi \to \psi \in \Pi$, then either $\psi \in \Pi$ or $\varphi \in \Theta$. If $\varphi \to \psi \in \Theta$, then $\varphi \in \Pi$ and $\psi \in \Theta$.*

Proof. The proof of this is as above, but makes use of the rules used for conditionals.

□

Lemma 6.13. *If $\exists x\varphi \in \Pi$, then $\varphi[t/x] \in \Pi$, for some $t \in P$. If $\exists x\varphi \in \Theta$, then either $\varphi[t/x] \in \Theta$ for all $t \in P$, or $P = \varnothing$.*

Proof. Let $\exists x\varphi \in \Pi$, then there is some sequent $A_j : \Gamma_j \Rightarrow \Sigma_j : B_j$, such that $\exists x\varphi \in \Gamma_j$, created at stage n, and $\exists x\varphi \notin L$. At stage $n+1$, t would have been introduced and the name-sequent $A_m, t : \Gamma_m, \exists x\varphi, \varphi[t/x] \Rightarrow \Sigma_m : B_m$ put in the branch.

Let $\exists x\varphi \in \Theta$, then at stage n, $\exists x\varphi$ would have been in Σ_j in the sequent $A_j : \Gamma_j \Rightarrow \Sigma_j : B_j$. Let $P \neq \varnothing$ and $t \in P$. There is a stage, k, and set A_l, such that $t \in A_l$. By fact 2, at stage $k+n$, there is a sequent $A_m : \Gamma_m \Rightarrow \Sigma_m : B_m$, such that $t \in A_m$ and $\exists x\varphi \in \Sigma_m$. At some point, the pair $\langle \exists x\varphi, t\rangle$ would have been considered. So the branch contains either $A_m : \Gamma_m \Rightarrow \varphi[t/x], \exists x\varphi, \Sigma_m : B_m$ or $A_m : \Gamma_m \Rightarrow \exists x\varphi, \Sigma_m : B_m, t$. But since the branch is open the latter is impossible. So $\varphi[t/x] \in \Sigma_m$, and by fact 2, $\varphi[t/x] \in \Theta$. □

6.7.5 Building a Model

The model will be build in the following way. Consider the set of sentences, of the form $t_i = t_j \in \Pi$. Define a relation O, such that if $t_i = t_j \in \Pi$, then Ot_i, t_j, and for any $t \in P$, which by lemma 6.8 includes any term appearing in an atomic sentence in Π. Let Ott. O holds of terms by these two clauses only.

Lemma 6.14. *O is an equivalence relation.*

Proof. O was forced to be reflexive. Let Ot_i, Ot_j. If t_i is not t_j, then $t_i = t_j \in \Pi$. Let Γ_m be the first appearance of this sentence, at stage n. At stage $n+1$, it would have been considered in the set Γ_k, and $t_j = t_i$ added to Γ_k. So $t_j = t_i \in \Pi$, and thus Ot_j, t_i. Suppose that Ot_i, t_j and Ot_j, t_k. The only way that these could hold is if $t_i = t_j$ and $t_j = t_k$ are in Π. Let stage n be the first stage at which they both appear, and Γ_k, the set in which $t_j = t_k$ is considered at stage $n+1$. At that point, $t_i = t_k$ would have been added to Γ_k, and so is in Π. But then Ot_i, t_k. □

A model M is built in the following way. Let $[t] = \{c : Otc\}$, and $D_e = \{[t] : t \in P\}$. Let $D_n = \{t : t \notin \Pi$. Let $D_M = D_e \cup D_n$. $D_e \cap D_n$ since no term is in both P and Q. For a term t if $t \in D_e$, then $I_M(t) = [t]$, otherwise, $I_M(t) = t$. For a predicate F, if $Ft_1, \ldots, t_n \in \Pi$, then let $\langle I_M(t_1), \ldots, I_M(t_n) \in I_M(F)$, and let nothing else be in $I_M(F)$. Finally let $I_M(t_i = t_j) = 1$ iff $I_M(t_i) = I_M(t_j)$. Let the rest of the model be constructed according to section 6.7.1

Lemma 6.15. *If $t \in P$ then $I_M(t) \in D_e$ and if $t \in Q$ then $I_M(t) \in D_n$.*

Proof. Let $t \in P$. So $I_M(t) = [t]$, but then $I_M(t) \in D_e$. Let $t \in Q$. So $I_M(t) = t$. Since $t \notin P$, $t \in D_n$, so $I_M(t) \in D_n$. □

Lemma 6.16. *If $\varphi \in \Pi$ then $I_M(\varphi) = 1$ and if $\varphi \in \Theta$ then $I_M(\varphi) = 0$.*

Proof. This is proved by induction on the rank of φ.

Case 1 (φ is atomic)*.* Let φ be $Ft_1, \ldots, t_n$. Let $\varphi \in \Pi$. This case holds by definition above. Let $\varphi \in \Theta$, but $\langle I_M(t_1), \ldots, I_M(t_n)\rangle \in I_M(F)$. This could only happen if $Ft_1, \ldots, t_n$ were also in Π, but this cannot happen by lemma 6.9.

Case 2 (φ is $\lambda x(\psi)t$)*.* Let $\varphi \in \Pi$. By lemma 6.10, $\varphi[t/x] \in \Pi$ and $t \in P$. By IH, $I_M(\varphi[t/x]) = 1$, and by lemma 6.15, $I_M(t) \in D_e$. So $I_M(\lambda x(\psi)t) = 1$.

Let $\varphi \in \Theta$. By lemma 6.10, either $\psi[t/x] \in \Theta$ or $t \in Q$. In the first case, by IH, $I_M(\psi[t/x] = 0$. But then $I_M(\varphi) = 0$. In the second case, by lemma 6.15, $I_M(t) \in D_n$, so $I_M(\varphi) = 0$.

Case 3 (φ is $\neg\psi$)*.* Let $\varphi \in \Pi$. By lemma 6.11, $\psi \in \Theta$. So by IH $I_M(\psi) = 0$. But then $I_M(\varphi) = 1$. Similarly, let $\varphi \in \Theta$. By lemma 6.11, $\psi \in \Pi$. By IH, $I_M(\psi) = 1$ so $I_M(\varphi) = 0$.

Case 4 (φ is $\psi \to \theta$)*.* This case is omitted as it follows the general pattern of the above.

Case 5 (φ is $\exists x\psi$)*.* Let $\varphi \in \Pi$. By lemma 6.13, there is a term, t such that $\psi[t/x] \in \Pi$. Let M' be the extension of M by w such that $I_{M'}(w) = I_M(t)$. By lemma 6.3, $I_{M'}(psi[w/x]) = I_M[t/x]$. But then $I_M(\varphi) = 1$. Let $\varphi \in \Theta$, and suppose that there is an extension of M, N, by w such that $I_N(\psi[w/x]) = 1$. By lemma 6.13, either $P = \varnothing$ or for any $t \in P$, $\psi[t/x] \in \Theta$. Suppose that $P = \varnothing$. Then $D_n = \varnothing$. But then there are no extensions of M. So $\exists x\varphi$ is false. Let $P \neq \varnothing$ and N be an extension of M by w, such that $I_N(w) = [t]$ for some $t \in P$. Since $I_M(\psi[t/x]) = 0$ and

$I_M(t) = I_N(w)$ and they agree everywhere else, by lemma 6.3, $I_N(\psi[t/x][w/t]) = 0$, which is equivalent to $I_N(\psi[w/x]) = 0$. Since that was N was an arbitrary extension, this holds for all extensions of M. Thus, $I_M(\varphi) = 0$.

□

$\vdash_{cf} A : \Gamma \Rightarrow \Sigma : B$ indicates that $A : \Gamma \Rightarrow \Sigma : B$ is provable without using either Cut(s) or Cut(t).

Lemma 6.17. *If* $\vDash A : \Gamma \Rightarrow \Sigma : B$, *then* $\vdash_{cf} A : \Gamma \Rightarrow \Sigma : B$

Proof. It follows from lemma 6.15 and lemma 6.16 that M is a counter-example to $P : \Pi \Rightarrow \Theta : Q$. From fact 2, $C \subseteq P$, $\Delta \subseteq \Pi$, $\Lambda \subseteq \Theta$, and $D \subseteq Q$. So M is also a counter-example to $C : \Delta \Rightarrow \Lambda : D$. Generalizing, it follows that for any sequent not derivable in the cut-free system has a model. □

Theorem 6.7.2. *If* $\vdash A : \Gamma \Rightarrow \Sigma : B$, *then* $\vdash_{cf} \Gamma \Rightarrow \Sigma : B$

Proof. Let $\vdash A : \Gamma \Rightarrow \Sigma : B$. By theorem 7.5.1, there are no counter-examples to $A : \Gamma \Rightarrow \Sigma : B$. Suppose that $\nvdash_{cf} A : \Gamma \Rightarrow \Sigma : B$. But then there is a counter-example to $A : \Gamma \Rightarrow \Sigma : B$. □

6.8 Failure of Cut Admissibility for a logic with AR and AL

6.8.1 Model Theory

A model M is a pair, $\langle D_M, I_M\rangle$, where D_M is a non-empty set that is partitioned into D_e and D_n. I_M is a function obeying the following clauses:

- For any term t, $I_M(t) \in D_e \cup D_n$.
- For any n-ary predicate, F, $I_M(F) = \langle F^+, F^-\rangle$, such that $F^+ \subseteq D_e^n$, $F^- \subseteq D_e^n$ and $F^+ \cap F^- = \varnothing$ and $F^+ \cup F^- = D_e^n$.
- For any sentence φ, $I_M(\varphi)$ is given by
  - If φ is atomic, $Ft_1, \ldots, t_n$, if $\langle I_M(t_1), \ldots, I_M(t_n)\rangle \in F^+$ then $I_M(Ft_1, \ldots, t_n) = 1$, and if $\langle I_M(t_1), \ldots, I_M(t_n)\rangle \in F^-$ then $I_M(Ft_1, \ldots, t_n) = 0$.
  - If φ is $\neg\psi$, then $I_M(\varphi) = 1$ iff $I_M(\psi) = 0$.
  - If φ is $\psi \rightarrow \theta$, then $I_M(\varphi) = 1$ iff $I_M(\psi) = 0$ or $I_M(\theta) = 1$.
  - If φ is $\exists x\psi$, then $I_M(\varphi) = 1$ iff there is an expansion of M, M', by a witness, w, such that $I_{M'}(\psi[w/x]) = 1$.

Counter-example is defined as above. Let $\vdash A : \Gamma \Rightarrow \Sigma : B$ indicate that $A : \Gamma \Rightarrow \Sigma$ is provable in the cut-free system including the rules from fig. 6.2, AL, and AR. Let $\vDash A : \Gamma \Rightarrow \Sigma$ indicate that there is no counter-example in the above sense to $A : \Gamma \Rightarrow \Sigma : B$. Call this system RL.

Theorem 6.8.1. *If* $\vdash A : \Gamma \Rightarrow \Sigma : B$ *then* $\models A : \Gamma \Rightarrow \Sigma : B$.

Proof. The proof of this is much like the one above. For the axioms no model can assign a sentence both 1 and 0. The only difference between this system and the above is that AL. This is the only case that will be considered. The rest of the proof is similar to theorem 7.5.1. Let M be a counterexample to $A : \Gamma \Rightarrow Ft_1, \ldots, t, \ldots, t_n, \Sigma : B$. So $I_M(Ft_1, \ldots, t, \ldots, t_n) = 0$. Thus, $\langle I_M(t_1), \ldots, I_M(t), \ldots, I_M(t_n)\rangle \in F^-$. But then $I_M(t) \in D_e$. So M is a counter-example to $A, t : \Gamma \Rightarrow \Sigma : B$. □

Lemma 6.18. *There is a counter-example to* $\neg\exists x\neg Fx \Rightarrow \exists xFx$.

Proof. Let $D_M = \{1\}$ and $D_e = \varnothing$. It follows that there is no extension of M. So $I_M(\exists xFx) = 0$ and $I_M(\exists x\neg Fx) = 0$. But then $I_M(\neg\exists x\neg Fx) = 1$. □

Theorem 6.8.2. *Cut is not admissible for system RL.*

Proof. Consider the following two deductions:

$$\dfrac{\dfrac{t : \;\Rightarrow\; : t}{: Ft \Rightarrow\; : t}\,\text{AR} \qquad : Ft \Rightarrow Ft :}{: Ft \Rightarrow \exists xFt}\,\text{R}\exists$$

$$\dfrac{\dfrac{\dfrac{t : \;\Rightarrow\; : t}{: \;\Rightarrow Ft : t}\,\text{AL} \qquad \dfrac{: Ft \Rightarrow Ft :}{: \;\Rightarrow Ft : \neg Ft}\,\text{R}\neg}{: \;\Rightarrow \exists x\neg Ft, Ft}\,\text{R}\exists}{: \neg\exists x\neg Fx \Rightarrow Ft}\,\text{L}\neg$$

□

Since neither of these derivations uses Cut(s), there is no counter-example to their end-sequents. But there is a counter-example to $: \neg\exists x\neg Fx \Rightarrow \exists xFx$.

Figure 6.3: Negative Free Logic

STRUCTURAL RULES

$$\text{Id(s)}\ \frac{}{:\varphi \Rightarrow \varphi :} \qquad \text{Id(t)}\ \frac{}{c: \quad \Rightarrow \quad :c}$$

$$\text{WL(s)}\ \frac{A:\Gamma \Rightarrow \Sigma : B}{A:\Gamma,\varphi \Rightarrow \Sigma : B} \qquad \text{WR(s)}\ \frac{A:\Gamma \Rightarrow \Sigma : B}{A:\Gamma \Rightarrow \varphi,\Sigma : B}$$

$$\text{WL(t)}\ \frac{A:\Gamma \Rightarrow \Sigma : B}{A,c:\Gamma \Rightarrow \Sigma : B} \qquad \text{WR(t)}\ \frac{A:\Gamma \Rightarrow \Sigma : B}{A:\Gamma \Rightarrow \Sigma : B,c}$$

$$\text{Cut(s)}\ \frac{A:\Gamma \Rightarrow \varphi,\Sigma : B \qquad A:\Gamma,\varphi \Rightarrow \Sigma : B}{A:\Gamma \Rightarrow \Sigma : B}$$

$$\text{Cut(t)}\ \frac{A:\Gamma \Rightarrow \Sigma : B,c \qquad A,c:\Gamma \Rightarrow \Sigma : B}{A:\Gamma \Rightarrow \Sigma : B}$$

OPERATIONAL RULES

$$\text{R=}\ \frac{}{: \quad \Rightarrow t = t :}$$

$$\text{L=}\ \frac{A:\Gamma \Rightarrow \Sigma : B(t_0/t_1)}{A:\Gamma,\varphi,t_i = t_j \Rightarrow \Sigma : B}\ i,j \in \{0,1\}$$

$$\text{L}\lambda\ \frac{A,t:\Gamma,\varphi[t/x] \Rightarrow \Sigma : B}{A:\Gamma,\lambda x(\varphi)t \Rightarrow \Sigma : B} \qquad \text{R}\lambda\ \frac{A:\Gamma \Rightarrow \varphi[t/x],\Sigma : B \qquad A:\Gamma \Rightarrow \Sigma : B,t}{A:\Gamma \Rightarrow \lambda x(\varphi)t,\Sigma : B}$$

$$\text{L}\neg\ \frac{\Gamma \Rightarrow \varphi,\Sigma}{\Gamma,\neg\varphi \Rightarrow \Sigma} \qquad \text{R}\neg\ \frac{\Gamma,\varphi \Rightarrow \Sigma}{\Gamma \Rightarrow \neg\varphi,\Sigma}$$

$$\text{L}\!\rightarrow\ \frac{\Gamma \Rightarrow \varphi,\Sigma \qquad \Gamma,\psi \Rightarrow \Sigma}{\Gamma,\varphi \rightarrow \psi \Rightarrow \Sigma} \qquad \text{R}\!\rightarrow\ \frac{\Gamma,\varphi \Rightarrow \psi,\Sigma}{\Gamma \Rightarrow \varphi \rightarrow \psi,\Sigma}$$

$$^{1}\text{L}\exists\ \frac{A,c:\Gamma,\varphi(c) \Rightarrow \Sigma : B}{A:\Gamma,\exists x\varphi(x) \Rightarrow \Sigma : B} \qquad \text{R}\exists\ \frac{A:\Gamma \Rightarrow \Sigma : B,c \qquad A:\Gamma \Rightarrow \varphi(c),\Sigma : B}{A:\Gamma \Rightarrow \exists x\varphi(x),\Sigma : B}$$

t does not appear in the conclusion of L∃
$A:\Gamma \Rightarrow \Sigma : B(t_0/t_1)$ is schematic for any replacement of t_0 by t_1 in $A:\Gamma \Rightarrow \Sigma : B$

Chapter 7

PHIL: A Logic for Contingentism

Abstract. This chapter develops a Predicative Higher-Order Intensional Logic, PHIL, that can underwrite a contingentist account of the nature of possibility. Williamson has argued that such a logic would suffer from being ad hoc. Its first-order quantifiers would be free but to maintain strength its second-order quantifiers could not also be free. Prior has argued that any contingentist logic must deny the interdefinability of necessity and possibility and the rule of necessitation. This chapter shows that PHIL is a response to both challenges. In the latter half of the chapter it is shown that the calculus used to specify PHIL is sound and complete for a set of second-order Henkin models, that the rule of Cut is admissible for that calculus, and that the calculus uniquely characterizes the logical vocabulary of PHIL. The chapter concludes that PHIL is independently well-motivated in addition to being able to solve the above challenges to a contingentist logic.

Keywords. Contingentism, Second-Order Modal Logic, Prior, Williamson

Necessitism has been described by Williamson [74] as the view that necessarily everything is necessarily something, or that what does exist exists in every possible world and what exists in any possible world exists in the actual world. Contingentism is the denial of this claim. The contingentist view is naturally paired with "free" first-order quantifiers, i.e. quantifiers whose range is restricted in some way. In this case the range of a quantifier is restricted to those beings that exist at the world in which it is being evaluated.[1] A logic is contingentist when it does not prove $BF + CBF$

$$\exists x \Diamond \varphi \leftrightarrow \Diamond \exists x \varphi \tag{BF + CBF}$$

The core contingentist claim is that there are individuals that exist that could have failed to exist and some individuals that do not exist though they could have existed.

The logic developed solves two problems for contingentists. The first is a response to an argument given by Prior [43]. Prior held that a being a exists by definition when there are true facts about that being, i.e. $E!a =_{df} \exists f fa$. In other words a thing exists just when something is true of it.[2] There are concerns with this definition of existence in the extensional case. For instance, it is true that Pegasus is not winged, i.e. $\neg Wp$, but it follows from the truth of that sentence that $\exists f fp$ and from there to the existence of Pegasus. The solution to these issues is entailed by the solution to a further problem that occurs when this definition of existence is embedded in a modal logic. Prior [43] argues that adopting the above definition of existence requires that the rules of necessitation and the interdefinability of $\Box$ and $\Diamond$ be abandoned. The

[1] A sequent calculus for free first-order quantification can be found in Gratzl [19] and Restall [57]. This chapter follows Restall [57] in its formulation for first-order quantification.

[2] This definition was accepted by Prior [43].

logic of this chapter holds that definition of existence without giving up the standard $S5$ modal logic where $\Box$ and $\Diamond$ are interdefinable.

A second problem for contingentists has been raised by Williamson [74]. Williamson [74] argues that a contingentist logic must have free first-order quantifiers. This is taken for granted in the system presented below. He then argues that it is prima facie ad-hoc that first-order quantifiers and second-order quantifiers should differ in their treatment, i.e. that a contingentist logic with free first-order quantifiers but classical second-order quantifiers is unmotivated. The problem with free second-order quantifiers is that important principles governing the interaction of $\Box$ and other vocabulary are underivable. For instance Williamson [74] highlights the following two sentences as being underivable in a logic with free second-order quantifiers

Gen $(Taa \wedge \Diamond\neg Taa) \rightarrow \exists f(fa \wedge \Diamond\neg fa)$

Conj $\forall f\forall g\exists h\Box\forall x(hx \leftrightarrow (fx \wedge gx))$

The logic given by fig. 7.1 and fig. 7.2 is a Predicative Higher-order Intensional Logic, called PHIL. Both **Gen** and **Conj** are valid in PHIL but the second-order quantifiers of that logic are nonetheless free.

Not only does PHIL avoid both of these pitfalls, it is cut-admissible and uniquely characterizes all of the expressions for which there are left and right rules in fig. 7.2. Because PHIL has these features it is well-motivated independently of its application to the problems for contingentism raised above.[3] Section 7.1 is an interpretation and explanation of the formalism. Section 7.2 shows how PHIL solves the above

[3]For more on the importance of these features of a calculus see Belnap [4], Prawitz [41], and Humberstone [21].

issues for contingentists. Sections 7.3 to 7.6 establish that PHIL has desirable logical properties such as a sound and complete model-theory, cut-admissibility, and the unique characterization of expressions featuring in left and right introduction rules. The chapter concludes with a discussion of what PHIL can be used to accomplish and what questions PHIL leaves unanswered for future research.

7.1 Interpretation of PHIL

This chapter takes a proof-theoretic starting point for the investigation into a logic. This is justified by what is taken to be the primary object of logical investigation, a position. Following Restall [50, 53, 57], a position is an ordered pair of sets of sentences, those that the position asserts and those that it denies. If Γ and Σ are sets of sentences then $\Gamma \Rightarrow \Sigma$ is a position. $\Gamma \Rightarrow \Sigma$ is the position one takes up by asserting all of Γ and denying all of Σ. For example, if $\Gamma = \{\text{It is raining}\}$ and $\Sigma = \{\text{It is sunny}\}$ then $\Gamma \Rightarrow \Sigma$ is the position that asserts that it is raining and denies that it is sunny.

Positions can be either coherent or incoherent. For instance, it is always incoherent to univocally assert and deny the same sentence. This is why any position of the form $\varphi \Rightarrow \varphi$ is incoherent. Rules of a sequent calculus define which positions are incoherent. The rule of Identity ensures that any position that asserts and denies the same sentence is incoherent. The standard L$\neg$ rule governing the introduction of a negation on the left of a sequent

$$\text{L}\neg \frac{\Gamma \Rightarrow \varphi, \Sigma}{\Gamma, \neg\varphi \Rightarrow \Sigma}$$

says that if it is incoherent to deny φ while asserting all of Γ and denying all of Σ then it is incoherent to assert $\neg\varphi$ while asserting all of Γ and denying all of Σ. In general a rule of a calculus with premises $P_1, \ldots, P_n$ and conclusion C says that if P_i is incoherent for $1 \leqslant i \leqslant n$ then C is incoherent. Read contrapositively this indicates that if C is coherent then one of the P_i's is.

L$\neg$ along with Id guarantees that it is always incoherent to assert both a sentence and its negation. This is given by the deduction

$$\mathrm{L}\neg\,\frac{\mathrm{Id}\,\dfrac{}{\varphi \Rightarrow \varphi}}{\varphi, \neg\varphi \Rightarrow}$$

A deduction of a position guarantees that that position is incoherent. If a position is not deducible given a calculus then for all that has been said that position is coherent.[4]

In addition to rules governing specific expressions of a language, like L$\neg$ governs $\neg$, there are structural rules that govern the language more generally. These rules are justified by the theory of meaning that is being advanced. An example of such a rule is Id. Id is justified by appeal to the incoherence of univocally asserting and denying the same sentence. This is a feature of the practices of asserting and denying. The rules of thinning on the left and right

$$\mathrm{TL}\,\frac{\Gamma \Rightarrow \Sigma}{\Gamma, \varphi \Rightarrow \Sigma} \qquad\qquad \mathrm{TR}\,\frac{\Gamma \Rightarrow \Sigma}{\Gamma \Rightarrow \varphi, \Sigma}$$

are justified by the fact that if one is in an incoherent position asserting or denying more things cannot change that. If it is incoherent to assert that it is raining and

[4]Since a logic of the sort given in this chapter is not a full account of the inference rules of a language a position that is coherent relative to a calculus may turn out to be incoherent at a deeper level of analysis.

deny that it is raining then it is incoherent to assert that it is raining and that it is cloudy and deny that it is raining. In the calculus under consideration the Cut rule is admissible and so does not require a justification.

A consideration of positions alone is enough to pin down the meanings of the propositional connectives. Because both modality and quantification are also objects of study the notion of a position must be expanded to accommodate these features of language. First-order quantification is closely tied to names. An important use of a name, independently of its occurrence in a sentence is that it can be taken to denote or not to denote. A name position takes into account the acceptance (taking to denote) and rejection (taking to be non-denoting) of a name. If $\Gamma \Rightarrow \Sigma$ is a position and A and B are sets of names, then $A : \Gamma \Rightarrow \Sigma : B$ is the name-position one takes up by accepting all of A, asserting all of Γ, denying all of Σ, and rejecting all of B. Because only name-positions are considered in what follows they are from this point forward referred to as 'positions'.

First-order quantifiers take into account which names are accepted and which are not. The rule governing the introduction of a first-order quantifier on the left is

$$\mathrm{L}\exists_1 \frac{A, n : \Gamma, \varphi[n/x] \Rightarrow \Sigma : B}{A : \Gamma, \exists x\varphi \Rightarrow \Sigma : B}$$

where n does not occur in $A : \Gamma, \exists x\varphi \Rightarrow \Sigma : B$. Read from bottom to top $\mathrm{L}\exists_1$ indicates that if it is coherent to accept $A : \Gamma, \exists x\varphi \Rightarrow \Sigma : B$ then there is an expansion of the language of that position with a new term n such that the position $A, n : \Gamma, \varphi[n/x] \Rightarrow \Sigma : B$ is coherent. It is this feature of quantification that makes it free. An existential sentence $\exists x\varphi$ is coherent to assert only if there is an object that

can be named, by e.g. n, such that $\varphi[n/x]$ is coherent to assert. The language itself is free because there are coherent positions that take some names to fail to denote.

As mentioned above the notion of a position is further expanded to account for modality. On the account of logic under consideration modal expressions mark ways that positions can be related to one another. This is represented by means of a hyper-position. A hyper-position is a set of positions. If $A_0 : \Gamma_0 \Rightarrow \Sigma_0 : B_0, \ldots, A_n : \Gamma_n \Rightarrow \Sigma_n : B_n$ are positions then $(A_0 : \Gamma_0 \Rightarrow \Sigma_0 : B_0); \ldots; (A_n : \Gamma_n \Rightarrow \Sigma_n : B_n)$ is a hyper-position. A hyper-position $(A_0 : \Gamma_0 \Rightarrow \Sigma_0 : B_0); \ldots; (A_n : \Gamma_n \Rightarrow \Sigma_n : B_n)$ is incoherent when it is incoherent to take up the position $A_n : \Gamma_n \Rightarrow \Sigma_n : B_n$ and hold that each $A_i : \Gamma_i \Rightarrow \Sigma_i : B_i$ are coherent positions relative to that one. For instance the hyper-position $(: \varphi \Rightarrow \ :); (:\Rightarrow \Diamond\varphi :)$ is incoherent. One cannot coherently deny that φ is possible and hold that there are coherent positions that assert φ. If positions can be thought of as reckonings of how the world is, then hyper-positions can be thought of as reckoning of how descriptions of the world relate to one another.[5]

The above is an account of the author's preferred way of thinking . It may be more natural to for some readers to replace the notion of assertion with truth, denial with falsity, and a position with a set of names that denote, a set of true sentences, a set of false sentences, and a set of terms that do not denote. On this reading a hyper-position corresponds neatly to a set of possible worlds in an S5 model. This

[5]What are here called hyper-positions are elsewhere called 'hypersequents' the change in terminology allows for continuity between the positions of the propositional case and hyper-positions in the modal case. The apparatus of hypersequents as a method of generating a proof system for modal logic was first introduced by Mints [33]. It has more recently been used by Avron [2], Restall [51], and myself (see Chapters 2 and 3) to generate calculi for modal logics.

can be seen most clearly with definition 34.[6] The reason to prefer the reading given here is that it is easier to offer an account of what λ-operators do in this language. Since they play the key role in untangling the Priorian concern with a logic for contingentism the unconventional account above is used.

In order to be complete call the language under consideration $\mathscr{L}$. Let N and V_1 be denumerably infinite sets of names and first-order variables respectively. Let P_i and V_i be denumerably infinite sets of predicates and variables respectively for each arity $i \in \mathbb{N}$. Let $P = \cup_i P_i$ and $V_2 = \cup_i V_i$. Let $\mathscr{L}_1 \subset \mathscr{L}$ be the first-order fragment of $\mathscr{L}$ and $A \subset \mathscr{L}_1$ be the set of atomic sentences of $\mathscr{L}$. The set of sentences $\mathscr{L}$ is defined inductively along with the set of atomic sentences of $\mathscr{L}$ and first-order sentences $\mathscr{L}_1$. The set of sentences is given by the following definition:

1. If $t_1, \ldots, t_n \in N \cup V_1$ and $f \in P_n \cup V_n$ then $ft_1, \ldots, t_n \in A$.

2. If $\varphi, \psi \in \mathscr{L}_1$ then $\neg\varphi, \Diamond\varphi, (\varphi \to \psi)$, and $(\exists_1 v\varphi) \in \mathscr{L}_1$.

3. If $\varphi \in \mathscr{L}_1$ and $t_1, \ldots, t_n \in N$ then $\langle \lambda v_1, \ldots, v_n \varphi \rangle t_1, \ldots, t_n \in A$.

4. If $\varphi, \psi \in \mathscr{L}_1$ then $\neg\varphi, \Diamond\varphi, (\varphi \to \psi), (\exists_1 v\varphi)$, and $(\exists_2 f\varphi) \in \mathscr{L}$.

In cases where no clarity is lost parentheses are dropped. Clause (3) is responsible for the comprehension schema of this language. All instances of the comprehension schema are provable for which there is a λ-expression defined in clause (3). This, in particular, leaves out λ expressions such as $\langle \lambda x \exists f f x \rangle$ which accounts for the fact that PHIL is predicative.

[6] For an account of how to generate other modal logics by manipulation of the rules in fig. 7.1 see chapter 2

Figure 7.1 gives the structural rules of PHIL. The rules Id and Id_t indicate that it is incoherent to assert and deny the same sentence and that it is incoherent to accept and reject the same term. TL and TR indicate that if a position is coherent then removing an asserted (denied) sentence or accepted (rejected) term preserves coherence. The Cut rules indicate that if a hyper-position is coherent then for any term or sentence of the language and any position in that hyper-position, there is a coherent way of expanding the original hyper-position by that term or sentence. W indicates that removing the empty position from a hyper-position maintains coherence.[7] Finally, AL indicates that if it is coherent to assert an atomic sentence featuring a term t in a position then it is coherent to accept t. This amounts to the

[7] Another paper, chapter 2 shows that this is equivalent to requiring transitivity for the modal frames in the standard possible worlds account of modal logic.

Figure 7.1: Contingentist Predicative Second-Order Logic: Structural Rules

$$\text{Id(s)}\ \frac{}{(: Ft_1, \dots t_n \Rightarrow Ft_1, \dots, t_n :)} \qquad \text{Id(t)}\ \frac{}{(t : \ \Rightarrow \ : t)}$$

$$\text{TL(s)}\ \frac{G; (A : \Gamma \Rightarrow \Sigma : B); H}{G; (A : \Gamma, \varphi \Rightarrow \Sigma : B); H} \qquad \text{TR(s)}\ \frac{G; (A : \Gamma \Rightarrow \Sigma : B); H}{G; (A : \Gamma \Rightarrow \varphi, \Sigma : B); H}$$

$$\text{TL(t)}\ \frac{G; (A : \Gamma \Rightarrow \Sigma : B); H}{G; (A, t : \Gamma \Rightarrow \Sigma : B); H} \qquad \text{TR(t)}\ \frac{G; (A : \Gamma \Rightarrow \Sigma : B); H}{G; (A : \Gamma \Rightarrow \Sigma : t, B); H}$$

$$\text{Cut}\ \frac{G; (A : \Gamma, \varphi \Rightarrow \Sigma : B); H \qquad G; (A : \Gamma \Rightarrow \varphi, \Sigma : B); H}{G; (A : \Gamma \Rightarrow \Sigma : B); H} \qquad \text{W}\ \frac{G; H}{G; (: \ \Rightarrow \ :); H}$$

$$\text{AL}\ \frac{G; (A, t : \Gamma \Rightarrow \Sigma : B); H}{G; (A : \Gamma, Ft_1, \dots, t, \dots, t_n \Rightarrow \Sigma : B); H}$$

$$\text{Cut}_t\ \frac{G; (A, t : \Gamma \Rightarrow \Sigma : B); H \qquad G; (A : \Gamma \Rightarrow \Sigma : B, t); H}{G; (A : \Gamma \Rightarrow \Sigma : B); H}$$

claim that all terms featuring in atomic sentences denote and it entails that if a term does not denote then any atomic sentence in which it occurs is false.[8]

There are two more speech acts that are taken into account by this logic. Call a judgment any assertion or denial. Call a predication a sentence that directly connects a name with a predicate. All atomic sentences are predications but e.g. $\neg\varphi$ is not a predication. An assertion of a predication is called an act of predication.

The philosophical motivation for AL is that a judgment about a predication is importantly different from a judgment about a negation or other sentence. Judging that a predicate P is true of an entity o requires that that entity exist. AL is entailed by what Williamson [74] calls the *being constraint*: if a thing has properties then that thing exists. This fits well with the definition of existence discussed above. Importantly AL only applies to atomic predicates because only in judging an atomic sentence does one also either perform an act of predication or deny a predication. Judging a sentence φ to be false by asserting $\neg\varphi$ does not require anything to exist. One could assert, for instance, the sentence "Pegasus is not winged" not because one wants to judge that it is true of Pegasus that it is not winged but because Pegasus does not exist and so cannot be winged. This chapter accepts the being constraint in the following form: asserting a predication commits one to the existence of the subject of that predication. In another mode of speech one might read AL as saying that if anything is true of an object then that object exists. PHIL is distinguished from other logics of contingentism by its acceptance of this principle which Yagisawa [77] suggests contingentists deny.

[8]The logic is thus a *negative* free logic. See Lambert [25] or Lehmann [28] for details.

The operational rules of PHIL are given in fig. 7.2. In this figure, let $\{G_i\}_{1\leqslant i\leqslant n}$ be the set $\{G_1, \dots, G_n\}$, and β be any predicate or λ-expression. The rules for negation and conditional are standard sequent rules. L$\Diamond$ amounts to the claim that if it is coherent to assert $\Diamond\varphi$, it is coherent to expand one's hyper-position by a new position that asserts φ. R$\Diamond$ amounts to the claim that if it is coherent to deny $\Diamond\varphi$ then for any position in one's hyper-position it is coherent to deny φ.

Figure 7.2: Contingentist Predicative Second-Order Logic: Operational Rules

$$\text{L}\neg\ \frac{G; (A : \Gamma \Rightarrow \varphi, \Sigma : B); H}{G; (A : \Gamma, \neg\varphi \Rightarrow \Sigma : B); H} \qquad \text{R}\neg\ \frac{G; (A : \Gamma, \varphi \Rightarrow \Sigma : B); H}{G; (A : \Gamma \Rightarrow \neg\varphi, \Sigma : B); H}$$

$$\text{L}\rightarrow\ \frac{G; (A : \Gamma \Rightarrow \varphi, \Sigma : B); H \qquad G; (A : \Gamma, \psi \Rightarrow \Sigma : B); H}{G; (A : \Gamma, \varphi \rightarrow \psi \Rightarrow \Sigma : B); H} \qquad \text{R}\rightarrow\ \frac{G; (A : \Gamma, \varphi \Rightarrow \psi, \Sigma : B); H}{G; (A : \Gamma \Rightarrow \varphi \rightarrow \psi, \Sigma : B); H}$$

$$\text{L}\Diamond\ \frac{(: \varphi \Rightarrow\ :); (A : \Gamma \Rightarrow \Sigma : B); H}{(A : \Gamma, \Diamond\varphi \Rightarrow \Sigma : B); H} \qquad \text{R}\Diamond\ \frac{G; (A : \Gamma \Rightarrow \varphi, \Sigma : B); (C : \Delta \Rightarrow \Lambda : D); H}{G; (A : \Gamma \Rightarrow \Sigma : B); (C : \Delta \Rightarrow \Diamond\varphi, \Lambda : D); H}$$

$$\text{L}\exists_1\ \frac{G; (A, t : \Gamma, \varphi[t/x] \Rightarrow \Sigma : B); H}{G; (A : \Gamma, \exists x\varphi \Rightarrow \Sigma : B); H} \qquad \text{R}\exists_1\ \frac{G; (A : \Gamma \Rightarrow \varphi[t/x], \Sigma : B); H \qquad G; (A : \Gamma \Rightarrow \Sigma : B, t); H}{G; (A : \Gamma \Rightarrow \exists x\varphi, \Sigma : B); H}$$

$$\text{L}\lambda\ \frac{G; (A, c_1, \dots, c_n : \Gamma, \varphi[c_1/x] \dots [c_n/x] \Rightarrow \Sigma : B); H}{G; (A : \Gamma, \langle\lambda x_1, \dots x_n\varphi\rangle c_1, \dots, c_n \Rightarrow \Sigma : B); H}$$

$$\text{L}\exists_2\ \frac{G; (A : \Gamma, \varphi[F/f] \Rightarrow \Sigma : B); H}{G; (A : \Gamma, \exists f\varphi \Rightarrow \Sigma : B); H} \qquad \text{R}\exists_2\ \frac{G; (A : \Gamma \Rightarrow \varphi[\beta/f], \Sigma : B); H}{G; (A : \Gamma \Rightarrow \exists f\varphi, \Sigma : B); H}$$

$$\text{R}\lambda\ \frac{G; (A : \Gamma \Rightarrow \varphi[c_1/x_1] \dots, [c_n/x_n], \Sigma : B); H \qquad \{G; (A : \Gamma \Rightarrow \Sigma : B, c_i); H\}_{1\leqslant i\leqslant n}}{G; (A : \Gamma \Rightarrow \langle\lambda x_1, \dots, x_n\varphi\rangle c_1, \dots, c_n, \Sigma : B); H}$$

1. t does not appear in the conclusion of L$\exists_1$
2. F does not appear in the conclusion of L$\exists_2$

The rules for first-order quantification interact with the accepted and rejected terms of a position. This is because first-order quantifiers are usually taken to have ontological significance. If φ is a sentence and ε_1 and ε_2 expressions of the same syntactic category then $\varphi[\varepsilon_2/\varepsilon_1]$ is the result of replacing ε_2 for ε_1 in φ. L$\exists$ says that if it is coherent to assert $\exists x\varphi$, then it is coherent to introduce a term t that does not occur in one's position, accept t, and assert $\varphi[t/x]$, where $\varphi[t/x]$ is the result of replacing x everywhere by t in φ. R$\exists$ says that if it is coherent to deny $\exists x\varphi$ in a position then for any term t it is either coherent to deny $\varphi[t/x]$ or to reject t. As noted above predication is taken to be ontologically significant. If one judges that a predicate holds of a term then one is committed to that term having a denotation. This is the reason that Lλ and Rλ interact with accepted and rejected terms in the same way that first-order quantifiers do. A λ-expression e.g. $\lambda x\neg Fx$ is a predicate. Asserting $\langle\lambda x\neg Fx\rangle a$ is judging that the predicate $\lambda x\neg Fx$ holds of a. This is an act of predication and as such according to the being constraint discussed above requires that a exist. This is why the Lλ rule requires adding terms to the accepted terms of a sequent. Similarly, denying $\langle\lambda x\neg Fx\rangle a$ is a denial that $\lambda x\neg Fx$ holds of a. This denial would be justified if either a was not a non-F or a did not denote. This is why the Rλ rule branches. It is through the use of λ-expressions that this logic expresses the distinction between an act of judgment and an act of predication. All acts of predication are judgments but not vice-versa. It is also not the case, as standard accounts of λ-expressions assume, that the two are convertible, i.e. that every judgment can be considered as an act of predication. This is how this logic accounts for the logical coherence of asserting that Pegasus is not winged but denies

that Pegasus exists. The assertion that Pegasus is not winged is a judgment that cannot be converted into an act of predication.

The rules for second-order quantification are the standard proof-theoretic rules governing second-order quantifiers. The comprehension schema is built into the specification of $\mathscr{L}$ itself. The predicates that can be comprehended are those for which there is a λ-expression in the language.

Though the rules governing second-order quantification in fig. 7.2 are the standard ones, this is only to save space. The theory of second-order quantification at play in this logic is free. For ease of notation and discussion only positions are considered for this argument. Given theorem 7.6.3 and theorem 7.6.4 any result proved here about the logic that does not contain modal operators can be ported without troubles into the system that does contain modal logic.[9] Expand the notion of a hyper-position now to contain sets of λ-expressions. If A and B are sets of names, Γ and Σ sets of sentences, and η and θ sets of λ-expressions then $\eta \wr A : \Gamma \Rightarrow \Sigma : B \int \theta$ is a position. As in the first-order case, second-order quantification takes into account whether or not a λ-expression denotes.

One account of the denotation of a predicate is that a predicate term $\lambda x\varphi$ denotes iff there is some term to which it can be truly or falsely applied. The rules governing such a criterion are the following

$$\text{LTD}\ \frac{\eta \wr A : \Gamma, \varphi[t/x] \Rightarrow \Sigma : B \int \theta \qquad \eta \wr A : \Gamma \Rightarrow \varphi[t/x], \Sigma : B \int \theta}{\eta, \lambda x\varphi \wr A : \Gamma \Rightarrow \Sigma : B \int \theta}$$

[9]This follows from the fact that cut admissibility in this case entails that any part of the language conservatively extends the reset of the language.

$$\text{RTD}\ \frac{\eta \wr A : \Gamma, \varphi[t/x] \Rightarrow \varphi[t/x], \Sigma : B \int \theta}{\eta \wr A : \Gamma \Rightarrow \Sigma : B \int \lambda x\varphi, \theta}$$

where t does not occur in the conclusion of LTD. LTD and RTD in a classical setting entail that for any λ-expression, that λ expression denotes. This is because classical logic enforces bivalence and the denotation of λ-expressions depends on the truth of sentences that those λ-expressions feature in. In a three valued setting, such as the logic Q developed by Prior [43], LTD and RTD would not guarantee that every λ-expression denotes.

The corresponding rules for second-order quantification in such a context are

$$\text{L}\exists_2^*\ \frac{\eta, \lambda x F x \wr A : \Gamma, \varphi[\lambda x F x/f] \Rightarrow \Sigma : B \int \theta}{\eta \wr A : \Gamma, \exists f \varphi \Rightarrow \Sigma : B \int \theta}$$

$$\text{R}\exists_2^*\ \frac{\eta \wr A : \Gamma \Rightarrow \Sigma : B \int \beta, \theta \qquad \eta \wr A : \Gamma \Rightarrow \varphi[\beta/x], \Sigma : B \int \theta}{\eta \wr A : \Gamma \Rightarrow \exists f \varphi, \Sigma : B \int \theta}$$

where neither F nor $\lambda x F x$ appear in the conclusion of L$\exists_2^*$. These rules are equivalent to L$\exists_2$ and R$\exists_2$ given that $\eta \wr A : \Gamma \Rightarrow \Sigma : B \int \lambda x\varphi, \theta$ is derivable for any λ-expression $\lambda x\varphi$. The second-order quantifiers of PHIL are therefore free under one interpretation of what it would take for a λ-expression to fail to denote. Because the underlying logic is bivalent the account of what it takes for a λ-expression to denote entails that all λ-expressions denote. In order to avoid clumsier notation the rules L$\exists_2$ and R$\exists_2$ are used instead of L$\exists_2^*$, R$\exists_2^*$, LTD, and RTD.

7.2 Challenges to Contingentism

7.2.1 Prior's Arguments

As noted above Prior [43] argues that if "a exists" is defined as $\exists f f a$ then it is necessary to deny that $\Diamond$ and $\Box$ are interdefinable and to deny that necessitation holds. Both of these arguments are discussed in Menzel [32].

Prior first argues that on the given definition of existence if a exists then it is not the case that it is possible that a does not exist. Prior's argument is along the following lines: if a does not exist then there is something true of a, i.e. that it does not exist. Since all definitions are necessary, by the given definition of existence it follows that it is not possible that a does not exist. The rules governing classical λ-abstraction are CLA and CRA.

$$\text{CLA}\ \frac{G; (A : \Gamma, \varphi[a/x] \Rightarrow \Sigma : B); H}{\Gamma; (A : \Gamma, \langle \lambda x \varphi \rangle a \Rightarrow \Sigma : B); H} \qquad \text{CRA}\ \frac{G; (A : \Gamma \Rightarrow \varphi[a/x], \Sigma : B); H}{G; (A : \Gamma \Rightarrow \langle \lambda x \varphi \rangle a, \Sigma : B); H}$$

In a language with impredicative comprehension and classical λ-abstraction this argument can be formalized as follows

$$\text{R}\neg\,\dfrac{\text{L}\Diamond\,\dfrac{\text{W}\,\dfrac{\text{L}\neg\,\dfrac{\text{Cut}\,\dfrac{\text{R}\exists_2\,\dfrac{\text{CRA}\,\dfrac{\text{Id(s)}\,\dfrac{}{(: \neg\exists f f a \Rightarrow \neg\exists f f a :)}}{(: \neg\exists f f a \Rightarrow \langle \lambda x \neg\exists f f x \rangle a :)}}{(: \neg\exists f f a \Rightarrow \exists f f a :)} \qquad \text{R}\neg\,\dfrac{\text{Id(s)}\,\dfrac{}{(: \exists f f a \Rightarrow \exists f f a :)}}{(: \ \Rightarrow \exists f f a, \neg\exists f f a :)}}{(: \ \Rightarrow \exists f f a :)}}{(: \neg\exists f f a \Rightarrow \ :)}}{(: \neg\exists f f a \Rightarrow \ :); (: \ \Rightarrow \ :)}}{(: \Diamond\neg\exists f f a \Rightarrow \ :)}}{(: \ \Rightarrow \neg\Diamond\neg\exists f f a :)}$$

If $\Box$ and $\Diamond$ are inter-definable it follows that a necessarily exists. Prior rejects that conclusion but accepts the other rules that are applied in the argument. This leads him to reject the interdefinability of $\Diamond$ and $\Box$.

By the lights of PHIL this deduction fails at the first step. The application of CRA to $(: \neg\exists f fa \Rightarrow \neg\exists f fa :)$ is not an admissible rule of PHIL. CRA allows one to convert any judgment to an act of predication. In this particular case the judgment that a is nothing is converted to an assertion of the predication that nothing is true of a. These two are importantly distinct. The judgment that a has no properties – in this context, that a does not exist – does not require that a exists. Contrasted with this is that the assertion of the predication of a that it has no properties does entail that a exists. Because of the definition of λ-expressions above that guarantees that PHIL is predicative the position that predicates of a that it has no properties is not expressible in PHIL. However, in PHIL's more relaxed relative Contingentist Higher-order Impredicative Logic with Lambda-expressions, CHILL, the following derivation rules out the coherence of such an assertion in any position.

$$
\text{L}\lambda\,\dfrac{\text{L}\neg\,\dfrac{\text{R}\exists_2\,\dfrac{\text{R}\lambda\,\dfrac{\text{R}{\to}\,\dfrac{\text{TL(t)}\,\dfrac{\text{Id}\,\dfrac{}{: Fa \Rightarrow Fa :}}{a : Fa \Rightarrow Fa :}}{a : \quad \Rightarrow Fa \to Fa :} \qquad \text{Id(t)}\,\dfrac{}{a : \quad \Rightarrow \quad : a}}{a : \quad \Rightarrow \langle \lambda x.Fx \to Fx\rangle a}}{a : \quad \Rightarrow \exists f fa}}{a : \neg\exists f fa \Rightarrow \quad :}}{: \langle\lambda x \neg\exists f fx\rangle a \Rightarrow \quad :}
$$

Given the model theory of section 7.4 and theorem 7.5.1 it is possible to prove that the position that asserts that a does not have any properties without predicating that of a is not deducible and so is coherent. The position $(: \quad \Rightarrow \neg\Diamond\neg\exists f fa :)$ is

coherent for similar reasons. According to PHIL it is coherent to deny that it is not possible for a not to have a property.

The second argument that Prior gives hinges on the definition of existence given above. It is a law of logic that $Fa \to Fa$. It follows that if $Fa \to Fa$ then there is something that a is, i.e. $\exists f fa$. It then follows that $\Box(Fa \to Fa)$ entails $\Box\exists f fa$. But since $Fa \to Fa$ is a logical law by necessitation $\Box(Fa \to Fa)$ is also a logical law. It follows that $\Box\exists f fa$. This can be formalized as follows[10]

$$
\text{Cut}\;\dfrac{
\text{R}\Box\;\dfrac{\text{W}\;\dfrac{\text{R}{\to}\;\dfrac{\text{Id(s)}\;\dfrac{}{(: Fa \Rightarrow Fa :)}}{(: \quad \Rightarrow Fa \to Fa :)}}{(: \quad \Rightarrow Fa \to Fa :); (: \quad \Rightarrow \quad :)}}{(: \quad \Rightarrow \Box(Fa \to Fa) :)}
\qquad
\text{R}\Box\;\dfrac{\text{L}\Box\;\dfrac{\text{W}\;\dfrac{\text{R}\exists\;\dfrac{\text{CRA}\;\dfrac{\text{Id(s)}\;\dfrac{}{(: Fa \to Fa \Rightarrow Fa \to Fa :)}}{(: Fa \to Fa \Rightarrow \langle \lambda x.Fx \to Fx \rangle a :)}}{(: Fa \to Fa \Rightarrow \exists f fa :)}}{(: Fa \to Fa \Rightarrow \exists f fa :); (: \quad \Rightarrow \quad :)}}{(:\Rightarrow \exists f fa :); (: \Box(Fa \to Fa) \Rightarrow \quad :)}}{(: \Box(Fa \to Fa) \Rightarrow \Box\exists f fa :)}
}{(: \quad \Rightarrow \Box\exists f fa :)}
$$

Prior diagnoses the issue with this deduction to be the left most instance of R$\Box$ which in natural deduction settings is an instance of necessitation. The problem with this deduction is similar to the above problem. Both deductions require an application of CRA which is not an admissible rule of PHIL. As above there are counter-examples to $(: \quad \Rightarrow \Box\exists f fa :)$ in the model theory given in section 7.4. Given this, theorem 7.5.1 entails that $(: \quad \Rightarrow \Box\exists f fa :)$ is not provable according to PHIL.

The logic proposed in this chapter diagnoses the problem as the instance of CRA applied in the above deduction. The corresponding instance of Lλ is not valid. As above a model where a does not exist is a counter-example to $(: \quad \Rightarrow \Box\exists f fa :)$

[10] For the definable rules governing $\Box$ see section 7.3.1. The derivable rules governing the other extensional connectives e.g. $\wedge$ and $\vee$ are the familiar rules embedded in hyper-positions.

7.2.2 Williamson's Challenge

Williamson [74] argues that a logic with free first-order quantification ought also to use free second-order quantification. A logic with free second-order quantifiers, Williamson argues, is too weak to capture normal intuitions govering valid arguments of second-order modal logic. In particular, he cites Gen and Conj as examples.

It was argued above that the second-order quantifiers of this logic are free. It is a theorem of PHIL that all λ-expressions denote. This is entailed by the account of denotation for λ-expressions that is assumed and the fact that the logic is bivalent. Given this both Gen and Conj are valid. This is given by the following two deductions.

Gen $(Taa \wedge \Diamond\neg Taa) \rightarrow \exists f(fa \wedge \Diamond\neg fa)$

$$\text{L}\wedge\,\dfrac{\text{L}\Diamond\,\dfrac{\text{R}\exists_2\,\dfrac{\text{R}\wedge\,\dfrac{\text{R}\lambda\,\dfrac{\text{TL(s)}\,\dfrac{\text{W}\,\dfrac{\text{TR(t)}\,\dfrac{\text{Id(s)}\,\dfrac{}{(:Taa \Rightarrow Taa:)}}{(a:Taa \Rightarrow Taa:)}}{(:\ \Rightarrow\ :);(a:Taa \Rightarrow Taa:)}}{(:\neg Taa \Rightarrow\ :);(a:Taa \Rightarrow Taa:)} \quad \text{TL(s)}\,\dfrac{\text{W}\,\dfrac{\text{TL(s)}\,\dfrac{\text{Id(t)}\,\dfrac{}{(a:\ \Rightarrow\ :a)}}{(a:Taa \Rightarrow\ :a)}}{(:\ \Rightarrow\ :);(a:Taa \Rightarrow\ :a}}{(:\neg Taa \Rightarrow\ :);(a:Taa \Rightarrow\ :a}}{(:\neg Taa \Rightarrow\ :);(a:Taa \Rightarrow \langle\lambda x Tax\rangle a:)} \quad \text{R}\Diamond\,\dfrac{\text{R}\neg\,\dfrac{\text{L}\lambda\,\dfrac{\text{L}\neg\,\dfrac{\text{TL(t)}\times 2\,\dfrac{\text{TL(s)}\,\dfrac{\text{W}\,\dfrac{\text{Id(s)}\,\dfrac{}{(:Taa \Rightarrow Taa:)}}{(:Taa \Rightarrow Taa:);(:\ \Rightarrow\ :)}}{(:Taa \Rightarrow Taa:);(:Taa \Rightarrow\ :)}}{(a:Taa \Rightarrow Taa:);(a:Taa \Rightarrow\ :)}}{(a:Taa,\neg Taa \Rightarrow\ :);(a:Taa \Rightarrow\ :)}}{(:\langle\lambda x Tax\rangle a,\neg Taa \Rightarrow\ :);(a:Taa \Rightarrow\ :)}}{(:\neg Taa \Rightarrow \neg\langle\lambda x Tax\rangle a:);(a:Taa \Rightarrow\ :)}}{(:\neg Taa \Rightarrow\ :);(a:Taa \Rightarrow \Diamond\neg\langle\lambda x Tax\rangle a:)}}{(:\neg Taa \Rightarrow\ :);(Taa \Rightarrow \langle\lambda x Tax\rangle a \wedge \Diamond\neg\langle\lambda x Tax\rangle a:)}}{(:\neg Taa \Rightarrow\ :);(Taa \Rightarrow \exists f(fa \wedge \Diamond\neg fa):)}}{(Taa,\Diamond\neg Taa \Rightarrow \exists f(fa \wedge \Diamond\neg fa):)}}{(Taa \wedge \Diamond\neg Taa \Rightarrow \exists f(fa \wedge \Diamond\neg fa):)}$$

Conj $\forall f\forall g\exists h \Box \forall x(hx \leftrightarrow (fx \wedge gx))$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{(: Fa \wedge Ga \Rightarrow Fa \wedge Ga :)}\,\text{Id(s)}}{(a : Fa \wedge Ga \Rightarrow Fa \wedge Ga :)}\,\text{TL(s/t)}}{(: \langle \lambda y.Fy \wedge Gy \rangle a \Rightarrow Fa \wedge Ga :)}\,\text{L}\lambda}{(a : \langle \lambda y.Fy \wedge Gy \rangle a \Rightarrow Fa \wedge Ga :)}\,\text{TL(t)} \quad (1)$$

$$\dfrac{\dfrac{}{(: Fa \wedge Ga \Rightarrow Fa \wedge Ga :)}\,\text{Id(s)}}{(a : Fa \wedge Ga \Rightarrow Fa \wedge Ga :)}\,\text{TL(t)} \quad (2)$$

$$\dfrac{\dfrac{}{(a : \ \Rightarrow \ : a)}\,\text{Id(t)}}{(a : Fa \wedge Ga \Rightarrow \ : a)}\,\text{TL(s)} \quad (3)$$

$$\dfrac{(2) \qquad (3)}{(a : Fa \wedge Ga \Rightarrow \langle \lambda y.Fy \wedge Gy \rangle a :)}\,\text{R}\lambda \quad (4)$$

$$\dfrac{(1) \qquad (4)}{(a : \ \Rightarrow \langle \lambda y.Fy \wedge Gy \rangle a \leftrightarrow Fa \wedge Ga :)}\,\text{R}\leftrightarrow$$

$$\dfrac{}{(: \ \Rightarrow \forall x(\langle \lambda y.Fy \wedge Gy \rangle x \leftrightarrow Fx \wedge Gx) :)}\,\text{R}\forall$$

$$\dfrac{}{(: \ \Rightarrow \forall x(\langle \lambda y.Fy \wedge Gy \rangle x \leftrightarrow Fx \wedge Gx) :); (: \ \Rightarrow \ :)}\,\text{W}$$

$$\dfrac{}{(: \ \Rightarrow \Box \forall x(\langle \lambda y.Fy \wedge Gy \rangle x \leftrightarrow Fx \wedge Gx) :)}\,\text{R}\Box$$

$$\dfrac{}{(: \ \Rightarrow \exists h \, \Box \, \forall x(hx \leftrightarrow Fx \wedge Gx) :)}\,\text{R}\exists_2$$

$$\dfrac{}{(: \ \Rightarrow \forall fg \exists h \, \Box \, \forall x(hx \leftrightarrow (fx \wedge gx)) :)}\,\text{R}\forall_2 \times 2$$

As mentioned above, on the account of what it takes for a λ-expression to denote the second-order quantifiers of this chapter are free. Second-Order quantifiers range only over predicates that denote. In order for a predicate to denote it must either be true or false of a term. However, because PHIL is bivalent these free second-order quantifier collapse with the classical (Henkin) second-order quantifiers in this case. This is what accounts for the validity of the above two deductions. Similar deductions show that in general a comprehension schema for which there is a corresponding λ-expression is valid.

7.3 Important Results

7.3.1 Interdefinabiliy of Necessity and Possibility

The modal operators $\Box$ and $\Diamond$ are interdefinable in PHIL. The rules governing $\Box$ in a hyper-position calculus for S5 are[11]

$$\text{L}\Box \, \dfrac{G; (A : \Gamma, \varphi \Rightarrow \Sigma : B); (C : \Delta \Rightarrow \Lambda : D); H}{G; (A : \Gamma \Rightarrow \Sigma : B); (C : \Delta, \Box\varphi \Rightarrow \Lambda : D); H} \qquad \text{R}\Box \, \dfrac{(: \ \Rightarrow \varphi :); (A : \Gamma \Rightarrow \Sigma : B); H}{(A : \Gamma \Rightarrow \Box\varphi, \Sigma : B); H}$$

[11] These rules can be found in Restall [53] or Chapters 2 and 3 of this work.

Theorem 7.3.1. $\vdash G; (A : \Gamma, \Box\varphi \Rightarrow \Sigma : B); H$ *iff* $\vdash G; (A : \Gamma, \neg\Diamond\neg\varphi \Rightarrow \Sigma : B); H$ *and* $\vdash G; (A : \Gamma \Rightarrow \Box\varphi, \Sigma : B); H$ *iff* $\vdash G; (A : \Gamma \Rightarrow \neg\Diamond\neg\varphi, \Sigma : B); H$

Proof. Call the following deductions δ_1 and δ_2 respectively

$$
\begin{prooftree}
\AxiomC{}
\LeftLabel{Id(s)}
\UnaryInfC{$(: \varphi \Rightarrow \varphi :)$}
\LeftLabel{W}
\UnaryInfC{$(: \varphi \Rightarrow \varphi :); (: \quad \Rightarrow \quad :)$}
\LeftLabel{L$\Box$}
\UnaryInfC{$(: \quad \Rightarrow \varphi :); (: \Box\varphi \Rightarrow \quad :)$}
\LeftLabel{L$\neg$}
\UnaryInfC{$(: \neg\varphi \Rightarrow \quad :); (: \Box\varphi \Rightarrow \quad :)$}
\LeftLabel{L$\Diamond$}
\UnaryInfC{$(: \Diamond\neg\varphi, \Box\varphi \Rightarrow \quad :)$}
\LeftLabel{R$\neg$}
\UnaryInfC{$(: \Box\varphi \Rightarrow \neg\Diamond\neg\varphi :)$}
\LeftLabel{W + TL + TR}
\UnaryInfC{$G; (A : \Gamma, \Box\varphi \Rightarrow \neg\Diamond\neg\varphi, \Sigma : B); H$}
\end{prooftree}
$$

$$
\begin{prooftree}
\AxiomC{}
\LeftLabel{Id(s)}
\UnaryInfC{$(: \varphi \Rightarrow \varphi :)$}
\LeftLabel{W}
\UnaryInfC{$(: \varphi \Rightarrow \varphi :); (: \quad \Rightarrow \quad :)$}
\LeftLabel{R$\neg$}
\UnaryInfC{$(: \quad \Rightarrow \neg\varphi, \varphi :); (: \quad \Rightarrow \quad :)$}
\LeftLabel{R$\Diamond$}
\UnaryInfC{$(: \quad \Rightarrow \varphi :); (\quad \Rightarrow \Diamond\neg\varphi :)$}
\LeftLabel{R$\Box$}
\UnaryInfC{$(: \quad \Rightarrow \Diamond\neg\varphi, \Box\varphi :)$}
\LeftLabel{L$\neg$}
\UnaryInfC{$(: \neg\Diamond\neg\varphi \Rightarrow \Box\varphi :)$}
\LeftLabel{W + TL + TR}
\UnaryInfC{$G; (A : \Gamma, \neg\Diamond\neg\varphi \Rightarrow \Box\varphi, \Sigma : B); H$}
\end{prooftree}
$$

Suppose $\vdash G; (A : \Gamma, \Box\varphi \Rightarrow \Sigma : B); H$. Cutting this deduction with δ_2 yields a deduction of $G; (A : \Gamma, \neg\Diamond\neg\varphi \Rightarrow \Sigma : B); H$. Suppose that $\vdash G; (A : \Gamma, \neg\Diamond\neg\varphi \Rightarrow \Sigma : B); H$. Cutting this with δ_1 yields a deduction of $G; (A : \Gamma, \Box\varphi \Rightarrow \Sigma : B); H$. The other conjunct of the theorem is similar. $\square$

Theorem 7.3.2. $\vdash G; (A : \Gamma, \neg \Box \neg\varphi \Rightarrow \Sigma : B); H$ *iff* $\vdash G; (A : \Gamma, \Diamond\varphi \Rightarrow \Sigma : B); H$ *and* $\vdash G; (A : \Gamma \Rightarrow \neg \Box \neg\varphi, \Sigma : B); H$ *iff* $\vdash G; (A : \Gamma \Rightarrow \Diamond\varphi, \Sigma : B); H$

Proof. Similar to the proof of theorem 7.3.1. $\square$

Theorem 7.3.1 and theorem 7.3.2 establish that unlike System Q – developed by Prior [43] – in the logic given by fig. 7.1 and fig. 7.2 the operators $\Box$ and $\Diamond$ are interdefinable.

7.3.2 Uniqueness

PHIL uniquely characterizes all of the operators that are given by rules of those figures. The cases of the extensional connectives $\neg$ and $\rightarrow$ are familiar. The cases of $\Diamond$, λ, the first-order quantifier, and the second-order existential quantifier are given below.

Lemma 7.1. *Let $\mathscr{L}$ be expanded by $\hat{\Diamond}$ that is governed by the rules $L\hat{\Diamond}$ and $R\hat{\Diamond}$ that result from replacing every occurrence of $\Diamond$ by $\hat{\Diamond}$ in $L\Diamond$ and $R\Diamond$. The following two facts hold*

1. $\vdash G; (A : \Gamma, \Diamond\varphi \Rightarrow \Sigma : B); H$ *iff* $\vdash G; (A : \Gamma, \hat{\Diamond}\varphi \Rightarrow \Sigma : B); H$
2. $\vdash G; (A : \Gamma \Rightarrow \Diamond\varphi, \Sigma : B); H$ *iff* $\vdash G; (A : \Gamma \Rightarrow \hat{\Diamond}\varphi, \Sigma : B); H$

Proof. Let δ_1 and δ_2 be the following deductions

$$\mathrm{W}+\mathrm{TL}+\mathrm{TR}\;\dfrac{\mathrm{L}\hat{\Diamond}\;\dfrac{\mathrm{R}\Diamond\;\dfrac{\mathrm{W}\;\dfrac{\mathrm{Id}\;\dfrac{\delta_1}{(:\varphi \Rightarrow \varphi :)}}{(:\varphi \Rightarrow \varphi :); (:\quad \Rightarrow \quad :)}}{(:\varphi \Rightarrow \quad :); (:\quad \Rightarrow \Diamond\varphi :)}}{(:\hat{\Diamond}\varphi \Rightarrow \Diamond\varphi :)}}{G; (A : \Gamma, \hat{\Diamond}\varphi \Rightarrow \Diamond\varphi, \Sigma : B); H}$$

$$
\text{W + TL + TR}\;\dfrac{\text{L}\Diamond\;\dfrac{\text{R}\hat{\Diamond}\;\dfrac{\text{W}\;\dfrac{\text{Id}\;\dfrac{\delta_2}{(:\varphi\Rightarrow\varphi:)}}{(:\varphi\Rightarrow\varphi:);(:\quad\Rightarrow\quad:)}}{(:\varphi\Rightarrow\quad:);(:\quad\Rightarrow\hat{\Diamond}\varphi:)}}{(:\Diamond\varphi\Rightarrow\hat{\Diamond}\varphi:)}}{G;(A:\Gamma,\Diamond\varphi\Rightarrow\hat{\Diamond}\varphi,\Sigma:B);H}
$$

Case 1. For the left to right direction let $\vdash G;(A:\Gamma,\Diamond\varphi\Rightarrow\Sigma:B);H$. Cutting this deduction with δ_1 yields a deduction of $G;(A:\Gamma,\hat{\Diamond}\varphi\Rightarrow\Sigma:B);H$. For the right to left direction let $\vdash G;(A:\Gamma,\hat{\Diamond}\varphi\Rightarrow\Sigma:B);H$. Cutting this deduction with δ_2 yields a deduction of $G;(A:\Gamma,\Diamond\varphi\Rightarrow\Sigma:B);H$.

Case 2. For the left to right direction let $\vdash G;(A:\Gamma\Rightarrow\Diamond\varphi,\Sigma:B);H$. Cutting this with δ_2 yields a deduction of $G;(A:\Gamma\Rightarrow\hat{\Diamond}\varphi,\Sigma:B);H$. For the right to left direction let $\vdash G;(A:\Gamma\Rightarrow\hat{\Diamond}\varphi,\Sigma:B);H$. Cutting this with a δ_1 yields a deduction of $G;(A:\Gamma\Rightarrow\Diamond\varphi,\Sigma:B);H$.

□

Lemma 7.2. *Let $\mathscr{L}$ be expanded by $\hat{\lambda}$ that is governed by the rules $L\hat{\lambda}$ and $R\hat{\lambda}$ that result from replacing every occurrence of λ by $\hat{\lambda}$ in $L\lambda$ and $R\lambda$. The following two facts hold*

1. $\vdash G;(A:\Gamma,\langle\lambda x_1,\ldots,x_n\varphi\rangle t_1,\ldots,t_n\Rightarrow\Sigma:B);H$ *iff* $\vdash G;(A:\Gamma,\langle\hat{\lambda}x_1,\ldots,x_n\varphi\rangle t_1,\ldots,t_n\Rightarrow\Sigma:B);H$
2. $\vdash G;(A:\Gamma\Rightarrow\langle\lambda x_1,\ldots,x_n\varphi\rangle t_1,\ldots,t_n,\Sigma:B);H$ *iff* $\vdash G;(A:\Gamma\Rightarrow\langle\hat{\lambda}x_1,\ldots,x_n\varphi\rangle t_1,\ldots,t_n,\Sigma:B);H$

Proof. This proof is similar to lemma 7.1. The key to the lemma is proving that $\vdash G;(A:\Gamma,\langle\lambda x_1,\ldots,x_n\varphi\rangle t_1,\ldots,t_n\Rightarrow\langle\hat{\lambda}x_1,\ldots,x_n\varphi\rangle t_1,\ldots,t_n,\Sigma:B);H$ and

$\vdash G; (A : \Gamma, \langle\hat{\lambda}x_1, \dots, x_n\varphi\rangle t_1, \dots, t_n \Rightarrow \langle\lambda x_1, \dots, x_n\varphi\rangle t_1, \dots, t_n, \Sigma : B); H$. This is given by the following two deductions.

$$\mathrm{W{+}TL{+}TR}\,\dfrac{\mathrm{L}\hat{\lambda}\,\dfrac{\mathrm{R}\lambda\,\dfrac{\mathrm{TL(t)}\,\dfrac{\mathrm{Id(s)}\,\dfrac{}{(: \varphi[t_1/x_1]\dots[t_n/x_n] \Rightarrow \varphi[t_1/x_1]\dots[t_n/x_n] :)}}{(t_1, \dots, t_n : \varphi[t_1/x_1]\dots[t_n/x_n] \Rightarrow \varphi[t_1/x_1]\dots[t_n/x_n] :)} \quad \mathrm{TL(s/t)}\,\dfrac{\mathrm{Id(t)}\,\dfrac{}{(t_1 : \ \Rightarrow \ : t_1)}}{(t_1, \dots, t_n : \varphi[t_1/x_1]\dots[t_n/x_n] \Rightarrow \ : t_1)} \quad \dots \quad \mathrm{TL(s/t)}\,\dfrac{\mathrm{Id(t)}\,\dfrac{}{(t_n : \ \Rightarrow \ : t_n)}}{(t_1, \dots, t_n : \varphi[t_1/x_1]\dots[t_n/x_n] \Rightarrow \ : t_n)}}{(t_1, \dots, t_n : \varphi[t_1/x_1]\dots[t_n/x_n] \Rightarrow \langle\lambda x_1, \dots, x_n\varphi\rangle t_1, \dots, t_n :)}}{(: \langle\hat{\lambda}x_1, \dots, x_n\varphi \Rightarrow t_1, \dots, t_n\rangle \Rightarrow \langle\lambda x_1, \dots, x_n\varphi\rangle t_1, \dots, t_n :)}}{G; (A : \Gamma, \langle\hat{\lambda}x_1, \dots, x_n\varphi\rangle t_1, \dots, t_n \Rightarrow \langle\lambda x_1, \dots, x_n\varphi\rangle t_1, \dots, t_n, \Sigma : B); H}$$

$$\mathrm{W{+}TL{+}TR}\,\dfrac{\mathrm{L}\lambda\,\dfrac{\mathrm{R}\hat{\lambda}\,\dfrac{\mathrm{TL(t)}\,\dfrac{\mathrm{Id(s)}\,\dfrac{}{(: \varphi[t_1/x_1]\dots[t_n/x_n] \Rightarrow \varphi[t_1/x_1]\dots[t_n/x_n] :)}}{(t_1, \dots, t_n : \varphi[t_1/x_1]\dots[t_n/x_n] \Rightarrow \varphi[t_1/x_1]\dots[t_n/x_n] :)} \quad \mathrm{TL(s/t)}\,\dfrac{\mathrm{Id(t)}\,\dfrac{}{(t_1 : \ \Rightarrow \ : t_1)}}{(t_1, \dots, t_n : \varphi[t_1/x_1]\dots[t_n/x_n] \Rightarrow \ : t_1)} \quad \dots \quad \mathrm{TL(s/t)}\,\dfrac{\mathrm{Id(t)}\,\dfrac{}{(t_n : \ \Rightarrow \ : t_n)}}{(t_1, \dots, t_n : \varphi[t_1/x_1]\dots[t_n/x_n] \Rightarrow \ : t_n)}}{(t_1, \dots, t_n : \varphi[t_1/x_1]\dots[t_n/x_n] \Rightarrow \langle\hat{\lambda}x_1, \dots, x_n\varphi\rangle t_1, \dots, t_n :)}}{(: \langle\lambda x_1, \dots, x_n\varphi\rangle t_1, \dots, t_n\rangle \Rightarrow \langle\hat{\lambda}x_1, \dots, x_n\varphi\rangle t_1, \dots, t_n :)}}{G; (A : \Gamma, \langle\lambda x_1, \dots, x_n\varphi\rangle t_1, \dots, t_n \Rightarrow \langle\hat{\lambda}x_1, \dots, x_n\varphi\rangle t_1, \dots, t_n, \Sigma : B); H}$$

The rest of this proof is similar to lemma 7.1. $\square$

Lemma 7.3. *Let $\mathscr{L}$ be expanded by $\hat{\exists}_1$ that is governed by the rules $L\hat{\exists}_1$ and $R\hat{\exists}_1$ that result from replacing every occurrence of $\exists_1$ by $\hat{\exists}_1$ in $L\exists_1$ and $R\exists_1$. The following two facts hold*

1. $\vdash G; (A : \Gamma, \exists_1 x\varphi \Rightarrow \Sigma : B); H$ *iff* $\vdash G; (A : \Gamma, \hat{\exists}_1 x\varphi \Rightarrow \Sigma : B); H$

2. $\vdash G; (A : \Gamma \Rightarrow \exists_1 x\varphi, \Sigma : B); H$ *iff* $\vdash G; (A : \Gamma \Rightarrow \hat{\exists}_1 x\varphi, \Sigma : B); H$

Proof. This proof is similar to the above. Let t be a term not appearing in $\exists_1 x\varphi$. The relevant deductions are

$$\mathrm{W{+}TL{+}TR}\,\dfrac{\mathrm{L}\hat{\exists}_1\,\dfrac{\mathrm{R}\exists_1\,\dfrac{\mathrm{TL(t)}\,\dfrac{\mathrm{Id(s)}\,\dfrac{}{(: \varphi[t/x] \Rightarrow \varphi[t/x] :)}}{(t : \varphi[t/x] \Rightarrow \varphi[t/x] :)} \quad \mathrm{TL(s)}\,\dfrac{\mathrm{Id(t)}\,\dfrac{}{(t : \ \Rightarrow \ : t)}}{(t : \varphi[t/x] \Rightarrow \ : t)}}{(t : \varphi[t/x] \Rightarrow \exists_1 x\varphi :)}}{(: \hat{\exists}_1 x\varphi \Rightarrow \exists_1 x\varphi :)}}{G; (A : \Gamma, \hat{\exists}_1 x\varphi \Rightarrow \exists_1 x\varphi, \Sigma : B); H}$$

$$
\text{W+TL+TR}\,\dfrac{\text{L}\exists_1\,\dfrac{\text{R}\hat{\exists}_1\,\dfrac{\text{TL(t)}\,\dfrac{\text{Id(s)}\,\dfrac{}{(:\varphi[t/x]\Rightarrow\varphi[t/x]:)}}{(t:\varphi[t/x]\Rightarrow\varphi[t/x]:)}\qquad \text{TL(s)}\,\dfrac{\text{Id(t)}\,\dfrac{}{(t:\quad\Rightarrow\quad:t)}}{(t:\varphi[t/x]\Rightarrow\quad:t)}}{(t:\varphi[t/x]\Rightarrow\hat{\exists}_1 x\varphi:)}}{(:\exists_1 x\varphi\Rightarrow\hat{\exists}_1 x\varphi:)}}{G;(A:\Gamma,\exists_1 x\varphi\Rightarrow\hat{\exists}_1 x\varphi,\Sigma:B);H}
$$

□

Lemma 7.4. *Let $\mathscr{L}$ be expanded by $\hat{\exists}_2$ that is governed by the rules $L\hat{\exists}_2$ and $R\hat{\exists}_2$ that result from replacing every occurrence of $\exists_2$ by $\hat{\exists}_2$ in $L\exists_2$ and $R\exists_2$. The following two facts hold*

1. $\vdash G;(A:\Gamma,\exists_2 x\varphi\Rightarrow\Sigma:B);H$ *iff* $\vdash G;(A:\Gamma,\hat{\exists}_2 x\varphi\Rightarrow\Sigma:B);H$
2. $\vdash G;(A:\Gamma\Rightarrow\exists_2 x\varphi,\Sigma:B);H$ *iff* $\vdash G;(A:\Gamma\Rightarrow\hat{\exists}_2 x\varphi,\Sigma:B);H$

Proof. Let F be a predicate of the proper arity that does not occur in φ. The following are the relevant deductions in this case

$$
\text{W + TL + TR}\,\dfrac{\text{L}\hat{\exists}_2\,\dfrac{\text{R}\exists_2\,\dfrac{\text{Id(s)}\,\dfrac{}{(:\varphi[f/F]\Rightarrow\varphi[f/F]:)}}{(:\varphi[f/F]\Rightarrow\exists_2 f\varphi:)}}{(:\exists_2 f\varphi\Rightarrow\hat{\exists}_2 f\varphi:)}}{G;(A:\Gamma,\exists_2 f\varphi\Rightarrow\hat{\exists}_2 f\varphi,\Sigma:B);H}
$$

$$
\text{W + TL + TR}\,\dfrac{\text{L}\exists_2\,\dfrac{\text{R}\hat{\exists}_2\,\dfrac{\text{Id(s)}\,\dfrac{}{(:\varphi[f/F]\Rightarrow\varphi[f/F]:)}}{(:\varphi[f/F]\Rightarrow\hat{\exists}_2 f\varphi:)}}{(:\hat{\exists}_2 f\varphi\Rightarrow\exists_2 f\varphi:)}}{G;(A:\Gamma,\hat{\exists}_2 f\varphi\Rightarrow\exists_2 f\varphi,\Sigma:B);H}
$$

□

Theorem 7.3.3. *PHIL uniquely characterizes all of the operators that are given by rules of those figures.*

Proof. This is proved by induction on the depth of the occurrence of the operator in question given in the sentence in question. The base case is given by lemmas 7.1 to 7.4. The inductive cases follow from an application of the inductive hypothesis and the corresponding rule governing the main operator of the sentence in question. $\square$

7.4 Models

A model, M, for $\mathscr{L}$ is an ordered triple, $M = \langle W^M, D^M, I^M \rangle$, where W^M is a set of worlds, D^M is a set of sequences, and I^M is an interpretation function.

With each world, $w \in W^M$, there is associated a set of domains. $d_w^M(0) \subseteq D^M$ is the set of individuals that exist at w. For each $i \geqslant 1$, let $d_w^M(i) \subseteq \wp((D^M)^i)$. The set $d_w^M(i)$ is fully defined inductively alongside the satisfaction relation below. $d_w^M(i)$ is the set of n-ary relations existing at w.

For each name, $n \in N$, $I^M(n) \in D^M$. For each n-ary predicate $F \in P$, $I^M(w, F) \subseteq d_w^M(0)^n$. Let Σ^M be the set of functions, σ constrained by:

- $\sigma(\varepsilon) \in D^M$, if ε is first-order.
- $\sigma(w, \varepsilon) \in d_w^M(i)$, if ε is an i-ary second-order variable.

If σ and $\sigma' \in \Sigma^M$, then σ' is an ε-variant of σ, $\sigma \sim_\varepsilon \sigma'$ iff the only place at which σ and σ' differ, if at all, is in the assignment to ε. Generalizing, $\sigma \sim_{\varepsilon_1,\ldots,\varepsilon_n} \sigma'$ iff the only places at which σ and σ' differ, if at all, is in their assignments to $\varepsilon_1, \ldots, \varepsilon_n$.

A denotation function δ is defined relative to a model, M, world, w, and function

$\sigma \in \Sigma^M$:

$$\delta^M_\sigma(w, \varepsilon) = \begin{cases} I_M(\varepsilon), & \text{if } \varepsilon \in N \\ I_M(w, \varepsilon), & \text{if } \varepsilon \in P \\ \sigma(w, \varepsilon), & \text{if } \varepsilon \in V_i \end{cases}$$

Satisfaction of a model relative to a world and variable assignment is defined inductively alongside the rest of the definition of I^M and D^M. This is to account for the interpretation of λ-expressions. φ_i is used to refer to a first-order sentence with i-many iterations of λ-expressions, i.e. i-many occurrences of $\langle$. Satisfaction of a sentence, φ_i by a model, M, relative to a world, w and sequence, σ is defined by double induction, the outer induction is over i, and the inner is over the rank of φ_i.

Case 1. (First-Order Sentence: Base Case).

- If φ_0 is $Ft_1, \ldots, t_n$, $M, w, \sigma \models \varphi_0$ iff $\langle \delta^M_\sigma(w, t_1), \ldots, \delta^M_\sigma(w, t_n) \rangle \in \delta^M_\sigma(w, F)$.
- If φ_0 is $\neg\psi_0$, $M, w, \sigma \models \varphi_0$ iff $M, w, \sigma \not\models \psi_0$.
- If φ_0 is $\psi_0 \rightarrow \theta_0$, then $M, w, \sigma \models \varphi_0$ iff $M, w, \sigma \not\models \psi_0$ or $M, w, \sigma \models \theta_0$.
- If φ_0 is $\Diamond\psi_0$, then $M, w, \sigma \models \varphi_0$ iff there is a world, $v \in W^M$ such that $M, v, \sigma \models \psi_0$.
- If φ_0 is $\exists x \psi_0$, then $M, w, \sigma \models \varphi_0$ iff there is a $\sigma' \sim_x \sigma$ such that $\delta^M_{\sigma'}(w, x) \in d^M_w(0)$ and $M, w, \sigma' \models \psi_0$.
- $\delta^M_\sigma(w, \lambda x_1, \ldots, x_n \varphi_0) = \{\langle \delta^M_{\hat{\sigma}}(x_1), \ldots, \delta^M_{\hat{\sigma}}(x_n) \rangle : \sigma \sim_{x_1, \ldots, x_n} \hat{\sigma} \& \delta^M_{\hat{\sigma}}(x_i) \in d^M_w(0), 1 \leqslant i \leqslant n \& M, w, \hat{\sigma} \models \varphi_0\}$.

- $\delta^M_\sigma(w, \lambda x_1, \ldots, x_n\varphi_0) \in d^M_w(n)$.

Case 2. (First-Order Sentence: Inductive Case).

- If φ_i is $\langle\lambda x_1, \ldots, x_n\psi_{i-1}\rangle t_1, \ldots t_n$, then $M, w, \sigma \models \varphi_i$ iff $\langle\delta^M_\sigma(w, t_1), \ldots, \delta^M_\sigma(w, t_n)\rangle \in \delta^M_\sigma(w, \lambda x_1, \ldots, x_n\psi_{i-1})$.
- If φ_i is $\neg\psi_i$, $M, w, \sigma \models \varphi_i$ iff $M, w, \sigma \not\models \psi_i$.
- If φ_i is $\psi_j \rightarrow \theta_k$, then $M, w, \sigma \models \varphi_i$ iff $M, w, \sigma \not\models \psi_j$ or $M, w, \sigma \models \theta_k$.
- If φ_i is $\Diamond\psi_i$, then $M, w, \sigma \models \varphi_i$ iff there is $v \in W^M$ such that $M, v, \sigma \models \psi_i$.
- If φ_i is $\exists x\psi_i$, then $M, w, \sigma \models \varphi_i$ iff there is a $\sigma' \sim_x \sigma$ such that $\delta^M_{\sigma'}(w, x) \in d^M_w(0)$ and $M, w, \sigma' \models \psi_i$.
- $\delta^M_\sigma(w, \lambda x_1, \ldots, x_n\varphi_i) = \{\langle\delta^M_{\hat{\sigma}}(x_1), \ldots, \delta^M_{\hat{\sigma}}(x_n)\rangle : \sigma \sim_{x_1,\ldots,x_n} \hat{\sigma} \& \delta^M_{\hat{\sigma}}(x_i) \in d^M_w(0), 1 \leqslant i \leqslant n \& M, w, \hat{\sigma} \models \varphi_i\}$.
- $\delta^M_\sigma(w, \lambda x_1, \ldots, x_n\varphi_i) \in d^M_w(n)$.

Case 3. (Second-Order Sentence).

- Nothing is an element of $d^M_w(n)$ except by virtue of the above clauses.
- If φ is $\exists f\psi$, then $M, w, \sigma \models \varphi$ iff there is a $\sigma' \sim_f \sigma$, such that $\delta^M_{\sigma'} \in d^M_w(n)$ and $M, w, \sigma' \models \psi$.

Definition 32 (True-in-a-model)**.** A sentence φ is true-in-a-model, M, at a world, w, $M, w \models \varphi$ iff for all sequences σ, $M, w, \sigma \models \varphi$.

Definition 33 (Counter-Example to a Position)**.** A model, M, is a counter-example to a position, $A : \Gamma \Rightarrow \Sigma : B$ at a world, w, and sequence σ, $M, w, \sigma \not\models A : \Gamma \Rightarrow \Sigma : B$ iff for each $c \in A$, $\delta^M_\sigma(c) \in d^M_w(0)$, for each $\varphi \in \Gamma$, $M, w, \sigma \models \varphi$, for each $\varphi \in \Sigma$, $M, w, \sigma \not\models \varphi$, and for each $c \in B$, $\delta^M_\sigma(c) \notin B$. If this holds for all sequences, the reference to σ is dropped.

Definition 34 (Counter-example)**.** A model M is a counter-example to a hyper-position G, at a sequence, σ, $M, \sigma \not\models G$, iff for each $S \in G$, there is a world, w, such that $M, w, \sigma \not\models S$. If this holds for all sequences, the reference to σ is dropped.

Fact 3. If there are no free variables in φ, and $M, w, \sigma \models \varphi$, then $M, w \models \varphi$.

Fact 4. Since λ-expressions do not have any free variables, $\delta^M_\sigma(w, \lambda x_1, \ldots, x_n\varphi) = \delta^M_{\sigma'}(w, \lambda x_1, \ldots, x_n)$ for all σ and σ'.

This section establishes that the set of models given above is sound for the proof theory given in fig. 7.1 and fig. 7.2. The theorem is established by induction on the length of deductions.

7.5 Soundness

Lemma 7.5. *Let $\delta^M_\sigma(w, t) = \delta^M_\sigma(w, c)$ for all $w \in W^M$. For any φ in which c occurs unbound, for any $w \in W^M$, $M, w, \sigma \models \varphi$ iff $M, w, \sigma \models \psi[t/c]$*

Proof. This is proved by induction on the rank of φ.

Case 1 (φ is $Ft_1, \ldots, t_n$). $M, w, \sigma \models Ft_1, \ldots, t_n$ iff $\langle \delta^M_\sigma(w, t_1), \ldots, \delta^M_\sigma(w, t_n) \rangle \in \delta^M_\sigma(w, F)$. Since $\delta^M_\sigma(w, t) = \delta^M(w, c)$ for all $w \in W^M$,

$$\langle \delta^M_\sigma(w, t_1), \ldots, \delta^M_\sigma(w, t_n) \in \delta^M_\sigma(w, F)\Big[\delta^M_\sigma(w, t)/\delta^M_\sigma(w, c)\Big]$$

But this is the case iff $M, w, \sigma \models Ft_1, \ldots, t_n[t/c]$.

Case 2 (φ is $\neg\psi$). $M, w, \sigma \models \neg\psi$ iff $M, w, \sigma \not\models \psi$. By IH, $M, w, \sigma \not\models \psi$ iff $M, w, \sigma \not\models \psi[t/c]$, iff $M, w, \sigma \models \neg\psi[t/c]$.

Case 3 (φ is $\psi \to \theta$). $M, w, \sigma \models \psi \to \theta$ iff $M, w, \sigma \not\models \psi$ or $M, w, \sigma \models \theta$. Since by IH, $M, w, \sigma \not\models \psi$ iff $M, w, \sigma \not\models \psi[t/c]$ and $M, w, \sigma \models \theta$ iff $M, w, \sigma \models \theta[t/c]$, $M, w, \sigma \not\models \psi$ or $M, w, \sigma \models \theta$ iff $M, w, \sigma \models \psi \to \theta[t/c]$.

Case 4 (φ is $\Diamond\psi$). $M, w, \sigma \models \Diamond\psi$ iff there is a world $v \in W^M$ such that $M, v, \sigma \models \psi$. Since $\delta^M_\sigma(v, t) = \delta^M_\sigma(v, c)$ for all $v \in W^M$, by IH, $M, v, \sigma \models \psi$ iff $M, v, \sigma \models \psi[t/c]$. $M, v, \sigma \models \psi[t/c]$ iff $M, w, \sigma \models \Diamond\psi[t/c]$.

Case 5 (φ is $\langle \lambda x_1, \ldots, x_n \psi \rangle s_1, \ldots, s_n$). $M, w, \sigma \models \langle \lambda x_1, \ldots, x_n \psi \rangle s_1, \ldots, s_n$ iff $\langle \delta^M_\sigma(w, s_1), \ldots, \delta^M_\sigma(w, s_n) \rangle$ $\delta^M_\sigma(w, \lambda x_1, \ldots, x_n \psi)$. $\delta^M_\sigma(w, \lambda x_1, \ldots, x_n \psi) = \{\langle \delta^M_{\hat\sigma}(w, x_1), \ldots, \delta^M_{\hat\sigma}(w, x_n) \rangle : \sigma \sim_{x_1, \ldots, x_n} \hat\sigma \& \delta^M_{\hat\sigma}(w, x_i \in d^M_w(0),\ 1 \leqslant i \leqslant n\ \&\ M, w, \hat\sigma \models \psi\}$.

Let $M, w, \sigma \models \langle \lambda x_1, \ldots, x_n \psi \rangle s_1, \ldots, s_n$. So $\langle \delta^M_\sigma(w, s_1), \ldots, \delta^M_\sigma(w, s_n) \rangle \in \{\langle \delta^M_{\hat\sigma}(w, x_1), \ldots, \delta^M_{\hat\sigma}(w, x_n)$ $\sigma \sim_{x_1, \ldots, x_n} \hat\sigma \& \delta^M_{\hat\sigma}(w, x_i \in d^M_w(0),\ 1 \leqslant i \leqslant n\ \&\ M, w, \hat\sigma \models \psi\}$. Since that set is non-empty, let σ' be such that $\delta^M_\sigma(w, s_i) = \delta^M_{\sigma'}(w, x_i) \in d^M_w(0)$ for $1 \leqslant i \leqslant n$, $\sigma \sim_{x_1, \ldots, x_n} \sigma'$ and $M, w, \sigma' \models \psi$. Since neither '$t$' nor '$c$' is one of $x_1, \ldots, x_n$, $\delta^M_\sigma(w, t) = \delta^M_\sigma(w, c) = \delta^M_{\sigma'}(w, t) = \delta^M_{\sigma'}(w, t)$ for all $w \in W^M$. So by IH, $M, w, \sigma' \models \psi[t/c]$. It follows that $\langle \delta^M_{\sigma'}(w, x_1), \ldots, \delta^M_{\sigma'}(w, x_n) \in \{\langle \delta^M_{\hat\sigma}(w, x_1), \ldots, \delta^M_{\hat\sigma}(w, x_n) \rangle : \sigma \sim_{x_1, \ldots, x_n} \hat\sigma \& \delta^M_{\hat\sigma}(w, x_i \in$

$d_w^M(0)$, $1 \leqslant i \leqslant n$ & $M, w, \hat{\sigma} \models \psi[t/c]\} = \delta_\sigma^M(w, \lambda x_1, \ldots, x_n \psi[t/c])$. Since for each s_i, $\delta_\sigma^M(w, s_i) = \delta_{\sigma'}^M(w, x_i)$, $\langle \delta_\sigma^M(w, s_1), \ldots, \delta_\sigma^M(w, s_n) \rangle \in \delta_\sigma^M(w, \lambda x_1, \ldots, x_n \psi[t/c])$. Since $\delta_\sigma^M(w, t) = \delta_\sigma^M(w, c)$,

$$\langle \delta_\sigma^M(w, s_1), \ldots, \delta_\sigma^M(w, s_n) \rangle \in \delta_\sigma^M(w, \lambda x_1, \ldots, x_n \psi[t/c]) \left[\delta_\sigma^M(w, t) / \delta_\sigma^M(w, c) \right]$$

So $M, w, \sigma \models \langle x_1, \ldots, x_n \psi \rangle s_1, \ldots, s_n[t/c]$.

The converse direction is analogous.

Case 6 (φ is $\exists x \psi$). $M, w, \sigma \models \exists x \psi$ iff there is a σ' such that $M, w, \sigma' \models \psi$ and $\sigma(w, x) \in d_w^M(0)$ and $\sigma' \sim_x \sigma$. By IH, $M, w, \sigma' \models \psi$ iff $M, w, \sigma' \models \psi[t/c]$. Since neither '$t$' nor '$c$' are '$x$', $M, w, \sigma' \models \psi[t/c]$ iff $M, w, \sigma \models \exists x \psi[t/c]$.

Case 7 (φ is $\exists f \psi$). $M, w, \sigma \models \exists f \psi$ iff there is a $\sigma' \sim_f \sigma$, $\delta_{\sigma'}^M(w, f) \in d_w^M(i)$, and $M, w, \sigma' \models \psi$. By IH,$M, w, \sigma' \models \psi$ iff $M, w, \sigma' \models \psi[t/c]$. Since neither '$t$' nor '$c$' are '$f$', the former holds iff $M, w, \sigma' \models \psi[t/c]$ and $\sigma' \sim_f \sigma$ and $\delta_{\sigma'}^M(w, f) \in d_w^M(i)$ iff $M, w, \sigma \models \exists f \psi[t/c]$

□

Lemma 7.6. *Let $\delta_\sigma^M(w, \varepsilon) = \delta_{\sigma'}^M(w, \varepsilon)$ for all unbound ε in φ. It follows that for any $w \in W$, $M, w, \sigma \models \varphi$ iff $m, w, \sigma' \models \varphi$.*

Proof. Let σ and σ' be such that $\delta_\sigma^M(w, \varepsilon) = \delta_{\sigma'}^M(\varepsilon)$ for all $w \in W^M$, and this holds for all unbound ε in φ. That for any $w \in W_M$, $M, w, \sigma\varphi$ iff $M, w, \sigma'\varphi$ is proved by induction on the rank of φ.

Case 1 (φ is $Ft_1, \ldots, t_n$). Let $w \in W_M$. $M, w, \sigma \models \varphi$ iff $\langle \delta^M_\sigma(w, t_1), \ldots, \delta^M_\sigma(w, t_n)\rangle \in \delta^M_\sigma(w, F)$. By assumption, $\delta^M_\sigma(w, t_i) = \delta^M_{\sigma'}(w, t_i)$ for all $1 \leqslant i \leqslant n$ and $\delta^M_\sigma(w, F) = \delta^M_{\sigma'}(w, F)$. So $M, w, \sigma \models \varphi$ iff $\langle \delta^M_{\sigma'}(w, t_1), \ldots, \delta^M_{\sigma'}(w, t_n)\rangle \in \delta^M_{\sigma'}(w, F)$ iff $M, w, \sigma' \models \psi$.

Case 2 (φ is $\neg\psi$). $M, w, \sigma \models \neg\psi$ iff $M, w, \sigma \not\models \psi$. By IH, $M, w, \sigma \not\models \psi$ iff $M, w, \sigma' \not\models \psi$. But $M, w, \sigma' \not\models \psi$ iff $M, w, \sigma' \models \neg\psi$.

Case 3 (φ is $\psi \rightarrow \theta$). $M, w, \sigma \models \psi \rightarrow \theta$ iff either $M, w, \sigma \not\models \psi$ or $M, w, \sigma \models \theta$. By IH, $M, w, \sigma \not\models \psi$ iff $M, w, \sigma' \not\models \psi$ and $M, w, \sigma \models \theta$ iff $M, w, \sigma' \models \theta$. Either $M, w, \sigma' \not\models \psi$ or $M, w, \sigma' \models \theta$ iff $M, w, \sigma' \models \psi \rightarrow \theta$.

Case 4 (φ is $\Diamond\psi$). $M, w, \sigma \models \psi$ iff there is a world, v, such that $M, v, \sigma \models \psi$. By IH, $M, v, \sigma \models \psi$ iff $M, v, \sigma' \models \psi$. There is a v such that $M, v, \sigma' \models \psi$ iff $M, w, \sigma' \models \Diamond\psi$.

Case 5 (φ is $\langle \lambda x_1, \ldots, x_n\psi\rangle s_1, \ldots, s_n$). $M, w, \sigma \models \langle \lambda x_1, \ldots, x_n\psi\rangle s_1, \ldots, s_n$ iff $\langle \delta^M_\sigma(w, s_1), \ldots, \delta^M_\sigma(w, s_n)\rangle$ $\{\langle \delta^M_{\hat\sigma}(w, x_1), \ldots, \delta^M_{\hat\sigma}(w, x_n)\rangle : \hat\sigma \sim_{x_1,\ldots,x_n} \sigma \& \delta^M_{\hat\sigma}(w, x_i) \in d^M_w(i)$ for all $1 \leqslant i \leqslant n \& M, w, \hat\sigma \models \psi\}$. Similarly, $M, w, \sigma' \models \langle \lambda x_1, \ldots, x_n\psi\rangle s_1 \ldots, s_n$ iff $\langle \delta^M_{\sigma'}(w, s_1), \ldots, \delta^M_{\sigma'}(w, s_n)\rangle \in \{\langle \delta^M_{\hat\sigma}(w, x_1), \ldots, \delta^M_{\hat\sigma}(w, x_n)\rangle : \hat\sigma \sim_{x_1,\ldots,x_n} \sigma' \& \delta^M_{\hat\sigma}(w, x_i) \in d^M_w(i)$ for all $1 \leqslant i \leqslant n \& M, w, \hat\sigma \models \psi\}$.

Suppose that $M, w, \sigma \models \langle \lambda x_1, \ldots, x_n\psi\rangle s_1, \ldots, s_n$. So $\delta^M_\sigma(w, \lambda x_1, \ldots, x_n\psi)$ is non-empty. Let σ^* be such that $\sigma^* \sim_{x_1,\ldots,x_n} \sigma$, $\delta^M_{\sigma^*}(w, x_i) \in d^M_w(0)$, for all $1 \leqslant i \leqslant n$, $M, w, \sigma^* \models \psi$, and $\delta^M_{\sigma^*}(w, x_i) = \delta^M_\sigma(w, s_i)$ for $1 \leqslant i \leqslant n$. Let $\sigma^1 \sim_{x_1,\ldots,x_n} \sigma'$ and be such that $\delta^M_{\sigma^*}(w, x_i) = \delta^M_{\sigma^1}(w, x_i)$, for all $w \in W^M$. Since for all unbound ε in $\langle \lambda x_1, \ldots, x_n\psi\rangle s_1, \ldots, s_n$, $\delta^M_\sigma(w, \varepsilon) = \delta^M_{\sigma'}(w, \varepsilon)$ and $\sigma \sim_{x_1,\ldots,x_n} \sigma^*$, $\delta^M_\sigma(w, \varepsilon) = \delta^M_{\sigma^*}(w, \varepsilon) = \delta^M_{\sigma^1}(w, \varepsilon) = \delta^M_{\sigma'}(w, \varepsilon)$. Since $\sigma^M_{\sigma^*}(w, x_i) = \sigma^M_{\sigma^1}(w, x_i)$ for all $1 \leqslant i \leqslant n$, for all $w \in W^M$, $\delta^M_{\sigma^*}(w, \varepsilon) = \delta^M_{\sigma^1}(w, \varepsilon)$ for all ε unbound in ψ. By IH, $M, w, \sigma^1 \models \psi$. From the other facts, $\delta^M_{\sigma^1}(w, x_i) \in d^M_w(0)$ for $1 \leqslant i \leqslant n$ and $\sigma^1 \sim_{x_1,\ldots,x_n} \sigma'$. So

$\langle \delta^M_{\sigma^1}(w, x_1), \ldots, \delta^M_{\sigma^1}(w, x_n)\rangle \in \delta^M_{\sigma'}(w, \lambda x_1, \ldots, x_n\psi)$. But $\delta^M_{\sigma^1}(w, x_i) = \delta^M_{\sigma}(w, s_i)$, for $1 \leqslant i \leqslant n$. So $M, w, \sigma' \models \langle \lambda x_1, \ldots, x_n\psi\rangle s_1, \ldots, s_n$.

The converse direction is similar.

Case 6 (φ is $\exists x\psi$). Let $M, w, \sigma \models \exists x\psi$. So there is a $\sigma^* \sim_x \sigma$ such that $\sigma^*(w, x) \in d^M_w(0)$ and $M, w, \sigma^* \models \psi$. Let $\sigma^1 \sim_x \sigma'$ be such that $\sigma^1(w, x) = \sigma^*(w, x)$, for all $w \in W^M$. By IH, $M, w, \sigma^1 \models \psi$, since $\delta^M_{\sigma^*}(w, \varepsilon) = \delta^M_{\sigma^1}(w, \varepsilon)$ for all ε unbound in ψ. Since $\sigma^1 \sim_x \sigma'$ and $\sigma^1(w, x) \in d^M_w(0)$, $M, w, \sigma' \models \exists x\psi$.

The converse direction, and second-order case are similar.

□

Lemma 7.7. *If M and N are models such that*

1. $W^M = W^N$,
2. $D^M = D^N$,
3. $d^M_w(i) = d^n_w(i)$ *for each i and $w \in W^M$, and*
4. *for all expressions, ε appearing in φ, all sequences σ, and worlds, w, $\delta^M_\sigma(w, \varepsilon) = \delta^N_\sigma(w, \varepsilon)$,*

then for all $w \in W^M$ and all σ, $M, w, \sigma \models \varphi$ iff $N, w, \sigma \models \varphi$.

Proof. Let M and N be such models, and φ such a sentence. The proof proceeds by induction on the rank of φ.

Case 1 (φ is $Ft_1, \ldots, t_n$). $M, w, \sigma \models Ft_1, \ldots, t_n$ iff $\langle \delta^M_\sigma(w, t_1), \ldots, \delta^M_\sigma(t_n)\rangle \in \delta^M_\sigma(w, F)$. By assumption $\delta^M_\sigma(w, \varepsilon) = \delta^N(\varepsilon)$ for all ε appearing in φ. So $\langle \delta^M_\sigma(w, t_1), \ldots, \delta^M_\sigma(t_n)\rangle \in \delta^M_\sigma(w, F)$ iff $\langle \delta^N_\sigma(w, t_1), \ldots, \delta^N_\sigma(t_n)\rangle \in \delta^N_\sigma(w, F)$ iff $N, w, \sigma \models Ft_1, \ldots, t_n$.

Case 2 (φ is $\neg\psi$). $M, w, \sigma \models \neg\psi$ iff $M, w, \sigma \not\models \psi$. By IH, $M, w, \sigma \not\models \psi$ iff $N, w, \sigma \not\models \psi$. But, $N, w, \sigma \not\models \psi$ iff $N, w, \sigma \models \neg\psi$.

Case 3 (φ is $\psi \to \theta$). $M, w, \sigma \models \psi \to \theta$ iff $M, w, \sigma \not\models \psi$ or $M, w, \sigma \models \theta$. By IH, $M, w, \sigma \not\models \psi$ iff $N, w, \sigma \not\models \psi$ and $M, w, \sigma \models \theta$ iff $N, w, \sigma \models \theta$. Finally, either $N, w, \sigma \not\models \psi$ or $N, w, \sigma \models \theta$ iff $N, w, \sigma \models \psi \to \theta$.

Case 4 (φ is $\Diamond\psi$). $M, w, \sigma \models \Diamond\psi$ iff there is a world, v, such that $M, v, \sigma \models \psi$. By IH, $M, v, \sigma \models \psi$ iff $N, v, \sigma \models \psi$. So $M, w, \sigma \models \Diamond\psi$ iff there is a world, $v \in W^N = W^N$ such that $N, v, \sigma \models \psi$ iff $N, w, \sigma \models \Diamond\psi$.

Case 5 (φ is $\langle \lambda x_1, \ldots, x_n \psi \rangle s_1, \ldots, s_n$). Let $M, w, \sigma \models \langle \lambda x_1, \ldots, x_n \psi \rangle s_1, \ldots, s_n$. So $\langle \delta^M_\sigma(w, s_1), \ldots, \langle \delta^M_\sigma(w, s_n) \rangle \in \{\langle \delta^M_{\hat\sigma}(w, x_1), \ldots, \delta^M_{\hat\sigma}(w, x_n) \rangle : \hat\sigma \sim_{x_1,\ldots,x_n} \sigma \& \delta^M_{\hat\sigma}(w, x_i) \in d^M_w(0)$ for $1, \leqslant i \leqslant n \& M, w, \hat\sigma \models \psi\}$. Since this set is non-empty, there is a σ' such that $\sigma' \sim_{x_1,\ldots,x_n} \sigma$, $\delta^M_{\sigma'}(w, x_i) \in d^M_w(0)$ for $1, \leqslant i \leqslant n$, $M, w, \sigma' \models \psi$, and $\delta^M_\sigma(w, s_i) = \delta^M_{\sigma'}(w, x_i)$ for $1 \leqslant i \leqslant n$. By IH, $N, w, \sigma' \models \psi$. Since $W^M = W^N$, $D^M = D^N$, $d^M_w(i) = d^n_w(i)$ for each i and $w \in W^M$, and for all expressions, ε appearing in φ, all sequences σ, and worlds, w, $\delta^M_\sigma(w, \varepsilon) = \delta^N_\sigma(w, \varepsilon)$, $\delta^M_{\sigma'}(w, x_i) = \delta^N_{\sigma'}(w, x_i)$ and $\sigma \sim_{x_1,\ldots,x_n} \sigma'$, $\langle \delta^N_{\sigma'}(w, x_1), \ldots, \delta^N_{\sigma'}(w, x_n) \rangle \in \{\langle \delta^N_{\hat\sigma}(w, x_1), \ldots, \delta^N_{\hat\sigma}(w, x_n) \rangle : \hat\sigma \sim_{x_1,\ldots,x_n} \sigma \& \delta^N_{\hat\sigma}(w, x_i) \in d^N_w(0)$ for $1, \leqslant i \leqslant n \& M, w, \hat\sigma \models \psi\}$. By the definition of $\delta^N_\sigma(w, \lambda x_1, \ldots, x_n \psi)$, $\langle \delta^N_{\sigma'}(w, x_1), \ldots, \delta^N_{\sigma'}(w, x_n) \rangle \in \delta^N_\sigma(w, \lambda x_1, \ldots, x_n \psi)$. Since $\delta^N_\sigma(w, s_i) = \delta^M_\sigma(w, s_i) = \delta^M_\sigma(w, x_i) = \delta^N_\sigma(w, x_i)$, for $1 \leqslant i \leqslant n$, $\langle \delta^N_{\sigma'}(w, s_1), \ldots, \delta^N_{\sigma'}(w, s_n) \rangle \in \delta^N_\sigma(w, \lambda x_1, \ldots, x_n \psi)$. So $N, w, \sigma \models \langle \lambda x_1, \ldots, x_n \psi \rangle s_1, \ldots, s_n$.

The converse case is similar.

Case 6 (φ is $\exists x \psi$). $M, w, \sigma \models \exists x \psi$ iff there is a $\sigma' \sim_x \sigma$ such that $\delta^M_{\sigma'}(w, x) \in d^M_w(0)$, and $M, w, \sigma' \models \psi$. Since for all σ^* and w, $\delta^M_{\sigma *}(w, x) = \delta^N_{\sigma *}(w, x)$, and $\sigma' \sim_x \sigma$, for

any ε unbound in ψ, for all σ^* and w, $\delta^M_{\sigma*}(w, \varepsilon) = \delta^N_{\sigma*}(w, \varepsilon)$, by IH $M, w, \sigma' \models \psi$ iff $N, w, \sigma' \models \psi$. But then $M, w, \sigma \models \exists x\psi$ iff there is a $\sigma' \sim_x \sigma$ such that $\delta^M_{\sigma'}(w, x) = \delta^N_{\sigma'}(w, x) \in d^N_w(0)$ and $N, w, \sigma' \models \psi$ iff $N, w, \sigma \models \exists x\psi$.

The second-order case is similar to the above case.

□

Lemma 7.8. *Let $t_1, \ldots, t_n$ not be any of $x_1, \ldots, x_n$. For any model, M, and world, $w \in W^M$, $M, w, \sigma \models \langle\lambda x_1, \ldots, x_n\varphi\rangle t_1, \ldots, t_n$ iff $M, w, \sigma \models \varphi[t_1/x_1] \ldots [t_n][x_n]$ and $\delta^M_\sigma(w, t_i) \in d^M_w(0)$ for $1 \leqslant i \leqslant n$.*

Proof.

Case 1 (Left-to-Right). Let $M, w, \sigma \models \langle\lambda x_1, \ldots, x_n\varphi\rangle t_1, \ldots, t_n$. So $\langle\delta^M_\sigma(w, t_1), \ldots, \delta^M_\sigma(w, t_n)\rangle \in \{\langle\delta^M_{\hat{\sigma}}(w, x_1), \ldots, \delta^M_{\hat{\sigma}}(w, x_n)\rangle : \hat{\sigma} \sim_{x_1,\ldots,x_n} \sigma \& \delta^M_{\hat{\sigma}}(w, x_i) \in d^M_w(0)$ for $1 \leqslant i \leqslant n \& M, w, \hat{\sigma} \models \psi\}$. Thus, there is a σ' such that

1. $\delta^M_\sigma(w, t_i) = \delta^M_{\sigma'}(w, x_i) \in d^M_w(0)$ for $1 \leqslant i \leqslant n$,
2. $\sigma' \sim_{x_1,\ldots,x_n} \sigma$, and
3. $M, w, \sigma \models \psi$.

by (1), $\delta^M_\sigma(w, t_i) \in d^M_w(0)$ for $1 \leqslant i \leqslant n$. The evaluation of terms is not relative to a world, i.e. $\delta^M_w(v) = \sigma(v)$ for a variable, and $\delta^M_\sigma(w, t) = I_M(t)$, for a term. Thus, $\delta^M_{\sigma'}(w, x_i) = \delta^M_\sigma(w, t_i)$ for each $w \in W^M$, for $1 \leqslant i \leqslant n$. Since $t_1, \ldots, t_n$ are not any of $x_1, \ldots, x_n$, $\delta^M_\sigma(w, t_i) = \delta^M_{\sigma'}(w, t_i)$ for all $w \in W^M$, for each $1 \leqslant i \leqslant n$. By lemma 7.5, $M, w, \sigma' \models \psi[t_1/x_1] \ldots [t_n/x_n]$. Since $x_1, \ldots, x_n$ do not occur in $\psi[t_1/x_1] \ldots [t_n/x_n]$, by lemma 7.6, $M, w, \sigma \models \psi[t_1/x_1] \ldots [t_n/x_n]$.

Case 2 (Right-to-Left). Let $M, w, \sigma \models \psi[t_1/x_1]\ldots[t_n/x_n]$ and $\delta_\sigma^M(w, t_i) \in d_w^M(0)$ for $1 \leqslant i \leqslant n$. Let $\sigma' \sim_{x_1,\ldots,x_n} \sigma$ be such that $\delta_{\sigma'}^M(v, x_i) = \delta_(^M v, t_i)$ for all $v \in W^M$, for each $1 \leqslant i \leqslant n$. Since $x_1, \ldots, x_n$ do not appear in ψ, by lemma 7.6, $M, w, \sigma' \models \psi[t_1/x_1]\ldots[t_n/x_n]$. By n-many applications of lemma 7.5, $M, w, \sigma' \models \psi[t_1/x_1]\ldots[t_n/x_n]$ iff $M, w, \sigma' \models \psi$. Since $\sigma' \sim_{x_1,\ldots,x_n}$ and $\delta_{\sigma'}^M(v, x_i) = \delta_(^M v, t_i) \in d_w^M(0)$, $\langle\delta_{\sigma'}^M(w, x_1), \ldots, \delta_{\sigma'}^M(w, x_n)\rangle \in \{\langle\delta_{\hat\sigma}^M(w, x_1), \ldots, \delta_{\hat\sigma}^M(w, x_n)\rangle : \hat\sigma \sim_{x_1,\ldots,x_n} \delta_{\hat\sigma}^M(w, x_i) \in d_w^M(0)$ for $1 \leqslant i \leqslant n \& M, w, \hat\sigma \models \psi\}$, i.e. $\langle\delta_{\sigma'}^M(w, x_1), \ldots, \delta_{\sigma'}^M(w, x_n)\rangle \in \delta_\sigma^M(w, \lambda x_1, \ldots, x_n\psi)$. So $\langle\delta_{\sigma'}^M(w, t_1), \ldots, \delta_{\sigma'}^M(w, t_n)\rangle \in \delta_\sigma^M(w, \lambda x_1, \ldots, x_n\psi)$, and $M, w, \sigma \models \langle\lambda x_1, \ldots, x_n\psi\rangle t_1, \ldots, t_n$.

□

Theorem 7.5.1. *If* $\vdash G$ *then* $\models G$.

Proof. This is proved by induction on the length of deductions. It is shown that there are no counter-examples to Id(s) and Id(t) for the base cases. For the inductive cases it is assumed that given a rule there is no counter-example to the premises, but there is a counter-example to the conclusion. In each case this is shown to lead to a contradiction. For each case let δ be the deduction of G, and I be the last rule of δ.

Case 1 (**Base Cases**).

Case 14 (I is an instance of $Id(s)$). In this case δ has the form

$$\frac{}{(: Ft_1, \ldots, t_n \Rightarrow Ft_1, \ldots, t_n :)}$$

Let M be such that $M \not\models (: Ft_1, \ldots, t_n \Rightarrow Ft_1, \ldots, t_n :)$. But this is impossible since then there is a world, $w \in W_M$, and σ such that $\langle\delta_\sigma^M(w, t_1), \ldots, \delta_\sigma^M(w, t_n)\rangle \in \delta_\sigma^M(w, F)$ and $\langle\delta_\sigma^M(w, t_1), \ldots, \delta_\sigma^M(w, t_n)\rangle \notin \delta_\sigma^M(w, F)$. But this is impossible.

Case 15 (I is an instance of $Id(t)$). In this case δ has the form

$$\frac{}{(t:\quad\Rightarrow\quad:t)}$$

Let M be a counter-example too $(t:\quad\Rightarrow\quad:t)$. But then there is a world, $w \in W_M$ such that $I_M(w,t) \in d_w^M(0)$ and $I_M(w,t) \notin d_w^M(0)$. This, however, is impossible.

Case 2 (**Inductive Cases**).

Case 16 (I is TL(s)). In this case δ has the form:

$$\frac{\begin{matrix}\vdots\\ G;(A:\Gamma \Rightarrow \Delta:B);H\end{matrix}}{G;(A:\Gamma,\varphi \Rightarrow \Delta:B);H}$$

Let M be a model such that $M \not\models G;(A:\Gamma,\varphi \Rightarrow \Delta:B);H$. For every position, $S \in G \cup H \cup \{(A:\Gamma,\varphi \Rightarrow \Delta:B)\}$, there is a world, w, such that $M,w \not\models S$. In particular, there is a world, w, such that $M,w \not\models (A:\Gamma,\varphi \Rightarrow \Delta:B)$. Since for every sentence, $\gamma \in \Gamma \cup \{\varphi\}$, $M,w \models \gamma$, for every sentence, $\gamma \in \Gamma$, $M,w \models \gamma$. So $M,w \models (A:\Gamma \Rightarrow \Delta:B)$. But then M is a counter-example to $G;(A:\Gamma \Rightarrow \Delta:B);H$. The cases of TL(t), TR(s), and TR(t) proceed in a similar way to this one.

Case 17 (I is Cut). In this case δ has the form:

$$\text{Cut}\ \frac{\begin{matrix}\vdots\\ G;(A:\Gamma,\varphi \Rightarrow \Delta:B);H\end{matrix}\qquad \begin{matrix}\vdots\\ G;(A:\Gamma \Rightarrow \varphi,\Delta:B);H\end{matrix}}{G;(A:\Gamma \Rightarrow \Delta:B);H}$$

Let M be a counter-example to $G;(A:\Gamma \Rightarrow \Delta:B);H$. Let w be a world, such that $M,w \not\models (A:\Gamma \Rightarrow \Delta:B)$. Given the law of excluded middle in the metalanguage, either $M,w \models \varphi$ or $M,w \not\models \varphi$. In the first case, for each sentence,

$\gamma \in \Gamma \cup \{\varphi\}$, $M, w \not\models \gamma$. Thus $M, w \not\models (A : \Gamma, \varphi, \Rightarrow \Delta : B)$. So M is a counter-example to $G; (A : \Gamma, \varphi \Rightarrow \Sigma : B); H$. In the second case, for each sentence, $\theta \in \Delta$, $M, w \not\models \theta$. Thus, $M, w \not\models (A : \Gamma \Rightarrow \varphi, \Delta : B)$, and M is a counter-example to $G; (A : \Gamma \Rightarrow \Sigma : B); H$.

Case 18 (I is W). In this case δ has the form

$$\mathrm{W}\,\frac{\begin{array}{c}\vdots\\ G; H\end{array}}{G; (: \quad \Rightarrow \quad :); H}$$

Let M be a counter-example to $G; (: \quad \Rightarrow \quad :); H$. So for each $S \in G \cup H \cup \{(: \quad \Rightarrow \quad :)\}$, there is a $w \in W_M$. From this it follows that for every position in $G \cup H$, there is a world, $w \in W_M$ such that $M \not\models G; H$.

Case 19 (I is AL). In this case δ has the following form

$$\frac{\begin{array}{c}\vdots\\ G; (A, t : \Gamma \Rightarrow \Delta : B); H\end{array}}{G; (A : \Gamma, Ft_1, \ldots t_n \Rightarrow \Delta : B); H}$$

Let M be a counter-example to $G; (A : \Gamma, Ft_1, \ldots, t, \ldots, t_n \Rightarrow \Delta : B); H$. So there is a sequence, $\sigma \in \Sigma$ and a world, $w \in W_M$, such that $M, w, \sigma \not\models (A : \Gamma, Ft_1, \ldots, t, \ldots, t_n \Rightarrow \Delta : B)$. In particular, this means that $\langle \delta^M_\sigma(t_1), \ldots, \delta^M_\sigma(t), \ldots, \delta^M_\sigma(t_n)\rangle \in I_M(w, F)$. Since $F \in P$, $I_M(w, F) \subseteq \wp(d_w(n))$, each element of $\langle \delta^M_\sigma(t_1), \ldots, \delta^M_\sigma(t), \ldots, \delta^M_\sigma(t_n)\rangle \in d_w(0)$. It follows that $\delta^M_\sigma(t) \in d^M_w(0)$. But then M is a counter-example to $G; (A, t : \Gamma \Rightarrow \Sigma : B); H$.

Case 20 (I is L$\neg$). δ has the form

$$\frac{\begin{array}{c}\vdots\\ G; (A : \Gamma \Rightarrow \varphi, \Delta : B); H\end{array}}{G; (A : \Gamma, \neg\varphi \Rightarrow \Delta : B); H}$$

Let M be a counter-example to $G; (A : \Gamma, \neg\varphi \Rightarrow \Delta : B); H$. There is a world, $w \in W_M$, such that $M, w \not\models (A : \Gamma, \neg\varphi \Rightarrow \Delta : B); H$. In particular, then $M, w \models \neg\varphi$, so $M, w \not\models \varphi$. From this it follows that $M, w \not\models (A : \Gamma \Rightarrow \varphi, \Delta : B)$. So M is a counter-example to $G; (A : \Gamma \Rightarrow \varphi, \Delta : B); H$. The case where I is R$\neg$ is similar.

Case 21 (I is L$\rightarrow$). In this case δ has the form

$$\dfrac{\begin{matrix}\vdots \\ G; (A : \Gamma \Rightarrow \varphi, \Delta : B); H\end{matrix} \qquad \begin{matrix}\vdots \\ G; (A : \Gamma, \psi \Rightarrow \Delta : B); H\end{matrix}}{G; (A : \Gamma, \varphi \rightarrow \psi \Rightarrow \Delta : B); H}$$

Let M be a counter-example to $G; (A : \Gamma, \varphi \rightarrow \psi \Rightarrow \Delta : B); H$. There is a world, $w \in W_M$, such that $M, w \not\models (A : \Gamma, \varphi \rightarrow \psi \Rightarrow \Delta : B); H$. So $M, w \models \varphi \rightarrow \psi$ and either $M, w \models \psi$ or $M, w \not\models \varphi$. In the first case, $M, w \not\models (A : \Gamma \Rightarrow \psi, \Delta : B)$. So M is a counter-example to $G : (A : \Gamma \Rightarrow \psi, \Delta : B)$. In the second case, $M, w \not\models (A : \Gamma, \varphi \Rightarrow \Delta : B)$, so M is a counter-example to $G; (A : \Gamma, \varphi \Rightarrow \Sigma : B); H$.

The case where I is R$\rightarrow$ is similar.

Case 22 (I is L$\Diamond$). In this case δ has the form

$$\dfrac{\begin{matrix}\vdots \\ (: \varphi \Rightarrow \quad :); G; (A : \Gamma \Rightarrow \Delta : B); H\end{matrix}}{G; (A : \Gamma, \Diamond\varphi \Rightarrow \Delta : B); H}$$

Let M be a counter-example to $G; (A : \Gamma, \Diamond\varphi \Rightarrow \Sigma : B); H$. There is a world $w \in W_M$, such that $M, w \models \Diamond\varphi$. From this it follows that there is a world, v, such that $M, v \models \varphi$. $M, v \models (: \varphi \Rightarrow \quad :)$. Thus, M is a counter-example to $(: \varphi \Rightarrow \quad :); G; (A : \Gamma \Rightarrow \Delta : B); H$.

Case 23 (I is R$\Diamond$). In this case δ has the form

$$\cfrac{\vdots \\ G; (C : \Pi \Rightarrow \varphi, \Theta : D); G'; (A : \Gamma \Rightarrow \Delta : B); H}{G; (C : \Pi \Rightarrow \Theta : D); G'; (A : \Gamma \Rightarrow \Diamond\varphi, \Delta : B); H}$$

Let M be a counter-example to $G; (C : \Pi \Rightarrow \Theta : D); G'; (A : \Gamma \Rightarrow \Diamond\varphi, \Delta : B); H$. There is a world, $w \in W_M$ such that $M, w \not\models \varphi$. Thus there is no world, $v \in W_M$, such that $M, v \models \varphi$. There is a world $u \in W_M$, such that $M, u \models (C : \Pi \Rightarrow \Theta : D)$. So $M, u \not\models \varphi$. But then $M, u \models (C : \Pi \Rightarrow \varphi, \Theta : D)$. So M is a counter-example to $G; (C : \Pi \Rightarrow \varphi, \Theta : D); G'; (A : \Gamma \Rightarrow \Delta : B); H$.

Case 24 (I is Lλ). In this case δ has the form

$$\cfrac{\vdots \\ G; (A, t_1, \ldots, t_n : \Gamma, \varphi[t/x_1] \ldots [t_n/_n] \Rightarrow \Sigma : B); H}{G; (A : \Gamma, \langle \lambda x_1, \ldots, x_n \varphi \rangle t_1, \ldots, t_n \Rightarrow \Sigma : B); H}$$

Let $M \not\models G; (A : \Gamma, \langle \lambda x_1, \ldots, x_n \varphi \rangle t_1, \ldots, t_n \Rightarrow \Sigma : B); H$. So there is a world, $w \in W^M$, such that $M, w \not\models (A : \Gamma, \langle \lambda x_1, \ldots, x_n \varphi \rangle t_1, \ldots, t_n \Rightarrow \Sigma : B)$. Let $\sigma \in \Sigma^M$. $M, w, \sigma \models \langle \lambda x_1, \ldots, x_n \varphi \rangle t_1, \ldots, t_n$. By lemma 7.8, $M, w, \sigma \models \psi[t_1/x_1] \ldots [t_n/x_n]$ and $\delta^M_\sigma(w, t_i)$ for $1 \leqslant i \leqslant n \in d^M_w(0)$. Since σ was arbitrary, this holds for any such σ. So $M, w \not\models (A, t_1, \ldots, t_n : \Gamma, \varphi[t_1/x_1] \ldots [t_n/x_n] \Rightarrow \Sigma : B)$. Thus M is a counter-example to $G; (A, t_1, \ldots, t_n : \Gamma, \varphi[t/x_1] \ldots [t_n/_n] \Rightarrow \Sigma : B); H$.

Case 25 (I is Rλ). In this case δ has the form

$$\cfrac{\begin{matrix} \vdots & & \vdots \\ G; (A : \Gamma \Rightarrow \varphi[t_1/x_1] \ldots [t_n/x_n], \Sigma); H & \qquad & \{G; (A : \Gamma \Rightarrow \Sigma : B, t_i); H\}_i \end{matrix}}{G; (A : \Gamma \Rightarrow \langle \lambda x_1, \ldots, x_n \varphi \rangle t_1, \ldots, t_n, \Sigma); H}$$

Let M be a counter-example to $G; (A : \Gamma \Rightarrow \langle \lambda x_1, \ldots, x_n \varphi \rangle t_1, \ldots, t_n, \Sigma); H$. There is a world, $w \in W^M$, such that $M, w, \sigma \not\models (A : \Gamma \Rightarrow \langle \lambda x_1, \ldots, x_n \varphi \rangle t_1, \ldots, t_n, \Sigma)$ for all $\sigma \in \Sigma^M$. Let $\sigma \in \Sigma^M$, $M, w, \sigma \not\models \langle \lambda x_1, \ldots, x_n \varphi \rangle t_1, \ldots, t_n$. By lemma 7.8, either

$M, w, \sigma \not\models \varphi[t_1/x_1]\ldots[t_n/x_n]$ or there is a term t_i in $\{t_1, \ldots, t_i\}$ such that $\delta^M_w(t_i) \notin d^M_w(0)$. Since $\varphi[t_1/x_1]\ldots[t_n/x_n]$ is closed, for any sequence, σ, either $M, w, \sigma \not\models \varphi[t_1/x_1]\ldots[t_n/x_n]$ and M is a counter-example to $G; (A : \Gamma \Rightarrow \varphi[t_1/x_1]\ldots[t_n/x_n], \Sigma); H$ or there is a t_i such that M is a counter-example to $G; (A : \Gamma \Rightarrow \Sigma : B, t_i); H$.

Case 26 (I is L$\exists_1$). In this case δ has the form

$$\frac{\begin{array}{c}\vdots\\ G; (A, t : \Gamma, \varphi[t/x] \Rightarrow \Sigma : B); H\end{array}}{G; (A : \Gamma, \exists x\varphi \Rightarrow \Sigma : B); H}$$

where t does not occur in $G; (A : \Gamma, \exists x\varphi \Rightarrow \Sigma : B); H$. Let M be a counter-example to $G; (A : \Gamma \Rightarrow \exists x\varphi \Rightarrow \Sigma : B); H$. There is a $w \in W^M$, such that for every σ, $M, w, \sigma \not\models (A : \Gamma, \exists x\varphi \Rightarrow \Sigma : B)$. So there is a $\sigma' \sim_x \sigma$, such that $\delta^M_\sigma(w, x) \in d^M_w(0)$ and $M, w, \sigma \models \psi$. Let N be a model, such that

1. $W^M = W^N$,

2. $D^M = D^N$,

3. $d^M_w(i) = d^n_w(i)$ for each i and $w \in W^M$, and

4. for all expressions, ε appearing in $G; (A : \Gamma, \exists x\varphi \Rightarrow \Sigma : B); H$, all sequences σ, and worlds, w, $\delta^M_\sigma(w, \varepsilon) = \delta^N_\sigma(w, \varepsilon)$.

and such that $\delta^M_\sigma(w, x) = \delta^N_\sigma(w, t)$ for all worlds, w and sequences, σ. By lemma 7.7, $N, w, \sigma \models \psi$. By lemma 7.5, $M, w, \sigma \models \psi[t/x]$ and $\delta^M_\sigma(w, t) \in d^M_\sigma(w, t)$. Since this holds for any σ, by several applications of lemma 7.7, $N, w \not\models (A, t : \Gamma, \varphi[t/x] \Rightarrow \Sigma : B)$. By several more applications of lemma 7.7, N is a counter-example to $G; (A, t : \Gamma, \varphi[t/x] \Rightarrow \Sigma : B); H$.

Case 27 (I is R$\exists_1$). In this case δ has the form

$$\dfrac{\begin{matrix}\vdots \\ G;(A:\Gamma \Rightarrow \varphi[t/x],\Sigma:B);H\end{matrix} \qquad \begin{matrix}\vdots \\ G;(A:\Gamma \Rightarrow \Sigma:B,t);H\end{matrix}}{G';(A:\Gamma \Rightarrow \exists x\varphi,\Sigma:B);H}$$

Let M be a counter-example to $G';(A:\Gamma \Rightarrow \exists x\varphi,\Sigma:B);H$. Let, $M,w,\sigma \not\models \exists x\psi$. So there is a no $\sigma' \sim_x \sigma$ such that $\delta^M_{\sigma'}(w,x) \in d^M_w(0)$ and $M,w,\sigma \models \psi$. Let t be a term, and σ' be such that $\sigma' \sim_x \sigma$ and $\delta^M_{\sigma'}(w,x) = \delta^M_{\sigma'}(w,t)$ for all $w \in W^M$. Either $M,w,\sigma' \not\models \psi$ or $\delta^M_{\sigma'}(w,x) \notin d^M_w(0)$. In the first case, by lemma 7.7, $M,w,\sigma' \not\models \psi[t/x]$. Since $\psi[t/x]$ is closed, $M,w \not\models \psi[t/x]$ and M is thus a counter-example to $G;(A:\Gamma \Rightarrow \varphi[t/x],\Sigma:B);H$. In the latter case, since t is a term, M is a counter-example to $G;(A:\Gamma \Rightarrow \Sigma:B,t);H$.

Case 28 (I is L$\exists_2$). In this case δ has the form

$$\dfrac{\begin{matrix}\vdots \\ G;(A:\Gamma,\varphi[F/f] \Rightarrow \Sigma:B);H\end{matrix}}{G;(A:\Gamma,\exists f\varphi \Rightarrow \Sigma:B);H}$$

where F does not occur in $G;(A:\Gamma,\exists f\varphi \Rightarrow \Sigma:B);H$. Let M be a counter-example to $G;(A:\Gamma,\exists f\varphi \Rightarrow \Sigma:B);H$. So there is a world, $w \in W^M$, and sequence, σ, such that for any sequence $\sigma' \sim_f \sigma$, $\delta^M_\sigma(w,f) \in d^M_w(i)$ where f is an i-ary variable, and $M,w,\sigma' \models \varphi$. Let N be a model such that

1. $W^M = W^N$,
2. $D^M = D^N$,
3. $d^M_w(i) = d^n_w(i)$ for each i and $w \in W^M$, and

4. for all expressions, ε appearing in $G; (A : \Gamma, \exists f\varphi \Rightarrow \Sigma : B); H$, all sequences σ, and worlds, w, $\delta^M_\sigma(w, \varepsilon) = \delta^N_\sigma(w, \varepsilon)$.

and such that $\delta^M_\sigma(w, f) = \delta^N_\sigma(w, F)$ for all worlds, w and sequences, σ. By lemma 7.7, $N, w, \sigma \models \psi$. By lemma 7.5, $N, w, \sigma \models \psi[F/f]$. Several applications of lemma 7.7 gives that N is a counter-example to $G; (A : \Gamma, \varphi[F/f] \Rightarrow \Sigma : B); H$

Case 29 (I is an instance of R$\exists_2$). In this case δ has the form

$$\frac{\begin{matrix}\vdots\\ G; (A : \Gamma \Rightarrow \varphi[\langle\lambda x_1, \ldots, x_n\psi\rangle/f], \Sigma : B); H\end{matrix}}{G; (A : \Gamma \Rightarrow \exists f\varphi, \Sigma : B); H}$$

Let M be a counter-example to $G; (A : \Gamma \Rightarrow \exists f\varphi, \Sigma : B); H$ and $\langle\lambda x_1, \ldots, x_n\psi\rangle$ an arbitrary λ-expression. There is a $w \in W^M$ such that for every sequence σ, $M, w, \sigma \not\models \exists f\psi$. So for any sequence, $\sigma' \sim_f \sigma$, $M, w, \sigma' \not\models \psi$. At some stage, $\delta^M_{\hat{\sigma}}(\langle\lambda x_1, \ldots, x_n\psi\rangle)$ was defined and added to $d^M_w(n)$. Let σ' be a sequence such that $\sigma \sim_f \sigma'$ and $\delta^M_{\sigma'}(w, f) = \delta^M_{\sigma'}(w, \lambda x_1, \ldots, x_n\psi)$ for each $w \in W^M$. By lemma 7.5, $M, w, \sigma' \models \varphi[\lambda x_1, \ldots, x_n\psi/f]$. Since $\varphi[\lambda x_1, \ldots, x_n\psi/f]$ is closed, this holds for any σ'. So M is a counter-example to $G; (A : \Gamma \Rightarrow \varphi[\langle\lambda x_1, \ldots, x_n\psi\rangle/f], \Sigma : B); H$

□

7.6 Completeness

The completeness theorem is shown by means of a Hintikka-like construction. Importantly, what is shown is that the logic that results from removing Cut from fig. 7.1 and fig. 7.2 is complete for the given model-theory. This fact is used to establish that the contingentist second-order logic presented in this chapter is cut-admissible.

The completeness proof relies on the theorem that Id holds for any sentence φ.

Theorem 7.6.1. $\vdash : \varphi \Rightarrow \varphi :$

Proof. This is proved by induction on the rank of φ. If φ is $Ft_1, \ldots, t_n$ then the deduction in question is an instance of Id. For the other cases, suppose that the result holds for formulas of rank less than φ. The cases where φ is $\neg\psi$ and $\psi \rightarrow \theta$ are omitted.

Case 1 (φ is $\Diamond\psi$).

$$\mathrm{L}\Diamond\ \dfrac{\mathrm{R}\Diamond\ \dfrac{\mathrm{W}\ \dfrac{\begin{array}{c}\text{IH}\\ (:\psi \Rightarrow \psi :)\end{array}}{(:\psi \Rightarrow \psi :);(:\quad \Rightarrow \quad :)}}{(:\psi \Rightarrow \quad);(:\quad \Rightarrow \Diamond\psi :)}}{(:\Diamond\psi \Rightarrow \Diamond\psi :)}$$

Case 2 (φ is $\exists x\psi$). Let w be a term not occurring in ψ.

$$\mathrm{L}\exists_1\ \dfrac{\mathrm{R}\exists_1\ \dfrac{\mathrm{TL(t)}\ \dfrac{\dfrac{\text{IH}}{(:\psi[w/x] \Rightarrow \psi[w/x] :)}}{(w : \psi[w/x] \Rightarrow \psi[w/x] :)} \qquad \mathrm{TL(s)}\ \dfrac{\mathrm{Id(t)}\ \dfrac{}{(w :\quad \Rightarrow \quad : w)}}{(w : \psi[w/x] \Rightarrow \quad : w)}}{(w : \psi[w/x] \Rightarrow \exists x\psi :)}}{(: \exists x\psi \Rightarrow \exists x\psi :)}$$

Case 3 (φ is $\langle\lambda x_1, \ldots, x_n\psi\rangle c_1, \ldots, c_n$).

$$\mathrm{L}\lambda\ \dfrac{\mathrm{R}\lambda\ \dfrac{\mathrm{TL(t)}{\times}n\ \dfrac{\dfrac{\text{IH}}{(:\psi[c_1/x_1]\ldots \Rightarrow \psi[c_1/x_1]\ldots[c_n/x_n] :)}}{(c_1, \ldots, c_n : \psi[c_1/x_1]\ldots[c_n/x_n] \Rightarrow \psi[c_1/x_1]\ldots[c_n/x_n] :)} \qquad \dfrac{\mathrm{TL(s)}\ \dfrac{\mathrm{Id(t)}\ \dfrac{}{\{(c_i :\quad \Rightarrow \quad : c_i)\}_i}}{\{(c_i : \psi[c_1/x_1]\ldots[c_n/x_n] \Rightarrow \quad : c_i)\}_i}}{\{(c_1, \ldots, c_n : \psi[c_1/x_1]\ldots[c_n/x_n] \Rightarrow \quad : c_i)\}_i}\ \mathrm{TL(t)}{\times}n\ -}{(c_1, \ldots, c_n : \psi[c_1/x_1]\ldots[c_n/x_n] \Rightarrow \langle\lambda x_1, \ldots, x_n\psi\rangle c_1, \ldots, c_n :)}}{(: \langle\lambda x_1, \ldots, x_n\psi\rangle c_1, \ldots, c_n \Rightarrow \langle\lambda x_1, \ldots, x_n\psi\rangle c_1, \ldots, c_n :)}$$

□

Case 4 (φ is $\exists_2 f\psi$). Let φ be a predicate not occurring in ψ.

$$\mathrm{L}\exists_2\ \dfrac{\mathrm{R}\exists_2\ \dfrac{\dfrac{\text{IH}}{(:\psi[F/f] \Rightarrow \psi[F/f] :)}}{(:\psi[F/f] \Rightarrow \exists f\psi :)}}{(: \exists f\psi \Rightarrow \exists f\psi :)}$$

The proof of completeness proceeds by showing how to construct a tree for any given unprovable hyper-position. A model is then defined from the open branch of this tree. Thus, for any unprovable sequent there is a model.

7.6.1 Building a Tree

Let G be an unprovable hyper-position. Let $S = \varphi_1, \ldots, \varphi_n \ldots$ be an enumeration of the sentences of the language, $N = c_1, \ldots, c_n, \ldots$ an enumeration of the names of the language, and $R = \xi_1, \ldots, \xi_n, \ldots$ an enumeration of the predicates of the language, i.e. an enumeration of the union of P and the set of λ-expressions. Consider the set $S \times N \times R$, and let there be an ordering on it. Let W_0 be a list, $w_1, \ldots, w_n, \ldots$ of first-order witnesses, and let W_i, $i \geqslant 1$ be a list of i-ary witnesses, $w_1^i, \ldots, w_n^i, \ldots$.

Let W_1 be an infinite set of first-order witnesses and let W_2 be an infinite set of second-order witnesses. Let $T_1 \cup W_1$ be ordered such that for there is no $w \in W_1$ such that there is a $t \in T_1$ and w is ordered earlier than t. Let there be a similar ordering on the set of predicates and λ-expressions and W_2. Call that set P.

A tree is constructed with G at its root. A branch of the tree with leaf G' is closed iff either there is a position, $C : \Delta \Rightarrow \Lambda : D \in G'$, such that $C \cap D \neq \varnothing$ or there is a position, $C : \Delta \Rightarrow \Lambda : D \in G'$, such that $\Delta \cap \Lambda \neq \varnothing$. A leaf is open iff it is not closed. At each stage of building the tree, consider each open leaf, $H; (A : \Gamma \Rightarrow \Sigma : B); H'$, in the tree. For each position, $A : \Gamma \Rightarrow \Sigma : B \in H; (A : \Gamma \Rightarrow \Sigma : B); H'$, and for each sentence $\varphi \in \Gamma \cup \Sigma$ do the following:

1. If φ is $F^n t_1, \ldots, t_n$, then

(a) if $F^n t_1, \ldots, t_n \Gamma$ then expand the tree by

$$\frac{H; (A, t_1, \ldots, t_n : \Gamma \Rightarrow \Sigma : B); H'}{H; (A : \Gamma \Rightarrow \Sigma : B); H}$$

2. If φ is $\neg\psi$, then

(a) if $\neg\psi \in \Gamma$ then expand the tree by

$$\frac{H; (A : \Gamma \Rightarrow \psi, \Sigma : B); H'}{H; (A : \Gamma \Rightarrow \Sigma : B); H'}$$

(b) if $\neg\psi \in \Sigma$ then expand the tree by

$$\frac{H; (A : \Gamma, \psi \Rightarrow \Sigma : B); H'}{H; (A : \Gamma \Rightarrow \Sigma : B); H'}$$

3. If φ is $\psi \to \theta$ then

(a) if $\psi \to \theta \in \Gamma$ then expand the tree by

$$\frac{H; (A : \Gamma \Rightarrow \psi, \Sigma : B); H' \qquad H; (A : \Gamma, \theta \Rightarrow \Sigma : B); H'}{H; (A : \Gamma \Rightarrow \Sigma : B); H'}$$

(b) if $\psi \to \theta \in \Sigma$ then expand the tree by

$$H; (A : \Gamma, \psi \Rightarrow \theta, \Sigma : B); H'$$

4. If φ is $\Diamond\psi$ then

(a) if $\Diamond\psi \in \Gamma$ and there is no position, $(C : \Delta \Rightarrow \Lambda : D)$ such that $\psi \in \Delta$ expand the tree by

$$(: \quad \Rightarrow \psi :); H; (A : \Gamma \Rightarrow \Sigma : B); H'$$

(b) if $\Diamond\psi \in \Sigma$, let $H; (A : \Gamma \Rightarrow \Sigma : B); H'$ be $(A_1 : \Gamma_1 \Rightarrow \Sigma_1 : B); \ldots, (A : \Gamma \Rightarrow \Sigma : B); \ldots; (A_n : \Gamma_n \Rightarrow \Sigma_n : B_n)$ and expand the tree by

$$(A_1 : \Gamma_1 \Rightarrow \Sigma_1, \psi : B); \ldots, (A : \Gamma \Rightarrow \Sigma, \psi : B); \ldots ; (A_n : \Gamma_n, \psi \Rightarrow \Sigma_n, \psi : B_n)$$

5. If φ is $\langle \lambda x_1, \ldots, x_n \psi \rangle c_1, \ldots, c_n$ then

   (a) if $\langle \lambda x_1, \ldots, x_n \psi \rangle c_1, \ldots, c_n \in \Gamma$ then expand the tree by

   $$\frac{H; (A, c_1, \ldots, c_n : \Gamma, \psi[c_1/x_1] \ldots [c_n/x_n] \Rightarrow \Sigma : B); H'}{H; (A : \Gamma \Rightarrow \Sigma : B); H'}$$

   (b) if $\langle x_1, \ldots, x_n \psi \rangle c_1, \ldots, c_n \in \Sigma$ then expand the tree by

   $$\frac{H; (A : \Gamma \Rightarrow \psi[c_1/x_1] \ldots [c_n/x_n], \Sigma : B); H' \qquad \{H; (A : \Gamma \Rightarrow \Sigma : B, c_i); H'\}_{1 \leqslant i \leqslant n}}{H; (A : \Gamma \Rightarrow \Sigma : B); H'}$$

6. If φ is $\exists x \psi$ then

   (a) if $\exists x \psi \in \Gamma$ then expand the tree by

   $$\frac{H; (A, w : \Gamma \Rightarrow \Sigma : B); H'}{H; (A : \Gamma \Rightarrow \Sigma : B); H'}$$

   where w is the least element of $T_1 \cup W_1$ not appearing in $H; (A : \Gamma \Rightarrow \Sigma : B); H'$.

   (b) if $\exists x \psi \in \Sigma$ then expand the tree by

   $$\frac{H; (A : \Gamma \Rightarrow \psi[t/x], \Sigma : B); H' \qquad H; (A : \Gamma \Rightarrow \Sigma : B, t); H'}{H; (A : \Gamma \Rightarrow \Sigma : B); H'}$$

   where t is the least element of $T_1 \cup W_1$ such that neither $\psi[t/x] \in \Sigma$ nor $t \in B$.

7. if φ is $\exists f \psi$ then

(a) if $\exists f \psi \in \Gamma$ expand the tree by

$$\frac{H; (A : \Gamma, \psi[W/f] \Rightarrow \Sigma : B); H'}{H; (A : \Gamma \Rightarrow \Sigma : B); H'}$$

where W is the least element of W_2 not appearing in $H; (A : \Gamma \Rightarrow \Sigma : B); H'$

(b) if $\exists f \psi$ then expand the tree by

$$\frac{H; (A : \Gamma \Rightarrow \psi[\varepsilon/f], \Sigma : B); H'}{H; (A : \Gamma \Rightarrow \Sigma : B); H'}$$

where ε is the least element of P such that $\psi[\varepsilon/f] \notin \Sigma$

For convenience the stages are referred to by number. When a particular sentence is considered at a stage, that is referred to as a step. Each step is a derivable rule of the calculus of fig. 7.1 and fig. 7.2. Anything generated by (1) is an instance of AL. Anything generated by (2) is either an instance of L$\neg$ or R$\neg$, any thing generated by (3) is either an instance of L$\rightarrow$ or R$\rightarrow$. Anything generated by (4a) is an instance of L$\Diamond$, anything generated by (5) is an instance of Lλ or Rλ, and anything generated by (6) or (7) is an instance of one of L$\exists_1$, L$\exists_2$, R$\exists_1$, R$\exists_2$. This leaves only (4b) which is not an instance of any rule. Since hyper-positions are sets, this is derivable by as many applications of R$\Diamond$ as are necessary. Thus, if every branch of the above construction closed, there would be a deduction of G. It would begin from closed leaves, which are derivable by theorem 7.6.1, and proceed by means of derivable rules of fig. 7.1 and fig. 7.2 to G. So there is a open branch of this tree after a stage for each natural number has been completed.

7.6.2 Building A Model

Consider a step a particular step in some stage. At this step some procedure of (1) - (7) above was applied. Call it i. Let J be the hyper-position occurring at prior to i and K be a hyper-position occurring directly above J in the tree. Let $S \in J$ and $S' \in K$. A predecessor relation is defined on S and S' via definition 35.

Definition 35 (Predecessor)**.** S is a predecessor of S', $S \prec S'$ iff either $S = S'$ or S' results from S by an application of i where i is not (4a).

Suppose that $S \prec S'$. If it is also the case that $S \prec S''$, then there is a stage and step, i, such that there are four possibilities:

- $S = S'$ and $S = S''$
- $S = S'$ and S'' results from S by an application of i
- S' results from S by an application of i and $S = S''$
- S' results from S by an application of i and S'' results from S by an application of i.

The second case is impossible. If S and S' are identical, they were identical before the application of i, since i applied only to one element of J, S. But this cannot be the case if S'' results from S by an application of i. Similar considerations rule out the third possibility. On either of the other possibilities, S' and S'' are identical. This establishes fact 5.

Fact 5. If $S \prec S'$, then for any S'' such that $S \prec S''$, $S'' = S'$.

Fact 5 legitimates the following notation, $\#S$, is the unique S' such that $S \prec S'$ in a construction of the above sort. It is important to note that there will be some positions that do not have predecessors, e.g. those positions in the root of the tree or generated as a result of an application (4a).

Because steps (1) - (7) can at most expand a position in a hyper-position. For any $(A : \Gamma \Rightarrow \Sigma : B) \prec (C : \Delta \Rightarrow \Lambda : D)$, $A \subseteq C$, $\Gamma \subseteq \Delta$, $\Sigma \subseteq \Lambda$, and $B \subseteq D$. A chain of a position S in a branch $Ch(S)$ is defined inductively

Definition 36 (Chain of S)**.** If S is a position in an open branch, then $S' \in Ch(S)$ if either $S' = \#S$ or there is an $S^* \in Ch(S)$ such that $S' = \#S^*$.

Let $UCh(S)$ be a potentially infinite position identified by the union of all the elements of positions occurring in the chain of S.

Definition 37 (Union of a Chain)**.** Let $Ch(S) = (A_1 : \Gamma_1 \Rightarrow \Sigma_1 : B_1), \ldots, (A_n : \Gamma_n \Rightarrow \Sigma_n : B_n), \ldots$

$$UCh(S) = (\bigcup_i A_i : \bigcup_i \Gamma_i \Rightarrow \bigcup_i \Sigma_i : \bigcup_i B_i)$$

For any $(A : \Gamma \Rightarrow \Sigma : B)$, if $(C : \Delta \Rightarrow \Lambda : D) \in Ch(A : \Gamma \Rightarrow \Sigma : B)$, $A \subseteq C$, $\Gamma \subseteq \Delta$, $\Sigma \subseteq \Lambda$, and $B \subseteq D$. It follows that for any positions S, S' and S^*, if $UCh(S) = S'$ and $S^* \in Ch(S)$, $UCh(S^*) = S'$.

Let G be an unprovable hyper-position and $\beta = G_1, \ldots, G_n, \ldots$ an open branch in a tree constructed in the manner of section 7.6.1 from G.

$$E(\beta) = \bigcup\{UCh(S) : \exists G_i(G_i \in \beta \& S \in G_i\}$$

$E(\beta)$ is a hyper-position. Suppose that there is a position in $E(\beta)$, $(A : \Gamma \Rightarrow \Sigma : B)$, such that $\varphi \in \Gamma \cup \Sigma$. If that were the case, then there is a step, n, with position, $(C : \Delta \Rightarrow \Lambda : D)$ with φ in either Δ or Λ. At some later stage, $n + m$ with position, $(C' : \Delta' \Rightarrow \Lambda' : D') \in Ch(S)$, φ would have to have been added to Λ or Δ respectively. But then β would have closed at $m + n$, contradicting the hypothesis that β is an open branch. Similar considerations rule out that there is any sequent $(A : \Gamma \Rightarrow \Sigma : B) \in E(\beta)$, such that there is a $t \in A \cap B$.

There are several other important features of $E(\beta)$. These features are summed up in lemmas 7.9 to 7.15.

Lemma 7.9. *Let* $(A : \Gamma \Rightarrow \Sigma : B) \in E(\beta)$. *If* $Fc_1, \ldots, c_n \in \Gamma$, *then for each* c_i, $1 \leqslant i \leqslant n$, $c_i \in A$.

Proof. Let $(A : \Gamma \Rightarrow \Sigma : B) \in E(\beta)$. Since $Fc_1, \ldots, c_n \in \Gamma$, there is a stage j with position $(C : \Delta \Rightarrow \Lambda : D)$ such that $Fc_1, \ldots, c_n \in \Delta$. At stage $j + 1$, by (1) of the procedure in section 7.6.1, $c_1, \ldots, c_n$ would be added to C. Since for each such $(C : \Delta \Rightarrow \Lambda : D)$, $C \subseteq A$, $c_1, \ldots, c_n \in A$. □

Lemma 7.10. *Let* $(A : \Gamma \Rightarrow \Sigma : B) \in E(\beta)$. *If* $\neg\varphi \in \Gamma$, *then* $\varphi \in \Sigma$, *and if* $\neg\varphi \in \Sigma$, *then* $\varphi \in \Gamma$.

Proof. Let $(A : \Gamma \Rightarrow \Sigma : B) \in E(\beta)$. Only the first conjunct will be proved since the proof of th second conjunct is similar. Let $\neg\varphi \in \Gamma$. So there is a stage, n, with a position, $(C : \Delta \Rightarrow \Lambda : D)$, such that $\neg\varphi \in \Delta$. At stage $n + 1$, with sequent $(C' : \Delta' \Rightarrow \Lambda' : D')$, by (2a) of the process in section 7.6.1, φ would have been added to Λ'. Since $\Lambda' \subseteq \Sigma$, $\varphi \in \Sigma$. □

Lemma 7.11. *Let* $(A : \Gamma \Rightarrow \Sigma : B) \in E(\beta)$*. If* $\varphi \to \psi \in \Gamma$*, then either* $\varphi \in \Sigma$ *or* $\psi \in \Gamma$*. If* $\varphi \to \psi \in \Sigma$ *then* $\varphi \in \Gamma$ *and* $\psi \in \Sigma$*.*

Proof. Let $(A : \Gamma \Rightarrow \Sigma : B) \in E(\beta)$. Let $\varphi \to \psi \in \Gamma$. So there is a stage, n, with position $(C : \Delta \Rightarrow \Lambda : D)$, such that $\varphi \to \psi \in \Delta$. At some following step or stage, with a position $(C' : \Delta' \Rightarrow \Lambda' : D')$, the tree branches adding $(C' : \Delta', \psi \Rightarrow \Lambda' : D')$ to the left and $(C' : \Delta' \Rightarrow \psi, \Delta' : D')$, to the right. β must be one of these branches. In the first case, since $\Delta' \cup \{\psi\} \subseteq \Gamma$, $\psi \in \Gamma$. In the second case, since $\Lambda' \cup \{\varphi\} \subseteq \Sigma$, $\varphi \in \Sigma$.

Let $\varphi \to \psi \in \Sigma$. There is a stage, n, with position, $(C : \Delta \Rightarrow \Lambda : D)$ such that $\varphi \to \psi \in \Lambda$. At a later stage, with position $(C' : \Delta' \Rightarrow \Lambda' : D')$, by (3b) the tree would have been extended by the position $(C' : \Delta', \varphi \Rightarrow \psi, \Lambda' : D')$. Since $\Delta' \cup \{\varphi\} \subseteq \Gamma$ and $\Lambda' \cup \{\psi\} \subseteq \Sigma$, $\varphi \in \Gamma$ and $\psi \in \Sigma$. □

Lemma 7.12. *Let* $(A : \Gamma \Rightarrow \Sigma : B) \in E(\beta)$*. If* $\Diamond\varphi \in \Gamma$*, then there is a* $(C : \Delta \Rightarrow \Lambda : D) \in E(\beta)$ *such that* $\varphi \in \Delta$*, and if* $\Diamond\varphi \in \Sigma$*, then for any* $(C : \Delta \Rightarrow \Lambda \in E(\beta)$*,* $\varphi \in \Lambda$*.*

Proof. Let $(A : \Gamma \Rightarrow \Sigma : B) \in E(\beta)$. Suppose that $\Diamond\varphi \in \Gamma$. So there is a stage, n, with position $(P : \Pi \Rightarrow \Theta : Q)$. At the following stage with position, $P' : \Pi' \Rightarrow \Theta' : Q') \in Ch(P : \Pi \Rightarrow \Theta : Q)$ and hyper-position J. At that stage either there is a position, $(P'' : \Pi'' \Rightarrow \Theta'' : Q'')$ such that $\varphi \in \Pi''$ or not. If it is, the result is given. If not, then by (4a) the tree would be expanded by $(: \varphi \Rightarrow \quad :); J$. Since $(: \varphi \Rightarrow \quad :); J \in \beta$ and $(: \varphi \Rightarrow \quad :) \in (: \varphi \Rightarrow \quad :); J$, $UCh(: \varphi \Rightarrow \quad :) \in E(\beta)$. Let $UCh(: \varphi \Rightarrow \quad :)$ be $(C : \Delta \Rightarrow \Lambda : D)$. Since $\{\varphi\} \subseteq \Delta$, $\varphi \in \Delta$ for some $(C : \Delta \Rightarrow \Lambda : D) \in E(\beta)$.

Suppose that $\Diamond\varphi \in \Sigma$. Let $(C : \Delta \Rightarrow \Lambda : D) \in E(\beta)$, but $\varphi \notin \Lambda$. There must be some stage, n, with position, $C' : \Delta' \Rightarrow \Lambda' : D'$ such that $UCh(C'\Delta' \Rightarrow \Lambda' : D') =$

$(C' : \Delta' \Rightarrow \Lambda' : D')$. At a later stage,$n + m$, there is a position $(A' : \Gamma' \Rightarrow \Sigma' : B')$ such that $UCh(A' : \Gamma' \Rightarrow \Sigma' : B') = (A : \Gamma \Rightarrow \Sigma : B)$. At stage $n + m + 1$, let $(P : \Pi \Rightarrow \Theta : Q) \in Ch(C' : \Delta' \Rightarrow \Lambda' : D')$. There is a position, $(A'' : \Gamma'' \Rightarrow \Sigma'' : B'') \in Ch(A' : \Gamma' \Rightarrow \Sigma' : B')$ at that stage, with $\Diamond\varphi \in \Sigma'$. Let the hyper-position at $m + n + 1$ be $(A'' : \Gamma'' \Rightarrow \Sigma'' : B''); \ldots; (P : \Pi \Rightarrow \Theta : Q); \ldots$ At stage $m + n + 1.j$, by (4b), the tree is extended by $(A'' : \Gamma'' \Rightarrow \varphi, \Sigma'' : B''); \ldots; (P : \Pi \Rightarrow \varphi, \Theta : Q); \ldots$. Since $UCh(P : \Pi \Rightarrow \varphi, \Theta : Q) = (C : \Delta \Rightarrow \Lambda : D)$, $\varphi \in \Lambda$. Thus, for all positions, $(C : \Delta \Rightarrow \Lambda : D) \in E(\beta)$, $\varphi \in \Lambda$. □

Lemma 7.13. *Let $(A : \Gamma \Rightarrow \Sigma : B) \in E(\beta)$. If $\langle\lambda x_1, \ldots, x_n\varphi\rangle c_1, \ldots, c_n \in \Gamma$ then $c_1, \ldots, c_n \in A$ and $\varphi[c_1/x_1] \ldots [c_n/x_n] \in \Gamma$ and if $\langle\lambda x\varphi\rangle c \in \Sigma$ then either one of $c_i \in B$ or $\varphi[c_1/x_1] \ldots [c_n/x_n] \in \Sigma$.*

Proof. Let $(A : \Gamma \Rightarrow \Sigma : B) \in E(\beta)$. Suppose that $\langle\lambda x_1, \ldots x_n\varphi\rangle c_1, \ldots, c_n \in \Gamma$. Then there is a stage, n, with position $(C : \Delta \Rightarrow \Lambda : D)$ such that $UCh(C : \Delta \Rightarrow \Lambda : D) = (A : \Gamma \Rightarrow \Sigma : B)$ and $\langle\lambda x_1, \ldots, x_n \in \Delta$. At some later stage, $n + 1$ let position $(P : \Pi \Rightarrow \Theta : Q) \in Ch(C : \Delta \Rightarrow \Lambda : D)$. At that point $\varphi[c_1/x_1] \ldots [c_n/x_n]$ would have been added to Π and $c_1, \ldots, c_n$ added to P. Thus, $\varphi[c_1/x_1] \ldots [c_n/x_n] \in \Gamma$ and $c_1, \ldots, c_n \in A$.

Let $\langle\lambda x_1, \ldots, x_n\varphi\rangle c_1, \ldots, c_n \in \Sigma$. There is a stage, n with hyper-position J and position $(C : \Delta \Rightarrow \Lambda : D) \in J$ such that $UCh(C : \Delta \Rightarrow \Lambda : D) = (A : \Gamma \Rightarrow \Sigma : B)$ where $\langle\lambda x_1, \ldots, x_n\varphi\rangle c_1, \ldots, c_n \in \Lambda$. By (5b) at stage $n + 1$ the tree expands by branching. The left-most branch has J with $(C : \Delta \Rightarrow \Lambda : D)$ replaced by $(C : \Delta, \varphi[c_1, x_1] \ldots [c_n/x_n] \Rightarrow \Lambda : D)$ as its leaf. The $i + 1^{th}$ branch has J with $(C : \Delta \Rightarrow \Lambda : D)$ replaced by $(C : \Delta \Rightarrow \Lambda : D, c_i)$ as a leaf. β must pass through one of these

branches. If it passes through the left-most branch then $\varphi[c_1, x_1]\dots[c_n/x_n] \in \Sigma$. If it passes through the $i+1^{th}$ branch then $c_i \in B$. $\square$

Lemma 7.14. *Let $(A : \Gamma \Rightarrow \Sigma : B) \in E(\beta)$. If $\exists x\varphi \in \Gamma$, then there is a witness, w, such that $w \in A$ and $\varphi[w/x] \in \Gamma$ and if $\exists x\varphi \in \Sigma$, then for any name or witness, t, appearing in $E(\beta)$, either $t \in B$ or $\varphi[t/x] \in \Sigma$.*

Proof. Let $(A : \Gamma \Rightarrow \Sigma : B) \in E(\beta)$. Suppose that $\exists x\varphi \in \Gamma$. There is a stage, n, such that the hyper-position under consideration is $J; (C : \Delta \Rightarrow \Lambda : D)$, where $UCh(C : \Delta \Rightarrow \Lambda : D) = (A : \Gamma \Rightarrow \Sigma : B)$. Either there is a term, w, such that $w \in C$ and $\varphi[w/x] \in \Delta$ or not. If so, the result is given. If not, at stage the tree would have been expanded by $J; (C, w : \Delta, \varphi[w/x] \Rightarrow \Lambda : D)$ where w is the least element of W_1 not occurring in $J; (C : \Delta \Rightarrow \Lambda : D)$. Since $(C, w : \Delta, \varphi[w/x] \Rightarrow \Lambda : D) \in Ch(C : \Delta \Rightarrow \Lambda : D)$, $UCh(C, w : \Delta, \varphi[w/x] \Rightarrow \Lambda : D) = (A : \Gamma \Rightarrow \Sigma : B)$. So $w \in A$ and $\varphi[w/x] \in \Gamma$.

Suppose that $\exists x\varphi \in \Sigma$. Suppose that there is a term, t, such that $t \notin B$, and $\varphi[t/x] \notin \Sigma$. Let t be the i^{th} term in the ordering of $T_1 \cup W_1$. There is a stage, n, such with hyper-position, $J; (C : \Delta \Rightarrow \Lambda : D)$ such that $\exists x\varphi \in \Lambda$ and $UCh(C : \Delta \Rightarrow \Lambda : D) = A : \Gamma \Rightarrow \Sigma : B$. Let $K; (P : \Pi \Rightarrow \Theta : Q)$ be the hyper-position under consideration at stage $n + i$ and $UCh(P : \Pi \Rightarrow \Theta : Q) = A : \Gamma \Rightarrow \Sigma : B$. By (6b) the tree branches. On one branch $\varphi[t/x]$ is added to Π, on the other t is added to Q. Since $E(\beta)$ passes through one of these branches, either $\varphi[t/x] \in \Gamma$ or $t \in B$. $\square$

Lemma 7.15. *Let $(A : \Gamma \Rightarrow \Sigma : B) \in E(\beta)$. If $\exists f\varphi \in \Gamma$ then there is a witness W of the same arity of f, such that $\varphi[W/f] \in \Gamma$ and if $\exists f\varphi \in \Sigma$, then for all predicates or λ-expressions, ξ, of the same arity as f, $\varphi[\xi/f] \in \Sigma$.*

Proof. Let $(A : \Gamma \Rightarrow \Sigma : B) \in E(\beta)$. Suppose that $\exists f\varphi \in \Gamma$. There is a stage, n, such that the hyper-position under consideration is $J; (C : \Delta \Rightarrow \Lambda : D)$, where $UCh(C : \Delta \Rightarrow \Lambda : D) = (A : \Gamma \Rightarrow \Sigma : B)$. Either there is a term, W, such that $\varphi[W/f] \in \Delta$ or not. If so, the result is given. If not, at stage the tree would have been expanded by $J; (C : \Delta, \varphi[W/f] \Rightarrow \Lambda : D)$ where W is the least element of W_2 not occurring in $J; (C : \Delta \Rightarrow \Lambda : D)$. Since $(C : \Delta, \varphi[W/f] \Rightarrow \Lambda : D) \in Ch(C : \Delta \Rightarrow \Lambda : D)$, $UCh(C : \Delta, \varphi[W/f] \Rightarrow \Lambda : D) = (A : \Gamma \Rightarrow \Sigma : B)$. So $\varphi[W/f] \in \Gamma$.

Suppose that $\exists f\varphi \in \Sigma$. Suppose that there is a predicate expression, ε, such that $\varphi[\varepsilon/f] \notin \Sigma$. Let ε be the i^{th} term in the ordering of P. There is a stage, n, such with hyper-position, $J; (C : \Delta \Rightarrow \Lambda : D)$ such that $\exists f\varphi \in \Lambda$ and $UCh(C : \Delta \Rightarrow \Lambda : D) = A : \Gamma \Rightarrow \Sigma : B$. Let $K; (P : \Pi \Rightarrow \Theta : Q)$ be the hyper-position under consideration at stage $n + i$ and $UCh(P : \Pi \Rightarrow \Theta : Q) = A : \Gamma \Rightarrow \Sigma : B$. By (7b) $\varphi[t/x]$ is added to Π. $\square$

Lemma 7.16. *$\delta^M_\sigma(w, F) = \delta^M_\sigma(w, \lambda x_1, \ldots, x_n F x_1, \ldots, x_n)$, for all σ and n-ary predicate F.*

Proof. Let $\langle o_1, \ldots, 0_n\rangle \in \delta^M_\sigma(w, F)$.

Since $I^M(w, F) \subseteq \delta^M_w(0)^n$, $o_i \in d^M_w(0)$ for $1 \leqslant i \leqslant n$. Let σ' be such that $\sigma(w, x_i) = o_i$ for $1 \leqslant i \leqslant n$ and $\sigma' \sim_{x_1,\ldots,x_n} \sigma$. It follows that $M, w, \sigma' \models Fx_1, \ldots, x_n$. So $\langle \sigma'(w, x_1), \ldots, \sigma'(w, x_n)\rangle \in \{\langle \hat{\sigma}(w, x_1), \ldots, \hat{\sigma}(w, x_n)\rangle : \sigma \sim_{x_1,\ldots,x_n} \hat{\sigma} \& \delta^M_{\hat{\sigma}}(w, x_i) \in d^M_w(0)$ for $1 \leqslant i \leqslant n \& M, w, \hat{\sigma} \models Fx_1, \ldots, x_n\}$, i.e. $\langle \sigma'(w, x_1), \ldots, \sigma'(w, x_n)\rangle \in \delta^M_\sigma(w, \lambda x_1, \ldots, x_n F x_1, \ldots, x_n)$. Since $\delta^M_{\sigma'}(w, x_i) = \sigma'(w, x_i) = o_i$ for $1 \leqslant i \leqslant n$, $\langle o_1, \ldots, o_n\rangle \in \delta^M_\sigma(w, \lambda x_1, \ldots, x_n F x_1, \ldots, x_n)$.

Let $\langle o_1, \ldots o_n \in \delta^M_\sigma(w, \lambda x_1, \ldots, x_n F x_1, \ldots x_n)$. So $\langle o_1, \ldots o_n\rangle \in \{\langle \delta^M_{\hat{\sigma}}(w, x_1), \ldots, \delta^M_{\hat{\sigma}}(w, x_n)\rangle :$

$\sigma \sim_{x_1,\ldots,x_n} \hat{\sigma} \& \delta^M_{\hat{\sigma}}(w, x_i) \in d^M_w(0)$ for $1 \leqslant i \leqslant n \& M, w, \hat{\sigma} \models Fx_1, \ldots, x_n\}$. Let $\sigma' \sim_{x_1,\ldots,x_n} \sigma$ and be such that $\sigma(w, x_i) = o_i$ for $1 \leqslant i \leqslant n$. So $\langle \delta^M_{\sigma'}(w, x_1), \ldots \delta^M_{\sigma'}(w, x_n) \rangle \in \{\langle \delta^M_{\hat{\sigma}}(w, x_1), \ldots, \delta^M_{\hat{\sigma}}(w, x_n) \rangle : \sigma \sim_{x_1,\ldots,x_n} \hat{\sigma} \& \delta^M_{\hat{\sigma}}(w, x_i) \in d^M_w(0)$ for $1 \leqslant i \leqslant n \& M, w, \hat{\sigma} \models Fx_1, \ldots, x_n\}$. Fromm this it follows that $M, w, \sigma' \models Fx_1, \ldots, x_n$. So $\langle \delta^M_{\sigma'}(w, x_1), \ldots, \delta^M_{\sigma'}(w, x_n) \rangle \in I_M(w, F)$. Thus, $\langle o_1, \ldots, o_n \rangle \in I_M(w, F)$. □

Lemma 7.17. *In models where D^M is countable, if $X \in d^M_w(n)$ then there is a closed λ-expression, $\lambda x_1, \ldots, x_n \varphi$, in an expanded language such that $\delta^M_\sigma(w, \langle \lambda x_1, \ldots, x_n) = X$.*

Proof. If $X \in d^M_w$ then there is a stage in the construction of the model at which it was added. If it was added at a base case, then there is an n-ary predicate, F, such that $I_M(w, F) = X$. By lemma 7.16, there is a closed lambda expression, $\lambda x_1, \ldots, \lambda x_n F x_1, \ldots, x_n$, such that $\delta^M_\sigma(w, \lambda x_1, \ldots, x_n F x_1, \ldots, x_n) = X$ for all σ.

Let S be a countable set of witnesses. If X was added to $d^M_w(0)$ at a later stage, then either there already is a closed λ-expression, $\lambda x_1, \ldots, x_n \varphi$ such that $\delta^M_\sigma(w, \lambda x_1, \ldots, x_n \varphi) = X$ or there is a λ-expression, $\lambda x_1, \ldots, x_n \varphi$ with several free variables, $y_1, \ldots, y_m$, and a sequence, σ, such that $\delta^M_\sigma(w, \lambda x_1, \ldots, x_n \varphi) = X$. Since the domain is countable, only countably many such sets will be added at any stage. Since sentences are finite and the set of witnesses is countable for any such λ-expression it is possible to do the following. Let $s_{i_1}, \ldots, s_{i_m}$ be unassigned witnesses and assign $I_M(v, s_{i_j} = \sigma^M_\sigma(v, y_j)$ for all $v \in W_M$. □

A model, M, of G is constructed from $E(\beta)$. For each position $(A_i : \Gamma_i \Rightarrow \Sigma_i : B_i) \in E(\beta)$, let there be a world $w_i \in W_M$, $d_w m(0) = A$, and for each n-ary atomic predicate, F, if $Ft_1, \ldots, t_n \in \Gamma_i$, then $\langle t_1, \ldots, t_n \rangle \in I_M(w, F)$ and $\langle t_1, \ldots, t_n \rangle \in d^M_w(n)$.

Lemma 7.18. *Let* $(A_1 : \Gamma_1 \Rightarrow \Sigma_1 : B_1); \ldots; (A_n : \Gamma_n \Rightarrow \Sigma_n : B_n); \ldots$ *be* $E(\beta)$. *for each* $(A_i : \Gamma_i \Rightarrow \Sigma_i : B_i) \in E(\beta)$ *there is a world,* $w \in W_M$*, such that for each* $c \in A_i$*,* $c \in d_w^M(0)$*, for each* $c \in B_i$*,* $c \notin d_w^M(0)$*, for each* $\varphi \in \Gamma_i$*,* $M, w \models \varphi$ *and for each* $\varphi \in \Sigma_i$*,* $M, w \not\models \varphi$*.*

Proof. It must be that for each $(A_i : \Gamma_i \Rightarrow \Sigma_i : B_i) \in E(\beta)$, if $c \in A_i$, then $c \in d_i^M(0)$ where i is the world introduced to correspond to $(A_i : \Gamma_i \Rightarrow \Sigma_i : B_i)$. This is because $d_i^M(0) = A_i$. Similarly, suppose that $c \in B_i$ and $c \in d_i^M(0)$. Since $d_i^M(0) = A_i$, $c \in A_i$. But if $c \in A_i \cap B_i$, the branch β would not be open, contradicting the assumption that it was.

The second claim is proved by induction on φ.

Case 1 (φ is atomic). Let φ be $Ft_1, \ldots, t_n$. Let $\varphi \in \Gamma_i$ for some $(A_i : \Gamma_i \Rightarrow \Sigma_i : B_i)$. Let $w_i \in W_M$, be the world added to correspond to that position. By lemma 7.9 for each t_i, $1 \leqslant i \leqslant n$, $t_i \in A$, and so $t_i \in d_i^M$ for $1 \leqslant i \leqslant n$. By the construction of M, $\langle t_1, \ldots, t_n \rangle \in I_M(i, F)$ iff $Ft_1, \ldots, t_n \in \Gamma_i$. Since $Ft_1, \ldots, t_n \in \Gamma_i$ by hypothesis, $I_M(\varphi) = 1$.

Let $\varphi \in \Sigma_i$. By the construction of, M, $\langle t_1, \ldots, t_n \rangle \in I_M(i, F)$ iff $Ft_1, \ldots, t_n \in \Gamma_i$. If $\varphi \in \Gamma_i$, then the branch β would have closed, so $\varphi \notin \Gamma_i$. It follows that $\langle t_1, \ldots, t_n \rangle \notin I_M(i, F)$, so $M, i \not\models \varphi$.

Case 2 (φ is $\neg\psi$). Let $\varphi \in \Gamma_i$. By lemma 7.10, $\psi \in \Sigma_i$. By IH, $M, i \not\models \psi$, so $M, i \models \neg\psi$.

Let $\varphi \in \Sigma_i$. By lemma 7.10, $\psi \in \Gamma_i$. By IH, $M, i \models \psi$, so $M, i \not\models \neg\psi$.

Case 3 (φ is $\psi \to \theta$). Let $\varphi \in \Gamma_i$. By lemma 7.11 either $\psi \in \Sigma_i$ or $\theta \in \Gamma_i$. In the first case, by IH, $M, i \not\models \psi$, so $M, i \models \psi \to \theta$. In the second, by IH, $M, i \models \theta$, so

$M, i \models \psi \to \theta$.

Let $\varphi \in \Sigma_i$. By lemma 7.11, $\psi \in \Gamma_i$ and $\theta \in \Sigma_i$. By IH, $M, i \models \psi$ and $M, i \not\models \theta$. So $M, i \not\models \psi \to \theta$.

Case 4 (φ is $\Diamond\psi$). Let $\varphi \in \Gamma_i$. By lemma 7.12, there is a position, $(A_j : \Gamma_j \Rightarrow \Sigma_j : B_j)$, such that $\psi \in \Gamma_j$. Let $j \in W_M$ be the world that corresponds to that position. By IH, $M, j \models \psi$. There is a world, $j \in W_M$, such that $M, j \models \psi$, so $M, i \models \Diamond\psi$.

Let $\varphi \in \Gamma_i$. By lemma 7.12 for any $(A_j : \Gamma_j \Rightarrow \Sigma_j : B_j) \in E(\beta)$, $\psi \in \Sigma_j$. So for any world, j, corresponding to a position, $(A_j : \Gamma_j \Rightarrow \Sigma_j : B_j) \in E(\beta)$, $M, j \not\models \psi$. But for any world, $w \in W_M$, there is a corresponding position. So for any world, w, $M, w \not\models \psi$. From that it follows that, $M, i \not\models \Diamond\psi$.

Case 5 (φ is $\langle \lambda x_1, \ldots, x_n \varphi \rangle c_1, \ldots, c_n$). Suppose that $\varphi \in \Gamma_i$ for $(A_i : \Gamma_i \Rightarrow \Sigma_i : B_i)$. By lemma 7.13, $c_1, \ldots, c_n \in A_i$ and $\psi[c_1/x_i], \ldots, [c_n/x_n] \in \Gamma_i$. Let σ be a sequence and $\sigma' \sim_{x_1, \ldots, x_n} \sigma$ such that $\sigma(x_j) = I_M(c_j)$ for $1 \leqslant j \leqslant n$. It follows that $\sigma(x_j) \in d_j^M(0)$ for $1 \leqslant j \leqslant n$. By lemma 7.7, $M, w, \sigma' \models \psi$. So $\langle \sigma'(x_1), \ldots, \sigma'(x_n) \rangle \in \{\langle \delta_{\hat{\sigma}}^M(i, x_1), \ldots, \delta_{\hat{\sigma}}^M(i, x_n) \rangle : \hat{\sigma} \sim_{x_1, \ldots, x_n} \sigma \& \delta_{\hat{\sigma}}^M(i, x_j) \in d_i^M(0)$ for $1 \leqslant j \leqslant n \& M, w, \hat{\sigma} \models \psi\}$, that is $\langle \sigma(x_1), \ldots, \sigma(x_n) \rangle \in I_M(i, \lambda x_1, \ldots, x_n \psi)$. Since $\sigma(x_j) = I_M(c_j)$ for $1 \leqslant j \leqslant n$, $\langle I_M(c_1), \ldots, I_M(c_n) \rangle \in I_M(i, \lambda x_1, \ldots, x_n \psi)$. Since σ was arbitrary, this can be done for any sequence. So $M, w \models \langle \lambda x_1, \ldots, x_n \varphi \rangle c_1, \ldots, c_n$.

Suppose that $\varphi \in \Sigma_i$. By lemma 7.13 either $\psi[c_1/x_1], \ldots, [c_n/x_n] \in \Sigma_i$ or for some c_j, $c_j \in B_i$. Suppose that $\langle \delta_\sigma^M(c_1), \ldots, \delta_\sigma^M(c_n) \rangle \in \delta_{\sigma'}^M(i, \langle \lambda x_1, \ldots, x_n \psi)$ for some $\sigma \sim_{x_1, \ldots, x_n} \sigma'$. So $I_M(c_j) \in d_w^M(0)$ for $1 \leqslant j \leqslant n$ and $M, w, \sigma' \models \psi$. Since $A_i = d_w^M(0)$ and $c_j = I_M(c_j$ for $1 \leqslant j \leqslant n$, $c_1, \ldots, c_n \in A_i$. Since $E(\beta)$ is open, there is no c_j such that $c_j \in B_i$. Thus, $\psi[c_1/x_1] \ldots [c_n/x_n] \in \Sigma_i$. By IH, $M, w \models \psi[c_1/x_1] \ldots [c_n/x_n]$.

So $M, w, \sigma' \models \psi[c_1/x_1]\ldots[c_n/x_n]$. But by lemma 7.7, this contradicts the fact that $M, w, \sigma' \models \psi$.

Case 6 (φ is $\exists x\psi$). Suppose that $\varphi \in \Gamma_i$. By lemma 7.14, there is a witness, $w_j \in A_i$ such that $\psi[w_j/x] \in \Gamma_i$. By IH, $M, w \models \psi[w_j/x]$. Let σ be a sequence. Let $\sigma' \sim_x \sigma$ be such that $\sigma'(x) = I_M(w_j)$. By lemma 7.7, $M, w, \sigma' \models \psi$. Since $\sigma' \sim_x \sigma$ and $\delta^M_\sigma(i, x) \in d^M_i(0)$, $M, w, \sigma \models \exists x\psi$. Since σ was arbitrary, this holds for any sequence.

Suppose that $\varphi \in \Sigma_i$. Suppose that there is a sequence, σ such that $M, i, \sigma \models \exists x\psi$. So there is a sequence, σ' such that $\sigma \sim_x \sigma'$, $\delta^M_{\sigma'}(i, x) \in d^M_i(0)$ and $M, i\sigma' \models \psi$. Since $A_i = d^M_i(0)$, there is a term or witness, t, such that $t \in A_i$. By lemma 7.14, $\psi[t/x] \in \Sigma_i$. IH entails that $M, w \not\models \psi[t/x]$. But by lemma 7.7, this contradicts that $M, w, \sigma' \models \psi$.

Case 7 (φ is $\exists f\psi$). Suppose that $\varphi \in \Gamma_i$. By lemma 7.15, there is a witness, W, such that $\psi[W/f] \in \Gamma_i$. By IH, $M, w \models \psi[W/f]$. Let $\sigma' \sim_W \sigma$ be such that $\sigma'(j, f) = I_M(j, W)$ for all $j \in W_M$. By lemma 7.7, $M, w, \sigma' \models \psi$. So there is a sequence, σ' such that $\sigma' \sim_f \sigma$ and $M, w, \sigma' \models \psi$. So $M, w, \sigma \models \exists f\psi$. Since σ was arbitrary, this holds for all sequences.

Let $\varphi \in \Sigma_i$. Let $M, w, \sigma \models \exists f\psi$. So there is a sequence, σ' such that $\sigma' \sim_f \sigma$ and $M, w, \sigma' \models \psi$. Since D_M is countable and contains witnesses for each element of D_M, by lemma 7.17, there is a closed λ-expression, $\lambda x_1, \ldots, x_n\theta$ such that $I_M(j, \lambda x_1, \ldots, x_n\theta) = \sigma^M_{\sigma'}(j, f)$, for all $j \in W_M$. By lemma 7.7, $M, w, \sigma' \models \psi[\lambda x_1, \ldots, x_n\theta/f]$. However, by lemma 7.15, $\psi[\lambda x_1, \ldots, x_n\theta/f] \in \Sigma_i$. But applying IH contradicts the fact that $M, w, \sigma' \models \psi[\lambda x_1, \ldots, x_n\theta/f]$.

□

Theorem 7.6.2. *M is a counter-example to G.*

Proof. Lemma 7.18 establishes that M is a counter-example to $E(\beta)$. For any position, $(C_i : \Delta_i \Rightarrow \Lambda_i : D_i) \in G$, there is a position, $(A_i : \Gamma_i \Rightarrow \Sigma_i : B_i) \in E(\beta)$, such that $C_i \subseteq A_i$, $\Delta_i \subseteq \Gamma_i$, $\Lambda_i \subseteq \Sigma_i$, and $D_i \subseteq B_i$. Lemma 7.18 shows that for any position, $(A_i : \Gamma_i \Rightarrow \Sigma_i : B_i) \in E(\beta)$, there is a world, $i \in W_M$ that is a counter-example to $(A_i : \Gamma_i \Rightarrow \Sigma_i : B_i)$. It follows that for position, $(C_i : \Delta_i \Rightarrow \Lambda_i : D_i)$, there is a world, $i \in W_M$, such that i is a counter-example to $(C_i : \Delta_i \Rightarrow \Lambda_i : D_i)$. Thus, M is a counter-example to G. □

It follows from theorem 7.6.2 that for any unprovable hyper-position, there is a model that counter-examples it.

Theorem 7.6.3. *If there is a deduction of G, then there is a deduction of G without use of the cut-rule.*

Proof. Theorem 7.6.2 did not depend on the cut rule. Thus, if a hyper-position is not provable using the cut rule, there is a model, M, such that $M \not\models G$. Let there be a deduction of G using Cut, but no deduction of G without Cut. Since there is no deduction of G without Cut, there is a model, M, such that $M \not\models G$. But since G is provable with Cut, by theorem 7.5.1 there is no model, M, such that $M \not\models G$. Since there is a contradiction, if there is a derivation of a hpyersequent that makes use of Cut, there is a derivation that does not. □

Theorem 7.6.4. *If there is a deduction of G, then there is a deduction of G without the use of the Cut_t rule.*

Proof. As with theorem 7.6.3 this follows from the fact that the proof of theorem 7.6.2 did not rely on Cut_t although that rule is sound. $\square$

7.7 Discussion

Prior [43] and Williamson [74] propose problems for a logic that can capture contingentism. Prior argues that if the existence of a being named by 'a' is expressed by the sentence '$\exists f fa$' then necessitation is not valid and the modal operators $\Box$ and $\Diamond$ are not inter-definable. Williamson argues that a contingentist logic requires free first-order quantification and therefore also requires free second-order quantification. However, he claims, if a logic with free second-order quantifiers is not strong enough to prove important instances of the modal comprehension schemas. Section 7.2 showed that PHIL addresses both of these concerns. Prior's arguments are shown to be fail because of his assumption that all judgments can be converted into acts of predication. Williamson's arguments are addressed directly. The second-order quantifiers of PHIL are free but the free and classical accounts of second-order quantification collapse because PHIL is bivalent.

In order for a calculus to be adequate it must meet two standards. It must uniquely characterize the expressions for which it has operational rules and it must be cut admissible. Uniqueness guarantees that the meaning of those expressions is fully specified, i.e. introducing another expression by the same rules will always result in the one being substitutable for the other while preserving validity. The cut-admissibility of PHIL entails that it has some other useful properties. For instance,

each of the fragments of the language of PHIL conservatively extends the rest of the language. A version of the generalized sub-formula property is also true of PHIL. From this it follows that proof-search in PHIL is somewhat constrained. These properties suggest that independently of the solutions to the above difficulties that PHIL offers, it is well-motivated and natural.

7.7.1 Existence

This chapter assumes that the existence of a being named by 'a' is captured by the sentence '$\exists f f a$'. This leaves open the question whether the sentence '$\exists f f a$' is, according to PHIL, an adequate account of existence. As discussed in section 7.1 the reading of $A : \Gamma \Rightarrow \Sigma : B$ is that one takes that position up by taking all of A to denote, asserting all of Γ, denying all of Σ, and taking all of B to be non-denoting. If it is coherent to assert that a being named by 'a' exists then it is coherent to take the term 'a' to denote. Conversely, if it is coherent to deny that a being named by 'a' exists then it is coherent to take 'a' not to denote. These claims can be interpreted, in another mode of speaking, as the sentence 'a exists' is true iff a exists where $A : \Gamma \Rightarrow \Sigma : B$ is read as saying that all of A exists, all of Γ is true, all of Σ is false, and all of B fails to exist. Theorem 7.5.1 and theorem 7.6.2 suggest that such a reading of positions may be appropriate for PHIL. Let $\mathscr{L}'$ be $\mathscr{L}$ expanded by a unary predicate $E!$, the existence predicate. The above remarks suggest that the rules governing $E!$ in PHIL are the following[12]

[12] As in section 7.1 only position as opposed to hyper-positions are considered in the discussion of the existence predicate. Any remarks made here can also be made about the full logic of hyper-positions as a result of theorem 7.6.4 and theorem 7.6.3.

$$\mathrm{L}E!\,\dfrac{A, t : \Gamma \Rightarrow \Sigma : B}{A : \Gamma, E!t \Rightarrow \Sigma : B} \qquad\qquad \mathrm{R}E!\,\dfrac{A : \Gamma \Rightarrow \Sigma : B, t}{A : \Gamma \Rightarrow E!t, \Sigma : B}$$

The question of whether the sentence '$\exists ffa$' adequately expresses that the being denoted by 'a' exists can now be reduced to the question of whether or not $\exists ffa$ is everywhere substitutable for $E!a$. The following two deductions show that this is the case

$$\mathrm{L}\exists_2\,\dfrac{\mathrm{AL}\,\dfrac{A, t : \Gamma \Rightarrow \Sigma : B}{A : \Gamma, Ft \Rightarrow \Sigma : B}}{A : \Gamma, \exists fft \Rightarrow \Sigma : B} \qquad\qquad \mathrm{R}\exists_2\,\dfrac{\mathrm{R}\lambda\,\dfrac{\mathrm{R}\vee\,\dfrac{\mathrm{R}\neg\,\dfrac{\mathrm{Id(s)}\,\dfrac{}{A : \Gamma, Ft \Rightarrow Ft, \Sigma : B}}{A : \Gamma \Rightarrow Ft, \neg Ft, \Sigma : B}}{A : \Gamma \Rightarrow Ft \vee \neg Ft, \Sigma : B} \qquad A : \Gamma \Rightarrow \Sigma : B, t}{A : \Gamma \Rightarrow \langle \lambda x Fx \vee \neg Fx \rangle t, \Sigma : B}}{A : \Gamma \Rightarrow \exists fft, \Sigma : B}$$

The open branch of each deduction is a valid place to apply R$E!$ or L$E!$. All the other branches of the deductions are closed. Therefore, in any deduction where an application of R$E!$ or L$E!$ occurs the right or left deduction respectively can replace the rule governing the existence predicate. The sentence '$\exists ffa$' adequately expresses the existence of a being denoted by 'a' in PHIL.

7.7.2 Truth and Being

This chapter has been primarily concerned with the logic of assertions, denials, judgments and acts of predication in a higher-order modal language. Alternatively, it could be interpreted as being about the coherence relations between existing beings, true sentences, false sentences, and non-existent beings. This is only an account of what combinations of such things are logically possible. This focus can address some issues for contingentism, in particular those raised by Prior [43] and Williamson [74]

The logical investigation leaves unanswered questions about the nature of truth and being with respect to modal discourse. Suppose that $A : \Gamma \Rightarrow \Sigma : B$ is a coherent position and, in fact, the true position. An investigation of logic alone does not answer the question of what makes it the case that $A : \Gamma \Rightarrow \Sigma : B$ has this privileged status amongst other coherent positions. This chapter leaves out the relation between the coherence of a position and a true position. This, in turn, leaves open the question of in virtue of what a sentence of the form $\Diamond(\exists f f a) \wedge \neg\exists f f a$ is true.

The modest aim of this chapter was to introduce PHIL and show that it could overcome several difficult problems facing contingentism and was well-behaved proof-theoretically. These features of PHIL suggest that it is a candidate for the right logic in which discourse about possibilities is to be conducted.

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