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Spectral Properties of the Hata Tree

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Spectral Properties of the Hata Tree

Antoni Brzoska, Ph.D.

University of Connecticut, 2017

ABSTRACT

The Hata tree is the unique self-similar set in the complex plane determined by the contractions $\phi_0(z) = c\bar{z}$ and $\phi_1(z) = (1-|c|^2)\bar{z}+|c|^2$, where c is a complex number such that $|c|, |1-c| \in (0,1)$. There are four main results in the paper. First, by applying linear algebra and spectral theory it is possible to construct a dynamical system that can compute the eigenvalues of the probabilistic Laplacian on graph approximations to the Hata tree. Conclusions are made about the spectrum of the Laplacian on the limiting graphs. Second, the Sabot theory (c.f. [29]) is applied to construct a simpler dynamical system to compute the eigenvalues of a class of normalized graph Laplacians (including the probabilistic Laplacian) on these approximating graphs. Third, it is possible to reconstruct the Hata tree as the union of two copies of a mixed affine nested fractal identified at a point. Using techniques from [13], some results are stated on the spectral asymptotics of the eigenvalue counting function of a certain class of Laplacians (not including the probabilistic Laplacian) on this mixed affine nested fractal. In the final part, a spectral analysis is performed on graph approximations to the Basilica Julia set of the polynomial $z^2 - 1$. In [5], the authors give a dynamical system that can be used to construct finite approximations and classify the different possible infinite blow-ups. In this paper, the techniques from the

first part are used to construct a dynamical system that can compute the eigenvalues of Laplacian operators on these finite graph approximations. In addition, it is shown that the spectrum of the Laplacian on blow-ups satisfying certain conditions is pure point.

Spectral Properties of the Hata Tree

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B.S. Mathematics, University of Connecticut, 2012

A Dissertation

Submitted in Partial Fulfillment of the

Requirements for the Degree of

Doctor of Philosophy

at the

University of Connecticut

2017

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2017

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Doctor of Philosophy Dissertation

Spectral Properties of the Hata Tree

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2017

ACKNOWLEDGMENTS

First, I want to thank my advisors Luke Rogers and Alexander Teplyaev for all their help, time and advice. I could not have finished graduate school and completed my thesis without it.

Second, I want to thank the UConn math department as a whole. I have come to know many people during my nine years at UConn. I am grateful for everything, big and small. I will always remember by time here at UConn with fondness.

Next, I want to thank my parents and my siblings for all of their love and support. They have been there for me in both the good times and the bad times.

Finally, I want to thank God. I dreamed of going to college to study math and to earn a doctorate when I was ten years old, and now that dream has become reality.

Contents

Chapter 1

Introduction

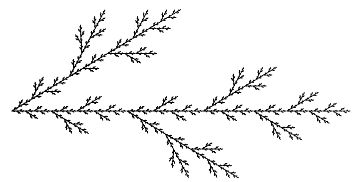
By the work of Kigami (c.f. [19]), one can construct Laplacian operators on a certain class of self similar sets known as post critically finite (p.c.f.) self similar sets. There is a proper definition for a post critical set, described by Hata in [14], but roughly it can be thought of as the set of boundary points. A self similar set is p.c.f. if its post critical set is finite. The operator itself is constructed by taking graph Laplacians on a sequence of appropriate graph approximations to the fractal and then carefully defining the limiting operator. This construction was first outlined by Kigami in [17]. Other well written introductions to these ideas can also be found in [33, 34].

In the 1980's, the physicists Rammal and Toulouse noticed certain relations between the eigenvalues of Laplacians on graph approximations to the Sierpinski gasket [26, 27]. In 1992, Fukushima and Shima computed the spectrum of the Laplacian on the limiting set [10]. The work relies upon formally establishing spectral similarity relations between Laplacians on successive graph approximations. Through these relations one obtains a quadratic polynomial that establishes a direct relationship

between the eigenvalues of the Laplacian operators. In general, this technique of relating eigenvalues of graph Laplacians on successive graph approximations is known as spectral decimation. In 1993, Shima gave a list of conditions on p.c.f. fractals that would guarantee that spectral decimation works [32]. In [35], Malozemov and Teplyaev show that this technique can be used on self-similar lattices known as M-point model graphs.

Unfortunately, the technique of spectral decimation only works for a limited class of self similar sets. In [29], Sabot generalized the technique of spectral decimation to a certain class of self-similar lattices. In particular, for this class of lattices, the eigenvalues can be related via a dynamical system that is not necessarily one dimensional. Sabot also makes some conclusions about the limiting distribution of the eigenvalues. In [16], Jordan applied the theory of Sabot to construct a three dimensonal dynamical system to compute the spectrum of the Laplacian on a fractal known as the pentagasket. In [15], Jordan directly constructs a one dimensional dynamical system to compute the eigenvalues of Laplacian operators on a certain sequence of self-similar lattices for which spectral decimation does not work. At the present moment, there do not exist many other examples of self-similar sets for which spectral decimation does not work but for which some sort of dynamical system to compute the eigenvalues of the Laplacian has been constructed.

One such self-similar set is the Hata tree. Let $\phi_0(z) = c\bar{z}$ and $\phi_1(z) = (1 - |c|^2)\bar{z} + |c|^2$ be defined on the complex plane, where c is a complex number such that $|c|, |1-c| \in (0,1)$. Let $C(\mathbb{C}) = \{X : X \subset \mathbb{C}, X \neq \varnothing, X \text{ compact}\}$. Define the function $\Phi(X)$ on $C(\mathbb{C})$, where $\Phi(X) = \cup_{i=0,1}\phi_i(X)$. The Hata tree is defined to be the unique fixed point of Φ. I.e., it is the unique set K satisfying $K = \Phi(K) = \phi_0(K) \cup \phi_1(K)$.



Hata tree

By the theory in [19], it is known that the Hata tree is a p.c.f self-similar fractal. The post critical set is $\{0, 1, c\}$ and in a sense is the boundary of the fractal.

Define Φ_n by

$$\Phi_n(X) = \underbrace{\Phi \circ \cdots \circ \Phi}_{n \text{ times}}(X).$$

Let $K_0 = \{t : 0 \leq t \leq 1\} \cup \{tc : 0 \leq t \leq 1\}$. Since Φ is a contractive map, the sets $K_n = \Phi_n(K_0)$ will converge to K in the Hausdorff metric. One can work with alternate approximations to the Hata tree. Let $V_0 = \{0, 1, c\}$ and $V_n = \Phi_n(V_0)$. There exists a natural graph structure on these sets of points. One can build a Laplacian on K by constructing graph Laplacians on the V_n and taking limits in the appropriate sense. In this paper, we will be primarily interested in the spectrum of this Laplacian operator, which will be found by analyzing the spectrum of the approximating graph Laplacians.

In chapter 2, we work with a sequence of graphs V_{-n} isomorphic to V_n known as the blow-ups of the Hata tree. Roughly, these graphs are formed by "gluing" together copies of the previous level in a manner determined by ϕ_0 and ϕ_1. In particular, we work with the blow-ups because it is more convenient to express the eigenfunctions on these graphs. By applying linear algebra and graph theory, it is possible to find a recursive system of polynomials whose roots give the eigenvalues of the probabilistic graph Laplacian at any given level. In particular, we can define polynomials recur-

sively (c.f. Proposition 2.3.3) and show that the the roots of

$$(1-\lambda)g^{(n)}g^{(n-1)} - \frac{1}{6}g_u^{(n)}g^{(n-1)} - \frac{1}{6}g^{(n)}g_u^{(n-1)}$$

are the eigenvalues of the level n probabilistic Laplacian. These eigenvalues are Neumann eigenvalues because no conditions are being imposed on the boundary. If one imposes a zero boundary condition on the three boundary points and restricts the Laplacian accordingly, then the corresponding (Dirichlet) eigenvalues are the roots of $g_w^{(n)}g_w^{(n-1)}$. It is possible to define the Neumann and Dirichlet Laplacian operators on the limiting infinite lattices. In this paper, it is proven that the spectrum of the limiting Neumann operator is contained in the closure of the union of the set of level 3 eigenvalues with the set

$$\bigcup_{k=4}^{\infty}(h^{(k)})^{-1}\{0\},$$

where

$$h^{(k)} = \frac{1}{3}g^{(n-1)}g^{(n-2)} + \frac{2}{3}[(1-\lambda)g^{(n-1)}g_w^{(n-2)} - \frac{1}{6}g^{(n-1)}g_{uw}^{(n-2)} - \frac{1}{6}g_u^{(n-1)}g_w^{(n-2)}].$$

Similarly, one can construct a Dirichlet operator on the infinite blow-up. However, analyzing the spectrum of this operator is more difficult than the Neumann case and is left as an open question.

At the end of the chapter, the theory of Dirichlet forms is used to define the probabilistic Neumann and Dirichlet Laplacian operators on the Hata tree itself. Note that Kigami's theory in [19] cannot be used directly, as he considers graph Laplacians as opposed to probabilistic Laplacians. It is then proven that the rescaled limit of the spectrum of the approximating Dirichlet operators is the spectrum of the Dirichlet

Laplacian on the Hata tree. Here the scaling factor is $F_{n+1}/(2^{n+1}+1)$, where F_{n+1} is the $n+1$st Fibonacci number. The proof depends on the uniform convergence of the approximating Green's functions.

In chapter 3, we define a class of normalized Laplacian operators on the blow-ups. In fact, with the appropriate choice of parameters we recover the probabilistic Laplacian. The theory in [29] is then adapted to our situation. In particular, we construct a dynamical system of two recursive sequences to compute the eigenvalues of these Laplacians. In particular, we specify polynomials $b_0, c_{-1}, c_0, e_{-1}, e_0$ and for $n \geq 0$ define

$$\begin{aligned} c_{n+1} &= c_n - \frac{c_{n-1}e_n^2}{c_{n-1}^2 + c_{n-1}\left(c_n + (b_0 - c_0) - \sum_{j=0}^{n-1} e_{j+1}\frac{e_j^2 - e_{j-1}^2}{e_{j-1}e_j}\right) - e_{n-1}^2}, \\ e_{n+1} &= -\frac{c_{n-1}e_{n-1}e_n}{c_{n-1}^2 + c_{n-1}\left(c_n + (b_0 - c_0) - \sum_{j=0}^{n-1} e_{j+1}\frac{e_j^2 - e_{j-1}^2}{e_{j-1}e_j}\right) - e_{n-1}^2}. \end{aligned}$$

Now define

$$D_n = c_{n-1}\left(c_n + (b_0 - c_0) - \sum_{j=0}^{n-1} e_{j+1}\frac{e_j^2 - e_{j-1}^2}{e_{j-1}e_j}\right)c_n - c_{n-1}e_n^2 - c_n e_{n-2}^2.$$

D_n will be a rational function, and for $n \geq 2$ the zeros of the numerator and denominator correspond to the Neumann and Dirichlet eigenvalues of the normalized Laplacian operator, respectively. It is also possible to apply the Sabot theory to make conclusions about the limiting distribution of eigenvalues, depending on the normalized Laplacian operator in question.

In [20], the authors give some results on the spectral asymptotics of the eigenvalue counting function of p.c.f. self-similar fractals, including the Hata tree. In chapter

4, we provide an alternate version of these results. The Laplacians considered here are the same Laplacians considered in [20], which differ slightly from the probabilistic/normalized Laplacians considered in the previous chapters. Recall that to obtain a graph approximation to the Hata tree, one can take two copies of the graph approximations on the previous level and "glue" them together at the appropriate point. However, it is possible to obtain the same approximation by looking at the previous approximation and appending edges in the appropriate places. By defining a notion of mixed affine nested fractals, it is possible to decompose the Hata tree as the union of two copies of a mixed affine nested fractal identified at the appropriate point. By adapting techniques from [13], it is possible to make some conclusions about the spectral asymptotics of this mixed affine nested fractal.

In chapter 5, we prove some miscellaneous results. In particular, by altering the construction of the mixed affine nested fractals considered in chapter 4, we obtain graph approximations with enough symmetry that the theory in [23] applies. For these graph approximations, the Sabot theory [29] also applies and for one specific case the theory is worked out. A spectral analysis is performed on a family of Cayley graph-like fractals. Finally, an example is given where by adding an extra contraction to the iterated function system of the Hata tree, it is possible to perform a tiling of the complex plane.

The results in chapter 6 are independent of the results of the previous chapters, and more background is given in the introduction of that chapter. In [5], the authors define a sequence of graph approximations $\{\Gamma_n\}_{n=0}^{\infty}$ to the Basilica Julia set of $z^2 - 1$. In the chapter, we consider a related sequence of graphs $\{G_n\}_{n=0}^{\infty}$. Essentially there are three main results. The methods of chapter 2 are used to construct a dynamical system (see Theorem 6.2.3) to compute the eigenvalues of Laplacian operators on

these graphs. It is shown in Theorem 6.4 that there is a gap in the limiting distribution of eigenvalues. Finally, certain infinite blow-ups satisfying the conditions in Assumption 1 are analyzed. In Theorem 6.5.4, it is shown that the spectrum of Laplacian operators is pure point and that the corresponding eigenfunctions are finitely supported.

Chapter 2

Computation of Eigenvalues

2.1 Graph Laplacians

Let $G = (V, E)$ denote a graph, where V is the set of vertices and E is the set of edges. An edge is a two-element subset of V. If two vertices x and y share an edge, we denote this by $x \sim y$. We say that a graph G is connected if for any pairs of vertices x and x' there exists a sequence of vertices $x = x_0, x_1, .., x_{n+1} = x'$ such that $x_i \sim x_{i+1}$ for $i = 0, .., n$. In what follows, all graphs are connected.

Suppose G has n vertices, indexed by $i = 1, ..., n$. The adjacency matrix of the graph, A, is defined as the matrix whose (i, j) entry is 1 if vertices i and j are connected by an edge and 0 otherwise. The degree matrix of G is defined as the diagonal matrix whose (i, i) entry equals the degree of vertex i. Let us denote the degree of a vertex x by d_x.

There exist various versions of graph Laplacians. The standard Laplacian is defined to be $L = D - A$. The normalized Laplacian is defined to be $N = D^{-1/2} L D^{-1/2}$.

The probabilistic Laplacian is defined to be $P = D^{-1}L$. It is a fact that N and P are unitarily equivalent via conjugation by $D^{1/2}$, $P = D^{-1/2}ND^{1/2}$, hence have the same eigenvalues.

In chapters 2 and 3, we will analyze the eigenvalues and eigenvectors of the probabilistic Laplacian on approximating graphs to the Hata tree. Given the previous fact and the fact that the normalized Laplacian is symmetric, it will convenient to work with the normalized Laplacian in some situations.

Given a normalized Laplacian N, we define the characteristic polynomial of N, $D(N)$, to be the determinant of the matrix $N - \lambda I$, where I is the identity matrix. The characteristic polynomial is a function of λ. By definition, the roots of $D(N)$ are the eigenvalues of N. Given a vertex i, we define N_i to be the matrix N with the row and column corresponding to i deleted. Let A be a subset of the set of indices corresponding to vertices of G. Then let N_A denote the matrix N with the rows and columns corresponding to the indices in A deleted. The following is a basic result on the eigenvalues of N.

Proposition 2.1.1. *Let N be the Laplacian on a finite graph G. Let $\sigma(N)$ denote the set of eigenvalues of N. Then $\sigma(N) \subset [0, 2]$.*

Proof. Observe that the order of the characteristic polynomial of N must equal the number of vertices of G. Let n denote the number of vertices of G. Let us enumerate the eigenvalues of N: $\lambda_1, \lambda_2, ..., \lambda_n$.

We will need the following fact which holds for any function f:

$$(f(x) - f(y))^2 \leq 2(f^2(x) + f^2(y)).$$

Then by the minimax principle and the previous fact

$$\lambda_n = \sup_f \frac{\sum_{x \sim y}(f(x) - f(y))^2}{\sum_x f^2(x) d_x} \leq 2.$$

Here, the supremum is over all functions f on the vertices of G. Observe that the previous inequality is in fact an equality if $f(x) = -f(y)$ for every edge $\{x, y\}$ in G.

By the minimax principle,

$$\lambda_1 = \inf_f \frac{\sum_{x \sim y}(f(x) - f(y))^2}{\sum_x f^2(x) d_x}.$$

It is clear that this infimum must be 0. This infimum is attained at any constant function. Again, since we are only working with connected graphs, the multiplicity of 0 as an eigenvalue is 1. Therefore, $\lambda_j > 0$ for $j > 0$. □

By definition, a graph G is bipartite if the vertex set V can be written as a disjoint union of two subsets A and B such that all edges of G connect one vertex of A with one vertex of B. Let us call the sets A and B bipartite components.

Proposition 2.1.2. *Let G be a graph. Let N be the Laplacian on G. Then G is bipartite if and only if for every λ that is an eigenvalue of N, the value $2 - \lambda$ is also an eigenvalue of G.*

Proof. Suppose G is a bipartite graph with bipartite components A and B. For any eigenfunction f with eigenvalue λ, consider the function g defined by

$$g(x) = \begin{cases} f(x) & : x \in A \\ -f(x) & : x \in B \end{cases}$$

This function is clearly an eigenfunction with eigenvalue $2-\lambda$. For the converse, observe that since 0 must be an eigenvalue, the value 2 must be as well. Let f be a non-zero eigenfunction corresponding to the eigenvalue 2. By an observation in the proof of the previous lemma, $f(x) = -f(y)$ for every edge $\{x, y\}$ in G. This can happen only if the graph G is bipartite. □

The following is a useful result in computing the characteristic polynomials of graph Laplacians.

Proposition 2.1.3. *Let G be a tree-like graph with finitely many vertices. Let N be the Laplacian on G. Fix a vertex j in G and let $j_1, j_2, ..., j_k$ be its neighboring vertices. Then*

$$D(N) = (1-\lambda)D(N_j) - \sum_{n=1}^{k} \frac{1}{d_j d_{j_n}} D(N_{j,j_n}).$$

Proof. Let us first look at the case of a vertex j with two neighbors j_1 and j_2. In this case the characteristic polynomial $D(N)$ can be written as

$$\begin{vmatrix} 1-\lambda & a & b \\ a^T & A-\lambda I & 0 \\ b^T & 0 & B-\lambda I \end{vmatrix}.$$

The removal of the vertex j partitions G into two connected components. Let A and B be the pieces of N corresponding to the connected components containing j_1 and j_2, respectively. The pieces a and b correspond to the connections between j and j_1 and j_2. In particular, a and b are matrices with a single row whose only non-zero entries are $-\sqrt{d_j d_{j_1}}^{-1}$ and $-\sqrt{d_j d_{j_2}}^{-1}$, respectively.

By linearity, we can rewrite the determinant as

$$\begin{vmatrix} 1-\lambda & 0 & 0 \\ a^T & A-\lambda I & 0 \\ b^T & 0 & B-\lambda I \end{vmatrix} + \begin{vmatrix} 0 & a & 0 \\ a^T & A-\lambda I & 0 \\ b^T & 0 & B-\lambda I \end{vmatrix} + \begin{vmatrix} 0 & 0 & b \\ a^T & A-\lambda I & 0 \\ b^T & 0 & B-\lambda I \end{vmatrix}.$$

The first determinant is a three by three triangular block matrix, with the upper block matrices begin zero. Its determinant is $(1-\lambda)D(A)D(B)$, which by definition is $(1-\lambda)D(N_j)$. The second determinant is a two by two triangular block matrix, with 0, a, a^T and A constituting one block. Its determinant will be the product of the determinants of the diagonal blocks. Taking the upper left block and performing a determinant expansion first along the row and then the column corresponding to j, we obtain $-\frac{1}{d_j d_{j_1}}D(A_{j_1})$. The determinant of the lower right block is $D(B)$. By definition, $D(N_{j,j_1}) = \frac{1}{d_j d_{j_1}}D(A_{j_1})D(B)$. The third matrix can be handled in a similar manner as the second after rewriting it as a triangular block matrix.

The case where j has an arbitrary finite number of neighboring vertices can be handled in the same way. This result is a simplified version of Theorem 2 in [31], which generalizes to the case of graphs with loops. $\square$

In a graph G, a vertex is called ultimate if it is of degree 1. A vertex is called penultimate if one of the vertices with whom it shares an edge is ultimate. The following result is commonly known as the Edge Principle.

Lemma 2.1.4. *Let G be a graph containing an ultimate vertex x. Let y be the corresponding penultimate vertex. Suppose f is an eigenfunction of N with the corresponding eigenvalue $\lambda \neq 1$. Then $f(x) = 0$ if and only if $f(y) = 0$.*

Proof. Since f is an eigenfunction, we have $(1-\lambda)f(x) = d_y^{-1/2} f(y)$. The result now follows. $\square$

Let G be a graph with n vertices. Suppose G is a tree, that is for any pair of vertices there is a unique sequence of edges that connect the two vertices. Let M be an $n \times n$ matrix whose rows and columns are associated with the vertices of G. In the literature, M is called a tree pattern matrix if for every pairs of vertices i and j that share an edge, the (i,j) entry of M is non-zero. The following theorem pertains to tree pattern matrices [6, 30].

Theorem 2.1.5. *Let M be a tree pattern matrix. Let f be an eigenvector of M. If every entry of f is non-zero, then the corresponding eigenvalue is of multiplicity one.*

For a tree pattern matrix M with eigenvalue λ, let $\mathrm{supp}(M,\lambda)$ denote the support of the eigenspace in G. The following theorem is likewise found in [30].

Theorem 2.1.6. *Let M be a tree pattern matrix with eigenvalue λ. Then the dimension of the eigenspace of M for eigenvalue λ equals the number of connected components of the subgraph of G induced by supp(M,λ) minus the number of vertices of G that are adjacent to supp(M,λ) but do not belong to this set.*

2.2 Approximating Graphs of the Hata Tree

Let V_0 denote the post-critical set $\{0,1,c\}$. The precise defintion of a post-critical set is omitted here, but can be found in [19]. In a sense, one can view V_0 as the boundary of the Hata tree. Let Φ be the function on compact sets in $\mathbb{C}$ defined by $\Phi(X) = \phi_1(X) \cup \phi_2(X)$. Recall that the Hata tree is the unique fixed point of Φ and

is denoted by K. We define the nth lattice of approximating points V_n by

$$V_n = \underbrace{\Phi \circ \cdots \circ \Phi}_{n \text{ times}}(V_0).$$

For $i_1...i_n \in W_n := \{0,1\}^n$, let us denote by $\phi_{i_1...i_n}$ the map $\phi_{i_1} \circ \cdots \circ \phi_{i_n}$. We can impose a graph structure on these approximating lattices as follows. On V_0, let $0 \sim 1$ and $0 \sim c$. If $x, y \in V_n$, then let $x \sim y$ if:

(i). there exists a $i_1...i_n \in W_n$ such that $x, y \in \phi_{i_1...i_n}(V_0)$, and

(ii). $\phi^{-1}_{i_1...i_n}(x) \sim \phi^{-1}_{i_1...i_n}(y)$ in V_0

Let $V_\infty = \cup_{n=0}^\infty V_n$. The closure of V_∞ in the complex plane is K. By the theory in [19], it is possible to construct a Laplacian operator on K by defining graph Laplacians on the approximating lattices V_n and taking limits in the appropriate sense.

We will define another sequence of graphs related to the Hata tree. Fix an infinite word $\omega = \omega_1\omega_2\omega_3 \cdots \in W_\infty := \{0,1\}^{\mathbb{N}}$. Define the lattice of points V_{-n} by

$$V_{-n} = \phi^{-1}_{\omega_1...\omega_n}(V_n) = \phi^{-1}_{\omega_n} \circ \cdots \phi^{-1}_{\omega_1}(V_n).$$

We define the blow-up $V_{-\infty}$ of the Hata tree by

$$V_{-\infty} = \cup_{n=0}^\infty V_{-n}.$$

The map $\phi_{\omega_1...\omega_n}$ gives us a one to one correspondence between V_n and V_{-n}. The map induces a natural graph structure on V_{-n}. In particular, for $x, y \in V_{-n}$ we say that $x \sim y$ if there exists $i_1....i_n \in W_n$ such that $x, y \in \phi^{-1}_{\omega_1...\omega_n} \circ \phi_{i_1...i_n}(V_0)$ and their preimages in V_0 are connected by an edge. Thus, there is a natural graph isomorphism

between V_{-n} and V_n. We denote by $(V_{-n})_{i_1\dots i_n}$ the cell in V_{-n} corresponding to $\phi^{-1}_{\omega_1\dots\omega_n} \circ \phi_{i_1\dots i_n}(V_0)$. Finally, we define the boundary ∂V_{-n} of V_{-n} by identifying the vertices corresponding to $0, 1, c$ under the isomorphism.

$V_{-\infty}$ is called the blow-up because of the way the sets V_{-n} fit into one another. In particular, we have $V_{-1} \subset V_{-2} \subset \cdots$. It is clear that for any choice of ω the graphs V_{-n} will be isomorphic. However, this is not necessarily the case for the infinite blow-up. For instance, $V_{-\infty}$ will have at most one boundary point.

Due to the graph isomorphism between V_{-n} and V_n, the corresponding graph Laplacians have the same matrix representations, and thus the same spectrum. Without loss of generality, in the next three sections we work with the graphs induced by $\omega = 000....$ The benefit of working with this blow-up is the ease in which one can express the eigenfunctions on V_{-n}.

2.3 Neumann Eigenvalues of Graph Laplacians

We will work with the graphs V_{-n} determined by $\omega = 000....$ Let us label the boundary points of V_{-n} corresponding to $0, c$ and 1 by z_n, c_n and w_n, respectively. The graph $V_{-(n+1)}$ can be obtained from V_{-n} as follows: Let V'_{-n} be a copy of V_{-n}, and identify the vertex c_n with the vertex z'_n of V'_{-n}. V_{-n} can naturally be embedded into $V_{-(n+1)}$ and V'_{-n} can be identified with $V_{-(n+1)} \backslash V_{-n}$. The boundary of $V_{-(n+1)}$ is given by $z_{n+1} = z_n$, $c_{n+1} = w_n$ and $w_{n+1} = w'_n$. Figure 2.3.1 gives a visual depiction of these graphs.

Let $N^{(n)}$ and $P^{(n)}$ denote the normalized and probabilistic Laplacian on V_{-n}. We call the spectrum of these operators the Neumann eigenvalues, since no condition has

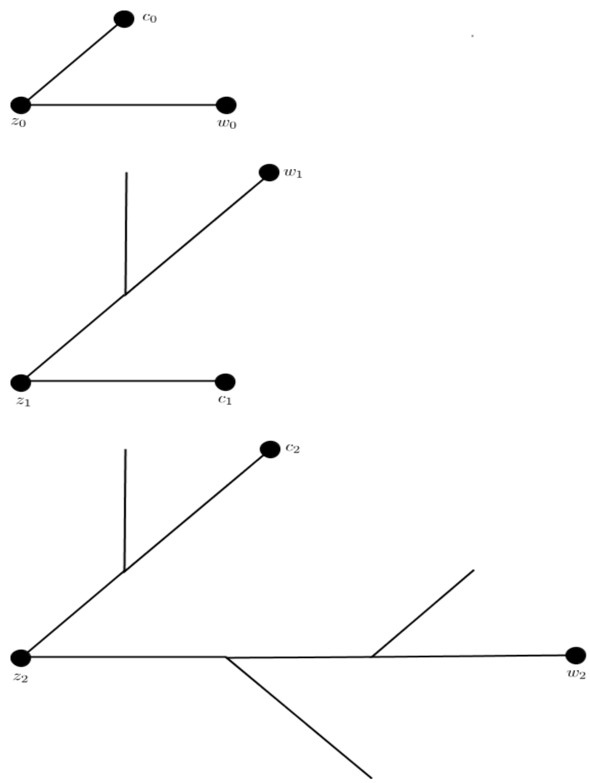


FIGURE 2.3.1: V_0, V_{-1} and V_{-2} for $\omega = 000...$

been imposed on the boundary points. The following is immediate from the results of section 2.1.

Proposition 2.3.1. $\sigma(N^{(n)}) \subset [0, 2]$. *In addition, the eigenvalues of* $N^{(n)}$ *are symmetric about 1.*

Proof. By Proposition 2.1.1, $\sigma(N^{(n)}) \subset [0, 2]$. To prove that the eigenvalues of $N^{(n)}$ are symmetric about 1, by Proposition 2.1.2 it suffices to prove that the graphs V_{-n} are bipartite.

The graph V_0 is clearly bipartite. We will prove that V_{-n} is bipartite by induction on n. Suppose V_{-n} is bipartite. The union of the bipartite component of V_{-n} containing c_n and that of the bipartite component of V'_{-n} containing z'_n is a bipartite component in $V_{-(n+1)}$. Naturally, its complement in $V_{-(n+1)}$ forms the other bipartite component. □

Here is another basic fact about the spectrum.

Proposition 2.3.2. *Let* λ *be an eigenvalue of* $N^{(n)}$. *Then* λ *is an eigenvalue of* $N^{(n+1)}$.

Proof. Without loss of generality let us work with the probabilistic Laplacian $P^{(n)}$. Let f be an eigenfunction with eigenvalue λ. Suppose that $f(z_n) \neq 0$. Define $k_n := f(c_n)/f(z_n)$. Let ι be the canonical isomorphism from V'_{-n} to V_{-n}. We extend f to $V_{-(n+1)}$ by setting $f(x) = k_n f(\iota(x))$ for $x \in V'_{-n}$. This extension is well defined at the "gluing" point $c_n = z'_n$ since $f(z'_n) = k_n f(z_n) = f(c_n)$.

For any $x \in V_{-(n+1)}$ such that $x \neq c_n$, it is clear that $(P^{(n+1)} - \lambda I)f(x) = 0$. It remains to verify that $(P^{(n+1)} - \lambda I)f(z'_n) = 0$. Recall that z'_n has three neighboring vertices. Let us label the two neighboring vertices in V'_{-n}: u'_n and u'_{n-1}, and the one

in V_{-n}: v_{n-1}. This labelling will agree with notation that will be used (and explained) later. Thus

$$\begin{aligned}
(P^{(n+1)} - \lambda I)f(z'_n) &= \frac{1}{3}\Big(f(u'_n) + f(u'_{n-1}) + f(v_{n-1})\Big) \\
&= \frac{2}{3}\Big(\frac{1}{2}f(u'_n) + \frac{1}{2}f(u'_{n-1})\Big) + \frac{1}{3}f(v_{n-1}) \\
&= \frac{2}{3}(P^{(n)} - \lambda I)f(z'_n) + \frac{1}{3}(P^{(n)} - \lambda I)f(c_n) \\
&= 0
\end{aligned}$$

and the extension of f is an eigenfunction of $P^{(n+1)}$ with eigenvalue λ.

Now suppose $f(z_n) = 0$. Then we can define a function f' on $V_{-(n+1)}$ such that $f'(\iota(x)) = f(x)$ for $x \in V'_{-n}$ and $f'(x) = 0$ for $x \in V_{-n}$. In a similar manner one can check that f' is an eigenfunction on $V_{-(n+1)}$ with eigenvalue λ. □

Let us examine the structure of the graph V_{-n} more closely for $n \geq 4$. The vertex z_n of V_{-n} is the unique vertex of degree 2, with every other vertex having either degree 1 or 3. This vertex divides the graph into two connected regions. Let us label the subgraphs of V_{-n} formed by removing z_n and the edges connected to z_n by A_n and B_n. Without loss of generality, let A_n be the subgraph containing more vertices. Note that B_n is isomorphic to the subgraph A_{n-1} of $V_{-(n-1)}$.

Furthermore, the subgraph A_n essentially consists of two copies of B_{n-1} and one copy of A_{n-1} joined together at the vertex z_{n-1}. Let us specify three other vertices of A_n: u_n, the vertex that was connected to z_n; w_n, the boundary vertex corresponding to the point 1 on the Hata tree; and v_n, the corresponding penultimate vertex. Figure 2.3.2 depicts these subgraphs and vertices in the case $n = 3$.

Let $A^{(n)}$ be the matrix $N^{(n)}$ containing only the rows and columns corresponding

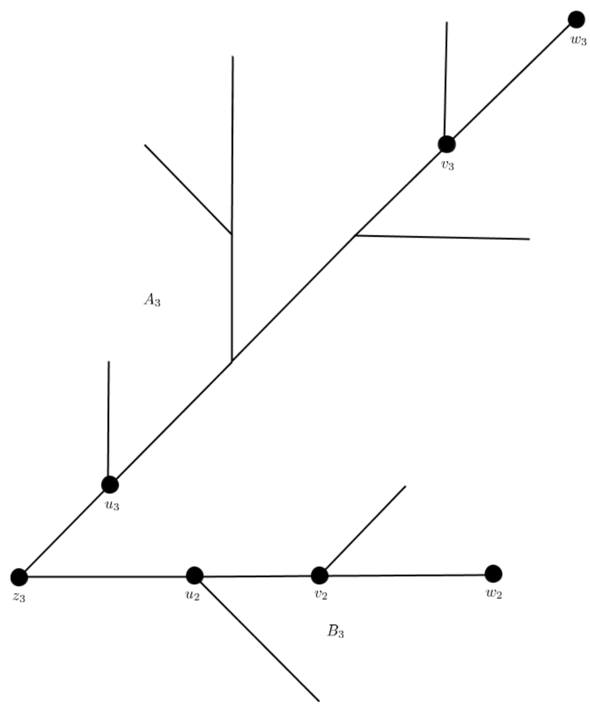


FIGURE 2.3.2: The graph V_{-3}

to the vertices in A_n. Let $g^{(n)} := D(A^{(n)})$ denote the characteristic polynomial of $A^{(n)}$. Let $A_S^{(n)}$ be the matrix $A^{(n)}$ but with the rows and columns corresponding to the vertices indexed by elements of a set S removed. Denote by $g_S^{(n)} := D(A_S^{(n)})$ the characteristic polynomial of $A_S^{(n)}$. The following polynomials for $n = 2$ can be easily computed. For brevity of notation, instead of writing u_2, we simply write u to denote the vertex u_2 of A_2. The same is done for v_2 and w_2.

$$
\begin{aligned}
g^{(2)}(\lambda) &= (1-\lambda)^5 - \frac{10}{9}(1-\lambda)^3 + \frac{2}{9}(1-\lambda) \\
g_u^{(2)}(\lambda) &= (1-\lambda)^4 - \frac{2}{3}(1-\lambda)^2 \\
g_w^{(2)}(\lambda) &= (1-\lambda)^4 - \frac{7}{9}(1-\lambda)^2 + \frac{1}{9} \\
g_{uw}^{(2)}(\lambda) &= (1-\lambda)^3 - \frac{1}{3}(1-\lambda) \\
g_{vw}^{(2)}(\lambda) &= (1-\lambda)^3 - \frac{1}{3}(1-\lambda) \\
g_{uvw}^{(2)}(\lambda) &= (1-\lambda)^2
\end{aligned}
$$

The same polynomials for $n = 3$ are computed. Again, for brevity of notation, instead of writing u_3 we simply write u. The same is done for v_3 and w_3.

$$
\begin{aligned}
g^{(3)}(\lambda) &= (1-\lambda)^{11} - \frac{22}{9}(1-\lambda)^9 + \frac{170}{81}(1-\lambda)^7 - \frac{20}{27}(1-\lambda)^5 + \frac{22}{243}(1-\lambda)^3 \\
g_u^{(3)}(\lambda) &= (1-\lambda)^{10} - 2(1-\lambda)^8 + \frac{100}{81}(1-\lambda)^6 - \frac{2}{9}(1-\lambda)^4 \\
g_w^{(3)}(\lambda) &= (1-\lambda)^{10} - \frac{19}{9}(1-\lambda)^8 + \frac{125}{81}(1-\lambda)^6 - \frac{37}{81}(1-\lambda)^4 + \frac{11}{243}(1-\lambda)^2 \\
g_{uw}^{(3)}(\lambda) &= (1-\lambda)^9 - \frac{5}{3}(1-\lambda)^7 + \frac{67}{81}(1-\lambda)^5 - \frac{1}{9}(1-\lambda)^3 \\
g_{vw}^{(3)}(\lambda) &= (1-\lambda)^9 - \frac{5}{3}(1-\lambda)^7 + \frac{23}{27}(1-\lambda)^5 - \frac{11}{81}(1-\lambda)^3 \\
g_{uvw}^{(3)}(\lambda) &= (1-\lambda)^8 - \frac{11}{9}(1-\lambda)^6 + \frac{1}{3}(1-\lambda)^4
\end{aligned}
$$

It was previously observed that the eigenvalues of the normalized and probabilistic Laplacians on V_{-n} are the same, with their eigenvectors related to one another via a transformation. In particular, the above polynomials are the same when computed using the probabilistic Laplacian as opposed to the normalized Laplacian.

One can exploit this recurrence in the construction of V_{-n}, in addition to Proposition 2.1.3, to find a recurrence in the characteristic polynomials of the Laplacian on V_{-n}.

Proposition 2.3.3. *Let $g^{(j)}, g_u^{(j)}, g_w^{(j)}, g_{uw}^{(j)}, g_{vw}^{(j)}, g_{uvw}^{(j)}$ for $j = 2, 3$ be defined as above.*

For $n \geq 4$, let

$$
\begin{aligned}
g^{(n)} &= (1-\lambda)g^{(n-1)}g^{(n-2)}g_w^{(n-2)} - \frac{1}{9}g_u^{(n-1)}g^{(n-2)}g_w^{(n-2)} - \frac{1}{9}g^{(n-1)}g_u^{(n-2)}g_w^{(n-2)} - \frac{1}{9}g^{(n-1)}g^{(n-2)}g_{vw}^{(n-2)} \\
g_u^{(n)} &= (1-\lambda)g^{(n-1)}g^{(n-2)}g_{uw}^{(n-2)} - \frac{1}{9}g_u^{(n-1)}g^{(n-2)}g_{uw}^{(n-2)} - \frac{1}{9}g^{(n-1)}g_u^{(n-2)}g_{uw}^{(n-2)} - \frac{1}{9}g^{(n-1)}g^{(n-2)}g_{uvw}^{(n-2)} \\
g_w^{(n)} &= (1-\lambda)g_w^{(n-1)}g^{(n-2)}g_w^{(n-2)} - \frac{1}{9}g_{uw}^{(n-1)}g^{(n-2)}g_w^{(n-2)} - \frac{1}{9}g_w^{(n-1)}g_u^{(n-2)}g_w^{(n-2)} - \frac{1}{9}g_w^{(n-1)}g^{(n-2)}g_{vw}^{(n-2)} \\
g_{uw}^{(n)} &= (1-\lambda)g_w^{(n-1)}g^{(n-2)}g_{uw}^{(n-2)} - \frac{1}{9}g_{uw}^{(n-1)}g^{(n-2)}g_{uw}^{(n-2)} - \frac{1}{9}g_w^{(n-1)}g_u^{(n-2)}g_{uw}^{(n-2)} - \frac{1}{9}g_w^{(n-1)}g^{(n-2)}g_{uvw}^{(n-2)} \\
g_{vw}^{(n)} &= (1-\lambda)g_{vw}^{(n-1)}g^{(n-2)}g_w^{(n-2)} - \frac{1}{9}g_{uvw}^{(n-1)}g^{(n-2)}g_w^{(n-2)} - \frac{1}{9}g_{vw}^{(n-1)}g_u^{(n-2)}g_w^{(n-2)} - \frac{1}{9}g_{vw}^{(n-1)}g^{(n-2)}g_{vw}^{(n-2)} \\
g_{uvw}^{(n)} &= (1-\lambda)g_{vw}^{(n-1)}g^{(n-2)}g_{uw}^{(n-2)} - \frac{1}{9}g_{uvw}^{(n-1)}g^{(n-2)}g_{uw}^{(n-2)} - \frac{1}{9}g_{vw}^{(n-1)}g_u^{(n-2)}g_{uw}^{(n-2)} - \frac{1}{9}g_{vw}^{(n-1)}g^{(n-2)}g_{uvw}^{(n-2)}
\end{aligned}
$$

Then $D(N^{(n)}) = (1-\lambda)g^{(n)}g^{(n-1)} - \frac{1}{6}g_u^{(n)}g^{(n-1)} - \frac{1}{6}g^{(n)}g_u^{(n-1)}$.

Proof. That $D(N^{(n)}) = (1-\lambda)g^{(n)}g^{(n-1)} - \frac{1}{6}g_u^{(n)}g^{(n-1)} - \frac{1}{6}g^{(n)}g_u^{(n-1)}$ holds follows by application of Proposition 2.1.3 to the vertex z_n and its neighbors in V_{-n}. The removal of vertex z_n divides V_{-n} into the two subgraphs A_n and A_{n-1}, whose vertices u_n and u_{n-1} each share an edge with z_n. By definition, the characteristic polynomials corresponding to these subgraphs are $g^{(n)}$ and $g^{(n-1)}$, respectively.

The six recurrence equations above follow by application of Proposition 2.1.3 to the vertex z_n and its neighbors on the graph A_n. Let us examine the equation for $g^{(n)}$, the other cases being similar. The removal of the vertex z_{n-1} from A_n divides this graph into three subgraphs. One subgraph is isomorphic to A_{n-1}. The vertex u_{n-1} in this subgraph shares an edge with z_{n-1} and is of degree 3. The second subgraph is isomorphic to A_{n-2}. The vertex u_{n-2} in this subgraph shares an edge with z_{n-1} and is also of degree 3. The third is isomorphic to A_{n-2} but with the vertex w_{n-2} removed. The vertex v_{n-2} shares an edge with z_n and is of degree 3. $\square$

Proposition 2.3.4. *Let $n \geq 4$. The eigenvalues in $\sigma(N^{(n)})$ not in $\sigma(N^{(n-1)})$ are a*

subset of the roots of the polynomial

$$h^{(n)} = \frac{1}{3} g^{(n-1)} g^{(n-2)} + \frac{2}{3}[(1-\lambda) g^{(n-1)} g_w^{(n-2)} - \frac{1}{6} g^{(n-1)} g_{uw}^{(n-2)} - \frac{1}{6} g_u^{(n-1)} g_w^{(n-2)}].$$

Furthermore, each such eigenvalue is an eigenvalue of multiplicity one. The remaining roots of $h^{(n)}$ are eigenvalues of $\sigma(N^{(n-1)})$.

Proof. As noted previously, we may simply work with the probabilistic Laplacian $P^{(n-1)}$. Fix a number λ such that $\lambda \notin \sigma(P^{(n-1)})$. Let f be the function on the vertices of $V_{-(n-1)}$ such that

$$(P^{(n-1)} - \lambda I) f(x) = \begin{cases} 1 & : x = c_{n-1} \\ 0 & : \text{otherwise} \end{cases}$$

Let f' be the function on the vertices of $V'_{-(n-1)}$ such that

$$(P^{(n-1)} - \lambda I) f'(x) = \begin{cases} -1 & : x = z'_{n-1} \\ 0 & : \text{otherwise} \end{cases}$$

By simple linear algebra, we know that $f(c_{n-1})$ has to equal the diagonal entry of the matrix $(P^{(n-1)} - \lambda I)^{-1}$ corresponding to c_{n-1}. This entry is $D(P^{(n-1)}_{c_{n-1}})/D(P^{(n-1)})$, where it is understood we evaluate this ratio of characteristic polynomials at λ. Similarly, $f'(z'_{n-1})$ must equal $-D(P^{(n-1)}_{z'_{n-1}})/D(P^{(n-1)})$. Since $\lambda \notin \sigma(P^{(n-1)})$, we know that $D(P^{(n-1)}) \neq 0$ at λ.

In a sense, f and f' are "almost" eigenfunctions of $P^{(n-1)}$ with eigenvalue λ. But by combining these two functions to be a function on V_{-n}, we may get an eigenfunction of $P^{(n)}$. Recall that the graph V_{-n} is the union of $V_{-(n-1)}$ and $V'_{-(n-1)}$ with the points c_{n-1} and z'_{n-1} identified. Define f'' on V_{-n} such that $f'' = 3f$ on $V_{-(n-1)}$ and

$f'' = \frac{3}{2}f'$ on $V'_{-(n-1)}$. Clearly, for any vertex x not equal to $c_{n-1} = z'_{n-1}$, we have $(P^{(n)} - \lambda I)f''(x) = 0$. The only question is the vertex z'_{n-1} itself. Recall that z'_{n-1} has two neighboring vertices in $V'_{-(n-1)}$: u'_{n-2} and u'_{n-1} and one neighboring vertex in $V_{-(n-1)}$: v_{n-2}. Thus

$$
\begin{aligned}
(P^{(n)} - \lambda I)f''(z'_{n-1}) &= \frac{1}{3}\big(f''(u'_{n-1}) + f''(u'_{n-2}) + f''(v_{n-2})\big) \\
&= \frac{2}{3}\Big(\frac{1}{2}(f''(u'_{n-1}) + f''(u'_{n-2}))\Big) + \frac{1}{3}f''(v_{n-2}) \\
&= \frac{1}{2}\Big(f'(u'_{n-1}) + f'(u'_{n-2})\Big) + f(v_{n-2}) \\
&= (P^{(n-1)} - \lambda I)f'(z'_{n-1}) + (P^{(n-1)} - \lambda I)f(c_{n-1}) \\
&= -1 + 1 = 0.
\end{aligned}
$$

So if f'' is well defined at z_n, that is $3f(c_{n-1}) = \frac{3}{2}f'(z_{n-1})$, or $\frac{2}{3}f(c_{n-1}) = \frac{1}{3}f'(z'_{n-1})$, we get that f'' is an eigenvector of $P^{(n)}$ with eigenvalue λ. More precisely, this condition is needed

$$\frac{2}{3}D(P^{(n-1)}_{c_{n-1}})/D(P^{(n-1)}) = -\frac{1}{3}D(P^{(n-1)}_{z'_{n-1}})/D(P^{(n-1)}).$$

The result follows since by Proposition 2.1.3

$$D(P^{(n-1)}_{c_{n-1}}) = D(N^{(n-1)}_{c_{n-1}}) = (1-\lambda)g^{(n-1)}g^{(n-2)}_w - \frac{1}{6}g^{(n-1)}g^{(n-2)}_{uw} - \frac{1}{6}g^{(n-1)}_u g^{(n-2)}_w.$$

By construction, the corresponding eigenvector is unique up to a constant. Thus, the corresponding eigenvalue is of multiplicity one.

Finally, let us consider the polynomial $h^{(n)}$. Its roots are either eigenvalues of $\sigma(P^{(n-1)})$ or not. If not, then one can follow the exact same construction above and show that this root must be an eigenvalue of $\sigma(P^{(n)})$. $\square$

We will now prove some nice corollaries about the eigenfunctions corresponding to "new eigenvalues".

Corollary 2.3.5. *Let λ be a root of $h^{(n)}$ that's not an eigenvalue of $N^{(n-1)}$. Let g be the corresponding eigenfunction of $N^{(n)}$. Then $g(z'_{n-1}) \neq 0$.*

Proof. Suppose $g(z'_{n-1}) = 0$. Take the restriction of g to $V'_{-(n-1)}$. Recall that the removal of z'_{n-1} divides $V'_{-(n-1)}$ into two connected components. After appropriately rescaling g on one of these connected components, we can ensure that

$$(N^{(n-1)} - \lambda I)g(z'_{n-1}) = 0.$$

Since $(N^{(n-1)} - \lambda I)g(x) = 0$ holds for all other $x \in V'_{-(n-1)}$, by definition g is an eigenfunction of $N^{(n-1)}$. This contradicts our assumption. □

Corollary 2.3.6. *Let λ be a root of $h^{(n)}$ that's not an eigenvalue of $N^{(n-1)}$. Let g and g' be the corresponding eigenfunction of $N^{(n)}$ and $P^{(n)}$, respectively. Then $g(z_{n-1}) = -\sqrt{2}g(c'_{n-1})$ and $g'(z_{n-1}) = -g'(c'_{n-1})$.*

Proof. Consider the eigenfunction f'' of $P^{(n)}$ corresponding to λ constructed in the proof of Proposition 2.3.4. Let $g(x) = d_x^{-1/2} f''(x)$ and $g' = f''$. g is an eigenfunction of $N^{(n)}$ corresponding to λ. Let us denote by $R_{i,j}$ the (i, j) entry of the matrix $(N^{(n-1)} - \lambda I)^{-1}$. By symmetry, $R_{i,j} = R_{j,i}$. By the work in the proof, we can deduce that

$$\begin{aligned} g(z_{n-1}) &= \frac{3}{\sqrt{2}} R_{z_{n-1},c_{n-1}}/D(N^{(n-1)}), \\ g(c'_{n-1}) &= -\frac{3}{2} R_{c_{n-1},z_{n-1}}/D(N^{(n-1)}). \end{aligned}$$

After a little algebra, the result follows. □

By Proposition 2.3.2, we know that eigenvalues carry over from one level to the next. Thus by Proposition 2.3.4, we can immediately deduce the following.

Theorem 2.3.7. *Let $n \geq 4$. Then $\sigma(N^{(n)}) = \sigma(N^{(3)}) \cup \bigcup_{k=4}^{n} (h^{(k)})^{-1}\{0\}$.*

The rest of the section is devoted to making conclusions about the eigenvalues and eigenfunctions of the Neumann Laplacians.

Proposition 2.3.8. *The multiplicity of 1 as an eigenvalue of $N^{(n)}$ is $2^{n-1}+1$ for $n \geq 3$.*

Proof. Let us work with the graph Laplacian $P^{(n)}$. Define a maximal path to be a sequence of vertices in V_{-n}, $\{v_1, ..., v_m\}$ such that:
(i). v_n is connected to v_{n+1} for $1 \leq n < m-1$;
(ii). v_1 and v_m are vertices of degree 1;
(iii). m is odd.
One can construct a function on V_{-n} with this path under consideration. Assign to the vertices of a path the values of the sequence $\{1, 0, -1, 0, 1, 0, -1, ...\}$, and a 0 to all vertices not on this path. Let us call such a function a path function. Such a function may be an eigenfunction of $P^{(n)}$ with eigenvalue 1.

Let $k \geq 4$ be arbitrary. Take a basis of eigenfunctions on V_{-k}. Let s_k be the penultimate vertex connected to c_k and let t_k be the other ultimate vertex connected to s_k. Take g_k to be a path function on $\{t_k, s_k, c_k\}$. It is easy to check that g_k is in fact an eigenfunction. Without loss of generality let g_k be in our basis.

Recall that $V_{-(k+1)} = V_{-k} \cup V'_{-k}$. Those eigenfunctions in our basis that are non-zero at the vertex $c_k = z'_k$ will not be eigenfunctions on $V_{-(k+1)}$ (after extending by

zero). However, we can modify them to be eigenfunctions by adding some multiple of g_k that makes the function vanish at c_k, and then extending by zero into V'_{-k}. g_k itself can be made into an eigenfunction on $V'_{-(k+1)}$ in the following manner. There is a unique sequence of vertices in V'_{-k} of length 5, where z'_k is the middle vertex, and the first and last vertices are of degree one. Let h'_k be a path function on this sequence. h'_k in fact is an eigenfunction. After potentially scaling h'_k by -1 so that g_k and h'_k coincide at $c_k = z'_k$, and we can extend g_k to $V_{-(k+1)}$ by setting g_k equal to h'_k on V'_{-k}.

Take a basis of eigenfunctions on V'_{-k}. Without loss of generality suppose h'_k is in this basis. Each basis element that is zero at z'_k can be made into an eigenfunction on $V_{-(k+1)}$ after extending by zero. For a basis that attains a non-zero value, one can extend the function into V_{-k} by setting the function equal to the multiple of g_k that coincides with the function at $c_k = z'_k$. The resulting function is an eigenfunction on $V_{-(k+1)}$.

Now take the union of the modified bases on V_{-k} and V'_{-k}. This set is not necessarily a basis of the eigenspace on $V_{-(k+1)}$, but it certainly spans the set. I.e., any eigenfunction of $\lambda = 1$ on $V_{-(k+1)}$ can be decomposed as a linear combination eigenfunctions from this union. This tells us that the support of the eigenspace of $P^{(k+1)}$ is the union of the supports of modified basis functions on V_{-k} and $V_{-(k+1)}$. Thus, we are now in a position to apply Theorem 2.1.6.

Let m_k denote the multiplicity of 1 as an eigenvalue of $P^{(k)}$. Let α_k denote the number of connected components of V_{-k} induced by $\mathrm{supp}(P^{(k)}, 1)$. Let β_k denote the number of vertices of V_{-k} adjacent to $\mathrm{supp}(P^{(k)}, 1)$ but that do not belong to the set.

Since the vertex $c_k = z'_k$ is in $\operatorname{supp}(P^{(k)}, 1)$, we have that

$$\alpha_{k+1} = 2\alpha_k - 1, \quad \beta_{k+1} = 2\beta_k,$$

and thus

$$m_{k+1} = \alpha_{k+1} - \beta_{k+1} = 2(\alpha_k - \beta_k) - 1 = 2m_k - 1.$$

We obtain the result by solving this recurrence. □

Proposition 2.3.9. *Let $n \geq 3$. Let λ be an eigenvalue of $N^{(k)}$ that is not an eigenvalue of $N^{(k-1)}$. If the corresponding eigenfunction on V_{-k} does not attain a zero, then λ is an eigenvalue of multiplicity one of $N^{(n)}$ for $n \geq k$.*

Proof. Let f_k be the eigenfunction on V_{-k}. By the method in Proposition 2.3.2, we can extend the eigenfunction to V_{-n} essentially by placing scaled copies of f_k on the appropriate subgraphs isomorphic to V_{-k}. Thus, these extensions can never attain a zero. By Theorem 2.1.5, λ must be an eigenvalue of multiplicity one of $N^{(n)}$. □

We end this section with a conjecture about the eigenfunctions of $P^{(n)}$ and a nice consequence of the conjecture.

Conjecture 2.3.10. *Let f_n be an eigenfunction of $P^{(n)}$ corresponding to $\lambda \neq 1$. Then f_n does not attain a zero on V_{-n}.*

One idea for a proof is as follows. Let λ be a "new" eigenvalue of $P^{(n)}$ and f_n the corresponding eigenfunction. If one supposed that f_n did in fact attain a zero at some vertex, then one could use the Edge Principle (Lemma 2.1.4) to deduce that f_n is zero at least on some subgraph of V_{-n}. It would be enough to show that f_n is identically zero on V_{n-1} or V'_{n-1}, as that would imply that the restriction to the

complement, if non-zero, is an eigenfunction of $P^{(n-1)}$. Thus, f_n needs to be non-zero everywhere, as well as on its extensions to V_{-m} for $m > n$.

Proposition 2.3.11. *Suppose Conjecture* 2.3.10 *holds. Let* $n \geq 4$. *Then the* $2^{n+1}+1$ *eigenvalues of* $N^{(n)}$ *can be decomposed as follows:*

(i). the eigenvalue 1 of multiplicity $2^{n-1}+1$,

(ii). $2^n - 2^{n-1}$ *eigenvalues of multiplicity one that are eigenvalues of* $N^{(n-1)}$,

(iii). $2^{n+1} - 2^n$ *eigenvalues of multiplicity one that are not eigenvalues of* $N^{(n-1)}$.

Proof. By Proposition 2.3.8, we have (i). By the conjecture and Proposition 2.3.9, we obtain (ii) by counting the eigenvalues of $N^{(n-1)}$ not equal to one. By Proposition 2.3.9 and a counting argument, we obtain (iii). □

2.4 Dirichlet Eigenvalues of Graph Laplacians

Let $P_0^{(n)}$ and $N_0^{(n)}$ denote the matrices $P^{(n)}$ and $N^{(n)}$ but with the rows and columns corresponding to the boundary removed. This matrix operation corresponds to imposing a zero boundary condition on the boundary. Therefore, we call the eigenvalues of these operators the Dirichlet eigenvalues. It is easy to see that the eigenvalues of $P_0^{(n)}$ and $N_0^{(n)}$ must coincide. Let V_{-n}^0 denote V_{-n} with the boundary points removed.

Proposition 2.4.1. *Let* $n \geq 3$. *Then the eigenvalues of* $N_0^{(n)}$ *are the roots of* $h_0^{(n)} := g_w^{(n)} g_w^{(n-1)}$.

Proof. $N_0^{(n)}$ is a reducible matrix. Its two irreducible components correspond to A_n with w_n removed and $B_n = A_{n-1}$ with w_{n-1} removed. By definition, the characteristic polynomials of these components are $g_w^{(n)}$ and $g_w^{(n-1)}$, respectively. □

As for the Neumann case, it is possible to count the multiplicity of the eigenvalue one. One finds that the multiplicity of 1 as an eigenvalue of $P_0^{(n)}$ is three less than the multplicity of 1 as an eigenvalue of $P^{(n)}$. Roughly, this corresponds to the removal of the three boundary points.

Proposition 2.4.2. *Let $n \geq 3$. Then the multiplicity of 1 as an eigenvalue of $P_0^{(n)}$ is $2^{n-1} - 2$.*

Proof. Take a basis $\mathscr{B}$ of eigenfunctions for the eigenspace of $\lambda = 1$ of $P^{(n)}$. By Proposition 2.3.8 there must be $2^{n-1} + 1$ such eigenfunctions. There are unique paths of length three in V_{-n} containing w_n and c_n, respectively, and a unique path of length five containing z_n. Let us take corresponding path functions f_n^1, f_n^2, f_n^3, and note that they must be eigenfunctions. Without loss of generality suppose they are in our basis.

For each of the remaining basis elements, let us modify them by adding some linear combination of f_n^1, f_n^2, f_n^3 that makes function vanish at $\{w_n, c_n, z_n\}$. Thus, $\mathscr{B} \backslash \{f_n^1, f_n^2, f_n^3\}$ is a linearly independent set and any eigenfunction on V_{-n}^0 can be written as a linear combination of functions from this set. Thus, $\mathscr{B} \backslash \{f_n^1, f_n^2, f_n^3\}$ is a basis for the eigenfunctions on V_{-n}^0. $\square$

2.5 The Laplacian on the Infinite Blow-up

Let $L^2(V_{-\infty})$ denote the set of functions on the infinite lattice $V_{-\infty}$ for which $||f||_2 := \sqrt{\sum_{x \in V_{-\infty}} d_x^2 f^2(x)} < \infty$. We can define a Laplacian operator $P^{(\infty)}$ on $L^2(V_{-\infty})$ in a pointwise manner

$$P^{(\infty)} f(x) = \frac{1}{d_x} \sum_{x \sim y} (f(x) - f(y)).$$

Recall that the lattices V_{-n} are nested, that is, $V_0 \subset V_{-1} \subset V_{-2} \subset \cdots$. The Laplacian operators $P^{(n)}$ on V_{-n} can thus be extended to operators on $L^2(V_{-\infty})$ in the natural way:

$$P^{(n)}f(x) = \begin{cases} \frac{1}{d_x}\sum_{x\sim y, y\in V_{-n}}(f(x)-f(y)) & : x \in V_{-n} \\ 0 & : x \notin V_{-n} \end{cases}$$

Here d_x denotes the degree of x in V_{-n}.

For an operator T on $L^2(V_{-\infty})$, we define its spectrum $\sigma(T)$ to be the set of complex numbers λ such that the operator $T - \lambda I$ does not have a bounded inverse. In particular, it is clear that all eigenvalues must be in the spectrum. However, not all points in the spectrum are necessarily eigenvalues. Let $||T|| := \sup\{||Tf||_2 : ||f||_2 \leq 1\}$. It is not hard to see that $||P^{(n)}|| = 2$ for all n or that $\{P^{(n)}f\}$ converges pointwise to $P^{(\infty)}f$ as $n \to \infty$ for any $f \in L^2(V_{-\infty})$. Furthermore, the sequence of operators $P^{(n)}$ converges strongly to $P^{(\infty)}$. That is, $||P^{(n)}f - P^{(\infty)}f||_2 \to 0$ as $n \to \infty$ for any function $f \in L^2(V_{-\infty})$. To see this, find N such that for $n \geq N$ we have $\sum_{x\in V_{-\infty}\setminus V_{-N}} d_x^2 f^2(x) < \epsilon$. Note that we have the crude operator bound $||P^{(n)} - P^{(\infty)}|| \leq 4$. Then for $n \geq N$, $||P^{(n)}f - P^{(\infty)}f|| < 4\epsilon$. The strong convergence allows us to deduce the following.

Theorem 2.5.1. *The spectrum of* $P^{(\infty)}$ *satisfies* $\sigma(P^{(\infty)}) \subseteq cl\big(\sigma(N^{(3)}) \cup \bigcup_{k=4}^{\infty}(h^{(k)})^{-1}\{0\}\big)$.

Proof. By Theorem 2.3.7, for $n \geq 4$ we know that

$$\sigma(P^{(n)}) = \sigma(N^{(3)}) \cup \bigcup_{k=4}^{n}(h^{(k)})^{-1}\{0\}.$$

By the strong convergence of $P^{(n)}$ to $P^{(\infty)}$, we know that every $\lambda \in \sigma(P^{(\infty)})$ is a limit

of eigenvalues $\lambda_n \in \sigma(P^{(n)})$ (c.f. [36]). Thus

$$\sigma(P^{(\infty)}) \subseteq \mathrm{cl}\big(\sigma(N^{(3)}) \cup \bigcup_{k=4}^{\infty} (h^{(k)})^{-1}\{0\}\big). \quad \square$$

By our previous work, we know that the spectrum of $P^{(n)}$ consists of eigenvalues. Note that for each nonzero eigenvalue, the corresponding eigenfunction on V_{-n} is extended to the infinite lattice by zero. Therefore, the dimensions of the corresponding eigenspaces on V_{-n} equals the dimension of the eigenspace on $V_{-\infty}$. For the eigenvalue zero, the dimension of its eigenspace is infinite. The eigenspace consists of all functions constant on V_{-n}, with no conditions outside of V_{-n}. However, deducing if something is an eigenvalue of $P^{(\infty)}$ is not as simple. In the rest of the section, we will make some conclusions about the elements in $\sigma(P^{(\infty)})$.

Proposition 2.5.2. $\lambda = 1$ *is an eigenvalue of* $P^{(\infty)}$ *of infinite multiplicity.*

Proof. Let us consider path eigenfunctions, as defined in Proposition 2.3.8. Take any path eigenfunction (say f) on V_{-n} that does not take a non-zero value at c_n and w_n and extend by zero to the infinite lattice. It is clear that $P^{(\infty)} f = f$ holds pointwise. It is also clear that the L^2 norm of f must be finite. So f is a proper eigenfunction of $P^{(\infty)}$. By Proposition 2.3.8, the multiplicity of 1 as an eigenvalue of $P^{(n)}$ is $2^{n-1} + 1$ for $n \geq 3$. Taking $n \to \infty$, we deduce that 1 is an eigenvalue of $P^{(\infty)}$ of infinite multiplicity. $\square$

Proposition 2.5.3. *Suppose* $\lambda \in \sigma(P^{(n)})$ *such that the corresponding eigenvector* f_n *satisfies: (i)* $f_n(z_n) \neq 0$ *and (ii)* $f_n(c_n) = 0$ *or* $f_n(w_n) = 0$*. Then* λ *is an eigenvalue of* $P^{(\infty)}$*.*

Proof. First, suppose both $f_n(c_n) = 0$ and $f_n(w_n) = 0$. Then by extending f_n by zero to $V_{-\infty}$, we obtain an eigenfunction of $P^{(\infty)}$.

Second, suppose $f_n(c_n) \neq 0$ but $f_n(w_n) = 0$. Let $k_n = f_n(c_n)/f_n(z_n)$. We will define an extension f_{n+1} of f_n on $V_{-(n+1)}$. In particular, set $f_{n+1}(x) = k_n f_n(\iota(x))$ for $x \in V'_{-n}$, where ι is the canonical isomorphism between V_{-n} and V'_{-n}. Thus, $f_{n+1}(c_{n+1}) = f_n(w_n) = 0$ and $f_{n+1}(w_{n+1}) = k_n f_n(w_n) = 0$. So by the first case, we are done.

Finally, suppose $f_n(c_n) = 0$ but $f_n(w_n) \neq 0$. Extend f_n to V'_{-n} by zero. Call the extension f_{n+1}. Then $f_{n+1}(c_{n+1}) = f_n(w_n) \neq 0$ and $f_{n+1}(w_{n+1}) = 0$. By the second case, we are done. □

For a bounded self-adjoint operator T on a Hilbert space X, we can decompose its spectrum $\sigma(T)$ into two disjoint pieces: the discrete spectrum $\sigma_{\text{discr}}(T)$ and the essential spectrum $\sigma_{\text{ess}}(T)$. By definition, λ is in the discrete spectrum if it is an isolated eigenvalue of finite multiplicity. That is, the dimension of the set $\{x \in X : Tx = \lambda x\}$ is finite and non-zero and there is an interval around λ that contains no eigenvalues apart from λ. The essential spectrum is defined to be the complement of the discrete spectrum.

The following result is due to Weyl [37].

Theorem 2.5.4. *Let T be a bounded self-adjoint operator on a Hilbert space X. $\lambda \in \sigma(T)$ if and only if there exists a sequence $\{x_k\}$ in X such that $||x_k|| = 1$ and*

$$\lim_{k\to\infty} ||Tx_k - \lambda x_k|| = 0.$$

Furthermore, λ is in the essential spectrum if there is a sequence satisfying this con-

dition, but such that it contains no strongly convergent subsequence.

We will use the previous theorem in the following result.

Proposition 2.5.5. *Suppose $\lambda \in \sigma(P^{(n)})$ such that the corresponding eigenvector f_n is not zero at z_n, c_n and w_n. Then λ is an eigenvalue of $P^{(\infty)}$ if and only if*

$$|f_n(w_n)/f_n(z_n)|^{-\frac{1+\sqrt{5}}{2}} > |f_n(c_n)/f_n(z_n)|. \tag{2.5.1}$$

If Equation 2.5.1 *is not satisfied, then λ belongs to the essential spectrum of $P^{(\infty)}$.*

Proof. Let $\lambda \in \sigma(P^{(n_0)})$ and let f_{n_0} be a corresponding eigenfunction that is non-zero at z_{n_0}, c_{n_0} and w_{n_0}. By Proposition 2.3.2, we can extend this eigenfunction to V_{-n}, $n > n_0$. Let us label the extended eigenfunction by f_n. Define f_∞ so that f_∞ coincides with f_n on V_{-n}. It is clear that $P^{(\infty)} f_\infty = \lambda f_\infty$. By Corollaries 2.3.5 and 2.3.6, f_{n_0} is non-zero at z'_{n_0-1} and c'_{n_0-1}. Thus, the support of f_{n_0} contains at least five vertices, the support of f_{n_0+1} cotains at least nine vertices, and in general the support of f_n contains at least $2^{n-n_0+2}+1$ vertices.

Without loss of generality let us scale the eigenfunction f_{n_0} so that $f_{n_0}(z_{n_0}) = 1$. Let $c = f_{n_0}(c_{n_0})$ and $w = f_{n_0}(w_{n_0})$. Recall that the extension algorithm of the eigenfunction f_n on V_{-n} is given by scaling f_n by $k_n = f(c_n)/f(z_n) = f(c_n)$ and placing this scaled copied on V'_{-n} under the natural isomorphism between V_{-n} and V'_{-n}. In the blow-up of the Hata tree that we are considering, z_n is a fixed point (i.e. z_n is the same point for all n). So $f_n(z_n) = 1$. Also by construction, $w_{n+1} = w'_n$ and $c_{n+1} = w_n$. Then clearly $f_{n+1}(c_{n+1}) = f_n(w_n)$ and $f_{n+1}(w_{n+1}) = k_n f_n(w_n) = f_n(c_n) f_n(w_n)$. After cleaning up the recurrence relations, we get that for $n \geq n_0 + 1$

$$f_n(w_n) = w^{F_{n-n_0+1}} c^{F_{n-n_0}}.$$

Here, the F_j's are the Fibonacci numbers; $F_0 = 0, F_1 = 1$ and $F_j = F_{j-1} + F_{j-2}$. Note that $F_j = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^j - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^j$. Thus, asymptotically $\log|f_n(w_n)|$ behaves like

$$\left(\frac{1+\sqrt{5}}{2}\right)^{n-n_0+1} \log|w| + \left(\frac{1+\sqrt{5}}{2}\right)^{n-n_0} \log|c|.$$

The above expression, and thus $|f_n(w_n)|$, converges to ∞ if and only if $|w|^{-\frac{1+\sqrt{5}}{2}} < |c|$. In this case, the L^2 norm of f_∞ cannot be finite. If $|w|^{-\frac{1+\sqrt{5}}{2}} = |c|$, then $|f_n(w_n)|$ will tend to one and the L^2 norm of f_∞ likewise cannot be finite. Finally, we consider the case when $|w|^{-\frac{1+\sqrt{5}}{2}} > |c|$. In this case, $|f_n(w_n)|$ decays to zero. Let K denote the maximum attained by f_{n_0}. The maximum of f_{n_0+k} on $V'_{-(n_0+k-1)}$ is then $Kf_{n_0+k-1}(c_{n_0+k-1})$. Thus, the square of the L^2 norm of f_∞ is bounded above by

$$K^2|V_{-n_0}| + K^2 \sum_{m=n_0}^{\infty} f_m(c_m)^2 |V'_{-m}| = K^2(2^{n_0+1}+1) + K^2 \sum_{m=n_0}^{\infty} f_m(c_m)^2(2^{m+1}+1).$$

Note that the summation is finite, since $\log|f_m(c_m)^2(2^{m+1}+1)|$ is equal to

$$2\left(\frac{1+\sqrt{5}}{2}\right)^{m-n_0} \left(\log|w| + \left(\frac{1+\sqrt{5}}{2}\right)^{-1} \log|c|\right) + \log(2^{m+1}+1).$$

The first term in the expression above dominates and converges to $-\infty$. So $f_m(c_m)^2(2^{m+1}+1)$ decays to zero exponentially and the summation of these terms is finite.

We now will prove the second statement in the proposition. So suppose $|w|^{-\frac{1+\sqrt{5}}{2}} \leq |c|$. Let us extend each function f_n to $V_{-\infty}$ by zero. Note that the resulting function will not be an eigenfunction of $P^{(\infty)}$. In particular, $(P^{(\infty)} - \lambda I)f_n = g_n$, where g_n is identically zero except at the vertices c_n, w_n and the four adjacent vertices outside of V_{-n}. Let us denote by x_n, y_n the two vertices adjacent to w_n outside of V_{-n}. Recall

that v_n is the vertex in V_{-n} attached to w_n. Then

$$\begin{aligned}(P^{(\infty)} - \lambda I)f_n(w_n) &= (1-\lambda)f_n(w_n) - \frac{1}{3}f_n(v_n) - \frac{1}{3}f_n(x_n) - \frac{1}{3}f_n(y_n) \\ &= (1-\lambda)f_n(w_n) - \frac{1}{3}f_n(v_n) \\ &= (1-\lambda)f_n(w_n) - f_n(v_n) + \frac{2}{3}f_n(v_n) \\ &= (P^{(n)} - \lambda I)f_n(w_n) + \frac{2}{3}f_n(v_n) \\ &= \frac{2}{3}f_n(v_n) = \frac{2}{3}(1-\lambda)^{-1}f_n(w_n).\end{aligned}$$

In a similar way, we can check that $(P^{(\infty)} - \lambda I)f_n(x_n) = (P^{(\infty)} - \lambda I)f_n(y_n) = -\frac{1}{3}f_n(w_n)$. After analyzing c_n and the vertices connected to it, we can deduce that

$$||g_n||_2^2 = (2 + 4(1-\lambda)^{-2})(f_n(c_n)^2 + f_n(w_n)^2).$$

Let $C_n := ||f_n||_2$. Since $|w|^{-\frac{1+\sqrt{5}}{2}} \le |c|$, we know that $C_n \to \infty$ as $n \to \infty$. So $||f_n/C_n||_2 = 1$ but f_n/C_n converges weakly to the zero function on $V_{-\infty}$. So $\{f_n/C_n\}$ contains no strongly convergent subsequence. By Theorem 2.5.4, to deduce that λ is in the essential spectrum of $P^{(\infty)}$, it suffices to prove that $||g_n/C_n||_2 \to 0$ as $n \to \infty$.

Let $k > 0$ be the smallest non-zero value that $|f_{n_0}|$ attains on V_{-n_0}. Note that $k = k'|f_{n_0}(w_{n_0})|$ for some $k' > 0$. The $|f_{n_0+1}|$ attains a non-zero minimum of $k'|f_{n_0}(w_{n_0})||f_{n_0}(c_{n_0})| = k'|f_{n_0+1}(w_{n_0+1})|$ on V'_{-n_0}. In general, $|f_n|$ attains a non-zero minimum of $k'|f_n(w_n)|$ on $V'_{-(n-1)}$. As there are at least $2^{n-n_0+1}+1$ vertices in $V'_{-(n-1)}$ where f_n is non-zero, we deduce that

$$C_n^2 \ge (2^{n-n_0+1}+1)(k')^2 f_n(w_n)^2.$$

Thus

$$\begin{aligned} ||g_n/C_n||_2^2 &\leq \frac{(2+4(1-\lambda)^{-2})(f_n(c_n)^2+f_n(w_n)^2)}{(2^{n-n_0+1}+1)(k')^2 f_n(w_n)^2} \\ &= \frac{2+4(1-\lambda)^{-2}}{(k')^2}\left(\frac{1}{2^{n-n_0+1}+1}+\frac{f_{n-1}(w_{n-1})^2}{(2^{n-n_0+1}+1)f_n(w_n)^2}\right). \end{aligned}$$

The first term tends to zero. For the second term, note that either $|f_n(w_n)|$ tends to one or $|f_n(w_n)|$ is an increasing sequence that converges to ∞. In either case, the second term tends to zero as $n \to \infty$. $\square$

Corollary 2.5.6. *Suppose f_n does not attain a zero on V_{-n} and Equation* 2.5.1 *is satsified. Then λ is an eigenvalue of multiplicity one.*

Proof. By Proposition 2.3.9, the extensions f_n to V_{-n} in the proof are unique, up to a constant. Thus, f_∞ is unique up to a constant. $\square$

Corollary 2.5.7. *Suppose Conjecture* 2.3.10 *holds. Then*

$$\sigma(P^{(\infty)}) = cl\big(\sigma(N^{(3)}) \cup \bigcup_{k=4}^{\infty}(h^{(k)})^{-1}\{0\}\big).$$

Proof. By Theorem 2.5.1, we know

$$\sigma(P^{(\infty)}) \subseteq \mathrm{cl}\big(\sigma(N^{(3)}) \cup \bigcup_{k=4}^{\infty}(h^{(k)})^{-1}\{0\}\big).$$

Let $\lambda \in \sigma(N^{(3)}) \cup \bigcup_{k=4}^{\infty}(h^{(k)})^{-1}\{0\}$. If $\lambda \neq 1$, by the proposition and the conjecture we know $\lambda \in \sigma(P^{(\infty)})$. By Proposition 2.5.2, $\lambda = 1$ is an eigenvalue of $P^{(\infty)}$. Thus,

$$\sigma(N^{(3)}) \cup \bigcup_{k=4}^{\infty}(h^{(k)})^{-1}\{0\} \subseteq \sigma(P^{(\infty)}).$$

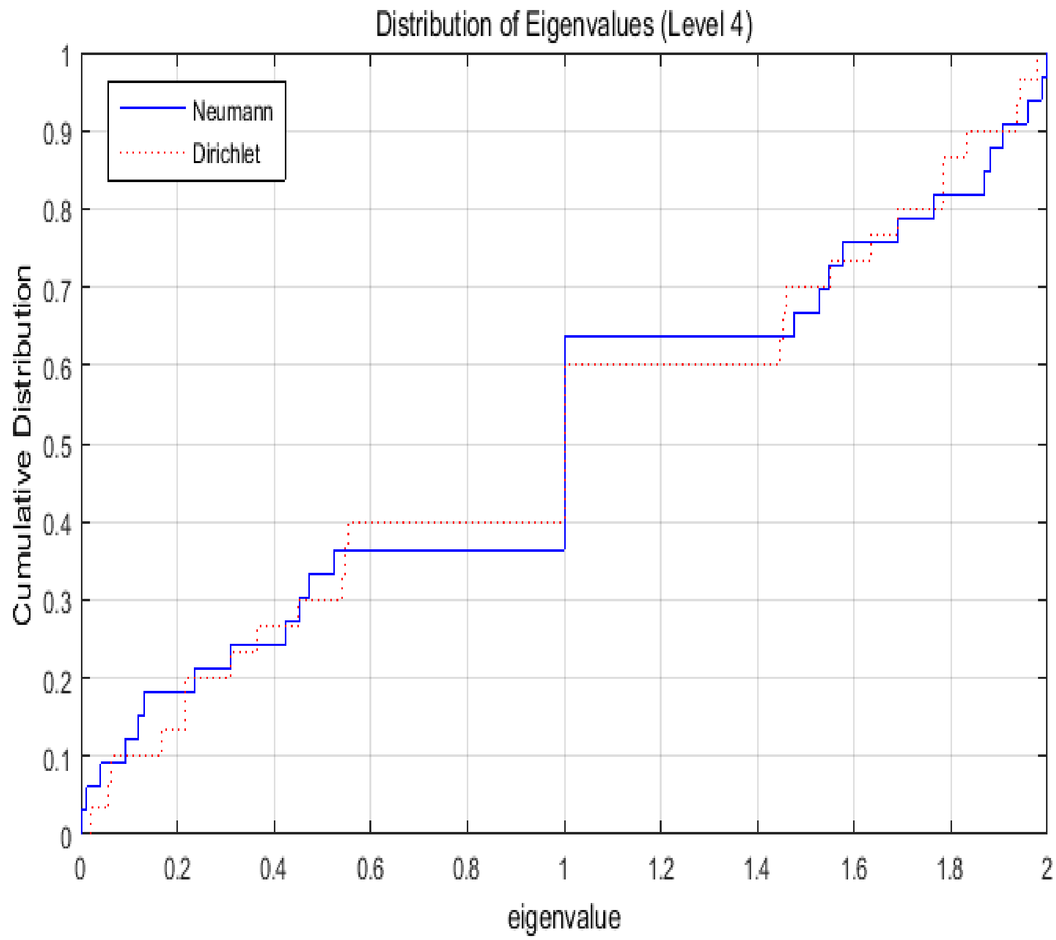


FIGURE 2.5.1: Distribution of Neumann and Dirichlet eigenvalues of level 4 graph Laplacian

Since the spectrum must be closed, the closure of the left hand side must also be in the spectrum of $P^{(\infty)}$. This gives the reverse containment. □

In a similar manner, one can construct a Dirichlet operator on the infinite lattice from the operators $P_0^{(n)}$ on the graph approximations V_{-n}. However, finding the precise spectrum of the Dirichlet operator is more challenging than it is for the Neumann case because in this case there are eigenvalues of the approximating operator $P_0^{(n)}$ that are not eigenvalues of $P_0^{(n+1)}$.

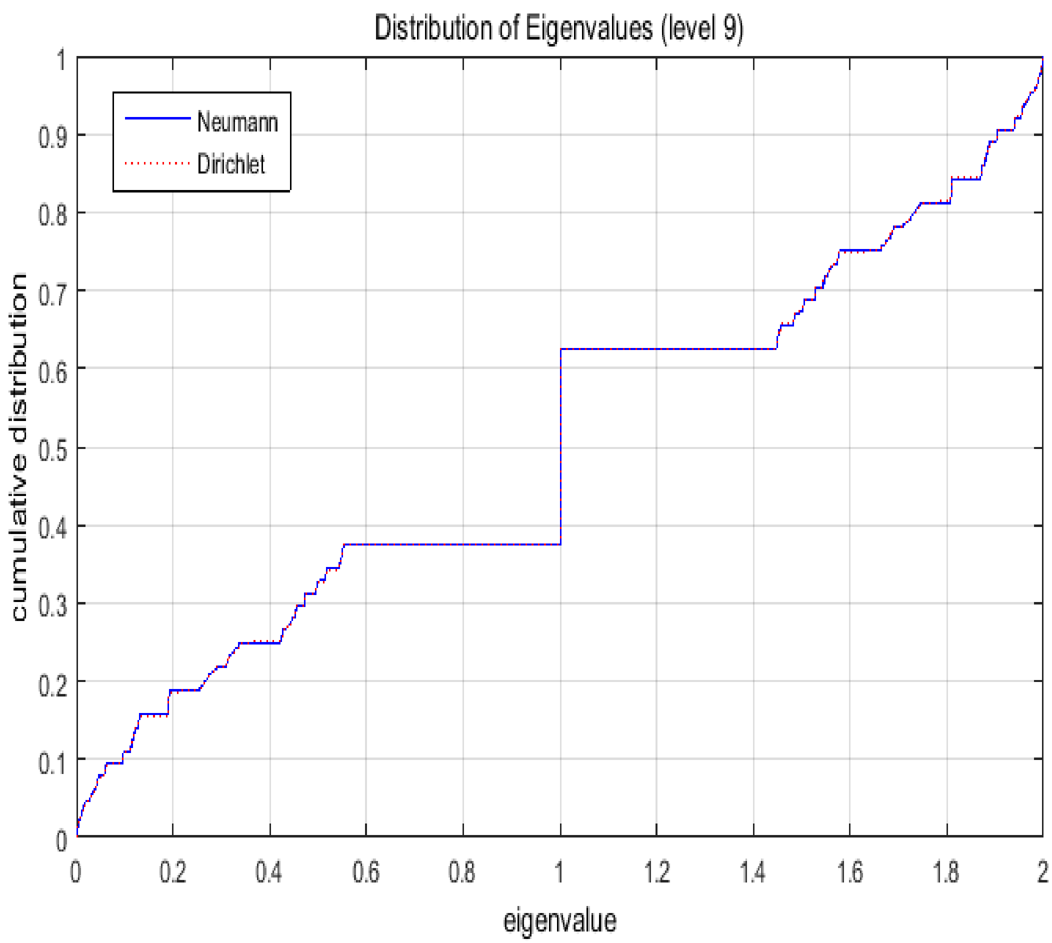


Figure 2.5.2: Distribution of Neumann and Dirichlet eigenvalues of level 9 graph Laplacian

2.6 The Laplacian on the Hata Tree

In this section, let $P^{(n)}$ denote the probabilistic Laplacian on the lattice V_n. It is possible to associate with the sequence of Laplacians $P^{(n)}$ a sequence of Dirichlet forms.

Definition 2.6.1. Let V be a finite set. Let $\ell(V)$ denote the set of real-valued functions on V. A symmetric bilinear form on $\ell(V)$, $\mathcal{E}$ is called a Dirichlet form on V if it satisfies:

(1) $\mathcal{E}(f,f) \geq 0$ for any $f \in \ell(V)$.

(2) $\mathcal{E}(f,f) = 0$ if and only if f is constant on V.

(3) For any $f \in \ell(V)$, $\mathcal{E}(f,f) \geq \mathcal{E}(\bar{f},\bar{f})$, where $\bar{f}$ is defined by

$$\bar{f}(x) = \begin{cases} 1 & : f(x) \geq 1 \\ f(p) & : 0 < f(x) < 1 \\ 0 & : f(x) \leq 0 \end{cases}$$

Let $E^{(n)}$ be the Dirichlet form on $\ell(V_n)$ defined by

$$E^{(n)}(f,g) = \sum_{x \sim y} (f(x) - f(y))(g(x) - g(y)).$$

There is a natural relationship between $E^{(n)}$ and $P^{(n)}$. Let μ_n be the discrete measure on V_n that assigns each vertex a weight equal to the degree of the vertex. Then the following relation holds

$$E^{(n)}(f,g) = \langle f, P^{(n)} g \rangle_{\mu_n},$$

where $\langle \cdot, \cdot \rangle_{\mu_n}$ is an inner product on $\ell(V_n)$ satisfying

$$\langle f, g \rangle_{\mu_n} = \sum_{x \in V_n} f(x)g(x)\mu_n(x).$$

We call $E^{(n)}(f, f)$, or more simply $E^{(n)}(f)$, the energy of the function f.

We define the resistance between two points x and y on V_n by

$$R^{(n)}(x, y) = \max\left\{ \frac{|f(x) - f(y)|^2}{E^{(n)}(f)} : f \in \ell(V_n), E^{(n)}(f) > 0 \right\}.$$

Proposition 2.6.2. *Let $x, y \in V_n$ such that $x \sim y$. Then $R^{(n)}(x, y) = 1$ and $R^{(k)}(x, y) = F_{k-n}$ or F_{k-n+1} for $k > n$, where $\{F_m\}_{m=0}^{\infty}$ is the Fibonacci sequence of numbers such that $F_0 = 1$, $F_1 = 1$ and for $m > 1$,*

$$F_m = F_{m-1} + F_{m-2}.$$

Proof. That $R^{(n)}(x, y) = 1$ is clear. For $k > n$, observe that the removal of the vertices x and y divides the graph V_k into three regions. Let A_k, B_k and C_k be the regions that have x, y, and both vertices, respectively, as "boundary" points. Define a function f_k on V_k such that $f_k = 1$ on $A_k \cup \{x\}$, $f_k = 0$ on $B_k \cup \{y\}$ and is linear on C_k. In particular, on C_k we linearly interpolate on the vertices that lie on the edge between x and y. Note that if there $M + 1$ such vertices (or M edges) the energy of f_k on this piece is $1/M$. For any subgraph attached to these vertices, we can define the function to be identically equal to the value assigned to the vertex. Thus, the only positive contribution to the energy will come from the edge between x and y.

So it suffices to understand how the edge between x and y evolves in graph ap-

proximations. In either V_{n+1} or V_{n+2}, this edge is replaced by a vertex of degree two with edges connecting to x and y, respectively. (In section 4.2, certain binary functions are defined on the edges of the graph approximation. These binary functions can be used to determine when edges are "appended". In particular, if the edge in V_n is assigned a one or zero, then the edge is appended in V_{n+1} or V_{n+2}, respectively.) The subsequent subgraphs are isomorphic to copies of the previous joined together in the obvious manner. Thus the number of edges in one subgraph is equal to the sum of those in the previous two. We can define the sequence $\{F_m\}_{m=0}^{\infty}$ as above to help us count the number of edges. Depending on the edge, $F_2 = 2$ will correspond to either V_{n+1} or V_{n+2}. For fixed k, we will have either F_{k-n} or F_{k-n+1} edges in the subgraph of interest. $\square$

By the proposition, determining the resistance between two vertices of V_n is the same as counting the number of edges on the shortest path connecting the vertices.

Corollary 2.6.3. *The resistance metric R_n coincides with the natural graph distance on V_n.*

Remark 2.6.4. The sequence $\{F_m\}_{m=0}^{\infty}$ is the Fibonacci sequence. Let $\psi = \dfrac{1+\sqrt{5}}{2}$. It is well known that

$$F_m = \frac{1}{\sqrt{5}}\big(\psi^{m+1} - (-\psi)^{-(m+1)}\big).$$

We may define a related metric R on the Hata tree by induction. Let $\varphi = \psi^{-1} = \frac{\sqrt{5}-1}{2}$ and set $R(0,1) = 1, R(0,c) = \varphi$. This defines R on V_0. We extend to V_n by setting

$$R(\phi_0(x), \phi_0(y)) = \varphi R(x,y), \quad R(\phi_1(x), \phi_1(y)) = \varphi R(x,y).$$

This is a well-defined geodesic metric on V_∞ because

$$R(0,|c|^2)+R(|c|^2,1)=\varphi^2+\varphi=1=R(0,1).$$

Evidently $R(x,y)$ is uniformly continuous for $V_\infty \times V_\infty \to \mathbb{R}$ so it extends to a metric on the closure of $K \times K$. Distance between vertices on some graph approximations are shown in Figure 2.6.1.

Lemma 2.6.5. *Let* $0 \le k \le n$. *Then*

$$\left|\frac{F_k}{F_n}-\varphi^{n-k}\right| \le 4\varphi^{n+k+1}.$$

Proof.

$$\begin{aligned}
\left|\frac{F_k}{F_n}-\varphi^{n-k}\right| &= \left|\frac{\psi^{k+1}-(-\psi)^{-(k+1)}}{\psi^{n+1}-(-\psi)^{-(n+1)}}-\varphi^{n-k}\right| \\
&= \left|\frac{\psi^{k+1}}{\psi^{n+1}}\frac{1+(-1)^k\varphi^{2(k+1)}}{1+(-1)^n\varphi^{2(n+1)}}-\varphi^{n-k}\right| \\
&= \left|\varphi^{n-k}\left[\frac{(-1)^k\varphi^{2(k+1)}-(-1)^n\varphi^{2(n+1)}}{1+(-1)^n\varphi^{2(n+1)}}\right]\right| \\
&= \left|\varphi^{n+k+1}\left[\frac{(-1)^k-(-1)^n\varphi^{2(n-k)}}{1+(-1)^n\varphi^{2(n+1)}}\right]\right| \\
&\le 4\varphi^{n+k+1},
\end{aligned}$$

where the last last inequality follows because $0<\varphi^2<\frac{1}{2}$, which implies that $(-1)^k-(-1)^n\varphi^{2(n-k)} \le 2$ and $1+(-1)^n\varphi^{2(n+1)} \ge \frac{1}{2}$. □

Proposition 2.6.6. *Let* $x, y \in V_\infty$. *Then* $\frac{1}{F_{n+1}}R^{(n)}(x,y)$ *converges to* $R(x,y)$ *as*

$n \to \infty$. Furthermore,

$$\left|\frac{R^{(n)}(x,y)}{F_{n+1}} - R(x,y)\right| \leq (4\sqrt{5})\varphi^{n-2-k}$$

for any pair of vertices $x, y \in V_k$ and $n \geq k$.

Proof. First, note that $R^{(n)}(0,1) = F_{n+1}$ by the previous proposition. Thus, scaling by $\frac{1}{F_{n+1}}$ ensures that the resistance between the vertices 0 and 1 is always one. Next, $R^{(n)}(0,c) = F_n$ and it is known that $\frac{F_n}{F_{n+1}} \to \varphi = R(0,c)$ as $n \to \infty$. If $x \sim y$ in V_k, $R(x,y) = \varphi^{k+1}$ or φ^k. The corresponding approximate resistances are $\frac{1}{F_{n+1}}R^{(n)}(x,y) = \frac{F_{n-k}}{F_{n+1}}$ and $\frac{F_{n-k+1}}{F_{n+1}}$, which converge to φ^{k+1} and φ^k respectively as $n \to \infty$.

In Proposition 2.6.2, it is made clear that $R^{(n)}(x,y)$ is either F_{n-k} or F_{n-k+1} for $n \geq k$. Then if $x \sim y$ in V_k and $m \geq n \geq k$, we have

$$\begin{aligned}
\left|\frac{R^{(n)}(x,y)}{F_{n+1}} - \frac{R^{(m)}(x,y)}{F_{m+1}}\right| &\leq \max\left(\left|\frac{F_{n-k}}{F_{n+1}} - \frac{F_{m-k}}{F_{m+1}}\right|, \left|\frac{F_{n-k+1}}{F_{n+1}} - \frac{F_{m-k+1}}{F_{m+1}}\right|\right) \\
&\leq \max\Big(|\varphi^{(n+1)-(n-k)} - \varphi^{(m+1)-(m-k)}| + 4\varphi^{2n+1-k} + 4\varphi^{2m+1-k}, \\
&\qquad |\varphi^{(n+1)-(n-k+1)} - \varphi^{(m+1)-(m-k+1)}| + 4\varphi^{2n+2-k} + 4\varphi^{2m+2-k}\Big) \\
&\leq \max\left(10\varphi^{2n+1-k}, 10\varphi^{2n+2-k}\right) \\
&\leq 10\varphi^{2n+1-k}.
\end{aligned}$$

We obtain the inequlity in the second line by applying Lemma 2.6.5 and the triangle inequality. The inequality in the third line again is obtained by applying the triangle inequality and noting that $m \geq n$.

So now let us consider the case when $x, y \in V_k$ but x is not necessarily connected

to y. There exists a minimal sequence of vertices and joining edges that connect x and y. The number of such edges is bounded above by the graph diameter, which in V_n is the number of edges in the minimal sequence connecting c and 1. This number is $R^{(n)}(c,0) + R^{(n)}(0,1) = F_n + F_{n+1} = F_{n+2}$, and $F_{n+2} \leq \frac{2}{\sqrt{5}}\psi^{n+3}$. So if $m \geq n \geq k$,

$$\left|\frac{R^{(n)}(x,y)}{F_{n+1}} - \frac{R^{(m)}(x,y)}{F_{m+1}}\right| \leq 10F_{n+2}\varphi^{2n+1-k} \leq (4\sqrt{5})\varphi^{n-2-k}.$$

However, knowing that $\frac{R^{(m)}(x,y)}{F_{m+1}} \to R(x,y)$ for $x \sim y$ in V_k, then

$$\left|\frac{R^{(n)}(x,y)}{F_{n+1}} - R(x,y)\right| \leq (4\sqrt{5})\varphi^{n-2-k}$$

just by sending $m \to \infty$. □

Denote the set $\mathcal{F}$ by

$$\mathcal{F} = \{f \in \ell(V_\infty) : \sup_n \mathcal{E}^{(n)}(f|_{V_n}, f|_{V_n}) < \infty\},$$

where $\mathcal{E}^{(n)} = F_{n+1}E^{(n)}$. We can define a Dirichlet form $\mathcal{E}$ on $\mathcal{F}$ where

$$\mathcal{E}(f,f) = \lim_{n\to\infty} \mathcal{E}^{(n)}(f|_{V_n}, f|_{V_n})$$

for $f \in \mathcal{F}$. The following is standard.

Lemma 2.6.7. *Let $f \in \ell(V_\infty)$. Then*

$$|f(y) - f(z)| \leq \sqrt{R(y,z)\mathcal{E}(f,f)}.$$

Since V_∞ is a dense subset of K, we can embed $\mathcal{F}$ into the set of continuous

functions on K. In order to define a Laplacian operator, we need to construct a Dirichlet form on an appropriate L^2 space.

Definition 2.6.8. Let X be a locally compact separable measure space. Let μ be a regular Borel measure on X such that $\mu(O) > 0$ for all open sets $O \subset X$. Let $\mathcal{F}$ be a dense subset of $L^2(X, \mu)$ and let $\mathcal{E}$ be a non-negative symmetric bilinear form on F. Then $(\mathcal{E}, \mathcal{F})$ is called a Dirichlet form on $L^2(X, \mu)$ if:
(1) For $\alpha > 0$, let $\mathcal{E}_\alpha(u, v) = \mathcal{E}(u, v) + \alpha\langle u, v\rangle_\mu$, where $\langle u, v\rangle_\mu = \int_X uv\, d\mu$. Then $(\mathcal{F}, \mathcal{E}_\alpha)$ is a Hilbert space.
(2) $\mathcal{E}$ satisfies the Markov property.

We will also need to define a measure on K in order to define our Laplacian. Let ν_n denote the probability measure on K with support in V_n that assigns each vertex equal weight. We define the measure ν to be the weak limit of the ν_n's. In particular, for $A \subset K$,

$$\nu(A) := \lim_{n\to\infty} \nu_n(A).$$

Proposition 2.6.9. *$(\mathcal{E}, \mathcal{F})$ is a local regular Dirichlet form on $L^2(K, \nu)$. In addition,*

$$\sup_{y,z\in K} |f(y) - f(z)| \leq (1 + \varphi)\sqrt{\mathcal{E}(f, f)}.$$

The proof is standard (c.f. Proposition 4.3.4) and is omitted. Given a Dirichlet form $(\mathcal{E}, \mathcal{F})$, by the machinery of functional analysis we can define a Laplacian on K.

Definition 2.6.10. Define the Laplacian Δ with respect to the measure ν to be the unique operator satisfying

$$\mathcal{E}(f, g) = \langle f, \Delta g\rangle_\nu.$$

The natural inclusion map from $(\mathcal{F}, \mathcal{E} + ||\cdot||_2)$ to $L^2(K,\nu)$ is a compact operator (c.f. Lemma 4.3.5). Thus, Δ is a compact operator whose spectrum consists of eigenvalues. Let

$$\mathcal{F}_0 = \{f \in \mathcal{F}, f|_{V_0} = 0\}.$$

In a similar manner, $(\mathcal{E}, \mathcal{F}_0)$ is a local regular Dirichlet form on $L^2(K,\nu)$. Let Δ_0 be the Laplacian associated to this Dirichlet form. By the following the proof of Proposition 3.4.8 in [19], it can be deduced that Δ_0 is invertible and $G_0 = \Delta_0^{-1}$ is a compact operator on $L^2(K,\nu)$ characterized by $\mathcal{E}(f, G_0 g) = \langle f, g\rangle_\nu$ for any $f \in \mathcal{F}_0$ and $g \in L^2(K,\nu)$.

In order to prove that the spectrum of $\Delta^{(n)}$ converges to that of Δ, we need the notion of a Green's function. The matrix representation of $P^{(n)}$ can be decomposed as

$$P^{(n)} = \begin{bmatrix} S^{(n)} & T^{(n)} \\ U^{(n)} & V^{(n)} \end{bmatrix},$$

where $S^{(n)} : \ell(V_0) \to \ell(V_0)$, $T^{(n)} : \ell(V_n \backslash V_0) \to \ell(V_0)$, $U^{(n)} : \ell(V_0) \to \ell(V_n \backslash V_0)$ and $V^{(n)} : \ell(V_n \backslash V_0) \to \ell(V_n \backslash V_0)$. By Lemma 3.5.1 in [19], the matrix $V^{(n)}$ is invertible. Let $D^{(n)}$ be the diagonal matrix with entries equal to the degrees of the vertices in $V_n \backslash V_0$. Let $G^{(n)} = (D^{(n)} V^{(n)})^{-1}$. Observe that for a function f on V_n such that $f|_{V_0} = 0$ we have

$$E^{(n)}(f, G^{(n)} g) = \langle f, V^{(n)} G^{(n)} g\rangle_{\mu_n} = \langle f, D^{(n)} V^{(n)} (D^{(n)} V^{(n)})^{-1} g\rangle = \langle f, g\rangle,$$

where the last inner product is the standard one. Thus, $G^{(n)}$ is the Green's function corresponding to $P^{(n)}$.

Computing $G^{(n)}$ directly is a lot of work computationally. Furthermore, it is not of much help in trying to define G_0. However, there is an alternative approach.

For $f \in \ell(V_n)$, we say that $h \in \ell(V_m)$, $m > n$, is the m-harmonic extension of f if $h|_{V_n} = f$ and h minimizes $\{E^{(m)}(g,g) : g|_{V_n} = f\}$. $h \in \mathcal{F}$ is said to be the harmonic extension of f if h minimizes $\{\mathcal{E}(g,g) : g|_{V_n} = f\}$. For a finite word w, let K_w denote the set $\phi_w(K)$.

Let $G = G^{(1)}$. Define $\Psi_\emptyset(x,y) = \sum_{x,y \in V_1 \setminus V_0} G_{pq}\psi_p(x)\psi_q(y)$, where $\psi_p(x)$ is the function on K that attains a one at $p \in V_1 \setminus V_0$, a zero at the other vertices of V_1 and is harmonically extended onto K. Let w be a finite word. Define the $|w|+1$ harmonic function

$$\Psi_w(x,y) = \begin{cases} \Psi_\emptyset((\phi_w)^{-1}(x), (\phi_w)^{-1}(y)) & : \text{ if } x, y \in K_w \\ 0 & : \text{ otherwise} \end{cases}$$

Set $\Psi_\emptyset^x(y) = \Psi_\emptyset(x,y)$. For $n > 1$, we construct a function $\Psi_{\emptyset,n}^x(y)$ from $\Psi_\emptyset^x(y)$ in the following manner. First, note that the removal of 0 and $|c|^2$ from K breaks it into four pieces, and thus V_n can be decomposed accordingly. By restricting $\Psi_\emptyset$ to V_1, extending harmonically to V_n, restricting to each component component and extending harmonically back onto K, we obtain four functions. Denote these four functions by p_n, q_n, r_n and s_n. Scale them by the appropriate resistance ($R^{(n)}(\phi_0(0), \phi_0(1))$, $R^{(n)}(\phi_0(0), \phi_0(c))$, $R^{(n)}(\phi_1(0), \phi_1(1))$ and $R^{(n)}(\phi_1(0), \phi_1(c))$, respectively) and add the functions together. This sum will not be zero on V_0, so we make an adjustment. Let t_n be the function that attains the aforementioned values on V_0, is extended harmonically onto V_n, and finally onto K. Let $\Psi_{\emptyset,n}^x$ be the result after subtracting t_n. By our

construction and Proposition 2.6.2, for any $f \in \ell(V_n)$ such that $f|_{V_0} = 0$ we have

$$E^{(1)}(f, \Psi^x_\emptyset) = E^{(n)}(f, \Psi^x_{\emptyset,n}). \tag{2.6.1}$$

In addition, $\Psi^x_{\emptyset,n}$ is zero on V_0. Set $\Psi_{\emptyset,n}(x, y) = \Psi^x_{\emptyset,n}(y)$.

For $n \geq |w| + 1$, define

$$\Psi_{w,n}(x, y) = \begin{cases} \Psi_{\emptyset,n-|w|-1}((\phi_w)^{-1}(x), (\phi_w)^{-1}(y)) & : \text{ if } x, y \in K_w \\ 0 & : \text{ otherwise} \end{cases}$$

Note that in the case where $n = |w| + 1$, $\Psi_{w,n} = \Psi_w$. For $f \in \mathcal{F}$, let f_n denote the function that is equal to f on V_n and extended harmonically onto K.

Lemma 2.6.11. *Let $f \in \mathcal{F}_0$. Then*

$$E^{(|w|+1)}(f, \Psi^x_w) = \begin{cases} f_{|w|+1}(x) - f_{|w|}(x) & : \textit{ if } x \in K_w \\ 0 & : \textit{ otherwise} \end{cases}$$

Furthermore, if $n > |w| + 1$,

$$E^{(n)}(f, \Psi^x_{w,n}) = \begin{cases} f_{|w|+1}(x) - f_{|w|}(x) & : \textit{ if } x \in K_w \\ 0 & : \textit{ otherwise} \end{cases}$$

Proof. Note that the second statement follows from the first by Equation 2.6.1. Thus, it suffices to prove the first statement. For $f \in \mathcal{F}_0$,

$$E^{(1)}(f, \Psi^x_\emptyset) = E^{(1)}(f - f_0, \Psi^x_\emptyset) = \sum_{p,q \in V_1 \setminus V_0} \mu_1(p)(f(p) - f_0(p)) V^{(1)}_{pq} \Psi^x_\emptyset(q)$$

$$= \sum_{p \in V_1 \setminus V_0} (f(p) - f_0(p))\psi_p(x) = f_1(x) - f_0(x).$$

Therefore, if $w \in W_{k-1}$, $x \in K_w$ and $z = F_w^{-1}(x)$, then

$$E^{(k)}(f, \Psi_w^x) = \sum_{v \in W_{k-1}} E^{(1)}(f \circ F_v, \Psi_w^x \circ F_v) = E^{(1)}(f \circ F_w, \Psi_\emptyset^z)$$

$$= ((f \circ F_w)_1(z) - (f \circ F_w)_0(z)) = f_k(x) - f_{k-1}(x). \quad \square$$

Fix $w \in W_{n-1}$. For $1 \le k \le n-1$, let $w[k] = w_1 w_2 ... w_k$. Suppose $x \in K_w$. Then by the previous lemma we know

$$\begin{aligned}
E^{(n)}(f, \Psi_{\emptyset,n}^x) &= f_1(x) - f_0(x) \\
E^{(n)}(f, \Psi_{w[1],n}^x) &= f_2(x) - f_1(x) \\
&\vdots \\
E^{(n)}(f, \Psi_{w[n-2],n}^x) &= f_{n-1}(x) - f_{n-2}(x) \\
E^{(n)}(f, \Psi_w^x) &= f_n(x) - f_{n-1}(x).
\end{aligned}$$

Set

$$g_n^x(y) = \frac{1}{F_{n+1}} \sum_{k=0}^{n-1} \sum_{w \in W_k} \Psi_{w,n}^x(y),$$

and let $g_n(x,y) = g_n^x(y)$. Putting everything together, we have

$$\mathcal{E}^{(n)}(f, g_n^x) = f_n(x) - f_0(x),$$

for $f \in \mathcal{F}_0$. Thus, $g_n(x,y)$ coincides with the corresponding entry of $G^{(n)}$ for $x, y \in V_n$.

Recall that removing 0 and $|c|^2$ divides K into four connected componenets. By re-

stricting $\Psi_\emptyset$ to V_1, extending harmonically to K, restricting to each of the components and then extending harmonically back onto K, we obtain four functions. Denote the four functions by p, q, r and s. Let $\Psi^x_{\emptyset,\infty}$ be the function obtained after scaling p, q, r, and s by $R(\phi_0(0),\phi_0(1))$, $R(\phi_0(0),\phi_0(c))$, $R(\phi_1(0),\phi_1(1))$ and $R(\phi_1(0),\phi_1(c))$, adding the functions together, and adjusting the sum by a function t that is equal to the sum on V_0 and extended harmonically onto K. For a finite word w, define

$$\Psi_{w,\infty}(x,y) = \begin{cases} \Psi_{\emptyset,\infty}((F_w)^{-1}(x),(F_w)^{-1}(y)) & : \text{ if } x,y \in K_w \\ 0 & : \text{ otherwise} \end{cases}$$

Proposition 2.6.12. *For $x, y \in K$, let*

$$g(x,y) = \lim_{n\to\infty} g_n(x,y).$$

This limit exists. Moreover, the convergence is uniform and

$$g(x,y) = \sum_{w\in W_\infty} \Psi^x_{w,\infty}(y).$$

Proof. By Proposition 2.6.6, it is clear that $\frac{1}{F_{n+1}}\Psi^x_{\emptyset,n}$ converges to $\Psi^x_{\emptyset,\infty}$ and that $\frac{1}{F_{n+1}}\Psi^x_{w,n}$ converges to $\Psi^x_{w,\infty}$ pointwise for a finite word w. Thus, if $\lim_{n\to\infty} g_n(x,y)$ exists, it must be $\sum_{w\in W_\infty}\Psi^x_{w,\infty}(y)$.

We will now estimate $||\frac{1}{F_{n+1}}\Psi^x_{\emptyset,n} - \Psi^x_{\emptyset,\infty}||_\infty$. By the triangle inequality,

$$\begin{aligned} \left|\left|\frac{R^{(n)}(\phi_0(0),\phi_0(c))}{F_{n+1}} q_n - R(\phi_0(0),\phi_0(c))q\right|\right|_\infty \quad \leq \quad & \left|\frac{R^{(n)}(\phi_0(0),\phi_0(c))}{F_{n+1}} - R(\phi_0(0),\phi_0(c))\right| ||q_n||_\infty \\ & + R(\phi_0(0),\phi_0(c))||q_n - q||_\infty. \end{aligned}$$

Let $C = \max_{x,y\in V_1} \Psi^x_\emptyset(y)$. Then $||q_n||_\infty \le C$. By Proposition 2.6.6,

$$\left|\frac{R^{(n)}(\phi_0(0),\phi_0(c))}{F_{n+1}} - R(\phi_0(0),\phi_0(c))\right| \le (4\sqrt{5})\varphi^{n-2}$$

and

$$||q_n - q||_\infty \le 2C(4\sqrt{5})\varphi^{n-2}.$$

Putting everything together,

$$\left|\left|\frac{R^{(n)}(\phi_0(0),\phi_0(c))}{F_{n+1}} q_n - R(\phi_0(0),\phi_0(c))q\right|\right|_\infty \le C(1+2R(\phi_0(0),\phi_0(c)))(4\sqrt{5})\varphi^{n-2}.$$

We can derive similar inequalities for p_n, r_n and s_n. For t_n, observe that

$$||t_n - t||_\infty \le \left|\left|\tfrac{R^{(n)}(\phi_0(0),\phi_0(1))}{F_{n+1}} p_n - R(\phi_0(0),\phi_0(1))p\right|\right|_\infty + \left|\left|\tfrac{R^{(n)}(\phi_0(0),\phi_0(c))}{F_{n+1}} q_n - R(\phi_0(0),\phi_0(c))q\right|\right|_\infty +$$

$$\left|\left|\tfrac{R^{(n)}(\phi_1(0),\phi_1(1))}{F_{n+1}} r_n - R(\phi_1(0),\phi_1(1))r\right|\right|_\infty + \left|\left|\tfrac{R^{(n)}(\phi_1(0),\phi_1(c))}{F_{n+1}} s_n - R(\phi_1(0),\phi_1(c))s\right|\right|_\infty.$$

Thus,

$$\left|\left|\frac{1}{F_{n+1}}\Psi^x_{\emptyset,n} - \Psi^x_{\emptyset,\infty}\right|\right|_\infty \le 8C(1+2\varphi)(4\sqrt{5})\varphi^{n-2}.$$

In a similar manner,

$$\left|\left|\frac{1}{F_{n+1}}\Psi^x_{w,n} - \Psi^x_{w,\infty}\right|\right|_\infty \le 8C(1+2\varphi^{|w|+1})(4\sqrt{5})\varphi^{n-2-|w|}.$$

Finally, using the fact that $\Psi^x_{w,n}(y)$ will be non-trivial for at most one $w \in W_k$, we

get

$$\begin{aligned}\left\|\frac{1}{F_{n+1}}\sum_{k=0}^{n-1}\sum_{w\in W_k}\Psi^x_{w,n}(y)-\sum_{w\in W_\infty}\Psi^x_{w,\infty}(y)\right\|_\infty &\le \left\|\sum_{k=0}^{n-1}\sum_{w\in W_k}\left(\frac{1}{F_{n+1}}\Psi^x_{w,n}(y)-\Psi^x_{w,\infty}(y)\right)\right\|_\infty \\ &\quad +\left\|\sum_{k=n}^{\infty}\sum_{w\in W_k}\Psi^x_{w,\infty}(y)\right\|_\infty \\ &\le \sum_{k=0}^{n-1}8C(1+2\varphi^{k+1})(4\sqrt{5})\varphi^{n-2-k}+C\sum_{k=n}^{\infty}\varphi^{k+1}.\end{aligned}$$

The right hand side converges to zero as $n \to 0$. □

The following proposition will show that $g(x,y)$ is the kernel of G_0.

Proposition 2.6.13. *For $f \in L^2(K,\nu)$,*

$$(G_\nu f)(y) = \int_K f(x)g(x,y)\,\nu(dx),$$

is well-defined for all $y \in K$ and $G_\nu f \in C(K)\cap\mathcal{F}_0$. Moreover, $G_\nu : L^2(K,\nu) \to C(K)$ is a compact operator. Also

$$\mathcal{E}(u, G_\nu f) = \int_K uf\,d\nu$$

for any $u \in \mathcal{F}_0$.

Proof. As the uniform limit of uniformly continuous functions, $g(x,y)$ is continuous. Thus, it is clear that $G_\nu f$ is well-defined and contained in $C(K)$. The fact that $G_\nu : L^2(K,\nu) \to C(K)$ is a compact operator can be deduced by a standard argument using Ascoli-Arzela's theorem.

Let

$$(G_{\nu_n}^{(n)} f)(x) = \int_K g_n(x,y) f(y)\, \nu_n(dy).$$

Like G_{ν_n}, $G_{\nu_n}^{(n)}$ is a well-defined compact operator from $L^2(K,\nu)$ to $C(K)$. By construction, we know that $\mathcal{E}^{(n)}(u, g_n^x) = u(x)$. So

$$\mathcal{E}^{(n)}\left(u, \sum_{x \in V_n} f(x) g_n^x\right) = f(x) \sum_{x \in V_n} u(x).$$

Thus, we can deduce

$$\mathcal{E}^{(n)}(u, G_{\mu_n}^{(n)} f) = \int_K uf\, d\nu_n.$$

Taking the limit as $n \to \infty$ we obtain

$$\mathcal{E}(u, G_\nu f) = \int_K uf\, d\nu.$$

This is due to the previous proposition and the fact that ν is the weak limit of the ν_n's. Since $G_{\nu_n}^{(n)} f$ is zero on V_0, so is $G_\nu f$. Thus, $G_\nu f \in \mathcal{F}_0$. □

Remark 2.6.14. Recall that the operator G_0 is characterized by $\mathcal{E}(f, G_0 g) = \langle f, g \rangle_\nu$. Comparing with the above proposition, G_0 and G_ν must coincide.

By the spectral theorem, if λ is an eigenvalue of $P_0^{(n)}$, then λ^{-1} is an eigenvalue of $G^{(n)}$. Observe that $G_{\nu_n}^{(n)}$ is an extension of $G^{(n)}$ from $\ell(V_n)$ to $L^2(K,\nu_n)$. An eigenvector of $G^{(n)}$ can be made into an eigenvector of $G_{\nu_n}^{(n)}$ by taking the harmonic extension onto K. Due to the scaling of energies and measures, the corresponding eigenvalue is scaled by $(2^{n+1}+1)/F_{n+1}$.

Proposition 2.6.15. *Let $K \subset \mathbb{R}^+$ be compact. The limit of $\dfrac{F_{n+1}}{2^{n+1}+1}\sigma(P_0^{(n)}) \cap K$ in the Hausdorff metric is $\sigma(\Delta_0) \cap K$.*

Proof. By Proposition 2.6.12, the kernel of $G_{\nu_n}^{(n)}$ converges uniformly to that of G_ν. Thus, it can be deduced that the operator $G_{\nu_n}^{(n)}$ converges in norm to that of G_ν. By standard functional analysis (c.f. [36]), since the operators are all bounded, $\sigma(G_{\nu_n}^{(n)})$ converges to $\sigma(G_\nu)$ in the Hausdorff metric.

Let $K \subset \mathbb{R}^+$ be compact. Let $K^{-1} = \{x^{-1} : x \in K\}$, which must also be compact. It is clear that $\sigma(G_{\nu_n}^{(n)}) \cap K^{-1}$ must converge in the Hausdorff metric to $\sigma(G_\nu) \cap K^{-1}$. Recall by the discussion preceding the proposition that $\lambda \in \dfrac{F_{n+1}}{2^{n+1}+1}\sigma(P_0^{(n)})$ if and only if $\lambda^{-1} \in \sigma(G_{\nu_n}^{(n)})$. Thus, $\dfrac{F_{n+1}}{2^{n+1}+1}\sigma(P_0^{(n)}) \cap K$ must converge in the Hausdorff metric to $\sigma(\Delta_0) \cap K$. □

Naturally, one would like to have a similar result for the Neumann Laplacian. However, this is left as an open problem. One possible strategy is to somehow define a Neumann's Green's function for the approximating operators and the Neumann Laplacian itself and to show that there is convergence in norm.

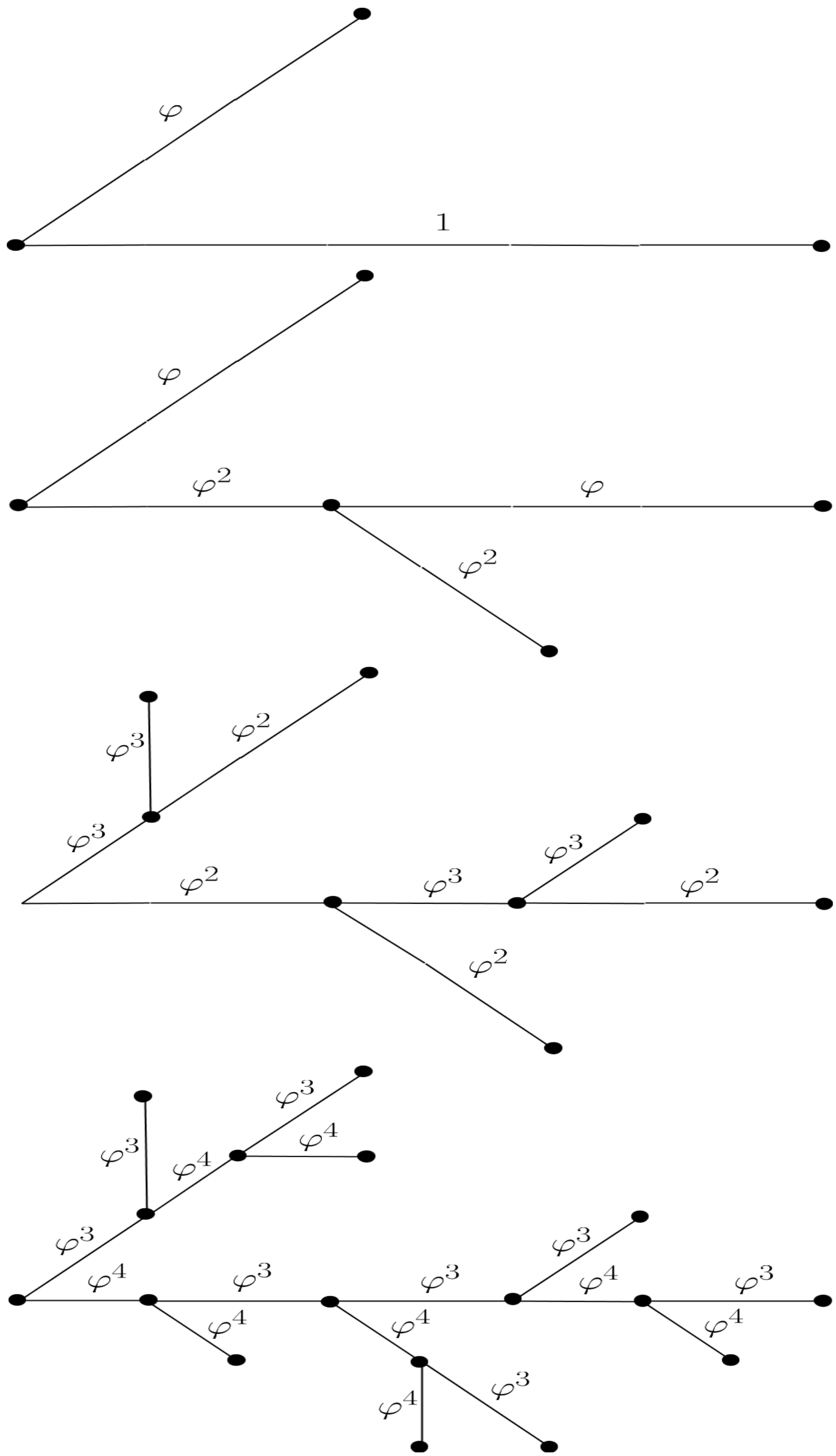


FIGURE 2.6.1: Values of R between vertices of V_n, $n = 0, 1, 2, 3$

Chapter 3

The Sabot Theory

3.1 Density of States

In [29], the author computes the spectrum of Laplacian operators on self-similar lattices that satisfy certain symmetries. Although the graphs V_{-n} do not satisfy these symmetry conditions, by carefully checking certain features it is possible to apply the theory.

First, we slightly modify the graphs V_{-n} determined by some infinite word $\omega \in \{0,1\}^{\mathbb{N}}$. In this chapter, we will assume that the graph V_0 is complete. I.e., we assume the existence of an extra edge between w_0 and c_0. The graphs V_{-n} are constructed in the same manner as in Section 2.2, and we will use the same notation.

We begin by constructing a set of symmetric operators and measures on the V_{-n}. Let $A = A^{(0)}$ be a linear symmetric operator on V_0. Its matrix representation is given by:

$$A^{(0)} = \left(\begin{smallmatrix} a & d & f \\ d & b & e \\ f & e & c \end{smallmatrix}\right).$$

Let the first, second and third rows correspond to c_0, z_0 and w_0, respectively.

For $i = 0, 1$, let A_i be a copy of A on the cell $(V_{-1})_i$. If $x \notin (V_{-1})_i$, then we let $A_i f(x) = 0$ for any function f on V_{-1}. We define the operator $A^{(1)}$ on V_{-1} as the sum of these two operators, i.e.,

$$A^{(1)} = \sum_{i=0,1} A_i.$$

In matrix form,

$$A^{(1)} = \begin{pmatrix} c & e & f & 0 & 0 \\ e & b & d & 0 & 0 \\ f & d & a+b & d & e \\ 0 & 0 & d & a & f \\ 0 & 0 & e & f & c \end{pmatrix}.$$

The first three and last three rows correspond to the two copies of A, and the third row in particular corresponds to the gluing point ($c_1 = z_1'$ if $\omega_1 = 0$ and $z_1 = c_1'$ if $\omega_1 = 1$. However, this is immaterial in the matrix representation.)

In general, let $A_{i_1 \ldots i_n}$ be a copy of A on the cell $(V_{-n})_{i_1 \ldots i_n}$. As before, if $x \notin (V_{-n})_{i_1 \ldots i_n}$, we let $A_{i_1 \ldots i_n} f(x) = 0$ for any function f on V_{-n}. We define the operator $A^{(n)}$ on V_{-n} by

$$A^{(n)} = \sum_{w \in W_n} A_w.$$

In a similar manner, we can construct a sequence of measures on V_{-n}. Let β be the measure on V_0. We define the measure β_n on V_{-n} by

$$\beta_n = \sum_{w \in W_n} b_w.$$

where β_w is a copy of β on $(V_{-n})_w$.

The sequence $A^{(n)}$ and b_n form an inductive sequence because for $n \geq p$, if

$\mathrm{supp}(f) \subset \mathrm{int}(V_{-p}) \cup (\partial V_{-p} \cap \partial V_{-n})$, then

$$A^{(n)} f = A^{(p)} f \text{ and } \int f d\,\beta_n = \int f\, db_p.$$

It is clear we can extend the measures β_n to a measure β_∞ on $V_{-\infty}$. To construct the operator $A^{(\infty)}$, first observe that we can can define $A^{(\infty)}$ on a compactly supported function f to equal $A^{(p)} f$ for some p such that $\mathrm{supp}(f) \subset V_{-p}$. It is then possible to extend the definition to an arbitrary function on $V_{-\infty}$.

Let $\langle \cdot, \cdot \rangle$ denote the usual scalar product on $\mathbb{R}^{V_{-n}}$. Let $H_+^{(n)}$ be the operator on $L^2(V_{-n}, \beta_n)$ defined by

$$\langle A^{(n)} f, g \rangle = -\int H_+^{(n)} f g \, d\beta_n \quad \forall f, g \in \mathbb{R}^{V_{-n}}.$$

The operator $H_+^{(n)}$ is semi-negative, self-adjoint, and can be viewed as a discrete difference operator with Neumann boundary condition on ∂V_{-n} since no conditions are imposed on the boundary points. We denote by $H_-^{(n)}$ the self-adjoint operator on $\mathbb{R}^{\mathrm{int}(V_{-n})}$ defined as the restriction of $H_+^{(n)}$ to $\mathbb{R}^{\mathrm{int}(V_{-n})} = \{f \in \mathbb{R}^{V_{-n}}, f|_{\partial V_{-n}} = 0\}$. Since the domain consists of functions that vanish at the boundary, $H_-^{(n)}$ is said to have a Dirichlet boundary condition.

There exists $K > 0$ such that $\langle Af, f \rangle \leq K \int f^2 \, db$ for all $f \in \mathbb{R}^{V_0}$. By definition, the same inequality must hold true for $A^{(n)}$, b_n and $f \in \mathbb{R}^{V_{-n}}$. So the sequence of operators $H_+^{(n)}$ is uniformly bounded for the operator norm on $L^2(V_{-n}, b_n)$ and can be extended into a semi-negative, self-adjoint operator $H_+^{(\infty)}$ on $\mathcal{D}_\infty = L^2(V_{-\infty}, b_\infty)$. Define $H_-^{(\infty)}$ as the restriction of $H_+^{(\infty)}$ to $\mathcal{D}^- = \{f \in \mathcal{D}_\infty^+, f|_{\partial V_{-\infty}} = 0\}$. We must

have

$$\langle A^{(\infty)} f, g\rangle = -\int H_{\pm}^{(\infty)} f g \, d\beta_{\infty} \quad \forall f, g \in \mathcal{D}_{\infty}^{\pm}.$$

Note that if $\partial V_{-\infty} = \emptyset$ then the operators $H_{+}^{(\infty)}$ and $H_{-}^{(\infty)}$ are equal and we simply write $H^{(\infty)}$.

The goal of the theory is to analyze the spectrum of $H_{\pm}^{(\infty)}$. This is done by analyzing the eigenvalues of $H_{\pm}^{(n)}$. Denote by

$$0 = \lambda_{n,1}^{+} > \lambda_{n,2}^{+} \geq \cdots \geq \lambda_{n,|V_{-n}|}^{+}.$$

the eigenvalues of $H_{+}^{(n)}$. Denote by

$$0 = \lambda_{n,1}^{-} > \lambda_{n,2}^{-} \geq \cdots \geq \lambda_{n,|\mathrm{int}(V_{-n})|}^{-}.$$

the eigenvalues of $H_{-}^{(n)}$. Let ν_n^{+} and ν_n^{-} be the counting measures of the Neumann and Dirichlet spectrum, respectively. I.e., we have

$$\nu_n^{\pm} = \sum_k \delta_{\lambda_{n,k}^{\pm}}.$$

Definition 3.1.1. If the limit

$$\lim_{n\to\infty} \frac{1}{2^n} \nu_n^{\pm}$$

exists and does not depend on the choice of the boundary condition then it is called the density of states and denoted by μ.

If $\lambda_{n,k}$ is an eigenvalue of both $H_{+}^{(n)}$ and $H_{-}^{(n)}$, then we say that is is a Neumann-Dirichlet (ND) eigenvalue. Let ν_n^{ND} denote the corresponding counting measure.

Definition 3.1.2. If the limit

$$\lim_{n\to\infty} \frac{1}{2^n} \nu_n^{ND}$$

exists then it is called the density of states of ND eigenvalues and denoted by μ^{ND}.

We now make two remarks. The first is that $\frac{1}{2^n}$ is an appropriate normalization factor since the number of Neumann and Dirichlet eigenvalues from level to level roughly doubles. The second is that the existence of the density of states must be established. Once we check the appropriate things, by the theory in [29] we will have existence.

3.2 The Maps T and R

Let V be a finite set and $V' \subset V$ a subset.

Definition 3.2.1. Let Q be a linear symmetric operator on V. Let $Q_{V\setminus V'}$ denote the operator with the rows and columns corresponding to $V\setminus V'$ removed. We define the trace of Q onto V' the $V' \times V'$ matrix $Q|_{V'}$, where

$$Q|_{V'} = ((Q^{-1})_{V\setminus V'})^{-1}.$$

The following is well known.

Proposition 3.2.2. *If Q has the following block decomposition on V*

$$Q = \begin{pmatrix} Q_{V\setminus V'} & B \\ B^T & Q_V \end{pmatrix},$$

then

$$Q|_{V'} = Q_{V\setminus V'} - BQ_V^{-1}B^T.$$

I.e., the trace of Q onto V' is given by taking the Schur complement of the above block matrix.

In order to be in a position to use the Sabot theory, we must check that the trace of an operator $A^{(n)}$ onto ∂V_{-n} can be found by iteration of a rational map. Let Sym_3 denote the space of 3×3 matrices. We define

$$\begin{aligned} T: \ \mathrm{Sym}_3 &\to \mathrm{Sym}_3 \\ Q &\mapsto (Q^{(1)})|_{\partial V_{-1}}. \end{aligned}$$

Here, $Q^{(1)}$ denotes the operator on V_{-1} constructed in the same way as $A^{(1)}$. In matrix notation, we have

$$\begin{pmatrix} a & d & f \\ d & b & e \\ f & e & c \end{pmatrix} \mapsto \begin{pmatrix} a^{(1)} & d^{(1)} & f^{(1)} \\ d^{(1)} & b^{(1)} & e^{(1)} \\ f^{(1)} & e^{(1)} & c^{(1)} \end{pmatrix},$$

where

$$a^{(1)} = c - \frac{af^2}{a^2+ab-d^2}, \quad b^{(1)} = b - \frac{ad^2}{a^2+ab-d^2}, \quad d^{(1)} = e - \frac{adf}{a^2+ab-d^2}, \quad e^{(1)} = \frac{d(-ae+df)}{a^2+ab-d^2}$$
$$c^{(1)} = \frac{-a^2c+cd^2+f(-2de+bf)+a(-bc+e^2+f^2)}{-a^2-ab+d^2}, \quad f^{(1)} = \frac{f(-ae+df)}{a^2+ab-d^2}.$$

It is possible to compute the trace of $Q^{(n)}$ onto V_{-n} by iterating n times the map T. Let us denote by $a^{(n)}, b^{(n)}, c^{(n)}, d^{(n)}, e^{(n)}, f^{(n)}$ the entries in $T^n(Q)$.

Proposition 3.2.3. *$(Q^{(n)})|_{\partial V_{-n}} = T^n(Q)$.*

Proof. The case $n = 1$ follows by definition. Suppose the statement is true for $n = k$.

The matrix $Q^{(k+1)}$ has the following block decomposition:

$$Q^{(k+1)} = \left(\begin{array}{ccccc|c|c}
c & 0 & 0 & 0 & 0 & & 0 \\
0 & b & 0 & 0 & 0 & A & \\ \cline{7-7}
0 & 0 & c+b & 0 & 0 & & \\ \cline{6-6}
0 & 0 & 0 & c & 0 & & B \\
0 & 0 & 0 & 0 & c & 0 & \\ \hline
\multicolumn{3}{c|}{A^T} & \multicolumn{2}{c|}{0} & C & 0 \\ \hline
\multicolumn{2}{c|}{0} & \multicolumn{3}{c|}{B^T} & 0 & D
\end{array}\right).$$

The upper left block corresponds to the boundaries of V_{-k} and V'_{-k} and row three in particular corresponds to $c_k = z'_k$. We now take the trace onto these five points, which in matrix terms corresponds to taking the Schur complement. Let $S = \partial V_{-k} \cup \partial V'_{-k}$. Then by linearity

$$Q|_S = \left(\begin{array}{c|c} \left(\begin{smallmatrix} c & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{smallmatrix}\right) - AC^{-1}A^T & 0 \\ \hline 0 & 0 \end{array}\right) + \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & \left(\begin{smallmatrix} b & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & c \end{smallmatrix}\right) - BD^{-1}B^T \end{array}\right).$$

The non-zero blocks in each bracketed term correspond to the trace of the copy of $A^{(k)}$ on V_{-k} and V'_{-k} onto ∂V_{-k} and $\partial V'_{-k}$, respectively. By the induction step, these terms equal $T^k(Q)$, up to a permutation of the rows. Thus,

$$\begin{aligned}
Q|_S &= \left(\begin{array}{c|c} T^k(Q) & 0 \\ \hline 0 & 0 \end{array}\right) + \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & T^k(Q) \end{array}\right) \\
&= \left(\begin{smallmatrix} c^{(k)} & e^{(k)} & f^{(k)} & 0 & 0 \\ e^{(k)} & b^{(k)} & d^{(k)} & 0 & 0 \\ f^{(k)} & d^{(k)} & a^{(k)} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{smallmatrix}\right) + \left(\begin{smallmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & b^{(k)} & d^{(k)} & e^{(k)} \\ 0 & 0 & d^{(k)} & a^{(k)} & f^{(k)} \\ 0 & 0 & e^{(k)} & f^{(k)} & c^{(k)} \end{smallmatrix}\right).
\end{aligned}$$

If we take the trace of $Q|_S$ onto $\partial V_{-(k+1)}$, then by definition this is $T\Big(\begin{pmatrix} a^{(k)} & d^{(k)} & f^{(k)} \\ d^{(k)} & b^{(k)} & e^{(k)} \\ f^{(k)} & e^{(k)} & c^{(k)} \end{pmatrix}\Big) = T^{k+1}\Big(\begin{pmatrix} a & d & f \\ d & b & e \\ f & e & c \end{pmatrix}\Big)$. $\square$

With this proposition established, the next goal is to construct the analogue of T on a Grassmann algebra. We begin by first defining the Grassmann algebra.

Definition 3.2.4. Identify V_0 with $\{1, 2, 3\}$. Let $\bar{E}$ and E be two copies of $\mathbb{C}^{V_0}$ with canonical basis $(\bar{\eta})_{x \in V_0}$ and $(\eta_x)_{x \in V_0}$. The Grassman algebra $\wedge(\bar{E} \oplus E)$ is defined by

$$\wedge(\bar{E} \oplus E) = \oplus_{k=0}^{6} (\bar{E} \oplus E)^k.$$

where $\wedge$ denotes the exterior product.

Denote by $\mathcal{A}$ the subalgebra generated by monomials containing the same number of variables $\bar{\eta}$ and η, i.e.,

$$\mathcal{A} = \oplus_{k=0}^{3} \bar{E}^{\wedge k} \wedge E^{\wedge k}.$$

A canonical basis for $\mathcal{A}$ is

$$\{1, \bar{\eta}_{i_1} \wedge \cdots \wedge \bar{\eta}_{i_k} \wedge \eta_{j_1} \wedge \cdots \wedge \eta_{j_k}, i_1 < \cdots < i_k, j_1 < \cdots < j_k, 1 \leq k \leq 3\}.$$

We can define a scalar product $\langle \cdot, \cdot \rangle$ on $\mathcal{A}$ that makes this basis orthonormal. To simplify notation, from now on we write $\eta_i \eta_j$ for $\eta_i \wedge \eta_j$.

If $Q \in \mathrm{Sym}_3$, we denote by $\bar{\eta} Q \eta$ the element of $\mathcal{A}$:

$$\bar{\eta} Q \eta = \sum_{i,j \in F} Q_{i,j} \bar{\eta}_i \eta_j.$$

We will be interested in terms of the form

$$\begin{aligned}
\exp(\bar{\eta}Q\eta) &= \sum_{k=0}^{\infty} \frac{1}{k!}\Big(\sum_{i,j} Q_{i,j}\bar{\eta}_i\eta_j\Big)^k \\
&= \sum_{k=0}^{n} \sum_{\substack{i_1<\cdots<i_k \\ j_1\cdots<j_k}} \det\big((Q)_{\substack{i_1<\cdots<i_k \\ j_1<\cdots<j_k}}\big)\bar{\eta}_{i_1}\eta_{j_1}\cdots\bar{\eta}_{i_k}\eta_{j_k}.
\end{aligned}$$

If Y is in $\mathcal{A}$, we denote by i_Y the interior product by Y, i.e., the linear operator $i_Y : \mathcal{A} \to \mathcal{A}$ defined by

$$\langle i_Y(X), Z\rangle = \langle X, YZ\rangle \quad \forall X, Z \in \mathcal{A}.$$

If V' is a subset of V_0, we denote by $\mathcal{A}_{V'}$ the subalgebra generated by the variables $(\bar{\eta}_x)_{x\in V'}, (\eta_x)_{x\in V'}$. We define the linear operator $R_{V_0\to V'}$ by

$$\begin{aligned}
R_{V\to V'} : \mathcal{A} &\to \mathcal{A}_{V'} \\
X &\mapsto i_{\prod_{x\in V_0\setminus V'}\bar{\eta}_x\eta_x}(X).
\end{aligned}$$

The following is Proposition 2.2 in [29].

Proposition 3.2.5. *Let $Q \in \mathrm{Sym}_3$. Then*

$$\det(Q) = \langle R_{V_0\to V'}(\exp(\bar{\eta}Q\eta), \prod_{x\in V'}\bar{\eta}_x\eta_x\rangle,$$

$$\det(Q_{V'}) = \langle R_{V_0\to V'}(\exp(\bar{\eta}Q\eta), 1\rangle,$$

and

$$\exp(\bar{\eta}Q|_{V'}\eta) = \frac{R_{V_0\to V'}(\exp(\bar{\eta}Q\eta)}{\det(Q_{V'})}$$

when $\det(Q_{V'}) \neq 0$.

We are now in a position to define a map $R : \mathcal{A} \to \mathcal{A}$ that corresponds to T. Let $\mathcal{A}_1$ be the counterpart to $\mathcal{A}$ for the set V_{-1}. Let $s_i, i = 0, 1$ denote the canonical injections from V_0 into $(V_{-1})_i$. These maps naturally induce the morphism $s_i : \mathcal{A} \to \mathcal{A}_1$ defined on the generators by $(\bar{\eta}_x, \eta_x) \mapsto (\bar{\eta}_{s_i(x)}, \eta_{s_i(x)})$. We define the map τ by

$$\begin{aligned} \tau : \mathcal{A} \ &\to \ \mathcal{A}_1 \\ X \ &\mapsto \ s_0(X)s_1(X). \end{aligned}$$

By previous definitions, the map τ satisfies

$$\exp(\bar{\eta}Q^{(1)}\eta) = \tau\big(\exp(\bar{\eta}Q\eta)\big).$$

We define the map R by

$$R = R_{V_{-1}\to\partial V_{-1}} \circ \tau.$$

Proposition 3.2.6. *(i). The map R is a homogeneous polynomial of degree 2.*
(ii). The following relation holds

$$R^n(\exp(\bar{\eta}Q\eta)) = \det\big(Q^{(n)}|_{\mathrm{int}(V_{-n})}\big)\exp(\bar{\eta}T^nQ\eta). \tag{3.2.1}$$

Proof. (i). τ is a homogeneous polynomial of degree 2 in the coefficients of X and τ is a linear map.
(ii). By Proposition 3.2.3 we know

$$T^n(Q) = Q^{(n)}|_{\partial V_{-n}}. \tag{3.2.2}$$

It suffices to prove that

$$R^n(\exp(\bar{\eta}Q\eta)) = i_{\prod_{x\in\mathrm{int}(V_{-n})}\bar{\eta}_x\eta_x}(\exp(\bar{\eta}Q^{(n)}\eta)). \tag{3.2.3}$$

because equation 3.2.1 is a direct consequence of equations 3.2.2 and 3.2.3 and Proposition 3.2.5.

We prove equation 3.2.3 by induction. Suppose equation 3.2.3 holds for n. Then by the induction step

$$R^n(\exp(\bar{\eta}Q\eta)) = i_{\prod_{x\in\mathrm{int}(V'_{-n})}}(\exp(\bar{\eta}Q'^{(n)}\eta)).$$

Observe that

$$\begin{aligned}\tau(R^n(\exp(\bar{\eta}Q\eta))) &= \left(i_{\prod_{x\in\mathrm{int}(V_{-n})}\bar{\eta}_x\eta_x}(\exp(\bar{\eta}Q^{(n)}\eta))\right)\left(i_{\prod_{x\in\mathrm{int}(V'_{-n})}}(\exp(\bar{\eta}Q'^{(n)}\eta))\right)\\ &= i_{\prod_{x\in\mathrm{int}(V_{-n})\cup\mathrm{int}(V'_{-n})}\bar{\eta}_x\eta_x}(\exp(\bar{\eta}Q^{(n+1)}\eta)).\end{aligned}$$

Each term in the expression above will be indexed by $x \in \partial V_{-n} \cup \partial V'_{-n}$. By identifying this set with V_{-1} and applying the map $R_{V_{-1}\to\partial V_{-1}}$, we prove the induction step. $\square$

As in [29], it is possible to embed Sym_3 into a certain projective space. Denote by $\mathcal{P}(\mathcal{A})$ the projetive space associated with $\mathcal{A}$ where $\pi : \mathcal{A} \to \mathcal{P}(\mathcal{A})$ is the canonical projection. Denote by $\mathbb{L}^3$ the closure in $\mathcal{P}(\mathcal{A})$ of elements of the form $\pi(\exp(\bar{\eta}Q\eta))$ for $Q \in \mathrm{Sym}_3$.

The map R on $\pi^{-1}(\mathbb{L}^3) \cup \{0\}$ is given by

$$R(Z1 + a\bar{\eta}_1\eta_1 + b\bar{\eta}_2\eta_2 + c\bar{\eta}_3\eta_3 + D_{1,2}\bar{\eta}_1\eta_1\bar{\eta}_2\eta_2 + D_{1,3}\bar{\eta}_1\eta_1\bar{\eta}_3\eta_3 + D_{2,3}\bar{\eta}_2\eta_2\bar{\eta}_3\eta_3 + D_{1,2,3}\bar{\eta}_1\eta_1\bar{\eta}_2\eta_2\bar{\eta}_3\eta_3) =$$
$$(a^2 + ZD_{1,2})1 + (aD_{1,3} + cD_{1,2})\bar{\eta}_1\eta_1 + (aD_{1,2} + bD_{1,2})\bar{\eta}_2\eta_2 + (aD_{1,3} + ZD_{1,2,3})\bar{\eta}_3\eta_3$$
$$+(aD_{1,2,3} + D_{1,2}D_{2,3})\bar{\eta}_1\eta_1\bar{\eta}_2\eta_2 + (D_{1,3}^2 + cD_{1,2,3})D_{1,3}\bar{\eta}_1\eta_1\bar{\eta}_3\eta_3 + (D_{1,2}D_{1,3} + bD_{1,2,3})\bar{\eta}_2\eta_2\bar{\eta}_3\eta_3$$
$$+(D_{1,3}D_{1,2,3} + D_{2,3}D_{1,2,3})\bar{\eta}_1\eta_1\bar{\eta}_2\eta_2\bar{\eta}_3\eta_3.$$

where $D_{1,2} = ab - d^2,\ D_{1,3} = ac - f^2,\ D_{2,3} = bc - e^2,\ D_{1,2,3} = abc - ae^2 - cd^2 - bf^2 + def.$

By Proposition 3.2.3, we know the map T is a well defined map on Sym_3. We also know by Proposition 3.2.6 that $\pi^{-1}(\mathbb{L}^3) \cup \{0\}$ is invariant by R. At this point, the rest of the theory in [29] naturally follows. First, one can express the counting measures of the eigenvalues of $H_{\pm}^{(n)}$ in terms of R. This will be pursued in the next section. More significantly, by Theorem 3.1 in [29] we know that the density of states must exist and that there exists an expression for the density. We refer to the text for the notation.

Theorem 3.2.7. *(i). The density of states is given by the following formula*

$$\mu = \frac{1}{2\pi}\Delta(G \circ \phi).$$

(ii). The density of Neumann-Dirichlet eigenvalues is given by

$$\mu^{ND} = \sum_{\lambda} \rho_\infty(\pi(\phi(\lambda)))\delta_\lambda.$$

We make one small remark about the Green function G. The map $G : \mathcal{A} \to$

$\mathbb{R} \cup \{-\infty\}$ is defined by

$$G(x) = \lim_{n\to\infty} \frac{1}{N^n} \ln ||R^n(x)||.$$

where N is the degree of the map R. In our situation, we take $N = 2$.

There is one more major result that comes as a natural consequence. Theorem 4.1 in [29], known as the Dichotomy Theorem, tells us that depending on the zeros of R, we either have $\mu^{ND} = \mu$ for any choice of (A, b) or $\mu^{ND} = 0$ for a generic choice of (A, b). This is interesting, because this tells us that for most situations we either have that the ND eigenvalues either contribute to all of the density of states or essentially do not exist.

3.3 Application

Let $A = \left(\begin{smallmatrix} a & d & 0 \\ d & b & e \\ 0 & e & c \end{smallmatrix}\right)$ be an operator on V_0. By setting the term in A corresponding to the edge between w_0 and c_0 equal to zero, the corresponding operators $A^{(n)}$ can be viewed as linear symmetric operators on the original graphs V_{-n}, as defined in section 2.2. Let $\beta := (\beta_a, \beta_b, \beta_c) \in \mathbb{R}^3_+$ denote a measure on V_0, where β_x corresponds to the vertex with diagonal entry x.

If we set $a = c = 1$, $b = 2$ and $d = e = -1$ in A and $\beta_a = \beta_b = \beta_c = 1$ in β (i.e. we have the uniform measure), then the operators $H^{(n)}_{\pm}$ that correspond to $A^{(n)}$ coincide with the probabilistic Laplacians $P^{(n)}$ and $P^{(n)}_0$.

We will outline a dynamical system that can be used to compute the eigenvalues

of $H_{\pm}^{(n)}$. The map T can be written as

$$T(a,b,c,d,e) = \left(c, b - \frac{ad^2}{a^2+ab-d^2}, c - \frac{ae^2}{a^2+ab-d^2}, e, -\frac{ade}{a^2+ab-d^2}\right). \quad (3.3.1)$$

Let $a_0 = a - \beta_a\lambda,\ b_0 = b - \beta_b\lambda,\ c_0 = c - \beta_c\lambda,\ d_0 = d,\ e_0 = e$. In general, let $(a_{n+1}, b_{n+1}, c_{n+1}, d_{n+1}, e_{n+1}) = T(a_n, b_n, c_n, d_n, e_n)$. Let

$$D(a,b,c,d,e) = abc - ae^2 - cd^2$$

denote the determinant of the matrix A. $D(a_n, b_n, c_n, d_n, e_n)$ will be a rational function of λ. By Proposition 3.2.6, the zeros of this rational function will be the eigenvalues of $H_+^{(n)}$ and the singularities in the denominator will be eigenvalues of $H_-^{(n)}$. Of course, it is possible that a certain value for λ will be both a zero of the numerator and denominator. Such a λ will be a ND eigenvalue. Thus, removable singularities will correspond to ND eigenvalues.

So far, we have established a five dimensional system to compute eigenvalues. It is possible to perform a dimension reduction by observing that $a_{n+1} = c_n$ and $d_{n+1} = e_n$. Define $c_{-1} = a_0$ and $e_{-1} = d_0$. Then

$$(b_{n+1}, c_{n+1}, e_{n+1}) = \left(b_n - \frac{c_{n-1}e_{n-1}^2}{c_{n-1}^2 + c_{n-1}b_n - e_{n-1}^2}, c_n - \frac{c_{n-1}e_n^2}{c_{n-1}^2 + c_{n-1}b_n - e_{n-1}^2}, -\frac{c_{n-1}e_{n-1}e_n}{c_{n-1}^2 + c_{n-1}b_n - e_{n-1}^2}\right).$$

is a dynamical system with only three recursive sequences, and by analyzing the roots and singularities of

$$D_n = c_{n-1}b_nc_n - c_{n-1}e_n^2 - c_ne_{n-2}^2,$$

we get the eigenvalues of $H_{\pm}^{(n)}$.

It is possible to perform another dimension reduction. In equation 3.3.1, the expressions for b_{n+1} and c_{n+1} are similar. By some algebra,

$$b_{n+1} - c_{n+1} = b_n - c_n + \frac{a_n(e_n^2 - d_n^2)}{a_n^2 + a_n b_n - d_n^2}.$$

Using the equivalent expression for e_{n+1} and the fact that $d_n = e_{n-1}$, we have that

$$b_{n+1} - b_n = (c_{n+1} - c_n) - e_{n+1}\frac{e_n^2 - e_{n-1}^2}{e_{n-1}e_n}.$$

By writing b_{n+1} as a telescoping sum, we can deduce that

$$\begin{aligned} b_{n+1} &= b_0 + \sum_{j=0}^{n}(b_{j+1} - b_j) \\ &= b_0 + \sum_{j=0}^{n}(c_{j+1} - c_j) - \sum_{j=0}^{n} e_{j+1}\frac{e_j^2 - e_{j-1}^2}{e_{j-1}e_j} \\ &= c_{n+1} + (b_0 - c_0) - \sum_{j=0}^{n} e_{j+1}\frac{e_j^2 - e_{j-1}^2}{e_{j-1}e_j}. \end{aligned}$$

Thus, the sequence b_n can be eliminated from our dynamical system.

$$\begin{aligned} c_{n+1} &= c_n - \frac{c_{n-1}e_n^2}{c_{n-1}^2 + c_{n-1}\big(c_n + (b_0 - c_0) - \sum_{j=0}^{n-1} e_{j+1}\frac{e_j^2 - e_{j-1}^2}{e_{j-1}e_j}\big) - e_{n-1}^2}, \\ e_{n+1} &= -\frac{c_{n-1}e_{n-1}e_n}{c_{n-1}^2 + c_{n-1}\big(c_n + (b_0 - c_0) - \sum_{j=0}^{n-1} e_{j+1}\frac{e_j^2 - e_{j-1}^2}{e_{j-1}e_j}\big) - e_{n-1}^2}. \end{aligned} \tag{3.3.2}$$

is a set of two recursive sequences encoding the dynamical system. The expression

$$D_n = c_{n-1}\left(c_n + (b_0 - c_0) - \sum_{j=0}^{n-1} e_{j+1}\frac{e_j^2 - e_{j-1}^2}{e_{j-1}e_j}\right)c_n - c_{n-1}e_n^2 - c_n e_{n-2}^2, \tag{3.3.3}$$

can be analyzed just as before to find the eigenvalues of $H_{\pm}^{(n)}$. We summarize our results with the following theorem.

Theorem 3.3.1. *Let* $b_0 = b - \lambda, c_{-1} = a - \lambda, c_0 = c - \lambda, e_{-1} = d$ *and* $e_0 = e$*. For* $n \geq 1$*, define the polynomials* c_n *and* e_n *by the equations in* 3.3.2*. Then the zeros of* D_n *in Equation* 3.3.3 *are the Neumann eigenvalues (the eigenvalues of* $H_{+}^{(n)}$*). The singularities are the Dirichlet eigenvalues (the eigenvalues of* $H_{-}^{(n)}$*). The removable singularities are the Neumann-Dirichlet eigenvalues.*

Chapter 4

Spectral Asymptotics

4.1 Mixed Affine Nested Fractals

For $l > 1$, an l-similitude is a map $\psi : \mathbb{C}^n \to \mathbb{C}^n$ such that

$$\psi(x) = l^{-1}U(x) + x_0.$$

where U is a unitary map and $x_0 \in \mathbb{C}^n$. Let $\{\psi_1, ..., \psi_m\}$ be a finite family of maps where ψ_i is an l_i-similitude. For $B \subset \mathbb{R}^n$, define

$$\Psi(B) = \cup_{i=1}^{m} \psi_i(B),$$

and let

$$\Psi_n(B) = \underbrace{\Psi \circ ... \circ}_{n \text{ times}} \Psi(B).$$

The map Ψ on the set of compact subsets of $\mathbb{R}^n$ has a unique fixed point S. This set is a self-similar set that satisfies $S = \Psi(S)$.

Since each ψ_i is a contraction, it has a unique fixed point. Let F' be the set of fixed points of the mappings $\psi_i, 1 \leq i \leq m$. A point $x \in F'$ is called an essential fixed point if there exist $i, j \in \{1, ..., m\}, i \neq j$ and $y \in F'$ such that $\psi_i(x) = \psi_j(y)$. Let F_0 denote the set of essential fixed points. Now define

$$\psi_{i_1 \ldots i_n}(B) = \psi_{i_1} \circ \ldots \circ \psi_{i_n}(B), \quad B \subset \mathbb{R}^n.$$

The set $(F)_{i_1 \ldots i_n} = \psi_{i_1 \ldots i_n}(F_0)$ is called an n-cell. Let E denote the simplex formed from the vertices in F_0. The set $(S)_{i_1 \ldots i_n} = \psi_{i_1 \ldots i_n}(E)$ is called an n-complex. The lattice of fixed points F_n is defined by

$$F_n = \Psi_n(F_0),$$

and the set F can be recovered from the essential fixed points by setting

$$S = \mathrm{cl}(\cup_{n=0}^{\infty} F_n).$$

We can now define the affine nested fractal.

Definition 4.1.1. The set S is an affine nested fractal if $\{\psi_1, ..., \psi_m\}$ satisfy:

(A1) (*Connectivity*) For any 1-cells C and C', there is a sequence $\{C_i : i = 0, ..., n\}$ of 1-cells such that $C_0 = C$, $C_n = C'$ and $C_{i-1} \cap C_i \neq \emptyset, i = 1, ..., n$.

(A2) (*Symmetry*) If $x, y \in F_0$, then reflection in the hyperplane $H_{xy} = \{z : |z - x| = |z - y|\}$ maps F_n to itself.

(A3) (*Nesting*) If $\{i_1, ..., i_n\}, \{j_1, ..., j_n\}$ are distinct sequences, then

$$\psi_{i_1 \ldots i_n}(E) \cap \psi_{j_1 \ldots j_n}(E) = \psi_{i_1 \ldots i_n}(F_0) \cap \psi_{j_1 \ldots j_n}(F_0).$$

(A4)(*Open set condition*) There is a non-empty, bounded, open set O such that the $\psi_i(O)$ are disjoint and $\cup_{i=1}^m \psi_i(O) \subset O$.

Observe that this definition is identical to the definition of the classical nested fractal, except that similitudes are allowed to have different scale factors.

We can construct a composite of affine nested fractals as follows. Let A be a finite set. For each $a \in A$, let

$$\psi^a = \{\psi_i^a : i = 1, ..., m_a\}.$$

denote a set of m_a similitudes in $\mathbb{R}^n$ that determines an affine nested fractal. Let us assume that the set of fixed points for each ψ^a is the same. As before, let us denote the set of fixed points by F_0. Let l_i^a denote the scaling factor of ψ_i^a. Let $S_a \subset \mathbb{R}^n$ denote the unique compact set that satisfies

$$S_a = \cup_{i=1}^{m_a} \psi_i^a(S_a).$$

By the open set condition, this set will have Hausdorff dimension the unique α such that

$$\sum_{j=1}^{m_a} (l_j^a)^{-\alpha} = 1.$$

In order to construct mixed fractals we also need an address space. The address of each branch in the tree is used to specify a set in our mixed fractal through the applications of the sets of similitudes determined by the address. Let $I_n = \cup_{k=0}^n \mathbb{N}^k$

and let $I = \cup_k I_k$ be the space of arbitrary length sequences. Let $\mathbf{i}$ denote a sequence in I. Write $\mathbf{i}, \mathbf{j}$ for the concatenation of two sequences $\mathbf{i}$ and $\mathbf{j}$. Let $[\mathbf{i}]_n$ denote the sequence of length n such that $\mathbf{i} = [\mathbf{i}]_n, \mathbf{k}$ for some sequence $\mathbf{k}$. Write $\mathbf{j} \leq \mathbf{i}$, if $\mathbf{i} = \mathbf{j}, \mathbf{k}$ for some $\mathbf{k}$. Let $|\mathbf{i}|$ denote the length of a sequence. Let $\mathbf{i}(m)$ denote the mth term of $\mathbf{i}$.

Our address space will be a subset T of the space I. Let T_n denote the sequences in T of length n. T must satisfy certain properties. We require that $T_0 = I_0$, the set consisting of the empty sequence. Let $U(\mathbf{i}), \mathbf{i} \in T$ be an A-valued function that indicates the type of nested fractal to be used. We require that $\mathbf{i} \in T$ if $[\mathbf{i}]_n \in T_n$ for each $n \leq |\mathbf{i}|$, where $[\mathbf{i}]_n \in T_n$ if

(i). $[\mathbf{i}]_{n-1} \in T_{n-1}$,

(ii). There is a j, $1 \leq j \leq m(U([\mathbf{i}]_{n-1}))$ such that $[\mathbf{i}]_{n-1}, j = [\mathbf{i}]_n$.

Let E be the complete set formed from the set of fixed points F_0. The empty sequence in T_0 corresponds to this set. For $\mathbf{i} \in T_n$, let $S_{\mathbf{i}}$ denote

$$(S)_{\mathbf{i}} = \psi_{\mathbf{i}}(E) = \psi_{\mathbf{i}(1)}^{U([\mathbf{i}]_0)} \circ ... \circ \psi_{\mathbf{i}(n)}^{U([\mathbf{i}]_{n-1})}(E).$$

The mixed affine nested fractal is then defined to be defined to be

$$S = \cap_{n=0}^{\infty} \cup_{\mathbf{i} \in T_n} (S)_{\mathbf{i}}.$$

The mth level approximation to S is defined as

$$S_m = \cap_{n=0}^{m} \cup_{\mathbf{i} \in T_n} (S)_{\mathbf{i}}.$$

We define the mth lattice of fixed points to be

$$F_m = \cup_{\mathbf{i}\in T_m} \psi_{\mathbf{i}}(F_0).$$

As a composition of affine nested fractals satisfying the open set condition, the set S does as well. By the results in [24], the Hausdorff dimension of S is

$$d_H(S) = \max_{a\in A} d_H(S_a).$$

4.2 Reconstruction of Hata Tree

The Hata tree can be constructed as a mixed affine nested fractal, with the address space determined in a natural way by binary functions and imposing an orientation on the edges of the approximating graphs. In order to formulate this construction more precisely, let us first analyze the approximating graphs.

By self similarity of the Hata tree, it suffices to understand a single edge and how it evolves in the graph approximations V_n. Without loss of generality let us pick the edge in V_0 that corresponds to the unit interval. In V_1, an new edge is “appended”. That is, a vertex (corresponding to the point $|c|^2$) is placed in the middle of the edge, dividing our original edge in two, and an ultimate vertex is attached to this middle vertex via a third edge. For brevity of notation, let us label the edge corresponding to $[0, |c|^2]$ by a and the edge corresponding to $[|c|^2, 1]$ by b. Since $\phi_1(K)$ is graph isomorphic to K, there is a correspondence between the unit interval and the edge b.

So in V_2, the edge b has another edge appended to it, but edge a remains unchanged. In V_2, a is part of a subgraph isomorphic to V_0. So in the next graph approximation, a has an edge appended to it. Thus, every edge in an approximating graph V_n will have an edge appended to it eventually.

In order to determine the general manner in which edges are appended, we need the notion of a binary function and orientation on the edges of the approximating graphs. On V_0, let us assign its two edges the orientation that goes from (the vertex corresponding to) 0 to 1 and c, respectively. We say that 0 is the "stem" vertex and 1 and c are the "leaf" vertices of these edges. If an edge is split in two, we call the two edges the stem and leaf, respectively. Let edges in future graph approximations inherit the orientation obtained from ϕ_0 and ϕ_1 in the natural way. Observe that for a new appended edge, the orientation will always go from a vertex placed on an "old" edge to the "new" vertex. Informally, this corresponds to how branches grow "out" from an actual tree. Figure 4.2.1 gives an illustration.

To construct our binary functions, let us first define the function on V_0. Let us assign the edge corresponding to the unit interval a 1, and the other edge a 0. Inductively, our binary function on V_n will assign an edge a 1 if it also appears in V_{n-1}. If it is one of two edges that compromised one whole edge in V_{n-1}, then the tail and head (with respect to the orientation) are assigned a 0 and 1, respectively. If the edge is new, then it is assigned a zero. Observe that the binary functions model when edges will have new edges appended to them, with edges assigned a 1 having an edge appended to them immediately in the next graph approximation. Edges assigned a 0 must wait until the subsequent approximation. To extend our analogy to an actual tree, binary functions can be viewed to model the age of sections of the tree. A new branch is assigned an age of 0, at maturity (one more graph approximation) it is

assigned an age of 1, and as part of it dies (the tail in this case) it is replaced by new wood.

Let us define the set of similitudes $\psi^a = \{\psi_1^a, \psi_2^a, \psi_3^a\}$ on $\mathbb{C}$ such that

$$\psi_1^a(z) = |c|^2 z; \quad \psi_2^a(z) = (1-|c|^2)(z-1)+1; \quad \psi_3^a(z) = |c|^2 + i|c|(1-|c|^2)z.$$

Let ψ^e consist of the identity map ψ_0^e. Although the identity map is not strictly speaking a similitude, let us ignore this fact for now. ψ^a and ψ^e share two essential fixed points, $F_0 = \{0, 1\}$. Let E be the unit interval, the simplex formed from the points in F_0. The goal is to define an address space T and a $\{a, e\}$ valued function U on T such that the resulting mixed affine nested fractal is isomorphic to half of the Hata tree. Specifically, remove the point 0 from the Hata tree and consider the connected component containing the unit interval. The resulting set will be isomorphic to this connected component.

We construct our address space T as follows: let T_0 consist of the empty sequence corresponding to E. Let $U(\emptyset) = a$, indicating that the set of similitudes ψ^a are to be applied to E. ψ_1^a and ψ_2^a map E onto itself, and the application of ψ_3^a to E creates the new "appended" edge. Let T_1 consist of the sequences $\{1\}, \{2\}, \{3\}$; where the first two sequences correspond to the stem and leaf parts of E, and third sequence corresponds to the appended edge. In general, for an edge with address $\mathbf{i}$, if another edge is appended to it, then let $\mathbf{i}, 1$ and $\mathbf{i}, 2$ be the addresses of the stem and leaf part of the branches, respectively. Let $\mathbf{i}, 3$ be the address of the new appended edge. If the edge remains intact in the next graph approximation, let the address of this same edge in the next graph approximation be $\mathbf{i}, 0$. By the previous discussion, we

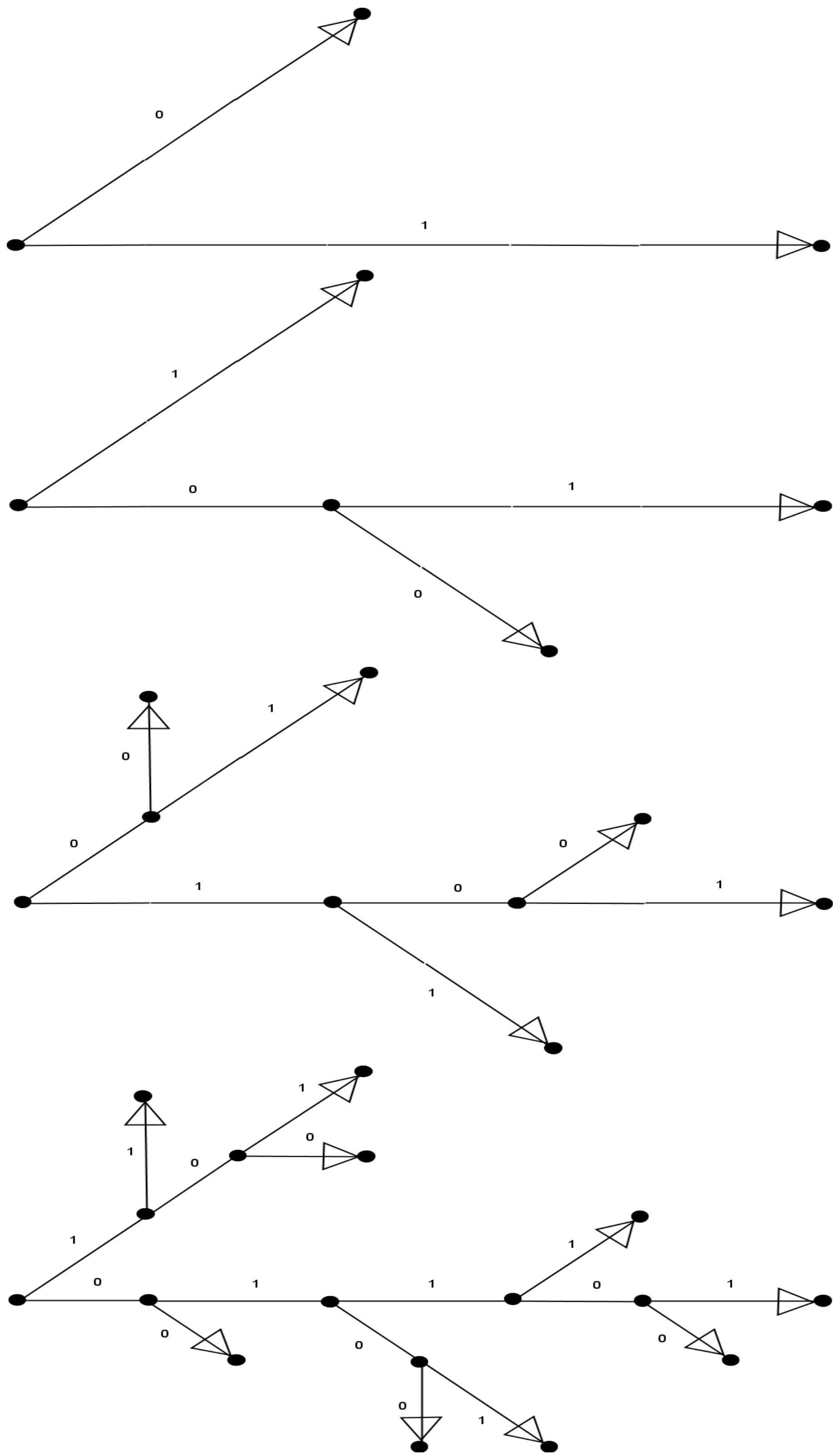


FIGURE 4.2.1: Binary Functions and Orientations on V_n, $n = 0, .., 3$

can extend the function U:

$$U([\mathbf{i}]_m) = \begin{cases} a & : \text{if } [\mathbf{i}]_m(m) = 0 \text{ or } 2 \\ e & : \text{if } [\mathbf{i}]_m(m) = 1 \text{ or } 3 \end{cases}$$

We define S to be the set determined by ψ^a, ψ^e, T, and U. The first few approximations are shown in Figure 4.2.2. Since the identity map in ψ^e not being a similitude, we cannot say yet that S is a mixed affine nested fractal. However, we can conclude the following.

Proposition 4.2.1. *Remove the point 0 from the Hata tree, and take the the connected component containing the unit interval. Then there exists an isometry between this connected component and S.*

Proof. Let us denote the connected component by C. Recall from the introduction that the sets K_n coverge to K, the Hata tree, in the Hausdorff metric. Let $C_m = K_m \cap C$. First, we must specify the metric with respect to which there is an isomorphism. Define the distance between any two points in a set to be the Euclidean distance of the corresponding geodesic within the set.

The graph approximations F_m of S were constructed to correspond to the evolution of the unit interval in the graph approximations V_m. Thus, the corresponding complex S_m is homeomorphic to C_m for all m.

By the self-similarity of the Hata tree, the lengths of the edges in V_m scale by certain constants. One can check that the length scale factors of the similitudes in ψ^a are the correct constants. Thus, we can conclude that S_m is isomorphic to C_m for all m, and by construction, the isomorphism carries over to the limit. $\square$

As in the proof, let C denote connected component of the Hata tree (with 0

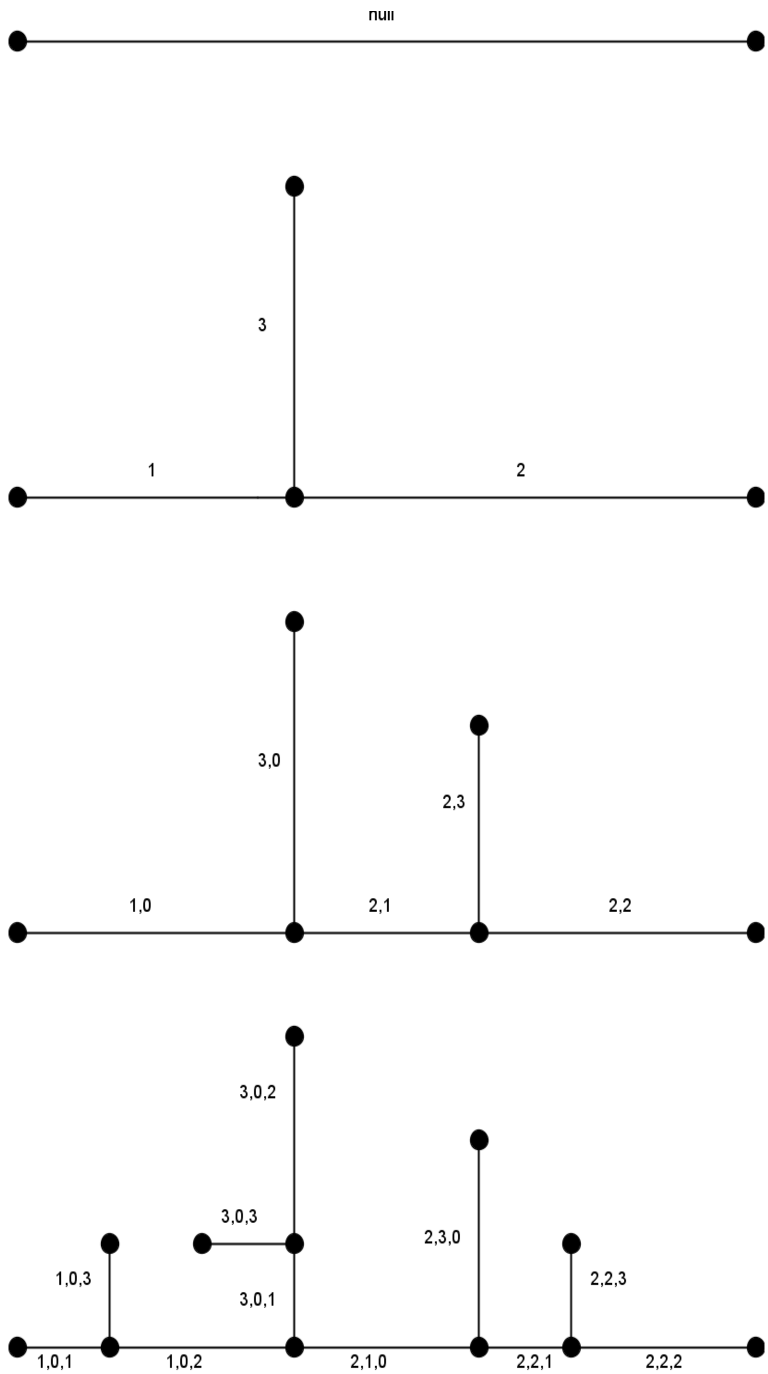


FIGURE 4.2.2: Addresses of Edges in Approximations S_n,$n = 0, .., 3$

removed) containing the unit interval. Let D denote the other connected component. By the self similarity of the Hata tree, there is clearly a homeomorphism between these connected components. In fact, if D is scaled by $|c|$ then we have isometry. We also have a homeomorphism between C_{m-1} and D_m for $m \geq 1$, with isometry if D_m is scaled by $|c|$. By these simple observations, we have the following.

Corollary 4.2.2. *Let S' be a copy of S. Identify the point 0 in these two copies. Let K_n denote the nth level approximation to the Hata tree. Then K_n is homeomorphic $S_n \cup S'_{n-1}$ for $n \geq 1$. If S' is scaled by $|c|$, then the homeomorphism extends to an isometry.*

As noted above, we have not yet established that S is a mixed affine nested fractal, due to the fact that the identity map in ψ^e is not strictly speaking a similitude. We now proceed to remedy this.

Let $\psi^b = \{\psi^b_j : j = 1, ..., 5\}$, where $\psi^b_j = \psi^a_2 \circ \psi^a_j$ for $j = 1, 2, 3$, $\psi^b_4 = \psi^a_1$, and $\psi^b_5 = \psi^a_3$. Let ψ^a be the same as above. ψ^a and ψ^b share two essential fixed points, 0 and 1. Let E be the simplex formed from the fixed points, namely the unit interval. We now construct an address space T^e and function U^e on T^e. Let T^e_0 consist of the empty sequence corresponding to E. Let $U^e(\emptyset) = b$. T^e_1 will consist of the sequences $\{j\}, j = 1, ..., 5$ corresponding to $\psi^b_j(e)$. We now extend U^e as such

$$U^e([\mathbf{i}]_m) = \begin{cases} a & : \text{if } [\mathbf{i}]_m(m) = 1, 3 \\ b & : \text{if } [\mathbf{i}]_m(m) = 2, 4, 5 \end{cases}$$

Let S^e denote the mixed affine nested fractal determined by ψ^a, ψ^b, T^e and U^e. It is not hard to see that S^e_n is the same set as S_{2n} for $n \geq 0$. Thus, $S^e = S$.

We will give another method to construct S. Let ψ^a and ψ^b be the same as above.

Let T_0^o consist of the empty sequence corresponding to the set E. Let $U^o(\emptyset) = a$. Then T_1^o will consist of the sequences $\{j\} : j = 1, 2, 3$ corresponding to $\psi_j^a(E)$. Let $U^e = U^o$ for all sequences not equal to $\emptyset$. Now let S^o be the corresponding mixed affine nested fractal. The approximations S_n^o are equal to S_{2n+1} for $n \geq 1$, so S^o is equal to S. (The superscripts e and o were chosen to correspond to the even and odd approximations to S, respectively.)

By the corollary and our previous work, we now have the following.

Theorem 4.2.3. *Identify the point 0 of S^o and S^e. Then K_n is homeomorphic to $S_n^e \cup S_n^o$ for $n \geq 1$.*

We conclude this section by computing the Hausdorff dimension of S^o and S^e and giving some remarks.

The Hausdorff dimension of K_a, the affine nested fractal determined by ψ^a, is

$$\begin{aligned}
d_H(K_a) &= \inf\{\alpha : \sum_{j=1}^{m_a} (l_j^a)^{-\alpha} = 1\} \\
&= \inf\{\alpha : (|c|^2)^\alpha + (1 - |c|^2)^\alpha + (|c|(1 - |c|^2))^\alpha = 1\} \\
&= \inf\{\alpha : |c|^\alpha \Big(|c|^\alpha + (1 - |c|^2)^\alpha\Big) + (1 - |c|^2)^\alpha = 1\} \\
&= \{\alpha : |c|^\alpha + (1 - |c|^2)^\alpha = 1\}.
\end{aligned}$$

The Haudsorff dimension of K_b coincides with $d_H(K_a)$:

$$\begin{aligned}
d_H(K_b) &= \inf\{\alpha : \sum_{j=1}^{m_b} (l_j^b)^{-\alpha} = 1\} \\
&= \inf\{\alpha : (|c|^2(1 - |c|^2))^\alpha + ((1 - |c|^2)^2)^\alpha + (|c|(1 - |c|^2)^2)^\alpha + (|c|^2)^\alpha + (|c|(1 - |c|^2))^\alpha = 1\} \\
&= \{\alpha : |c|^\alpha + (1 - |c|^2)^\alpha = 1\}.
\end{aligned}$$

Thus the Hausdorff dimension of S^o and S^e is:

$$d_H(S^o) = d_H(S^e) = \{\alpha : |c|^\alpha + (1 - |c|^2)^\alpha = 1\}.$$

The last line agrees with Moran's formula ([19]) for the Hausdorff dimension of the Hata tree, as one would expect.

In [12], the author defines random affine nested fractals, which according to the definitions presented here are mixed affine nested fractals such that the function $U(\mathbf{i})$ is determined by a random process. In [8], the authors work with V-variable Sierpinski gaskets, which are mixed affine nested fractals built from two classical versions of the gasket, where $U(\mathbf{i})$ is determined by a tree branching process. In [13], the authors work with graph-directed fractals. In fact, S^o and S^e are graph-directed fractals. The work done later in this chapter is based on these authors' work.

4.3 Dirichlet Forms and Laplacians

The goal in this section is to define a Dirichlet form and Laplacian on S^o and S^e. This will be done via the standard method of constructing a Dirichlet form on approximating lattices.

Definition 4.3.1. Let $\ell(V)$ be the set of real valued functions on a set V. A symmetric bilinear form $\mathcal{E}$ on $\ell(V)$ is called a Dirichlet form if it satisfies

(1) $\mathcal{E}(f, f) \geq 0$ for any $f \in \ell(V)$.

(2) $\mathcal{E}(f, f) = 0$ if and only if f is constant on V.

(3) $\mathcal{E}$ satisfies the Markov property. That is, for any $f \in \ell(V)$, $\mathcal{E}(f, f) \geq \mathcal{E}(\bar{f}, \bar{f})$, where $\bar{f}$ is defined by:

$$\bar{f}(p) = \begin{cases} 1 & : \text{if } f(p) \geq 1 \\ f(p) & : \text{if } 0 < f(p) \leq 1 \\ 0 & : \text{if } f(p) \leq 0 \end{cases}$$

By standard convention, we write $\mathcal{E}(f)$ for $\mathcal{E}(f,f)$. Note that one can recover $\mathcal{E}(f,g)$ from $\mathcal{E}(f)$ and $\mathcal{E}(g)$ by the polarization identity $\mathcal{E}(f,g) = \frac{1}{4}(\mathcal{E}(f+g) - \mathcal{E}(f-g))$.

We will alter notation introducted in the previous sections and let V_n^o and V_n^e denote the nth approximating lattices of fixed points of S^o and S^e, respectively. For points $y, z \in V_n^x$, we say that $y \sim z$ if there is a $\mathbf{i} \in T_n^x$ such that $(S^x)_{\mathbf{i}}$ has boundary points y and z. Let

$$\mathcal{E}_0^x(f,g) = \frac{1}{2} \sum_{\substack{y,z \in V_0^x \\ y \sim z}} (f(y) - f(z))(g(y) - g(z)), \quad f, g \in \ell(V_0^x), \quad x = o, e.$$

In order to construct a sequence of compatible Dirichlet forms, we need the notion of resistance between points. Without loss of generality let us define the resistance between the two points in $V_0^o = V_0^e = \{0, 1\}$ to be 1. For $\psi_j^a \in \psi^a$, define $\rho(a,j)$ to be the resistance scaling factor. We require that $0 < \rho(a,j) < 1$ for $j = 1, 2, 3$. We also require that $\rho(a,1) + \rho(a,2) = 1$. In a similar manner, we define $\rho(b,j)$ to be the resistance scaling factor for $\psi_j^b \in \psi^b$. Again, we require that $0 < \rho(b,j) < 1$ for $j = 1, .., 5$ and $\rho(b,1) + \rho(b,2) + \rho(b,4) = 1$. We define the resistance of the boundary points of $\psi_{[\mathbf{i}]_n}(E)$ to be

$$r_x([\mathbf{i}]_n) = \prod_{j=0}^{n-1} \rho\big(U([\mathbf{i}]_j), [\mathbf{i}]_n(j+1)\big).$$

We can then write

$$\mathcal{E}_n^x(f,g) = \sum_{\mathbf{i}\in T_n^x} r_x(\mathbf{i})^{-1}\mathcal{E}_0^x(f\circ\psi_{\mathbf{i}}, g\circ\psi_{\mathbf{i}}), \quad f,g\in \ell(V_n^x), \quad x=o,e.$$

By construction, the sequence of Dirichlet forms $\{\mathcal{E}_n^x\}$ is a compatible sequence on V_n^x. That is

$$\mathcal{E}_m^x(f,f) = \min\{\mathcal{E}_{m+1}^x(g,g) : g\in \ell(V_{m+1}^x), g|_{V_m^x} = f\}.$$

Let $V_\infty^x = \cup_{n=0}^\infty V_n^x$. For a function $f \in \ell(V_\infty^x)$, by the compatibility the sequence $\{\mathcal{E}_x^n(f|_{V_x^n}, f|_{V_x^n})\}$ is increasing. Denote by $\mathcal{F}^x$ the set

$$\mathcal{F}^x = \{f\in \ell(V_\infty^x) : \sup_n \mathcal{E}_n^x(f|_{V_n^x}, f|_{V_n^x}) < \infty\}.$$

We can then define a Dirichlet form $\mathcal{E}^x$ on $\mathcal{F}^x$, where

$$\mathcal{E}^x(f,f) = \lim_{n\to\infty} \mathcal{E}_n^x(f|_{V_n^x}, f|_{V_n^x}).$$

For $\mathbf{i}\in T_x^n$, the resistances $r_x(\mathbf{i})$ determine a metric on the vertices of V_n^x. Let us define the resistance between two points y,z in V_∞^x to be:

$$r_x(y,z) = (\inf\{\mathcal{E}^x(f,f) : f(x)=0, f(y)=1, f\in\mathcal{F}^x\})^{-1}.$$

Observe that if y,z are the endpoints of $(S^x)_{\mathbf{i}}$, then $r_x(y,z)$ will coincide with $r_x(\mathbf{i})$ (hence the slight abuse of notation). The following is a standard result on resistance metrics, c.f. [19].

Lemma 4.3.2. *Let $f \in \ell(V^x_\infty)$. Then*

$$|f(y) - f(z)| \leq \sqrt{r_x(y,z)\mathcal{E}^x(f,f)}.$$

Recall that V^x_∞ is a dense subset of S^x. So by the lemma, $\mathcal{F}^x$ can be embedded into the set of continuous functions on S^x.

In order to define a Laplacian operator, we need to construct a Dirichlet form on an appropriate L^2 space. We now precisely define such a Dirichlet form.

Definition 4.3.3. Let X be a locally compact separable measure space. Let μ be a regular Borel measure on X such that $\mu(O) > 0$ for all open sets $O \subset X$. Let $\mathcal{F}$ be a dense subset of $L^2(X,\mu)$ and let $\mathcal{E}$ be a non-negative symmetric bilinear form on F. Then $(\mathcal{E},\mathcal{F})$ is called a Dirichlet form on $L^2(X,\mu)$ if:
(1) For $\alpha > 0$, let $\mathcal{E}_\alpha(u,v) = \mathcal{E}(u,v) + \alpha\langle u,v\rangle_\mu$, where $\langle u,v\rangle_\mu = \int_X uv\,d\mu$. Then $(\mathcal{F},\mathcal{E}_\alpha)$ is a Hilbert space.
(2) $\mathcal{E}$ satisfies the Markov property.

We will construct a measure μ_x on S^x. There exist many possibilites, but we will construct a Bernoulli measure. Without loss of generality let $\mu_x(E) = 1$. For $\psi^a_j \in \psi^a$, let $u(a,j)$ be the measure scaling factor. We require that $0 < u(a,j) < 1$ for $j = 1,2,3$ and $u(a,1) + u(a,2) = 1$. Define $u(b,j)$ in a similar manner. We require that $u(b,1) + u(b,2) + u(b,4) = 1$. We define the measure of $(S^x)_{[\mathbf{i}]_n}$ to be

$$\mu_x([\mathbf{i}]_n) = \prod_{j=0}^{n-1} u\big(U([\mathbf{i}]_j), [\mathbf{i}]_n(j+1)\big).$$

We can now prove the following.

Proposition 4.3.4. *$(\mathcal{E}^x, \mathcal{F}^x)$ is a local regular Dirichlet form on $L^2(S^x, \mu_x)$. In addition, there exists a constant C such that*

$$\sup_{y,z \in S^x} |u(y) - u(z)| \leq C\sqrt{\mathcal{E}^x(u,u)}.$$

Proof. $\mathcal{F}^x$ can be embedded into the space of continuous functions on S^x. Since S^x is compact, it is clear that $\mathcal{F}^x \subset L^2(S^x, \mu_x)$. In addition, the Bernoulli measure μ_x satisfies the conditions of Definition 4.3.3.

As a limit of compatible Dirichlet forms, by Theorem 2.2.6 in [19], $(\mathcal{F}^x / \sim, \mathcal{E}^x)$ is a Hilbert space, where we quotient $\mathcal{F}^x$ by the set of constant functions. By standard arguments, it follows that $(\mathcal{F}^x, \mathcal{E}^x_\alpha)$ is a Hilbert space. By the same theorem, we can conclude that $\mathcal{E}^x$ satisfies the Markov property. Thus, the two conditions of Definition 4.3.3 are satisfied.

Recall that r_x is a resistance metric on V^x_∞. By following the methods of Kigami [19], this metric can be extended to S^x. By construction, r_x will have a finite diameter C. The inequality now follows from Lemma 4.3.2.

For definitions and fundamental results on Dirichlet forms, we refer to [9]. □

Now that we have constructed the Dirichlet form $(\mathcal{E}^x, \mathcal{F}^x)$, we can use the machinery of functional analysis to define a Laplacian on S^x.

Definition 4.3.5. Define the Laplacian Δ_x with respect to the measure μ_x to be the unique operator satisfying

$$\mathcal{E}^x(f, g) = -\langle \Delta_x f, g \rangle_{\mu_x}.$$

If one knows that Δ_x is a compact operator, then one can conclude that Δ_x has

a unique spectrum consisting of eigenvalues. To prove this fact, it suffices to prove that the natural inclusion from $\mathcal{F}^x$ into $L^2(\mathcal{F}^x, \mu_x)$ is compact.

Lemma 4.3.6. *The natural inclusion map from $(\mathcal{F}^x, \mathcal{E}^x + ||.||_2)$ to $L^2(\mathcal{F}^x, \mu_x)$ is a compact operator.*

Proof. Let U be a bounded set in the Banach space $(\mathcal{F}^x, \mathcal{E}^x + ||.||_2)$. By the inequality in the previous proposition, the set U is equicontinuous.

We will prove U is uniformly bounded. Let $h_p^x(z)$ where $z \in S_x$ and $p \in V_0^x$, denote the harmonic function such that h_p^x is 1 at p and 0 at the other point in V_0^x. Let $\bar{f}(z) = \sum_{p \in \{0,1\}} f(p) h_p^x(z)$. By the same inequality

$$|f(z) - \bar{f}(z)| \leq \sum_{p \in \{0,1\}} h_p^x(z)|f(z) - f(p)| \leq \sqrt{C\mathcal{E}^x(f, f)}.$$

Since the space of harmonic functions is finite dimensional, the L^2 and L^∞ norms are equivalent. Thus

$$\begin{aligned}
||f||_\infty &\leq ||f - \bar{f}||_\infty + ||\bar{f}||_\infty \\
&\leq ||f - \bar{f}||_\infty + ||\bar{f}||_2 \\
&\leq 2||f - \bar{f}||_\infty + ||f||_2 \\
&\leq 2\sqrt{C\mathcal{E}^x(f, f)} + ||f||_2.
\end{aligned}$$

So, there exists a constant C_2 such that for $f \in U$ we have $||f||_\infty \leq C_2$. Thus U is uniformly bounded.

By the Arzela-Ascoli Theorem, U is relatively compact in the space of continuous functions on S^x, and thus in $L^2(S^x, \mu_x)$. $\square$

4.4 The Multidimensional Renewal Theorem

In order to prove the spectral asymptotics in the following section, we will require a version of the renewal theorem. We now proceed to give some notation and state the theorems presented in [13], [21].

Let $M = [m_{ij}]$ be a matrix of Radon measures on $\mathbb{R}^+$. Let F be the corresponding matrix of distribution functions $F_{ij}(t) = \int_0^t m_{ij}(ds)$. Let $F_{ij}(t, t+h] = F_{ij}(t+h) - F_{ij}(t)$.

The indices of the matrix can be referred to as states and are the vertices of a graph G. The graph has a directed edge between states i and j if the measure m_{ij} is non-zero.

The operation of convolution of a function a with a measure b is denoted by

$$b * a(t) = a * b(t) = \int_0^t a(t-s)b(ds).$$

If a and b are both measures, then we can take the convolution of the distribution of a with the measure b. For two matrices of measure A and B, we denote by $C(t) = A * B(t)$ the matrix with entries $c_{ij} = \sum_k a_{ik} * b_{kj}(t)$. Let $\gamma(i, j)$ denote the directed path from i to j. The measure $m_{\gamma(i,j)}$ is defined by taking the convolution of the measures associated with each edge in the path.

For a matrix M, write $m_{\hat{i}i}$ for the ith column of M with the ith element removed. Similarly, we let $m_{i\hat{i}}$ denote the ith row with the ith element removed. Finally, we let M_{ii} denote M with the ith rows and columns removed.

Define the measure ν_1 by

$$\nu_1 = m_{11} + m_{1\hat{1}} * \sum_{k=0}^{\infty} (M_{11})^{*k} * m_{\hat{1}1}.$$

If $F(\infty)$ has maximum eigenvalue 1 and is irreducible, then ν_1 is a probability measure with support given by $\cup\{\text{supp}(m_\gamma) : \gamma \text{ is a simple cycle in } G\}$. If the support is contained in a discrete subgroup of $\mathbb{R}$, we call the measure lattice. Otherwise, it is called non-lattice. Finally, if ν_1 is non-lattice, by the irreducibility then ν_i is non-lattice for all i.

Theorem 4.4.1. *Assume that $F(t)$ is a matrix of measures in which $F(\infty)$ is irreducible, has maximum eigenvalue 1, $F_{ij}(0-) = 0$, $\int_0^\infty t\, dF_{ij}(t) < \infty$ for all i, j and for each j there is at least one i such that $F_{ij}(0) < F_{ij}(\infty)$. Let $V(t) = \sum_{k=0}^{\infty} F^{*k}(t)$ denote the matrix renewal measure, then if ν_1 is non-lattice,*

$$\lim_{t\to\infty} V(t, t+h] = Ah,$$

where

$$A = \frac{\boldsymbol{u}\boldsymbol{v}^T}{\boldsymbol{v}^T \mathcal{M} \boldsymbol{u}}$$

and $\boldsymbol{u}, \boldsymbol{v}$ are the unique normalized right and left 1-eigenvectors of $F(\infty)$ and $\mathcal{M}$ is the matrix of first moments of F. If ν_1 is lattice, with period T, then

$$\lim_{t\to\infty} [V_{ij}(t + \tau_{ij}, t + \tau_{ij} + T)] = AT$$

for any $\tau_{ij} \in \text{supp}(m_{\gamma(i,j)})$.

We also need the following result regarding the asymptotic behavior of the solution

of the renewal equation.

Theorem 4.4.2. *Let $\boldsymbol{z}(t)$ be directly Riemann integrable, and let F be a matrix of measure satisfying the assumptions of the previous theorem, then the renewal equation*

$$\boldsymbol{r}(t) = \boldsymbol{z}(t) + \boldsymbol{r} * F(t)$$

has a unique solution, bounded on finite intervals. If ν_1 is non-lattice, then

$$\boldsymbol{r}(t) \to \int_0^\infty \boldsymbol{z}(t)\, dtA, \text{ as } t \to \infty.$$

If ν_1 is lattice with period T, then

$$\boldsymbol{r}(t) = \lim_{n\to\infty} [r_i(t + \tau_{1i} + nT)] = \sum_k \boldsymbol{z}(t + kT)A$$

exists almost surely for every $t \in [0, T]$.

4.5 Asymptotics

We begin by defining the Dirichlet and Neumann eigenvalue problems for the Laplacian Δ_x on S^x.

The Dirichlet eigenvalues of Δ_x are defined to be the numbers λ such that

$$\Delta_x f = \lambda f,$$

where f is the corresponding eigenfunction that satisfies $f(x) = 0$ for $x \in V_0^x$ This problem can be reformulated in terms of the Dirichlet forms. Let $\mathcal{F}_0^x = \{f \in \mathcal{F}^x :$

$f(z) = 0, f \in V_0^x\}$. Let $\mathcal{E}_0^x(f,f) = \mathcal{E}^x(f,f)$ for $f \in \mathcal{F}_0^x$. Then λ is a Dirichlet eigenvalue with eigenfunction f if

$$\mathcal{E}_0^x(f,g) = \lambda\langle f,g\rangle_{\mu_x}$$

for all $g \in \mathcal{F}_0^x$. As Δ_x is compact, we can write the spectrum as an increasing sequence of eigenvalues: $0 < \lambda_0 < \lambda_1 \leq ....$ We define the corresponding eigenvalue counting function

$$N_0^x(z) = \max\{i : \lambda_i \leq z\}.$$

The analagous can be done for the Neumann eigenvalues. First, we must define the Neumann boundary condition. For the sequence of Dirichlet forms $\mathcal{E}_n^x$ there is a corresponding sequence of discrete Laplacians $\Delta_x^{(n)}$. If $z \in V_0^x$, we define the normal derivative of a function f at z to be

$$(du)_z = -\lim_{m\to\infty} \Delta_x^{(m)} f(z).$$

The existence of this limit is verified in [18]. λ is a Neumann eigenvalue with eigenfunction f if

$$\Delta^x f = \lambda f,$$

where $(df)_z = 0$ for $z \in V_0^x$. In terms of Dirichlet forms, λ is an Neumann eigenvalue with eigenfunction f if

$$\mathcal{E}^x(f,g) = \lambda\langle f,g\rangle_{\mu_x}$$

for all $g \in \mathcal{F}^x$. As before, the spectrum is a discrete sequence of eigenvalues: $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq ...$ and we define the corresponding eigenvalue counting function to

be

$$N^x(z) = \max\{i : \lambda_i \le z\}.$$

There exists a natural scaling of the Dirichlet form.

Lemma 4.5.1. *Let $f, g \in \mathcal{F}^e \cap \mathcal{F}^o$. Then*

$$\begin{aligned}
\mathcal{E}^o(f,g) &= \sum_{j=1,3} r_o(j)^{-1}\mathcal{E}^o(f \circ \psi_j^a, g \circ \psi_j^a) + r_o(2)^{-1}\mathcal{E}^e(f \circ \psi_2^a, g \circ \psi_2^a), \\
\mathcal{E}^e(f,g) &= \sum_{j=1,3} r_e(j)^{-1}\mathcal{E}^o(f \circ \psi_j^b, g \circ \psi_j^b) + \sum_{j=2,4,5} r_e(j)^{-1}\mathcal{E}^e(f \circ \psi_j^b, g \circ \psi_j^b).
\end{aligned}$$

Proof. Observe that these relations hold for $\mathcal{E}_1^x$ and $\mathcal{E}_2^x$:

$$\begin{aligned}
\mathcal{E}_2^o(f,g) &= \sum_{j=1,3} r_o(j)^{-1}\mathcal{E}_1^o(u \circ \psi_j^a, v \circ \psi_j^a) + r_o(2)^{-1}\mathcal{E}_1^e(f \circ \psi_2^a, g \circ \psi_2^a), \\
\mathcal{E}_2^e(u,v) &= \sum_{j=1,3} r_e(j)^{-1}\mathcal{E}_1^o(u \circ \psi_j^b, v \circ \psi_j^b) + \sum_{j=2,4,5} r_e(j)^{-1}\mathcal{E}_1^e(u \circ \psi_j^b, v \circ \psi_j^b).
\end{aligned}$$

To simplify notation, we understand that we take the appropriate restrictions. The same relations hold for $\mathcal{E}_n^x$ and $\mathcal{E}_{n+1}^x$. Take the limit as $n \to \infty$. □

The key relations for the eigenvalue counting functions are provided by the following.

Lemma 4.5.2. *Let $\theta_j^x = r_x(j)\mu_x(j)$. Let $z \ge 0$. Then*

$$N_0^e(\theta_2^o z) + \sum_{j=1,3} N_0^o(\theta_j^o z) \le N_0^o(z) \le N^o(z) \le N^e(\theta_2^o z) + \sum_{j=1,3} N^o(\theta_j^o z),$$

$$\sum_{j=2,4,5} N_0^e(\theta_2^e z) + \sum_{j=1,3} N_0^o(\theta_j^e z) \le N_0^e(z) \le N^e(z) \le \sum_{j=2,4,5} N^e(\theta_j^e z) + \sum_{j=1,3} N^o(\theta_j^e z).$$

Furthermore, there exists a finite constant M such that

$$N_0^x(z) \leq N^x(z) \leq N_0^x(z) + M.$$

Proof. Let us prove the first set of inequalities, as the proof for the second set is similar. Let

$$\tilde{\mathcal{F}}^o = \{u : S^o \backslash V_1^o \to \mathbb{R} : u \circ \psi_j^a \in \mathcal{F}^o \text{ for } j = 1, 3; \quad u \circ \psi_j^a \in \mathcal{F}^e \text{ for } j = 2\},$$

and for $u, v \in \tilde{\mathcal{F}}^o$ define

$$\tilde{\mathcal{E}}^o(u, v) = \sum_{j=1,3} r_o(j)^{-1} \mathcal{E}_1^o(u \circ \psi_j^a, v \circ \psi_j^a) + r_o(2)^{-1} \mathcal{E}_1^e(u \circ \psi_2^a, v \circ \psi_2^a).$$

We can define $\tilde{\mathcal{F}}^e$ and $\tilde{\mathcal{E}}^e$ in a similar manner.

It is easy to see that $\mathcal{F}^o \subset \tilde{\mathcal{F}}^o$. By the previous lemma, we know that $\mathcal{E}^o = \tilde{\mathcal{E}}^o$ when restricted to $\mathcal{F}^o \times \mathcal{F}^o$. The form $(\tilde{\mathcal{E}}^o, \tilde{\mathcal{F}}^o)$ is a local regular Dirichlet form on $L^2(S^o, \mu_o)$, and by adapting Lemma 4.3.6 it can be shown that the associated Laplacian operator has spectrum consisting of eigenvalues. Let f be an eigenfunction of $(\tilde{\mathcal{E}}^o, \tilde{\mathcal{F}}^o)$ with eigenvalue λ, thus

$$\tilde{\mathcal{E}}^o(f, g) = \lambda \langle f, g \rangle_{\mu_o} \text{ for all } g \in \tilde{\mathcal{F}}^o.$$

We can rewrite this using the scaling of the Dirichlet form in the previous lemma:

$$\textstyle\sum_{j=1,3} r_o(j)^{-1} \mathcal{E}^o(f \circ \psi_j^a, g \circ \psi_j^a) + r_o(2)^{-1} \mathcal{E}^e(f \circ \psi_2^a, g \circ \psi_2^a) = \lambda \sum_{j=1,3} \mu_o(j) \langle f \circ \psi_j^a, g \circ \psi_j^a \rangle_{\mu_o} + \lambda \mu_o(2) \langle f \circ \psi_2^a, g \circ \psi_2^a \rangle_{\mu_e}.$$

Thus for any $h_1 \in \mathcal{F}^o, h_2 \in \mathcal{F}^e$:

$$\begin{aligned}
\mathcal{E}^o(f \circ \psi_j^a, h_1) &= r_o(j)\mu_o(j)\lambda\langle f \circ \psi_j^a, h_1\rangle_{\mu_o}, j = 1, 3, \\
\mathcal{E}^e(f \circ \psi_2^a, h_2) &= r_o(2)\mu_o(2)\lambda\langle f \circ \psi_2^a, h_2\rangle_{\mu_e}.
\end{aligned}$$

This shows that $f \circ \psi_j^a$ is an eigenfunction of $(\mathcal{E}^o, \mathcal{F}^o)$ with eigenvalue $\theta_j^o\lambda$ for $j = 1, 3$ and $f \circ \psi_2^a$ is an eigenfunction of $(\mathcal{E}^e, \mathcal{F}^e)$ with eigenvalue $\theta_2^o\lambda$.

In addition, observe that the function

$$f_j(z) = \begin{cases} (f \circ \psi_j^a)(z) & : z \in \psi_j^a(S^o) \\ 0 & : \text{ otherwise} \end{cases}$$

is an eigenfunction of $(\tilde{\mathcal{E}}^o, \tilde{\mathcal{F}}^o)$ for $j = 1, 3$ and an eigenfunction of $(\tilde{\mathcal{E}}^e, \tilde{\mathcal{F}}^e)$ for $j = 2$, with eigenvalue λ. Let $\tilde{N}^x$ represent the eigenvalue counting function of $(\tilde{\mathcal{E}}^x, \tilde{\mathcal{F}}^x)$. Then by the properties described above:

$$\begin{aligned}
\tilde{N}^o(z) &= \#\{k : k \leq z\} \\
&= \sum_{j=1,2,3} \#\{\theta_j^o k : \theta_j^o k \leq \theta_j^o z\} \\
&= N^o(\theta_1^o z) + N^o(\theta_3^o z) + N^e(\theta_2^o z).
\end{aligned}$$

Since the domains of $(\tilde{\mathcal{E}}^x, \tilde{\mathcal{F}}^x)$ are larger than $(\mathcal{E}^x, \mathcal{F}^x)$, by a minimax argument we know that $N^x \leq \tilde{N}^x$. Thus, we get the right inequality.

Define $\tilde{\mathcal{F}}_0^x = \{f : f \in \mathcal{F}_0^x, f|_{V_1^x} = 0\}$ and define $\tilde{\mathcal{E}}_0^x = \mathcal{E}_0^x$ restricted to $\tilde{\mathcal{F}}_0^x \times \tilde{\mathcal{F}}_0^x$. By similar reasoning, if f is an eigenfunction of $(\tilde{\mathcal{E}}_0^o, \tilde{\mathcal{F}}_0^o)$ with eigenvalue λ, then f_j is an eigenfunction of $(\mathcal{E}^x, \mathcal{F}^x)$ (x depending on j) with eigenvalue $\theta_j^o\lambda$. This lets us

derive the left inequality.

That $N^x \leq N_0^x$ follows from a minimax argument since $\mathcal{F}_0^x \subset \mathcal{F}^x$. This gives the middle inequality.

Finally, the last statement is a standard consequence of Dirichlet-Neumann bracketing. Details can be found in [20]. $\square$

We are now in a position to find the asymptotic distribution of the spectrum of the Laplacians. Define the matrix R_s as follows:

$$R_s = \begin{bmatrix} \sum_{j=1,3}(\theta_j^o)^s & (\theta_2^o)^s \\ \sum_{j=1,3}(\theta_j^e)^s & \sum_{j=2,4,5}(\theta_j^e)^s \end{bmatrix}.$$

Let $\Phi(s)$ denote the spectral radius of R_s.

Theorem 4.5.3. *Let $-d_s/2$ be the solution to $\Phi(s) = 1$. Then*

$$0 < \liminf_{z\to\infty} N_0^x(z)z^{-d_s/2} \leq \limsup_{z\to\infty} N_0^x(z)z^{-d_s/2} < \infty,$$

$$0 < \liminf_{z\to\infty} N^x(z)z^{-d_s/2} \leq \limsup_{z\to\infty} N^x(z)z^{-d_s/2} < \infty.$$

Proof. R_s is a primitive matrix. By the Perron-Frobenius Theorem, for $s = -d_s/2$, the matrix R_s has an eigenvector $\begin{bmatrix} u_o \\ u_e \end{bmatrix}$ with eigenvalue 1. So

$$u_o = \sum_{j=1,3}(\theta_j^o)^{-d_s/2}u_o + (\theta_2^o)^{-d_s/2}u_e,$$

$$u_e = \sum_{j=1,3}(\theta_j^e)^{-s}u_o + \sum_{j=2,4,5}(\theta_j^e)^{-s}u_e.$$

Let $\beta^x(t) = \exp(-td_s/2)N^x(e^t), \beta_0^x(t) = \exp(-td_s/2)N_0^x(e^t)$. We can rewrite the first set of inequalities in Lemma 4.5.2 in terms of β as

$$\beta^o(t) \le \sum_{j=1,3} (\theta_j^o)^{-d_s/2}\beta^o(t - \log\theta_j^o) + (\theta_2^o)^{-d_s/2}\beta^e(t - \log\theta_j^o).$$

By iteration, we get

$$\beta^o(t) \le \sum_{\mathbf{i}\in T_n^o} (\theta_{\mathbf{i}}^o)^{-d_s/2}\beta^{\sigma(\mathbf{i})}(t - \log\theta_{\mathbf{i}}^o)$$

where we define $\theta_{\mathbf{i}}^o = r_o(\mathbf{i})\mu_o(\mathbf{i})$ and

$$\sigma(\mathbf{i}) = \begin{cases} o & : \text{ if } U([\mathbf{i}]_{n-1}) = a \\ e & : \text{ if } U([\mathbf{i}]_{n-1}) = b \end{cases}$$

This inequality will remain true if we replace T_n^o with the set

$$E_n^o = \{[\mathbf{i}]_m : \theta_{[\mathbf{i}]_m}^o < e^n \le \theta_{[\mathbf{i}]_{m-1}}^o\}.$$

Note that if $\mathbf{i} \in E_n^o$, then $\theta_{\mathbf{i}}^o \le e^n$. By repeated multiplication, our eigenvector satisfies

$$u_o = \sum_{\mathbf{i}\in E_n^o} (\theta_{\mathbf{i}}^o)^{-d_s/2} u_{\sigma(\mathbf{i})}.$$

Hence

$$\beta^o(t) \le \sum_{\mathbf{i}\in E_n^o} (\theta_{\mathbf{i}}^o)^{-d_s/2}\beta^{\sigma(\mathbf{i})}(t - \log\theta_{\mathbf{i}}^o).$$

Let $M = \max\log\theta_j^x$. Pick c such that $\beta^x(t) \le c$ for $t \in [0, M]$. Pick an n such that for $\mathbf{i} \in E_n^o$

$$t - \log(\theta_{\mathbf{i}}^o) \in [0, M].$$

Thus

$$\begin{aligned}
\beta^o(t) &\leq \sum_{\mathbf{i}\in E_n^o} (\theta_{\mathbf{i}}^o)^{-d_s/2} \beta^{\sigma(\mathbf{i})}(t - \log \theta_{\mathbf{i}}^o) \\
&\leq c \sum_{\mathbf{i}\in E_n^o} (\theta_{\mathbf{i}}^o)^{-d_s/2} \\
&\leq \frac{c}{\min_x u_x} \sum_{\mathbf{i}\in E_n^o} (\theta_{\mathbf{i}}^o)^{-d_s/2} u_{\sigma(\mathbf{i})} \\
&= \frac{c u_o}{\min_x u_x}.
\end{aligned}$$

This upper bound is independent of n. Therefore, the inequality holds for all $t > 0$. In a similar manner, we can analyze $N_0^o(z)$ to get a lower bound.

Finally, by the last statement of Lemma 4.5.2, we know there exist constants c_2, c_3 such that

$$c_2 N_0^o(z) \leq N^o(z) \leq c_3 N_0^o(z).$$

Putting everything together, we get the first set of inequalities. The second set follows in a similar manner. □

By the multidimensional renewal theorem of the previous theorem, we can improve our result. Let M be the matrix of measures

$$M(ds) = \begin{bmatrix} \sum_{j=1,3} (\theta_j^o)^{-d_s/2} \delta_{\log \theta_j^o}(ds) & (\theta_2^o)^{-d_s/2} \delta_{\log \theta_2^o}(ds) \\ \sum_{j=1,3} (\theta_j^e)^{-d_s/2} \delta_{\log \theta_j^e}(ds) & \sum_{j=2,4,5} (\theta_j^e)^{-d_s/2} \delta_{\log \theta_j^e}(ds) \end{bmatrix}.$$

Recall that the measure ν_1 was defined as

$$\nu_1 = m_{11} + m_{1\hat{1}} * \sum_{k=0}^{\infty} (M_{11})^{*k} * m_{\hat{1}1}.$$

In our situation

$$\nu_1 = m_{11} + m_{12} * \sum_{k=0}^{\infty} (m_{22})^{*k} * m_{21}.$$

Theorem 4.5.4. *If ν_1 is non-lattice, then*

$$\lim_{z\to\infty} N^x(z) z^{-d_s/2} = c_4(x),$$

$$\lim_{z\to\infty} N_0^x(z) z^{-d_s/2} = c_5(x),$$

where c_4, c_5 are constants depending on x. If ν_1 is lattice, then

$$\lim_{z\to\infty} N^x(z) z^{-d_s/2} - p_1^x(\log z) = 0,$$

$$\lim_{z\to\infty} N_0^x(z) z^{-d_s/2} - p_2^x(\log z) = 0,$$

where p_1^x, p_2^x are periodic functions depending on x.

Proof. Let

$$\mathbf{r}(t) = \begin{bmatrix} e^{-td_s/2} N_0^o(e^t) \\ e^{-td_s/2} N_0^e(e^t) \end{bmatrix}.$$

Let

$$\mathbf{z}(t) = \begin{bmatrix} e^{-td_s/2}[N_0^o(e^t) - \sum_{j=1,3} N_0^o(\theta_j^o e^t) - N_0^e(\theta_2^e e^t)] \\ e^{-td_s/2}[N_0^e(e^t) - \sum_{j=1,3} N_0^o(\theta_j^e e^t) - \sum_{j=2,4,5} N_j^e(\theta_j^e e^t)] \end{bmatrix}.$$

Let $F(t)$ be the matrix of distributions corresponding to $M(ds)$. We can now deduce that the renewal equation holds:

$$\mathbf{r}(t) = \mathbf{z}(t) + \mathbf{r} * F(t).$$

The results for $N_0^x(z)$ now follow from Theorem 4.4.2. By Theorem 4.5.3, these results can be extended to the $N^x(z)$. □

We conclude this section by doing some computations with the spectral dimension.

Example 4.5.5. Suppose the following relations hold for the resistance scaling factors: $\rho(b,j) = \rho(a,2)\rho(a,j)$ for $j = 1,2,3$ and $\rho(b,4) = \rho(a,1), \rho(b,5) = \rho(a,3)$. Suppose the analogous relations hold for the measure scaling factors u. The choice of these factors determine a resistance and measure on S^o and S^e that coincide with that constructed in [20].

Thus, $\theta_j^e = \theta_2^o\theta_j^o$ for $j = 1,2,3$ and $\theta_4^e = \theta_1^o, \theta_5^e = \theta_3^o$. By definition, $\Phi(s) = 1$ when $s = -d_s/2$. In our situation,

$$R_s = \begin{bmatrix} \sum_{j=1,3}(\theta_j^o)^s & (\theta_2^o)^s \\ \sum_{j=1,3}(\theta_j^o)^s & (\theta_2^o)^s \end{bmatrix}.$$

and the corresponding eigenvector is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Thus, we get the equation

$$\sum_{j=1}^{3}(\theta_j^o)^{-d_s/2} = 1.$$

This is the analogue to equation 3.5 in [20].

Chapter 5

Miscellaneous

5.1 Spectral Decimation

In this chapter we will work with operators that have the spectral decimation property. As such, the notion of spectral self-similarity will be introduced. We review some notation and results in [35], [23].

Let $\mathcal{H}$ and $\mathcal{H}_0$ be Hilbert spaces, and let J_0 be an isometry from $\mathcal{H}_0$ into $\mathcal{H}$. Let H and H_0 be bounded linear operators on $\mathcal{H}$ and $\mathcal{H}_0$ respectively. Let ϕ_0 and ϕ_1 be complex valued functions defined on $\Lambda \subset \mathbb{C}$.

Definition 5.1.1. *H is spectrally similar to H_0 with functions ϕ_0 and ϕ_1 and isometry J_0 if*

$$J_0^*(H - z)^{-1} J_0 = (\phi_0(z) H_0 - \phi_1(z))^{-1} \tag{5.1.1}$$

on $\mathcal{H}_0$ for any $x \in \Lambda_0$, where Λ_0 consists of those $z \in \Lambda$ for which both sides of relation 5.1.1 are well defined.

It is possible to decompose H in the following manner. Without loss of generality let $\mathcal{H}_0$ be a subspace of $\mathcal{H}$ and let $\mathcal{H}_1$ be the orthogonal complement to $\mathcal{H}_0$. Let P_0, P_1 be the orthogonal projectors from $\mathcal{H}$ onto $\mathcal{H}_0, \mathcal{H}_1$, respectively.

Define the operators $S : \mathcal{H}_0 \to \mathcal{H}_0$, $X : \mathcal{H}_0 \to \mathcal{H}_1$, $\bar{X} : \mathcal{H}_1 \to \mathcal{H}_0$, and $Q : \mathcal{H}_1 \to \mathcal{H}_1$ by $S = J_0^* H J_0$, $X = J_1^* H J_0$, $\bar{X} = J_0^* H J_1$ and $Q = J_1^* H J_1$. For $i = 1, 2$, denote the identity operator on $\mathcal{H}_i$ by I_i. Denote the resolvent set of an operator A by $\rho(A)$.

Lemma 5.1.2. *For $z \in \rho(H) \cap \rho(Q)$, relation* 5.1.1 *holds if and only if*

$$(S - z) - \bar{X}(Q - z)^{-1} X = \phi_0(z) H_0 - \phi_1(z).$$

Proof. For $z \in \rho(H) \cap \rho(Q)$, the following relation holds

$$J_0^*(H - z)^{-1} J_0((S - z) - \bar{X}(Q - z)^{-1} X) = I_0. \quad \square$$

By Corollary 3.4 in [23], it is possible to analytically extend ϕ_0 and ϕ_1 from $\Lambda_0 \cap \rho(Q)$ to its connected component in $\rho(Q)$ such that relation 5.1.1 holds.

Definition 5.1.3. The set $\mathscr{E} = \mathscr{E}(H, H_0) = \{z \in \mathbb{C} : z \notin \rho(Q) \text{ or } \phi_0(z) = 0\}$ is called the exceptional set for the operators H and H_0. If $\phi_0(z) \neq 0$ then define $R(z) := \phi_1(z)/\phi_0(z)$.

As before, let us assume that H and H_0 are finite dimensional self-adjoint spectrally similar operators. The following establishes the relation between the eigenprojectors of H and H_0 and is proved in [35], [23].

Theorem 5.1.4. *Suppose $z \notin \mathscr{E}(H, H_0)$. Then*

(1) $R(z) \in \rho(H_0)$ if and only if $z \in \rho(H)$.

(2) $R(z)$ is an eigenvalues of H_0 if and only if z is an eigenvalues of H. Moreover, there is a one-to-one map

$$f_0 \mapsto f = f_0 - \bar{X}(Q-z)^{-1} X f_0$$

from the eigenspace of H_0 corresponding to $R(z)$ onto the eigenspace of H corresponding to z.

In [23], the authors prove that a sequence of graphs with certain symmetries have the spectral decimation property. That is, the probabilistic Laplacians on two subsequence graphs are spectrally similar. We proceed to give the key lemma in the argument (which will be useful later), and then to define these graphs and give the main result.

For $\alpha \in \mathscr{A}$, let H^α and H_0^α be spectrally similar operators on $\mathscr{H}$ and $\mathscr{H}_0^\alpha$ respectively. Also suppose that the polynomials $\phi_0(z)$ and $\phi_1(z)$ do not depend on α.

Lemma 5.1.5. *Suppose that for a family of operators $\{L^\alpha\}_{\alpha\in\mathscr{A}}$, $\{R^\alpha\}_{\alpha\in\mathscr{A}}$ we have that $P_0 = \sum_{\alpha\in\mathscr{A}} L^\alpha P_0^\alpha R^\alpha$ and for each α, $P_1 L^\alpha = R^\alpha P_1 = P_1^\alpha$, $P_0 L^\alpha = L^\alpha P_0^\alpha$, $R^\alpha P_0 = P_0^\alpha R^\alpha$. Then the operators $H = \sum_\alpha L^\alpha H^\alpha R^\alpha$ and $H_0 = \sum_\alpha L^\alpha H_0^\alpha R^\alpha$ are spectrally similar with functions $\phi_0(z)$, $\phi_1(z)$.*

Definition 5.1.6. An M-point model graph G is a finite connected graph symmetric with respect to an M point set $\partial G = V_0 \subset V(G)$ if

(1) there are complete graphs G^s of M vertices such that $G = \cup_{s\in S} G^s$ where S is a finite set and $|S| \geq M \geq 2$;

(2) we have $G^s \cap G^{s'} = V(G^s) \cap V(G^{s'})$ for all distinct $s, s' \in S$, and this intersection is either empty or has only one point;

(3) we have $|G^s \cap \partial G| \leq 1$ for any $s \in S$;

(4) any bijection $\sigma : \partial G \to \partial G$ has an extension to a graph automorphism $\psi_\sigma : G \to G$, such that $\psi_\sigma G^s = G^{\bar{\sigma}s}$ for any bijection $\bar{\sigma} : S \to S$.

Definition 5.1.7. If an M-point model graph G is given then we define the corresponding self-similar symmetric sequence of finite graphs $\{G_n\}_{n=0}^{\infty}$ inductively as follows:

(1) G_0 is a complete graph of M vertices with $\partial G_0 = V(G_0)$;

(2) If $\partial G_n \subset V(G_n)$ is an M point set, then G_{n+1} is obtained by substituting each G^s in G by a copy G_n^s of G_n, so that $\partial G^s = V(G^s)$ is substituted by ∂G_n^s;

(3) ∂G_{n+1} is defined as ∂G after this substitution.

The following is the main result.

Theorem 5.1.8. *Let $\Delta_n = \Delta_{G_n}$ and $\Delta_\infty = \Delta_{G_\infty}$ be the probabilistic Laplacians on G_n and G_∞ respectively for a self-similar symmetric sequence of finite graphs. Then*

(1) For any $n \geq 0$, the operator Δ_{n+1} is spectrally similar to Δ_n with isometry U_n and rational functions $\phi_0(z)$ and $\phi_1(z)$ which do not depend on n. The exceptional set $\mathscr{E} = \mathscr{E}(\Delta_{n+1}, \Delta_n) = \mathscr{E}(\Delta_1, \Delta_0)$ also does not depend on n.

(2) Let $\mathscr{D}_n = \cup_{m=0}^{n} R^{-m}(\mathscr{E} \cup \sigma(\Delta_0))$, where R^{-m} is the preimage of order m under $R(z) = \phi_1(z)/\phi_0(z)$. Then $\sigma(\Delta_n) \subseteq \mathscr{D}_n$, where $\sigma(\cdot)$ is the spectrum of an operator.

(3) The operator Δ_∞ is spectrally self-similar with the isometry U_∞, rational functions $\sigma_0(z)$ and $\sigma_1(z)$ and the exceptional set $\mathscr{E}$.

$$\mathscr{J}(R) \subseteq \sigma(\Delta_\infty) \subseteq \mathscr{J}(R) \cup \mathscr{D}_\infty,$$

where $\mathscr{D}_\infty = \cup_{n=0}^{\infty} \mathscr{D}_n$ and $\mathscr{J}(R)$ is the Julia set of the rational function R.

5.2 Homogeneous Fractals

In section 4.2, it is possible to construct S by specifying a different address space. The lattices of approximating points will be different, and by defining graph Laplacians on these lattices and taking the limit in the appropriate manner, one can build a different Laplacian on S. In this section, we will show that the probabilstic Laplacian on specific lattices satisfy the spectral decimation property.

Let $\mathbf{j}$ be a length n sequence in $\{a, b\}$. Let $\mathbf{j}(k)$ denote the kth entry in $\mathbf{j}$. We define $S^{\mathbf{j}}$ to be the mixed affine nested fractal determined by repeatedly applying the set of similtudes $\psi^{\mathbf{j}(1)}$, .., $\psi^{\mathbf{j}(n)}$ repeatedly and in that order. Let $c(a) := 3, c(b) := 5$. The address space $T^{\mathbf{j}}$ is given by: $T^{\mathbf{j}}_0 = \{\emptyset\}$, $T^{\mathbf{j}}_1 = \{1, .., c(\mathbf{j}(1))\}$, and in general let $\mathbf{i} \in T^{\mathbf{j}}_m$ if $1 \leq \mathbf{i}(k) \leq c(\mathbf{j}(k \bmod n))$ for $1 \leq k \leq m$. Let $U^{\mathbf{j}}([\mathbf{i}]_k) = \mathbf{j}(k + 1 \bmod n)$. Together, ψ^a, ψ^b, $T^{\mathbf{j}}$, and $U^{\mathbf{j}}$ determine $S^{\mathbf{j}}$. In the literature such fractals are known as homogeneous fractals because each cell in one level undergoes the same transformation in the next level.

We begin by analyzing S^a. F^a_0 is the complete graph with two vertices corresponding to the points 0 and 1. The application of the set of similtudes ψ^a results in three copies of F^a_0 joined by identifying a boundary point in each copy. The unidentified points in two of the three copies will correspond to the new boundary (the vertices corresponding to 0 and 1). Thus F^a_1 satisfied the conditions of Definition 5.1.6. In general, F^a_{n+1} can be constructed from F^a_n in a similar manner. Thus, the graphs F^a_n are a self-similar symmetric sequence of 2-point model graphs.

Let $\mathbf{j}$ be a length m sequence in $\{a, b\}$. $F^{\mathbf{j}}_0$ will be a complete graph with two vertices. By applying the set of similtudes $\psi^{c(\mathbf{j}(1))}, .., \psi^{c(\mathbf{j}(m))}$ we end up with $c(\mathbf{j}(1)) \times \cdots \times c(\mathbf{j}(m))$ copies of $F^{\mathbf{j}}_0$ that are joined in a manner that satisfies Definition 5.1.6.

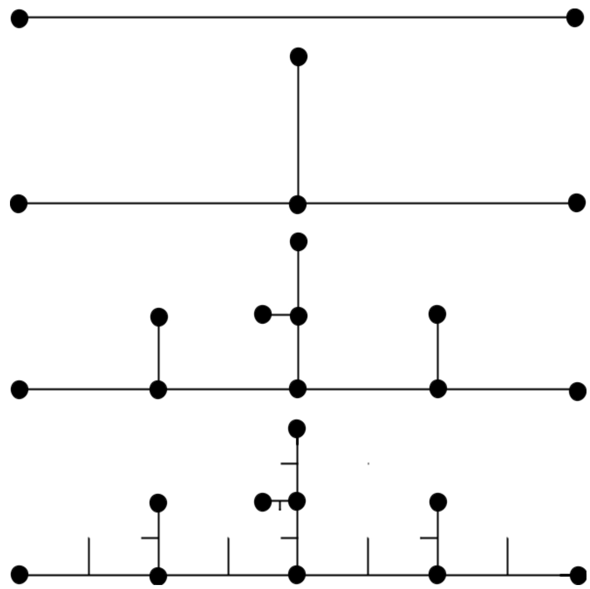


FIGURE 5.2.1: Graph Approximations of S^a

$F^{\mathbf{j}}_{m(n+1)}$ can be constructed from $F^{\mathbf{j}}_{mn}$ in a similar manner, and thus the graphs $F^{\mathbf{j}}_{mn}$ form a self-similar sequence of 2-point model graphs. We summarize this result below.

Proposition 5.2.1. *Let $S^{\boldsymbol{j}}$ be a homogeneous fractal. Let $m = |\boldsymbol{j}|$. Then the graphs $F^{\boldsymbol{j}}_{mn}$ for $n \geq 0$ form a self-similar symmetric sequence of 2-point model graphs.*

Thus, we are able to apply Theorem 5.1.8 to these fractals.

Example 5.2.2. Consider S^a. The probabilistic Laplacian $P^{(1)}_a$ on F^a_1 can be written as

$$P^{(1)}_a = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ -\frac{1}{3} & -\frac{1}{3} & 1 & -\frac{1}{3} \\ 0 & 0 & -1 & 1 \end{pmatrix}.$$

The upper left two by two block corresponds to ∂F^a_1 and the lower right two by two block corresponds to the interior vertices. By taking the Schur complement of the

matrix $P_a^{(1)} - \lambda I$, we get

$$\frac{1-\lambda}{3(1-\lambda)^2-1}P_a^{(0)} - \Big(-(1-\lambda) + \frac{2(1-\lambda)}{3(1-\lambda)^2-1}\Big)I.$$

Thus, $\phi_0^a(\lambda) = \frac{1-\lambda}{3(1-\lambda)^2-1}$, $\phi_1^a(\lambda) = -(1-\lambda) + \frac{2(1-\lambda)}{3(1-\lambda)^2-1}$ and $R_a(\lambda) = -3(1-\lambda)^2+3$. In [22], computations are done to find the eigenvalues of the probabilistic Laplacian on a sequence of approximating lattices related to F_n^a.

Example 5.2.3. Consider S^b. The probabilistic Laplacian $P_b^{(1)}$ on F_1^b can be written as

$$P_b^{(1)} = \begin{pmatrix} 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ -\frac{1}{3} & 0 & 1 & -\frac{1}{3} & -\frac{1}{3} & 0 \\ 0 & -\frac{1}{3} & -\frac{1}{3} & 1 & 0 & -\frac{1}{3} \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \end{pmatrix}.$$

The upper left two by two block corresponds to ∂F_1^b and the lower right two by two block corresponds to the interior vertices. By taking the Schur complement of the matrix $P_b^{(1)} - \lambda I$, we get

$$\frac{q(\lambda)}{3r(\lambda)}P_a^{(0)} - \Big(-(1-\lambda) + \frac{p(\lambda)+q(\lambda)}{3r(\lambda)}\Big)I,$$

where $p(\lambda) = \frac{2}{3} - \frac{8}{3}\lambda + 3\lambda^2 - \lambda^3$, $q(\lambda) = \frac{1}{3} - \frac{2}{3}\lambda + \frac{1}{3}\lambda^2$, and $r(\lambda) = \frac{1}{3} - \frac{22}{9}\lambda + \frac{47}{9}\lambda^2 - 4\lambda^3 + \lambda^4$. In our case, we have $\phi_0^b(\lambda) = \frac{q(\lambda)}{3r(\lambda)}$, $\phi_1^b(\lambda) = -(1-\lambda) + \frac{p(\lambda)+q(\lambda)}{3r(\lambda)}$ and $R_b(\lambda) = \phi_1^b(\lambda)/\phi_0^b(\lambda)$.

For a generic homogeneous fractal $S^{\mathbf{j}}$, we can obtain the decimation polynomial

$R_{\mathbf{j}}$ for the probabilistic Laplacians on the graphs $F^{\mathbf{j}}_{mn}$ from R_a and R_b alone. We will need the following lemma, which is a consequence of Lemma 3.10 in [23].

Lemma 5.2.4. *Let P be a probabilistic Laplacian on a graph G. Let G' be another graph formed by replacing each edge with a copy of F^x_1, $x = a, b$. Let P' be the probabilistic Laplacian on G'. Then P' is spectrally similar to P, with $\phi_0 = \phi^x_0$, $\phi_1 = \phi^x_1$ and $R = R_x$.*

Proof. One can index each edge in G by α. For each α, let H^α_0 be a copy of $P^{(0)}_x$ and H^α a copy of $P^{(1)}_x$. By the previous examples we know that $P^{(0)}_x$ is spectrally similar to $P^{(1)}_x$.

Let L^α and R^α be the inclusion and projection operators between the space $\mathscr{H}^\alpha_0$ associated with the edge α and the space $\mathscr{H}_0$ associated with G. We use the same notation to denote the inclusion and projection between $\mathscr{H}^\alpha$ (associated with the edge α with an appended edge) and $\mathscr{H}$ (associated with G'). Note that in matrix notation these operators will have a diagonal block with entries equal to the reciprocal of the degree of the corresponding vertex. By Lemma 5.1.5, we have that $P = H_0 = \sum_{\alpha \in \mathscr{A}} L^\alpha H^\alpha_0 R^\alpha$ is spectrally similar to $P' = H = \sum_{\alpha \in \mathscr{A}} L^\alpha H^\alpha R^\alpha$. $\square$

We are now in a position to prove the following.

Proposition 5.2.5. *Let $S^{\boldsymbol{j}}$ be a homogeneous fractal, where $m = |\boldsymbol{j}|$. Then $P^{(mn)}_{\boldsymbol{j}}$ is spectrally similar to $P^{(m(n+1))}_{\boldsymbol{j}}$ with decimation polynomial*

$$R_{\boldsymbol{j}} := R_{\boldsymbol{j}(1)} \circ \cdots \circ R_{\boldsymbol{j}(m)}.$$

Proof. By the previous lemma, we know that $P^{(k)}_{\mathbf{j}}$ is spectrall similar to $P^{(k+1)}_{\mathbf{j}}$ with decimation polynomial $R_{\mathbf{j}(k+1 \bmod m)}$. That is, if λ is an eigenvalue of $P^{(k+1)}_{\mathbf{j}}$, then

$R_{\mathbf{j}(k+1 \bmod m)}(\lambda)$ is an eigenvalue of $P_{\mathbf{j}}^{(k)}$. By chaining together the spectral similarities, we can compose the corresponding decimation polynomials to find $R_{\mathbf{j}}$. $\square$

5.3 Another Application of Sabot Theory

It is possible to apply the Sabot theory to the approximating lattices of S^a, as the necessary symmetry conditions are satisfied.

Let G represent the two element group representing the symmetries of the lattices F_n^a. Let Sym^G be the set of complex symmetric 2×2 matrices invariant under G. The set $\mathbb{C}^2$ can be decomposed into a sum of 2 irreducible representations $\mathbb{C}^2 = W_0 \oplus W_1$, where W_0 is the subspace of constant functions and W_1 its orthogonal complement. Hence, any Q in Sym^G can be written

$$Q = u_0 \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} + u_1 \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix},$$

where $u_0, u_1 \in \mathbb{C}$. We denote the above matrix by Q_{u_0,u_1}.

The operator $Q^{(1)}$ on F_1^a

$$Q^{(1)} = \begin{pmatrix} a & 0 & b & 0 \\ 0 & a & b & 0 \\ b & b & 3a & b \\ 0 & 0 & b & a \end{pmatrix}$$

can be constructed from $Q = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$ in a manner analagous to how F_1^a is constructed from F_0^a.

By taking the trace of $Q^{(1)}$ onto F_0^a, we get

$$Q^{(1)}|_{\partial F_1^a} = \begin{pmatrix} \frac{3a^3-2ab^2}{3a^2-b^2} & \frac{-ab^2}{3a^2-b^2} \\ \frac{-ab^2}{3a^2-b^2} & \frac{3a^3-2ab^2}{3a^2-b^2} \end{pmatrix}.$$

By rewriting in terms of u_0 and u_1, we define the map T on Sym^G by

$$T(u_0, u_1) = \Big(\frac{3u_0u_1(u_0+u_1)}{u_0^2+4u_0u_1+u_1^2}, \frac{1}{2}(u_0+u_1)\Big).$$

We know that the trace of a matrix $Q^{(n)}$ onto $\partial Q^{(n)}$ is given by $T^n(Q)$.

By the theory in Sabot, we can embed $\mathrm{Sym}^G \sim \mathbb{C}^2$ into the Lagrangian space $\mathbb{L}^G \sim \mathbb{P}^1 \times \mathbb{P}^1$ by the injection $Q_{u_0,u_1} \to ([u_0 : 1], [u_1 : 1])$. Recall that a point in $\mathbb{P}^1 \times \mathbb{P}^1$ can be represented by

$$([u_0 : v_0], [u_1 : v_1]).$$

Thus, the following map represents the compactification of T in Lagrangian space:

$$g([u_0 : v_0], [u_1 : v_1]) = ([3u_0u_1(u_1v_0+u_0v_1) : u_1^2v_0^2+4u_0u_1v_0v_1+u_0^2v_1^2], [u_0v_1+u_1v_0 : 2v_0v_1]).$$

Let $z = \frac{u_0}{u_1}$ and $\bar{z} = \frac{\bar{u}_0}{\bar{u}_1}$ where $(\bar{u}_0, \bar{u}_1) = T(u_0, u_1)$. Then we have $\bar{z} = g'(z)$ where

$$g'(z) = \frac{6z}{z^2+4z+1}.$$

In homogeneous coordinates in $\mathbb{P}^1$, g' is given by

$$g'([z_0 : z_1]) = [6z_0z_1 : z_0^2 + 4z_0z_1 + z_1^2].$$

Denote by $\hat{s}$ the rational map $\hat{s} : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ given by

$$\hat{s}([u_0 : v_0], [u_1 : v_1]) = [u_0v_1 : u_1v_0].$$

Then the following diagram is commutative.

$$\begin{tikzcd}
\mathbb{P}^1 \times \mathbb{P}^1 \arrow[r, "g"] \arrow[d, "\hat{s}"'] & \mathbb{P}^1 \times \mathbb{P}^1 \arrow[d, "\hat{s}"] \\
\mathbb{P}^1 \arrow[r, "g'"] & \mathbb{P}^1
\end{tikzcd}$$

So the map g' is birationally equivalent to g. By Proposition 4.6 in [29], the asymptotic degree d_∞ of g' is 2, less than $N = 3$ (the degree of the polynomial R). Thus, we are in case (i) of Theorem 4.1, i.e., we have $\mu^{ND} = \mu$ and for almost all blow-up the spectrum is pure point with compactly supported eigenfunctions.

In order to describe the density of states, it will be more practical to use an alternate version of the maps g and R. Let $p(a, b) = \det(Q^{(1)}_{\partial F_1^a}) = 3a^2 - b^2$. Define the map $\hat{R} : \mathbb{C}^2 \to \mathbb{C}^2$ by

$$\hat{R}(a, b) := p(a, b)\left(\frac{3a^3 - 2ab^2}{3a^2 - b^2}, \frac{-ab^2}{3a^2 - b^2}\right) = (3a^3 - 2ab^2, -ab^2).$$

Let $\hat{g}$ be the rational map on $\mathbb{P}^1$ induced from $\hat{R}$:

$$\hat{g}([a : b]) = [3a^3 - 2ab^2 : -ab^2].$$

We can rewrite $\hat{g}$ as $\hat{g}(z) = -3z^2 + 2$ where $z = \frac{b}{a}$. $\hat{g}$ represents an alternate compactification of T and $\hat{R}$ is the lift onto $\mathbb{C}^2$. Our goal will be to find the Green current of $\hat{g}$ and use it to write down the density of states.

Let $a_0 = a - \beta_a \lambda, b_0 = b$ and $(a_{n+1}, b_{n+1}) = \hat{R}(a_n, b_n)$. We have $\hat{R}(a_0, b_0) = (0, 0)$ if and only if $a_0 = 0$ which occurs when $\lambda = \frac{a}{\beta_a}$. In $\mathbb{P}^1$, a hypersurface is simply the root of a homogeneous polynomial. Thus

$$[D_1] = \left[\frac{a}{\beta_a}\right].$$

We have

$$\hat{R}^2(a_0, b_0) = (a_0^3(3a_0^2 - 2b_0^2)(27a_0^4 - 36a_0^2 b_0^2 + 10b_0^4), a_0^3 b_0^4(3a_0^2 - 2b_0^2)).$$

Notice that $\lambda = \frac{a}{\beta_a}$ will be a root of multiplicity three and the two solutions for λ in $\hat{g}((a - \beta_a\lambda)/b) = 0$ will be roots of multiplicity one of $(3a_0^2 - 2b_0^2)$. Thus

$$[D_2] = 3\left[\frac{a}{\beta_a}\right] + \Big[\lambda : \hat{g}((a - \beta_a\lambda)/b) = 0\Big].$$

In general, if λ' is a "new" root of $a_n b_n^2$, then each of the two solutions for λ in $\hat{g}((a - b_a\lambda)/b) = \lambda'$ will be roots of multiplicity one of $3a_n^3 - 2a_n b_n^2$. In addition, old roots increase in multiplicity by a factor of three. So

$$[D_n] = 3^{n-1}\left[\frac{a}{\beta_a}\right] + 3^{n-2}\Big[\lambda : \hat{g}((a - \beta_a\lambda)/b) = 0\Big] + \cdots + 1\Big[\lambda : \hat{g}^{n-1}((a - \beta_a\lambda)/b) = 0\Big].$$

So the density of states is given by

$$\mu = \mu^{ND} = \frac{1}{3}\delta_{\frac{a}{\beta_a}} + \sum_{k=0}^{\infty} \frac{1}{3^{k+2}} \Bigg(\sum_{\lambda : \hat{g}^{k+1}((a-\beta_a\lambda)/b)=0} \delta_\lambda \Bigg).$$

The density of states has total mass 1. Indeed, g has two inverse branches, so the set $\{\lambda : \hat{g}^{k+1}((a - \beta_a\lambda)/b) = 0\}$ has cardinality 2^{k+1}. Thus, the total mass is

$$\frac{1}{3} + \sum_{k=0}^{\infty} \frac{2^{k+1}}{3^{k+2}} = 1.$$

5.4 Spectral Analysis on a Family of Cayley Graph-like Fractals

In this section we perform a spectral analysis on graph approximations to a certain family of Cayley graph-like fractals. By an example in [32], it is possible to use the technique of spectral decimation to compute the eigenvalues of the probabilistic Laplacian on graph approximations. However, it is also possible to use techniques from Chapter 2 to compute multiplicities of eigenvalues and to compute eigenfunctions.

Let $\{e_1, .., e_n\}$ denote the canonical basis vectors for $\mathbb{R}^n$. For $i = 1, .., n$, define

$$\psi_i(x) = e_1 + r_i M_i(x - e_1),$$

where M_i is the linear transformation that maps e_j to $e_{j+i-1 \bmod n}$, $r_1 = \frac{1}{2}$, and

$0 < r_i < \frac{1}{2}$ for $2 \leq i \leq n$. For $i = n+1, .., 2n$, define

$$\psi_i(x) = -e_1 + r_i N_i(x + e_1),$$

where N_i is the linear transformation that maps $-e_j$ to $-e_{j+i-1 \text{ mod } n}$, $r_{n+1} = \frac{1}{2}$, and $0 < r_i < \frac{1}{2}$ for $n+2 \leq i \leq 2n$. This family of similtudes $\{\psi_i\}_{i=1}^{2n}$ determines a unique self similar set in $\mathbb{R}^n$. Specifically, define the map Ψ on compact subsets of $\mathbb{R}^n$ by

$$\Psi(B) = \cup_{i=1}^{2n} \psi_i(B).$$

Denote by S_n the fixed point of Ψ. This set is a self-similar in the sense that

$$S_n = \cup_{i=1}^{2n} \psi_i(S_n).$$

In fact, it is easy to check that S_n satisfies the definition of an affine nested fractal. This self similar set determines a visual representation of a Cayley graph on a free group of n generators.

The set of contractions $\{\psi_i\}_{i=1}^{2n}$ has two essential fixed points: e_1 and $-e_1$. Let us denote this set of essential fixed points by $V_{0,n}$. Let

$$V_{k,n} = \Psi_n(V_{0,n}) := \underbrace{\Psi \circ \cdots \circ \Psi}_{n \text{ times}}(V_{0,n}).$$

The set S_n can be recovered from these vertices. In particular,

$$S_n = \text{cl}(\cup_{k=0}^{\infty} V_{k,n}).$$

By imposing a graph structure on the sets of vertices $V_{k,n}$, we obtain the appropriate graph approximations. Write $x \sim y$ to denote that two vertices x and y are connected by an edge. In particular, the two vertices in V_0 can be connected by a single edge. In general, if $x \sim y$ in $V_{k,n}$, then $\psi_i(x) \sim \psi_i(y)$ in $V_{k+1,n}$ for $1 \leq i \leq 2n$. It is not hard to see that the sequence of graphs $V_{k,n}$ satisifed the definition of 2-point model graphs, as defined in [23]. Thus, we know by the results in [23] that it is possible to build a sequence of self similar probabilistic Laplacians on the graphs $V_{k,n}$ for any fixed n.

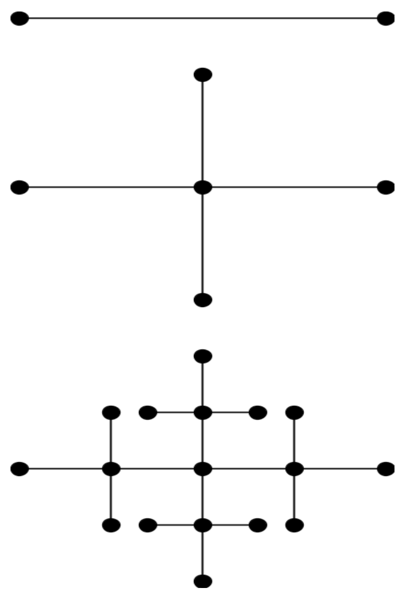


FIGURE 5.4.1: The graphs $V_{k,2}$, $k = 0, 1, 2$

For the graphs $V_{k,n}$, let $P_n^{(k)}$ denote the corresponding probabilistic Laplacian.

Lemma 5.4.1. *$P_n^{(1)}$ is spectrally similar to $P_n^{(0)}$.*

Proof. The graph $V_{0,n}$ is just the complete graph of two vertices. A matrix representation for $P_n^{(0)}$ is

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

The graph $V_{1,n}$ consists of the vertex of degree $2n$ corresponding to the point 0. It is connected to $\{\pm 2r_i e_i\}_{i=1}^n$, which are vertices of degree 1. A matrix representation for $P_n^{(1)}$ is then

$$\begin{bmatrix} 1 & 0 & -1 & 0 & \cdots & 0 \\ 0 & 1 & -1 & 0 & \cdots & 0 \\ -\frac{1}{2n} & -\frac{1}{2n} & 1 & -\frac{1}{2n} & \cdots & -\frac{1}{2n} \\ 0 & 0 & -1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & -1 & 0 & \cdots & 1 \end{bmatrix}.$$

Here, the upper left two by two block corresponds to the two boundary vertices $V_{0,n}$. The third row and column corresponds to the point 0. The remaining rows and columns correspond to $\{\pm e_i\}_{i=2}^n$. In order to establish the spectral similarity relation, we now compute the Schur complement of $P_n^{(1)} - \lambda I$ onto the two boundary points:

$$\begin{bmatrix} 1-\lambda & 0 \\ 0 & 1-\lambda \end{bmatrix} - \begin{bmatrix} -1 & 0 & \cdots & 0 \\ -1 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{2n} & \cdots & -\frac{1}{2n} \\ -1 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ -1 & 0 & \cdots & 1 \end{bmatrix}^{-1} \begin{bmatrix} -\frac{1}{2n} & -\frac{1}{2n} \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix}.$$

This reduces to

$$\begin{bmatrix} 1-\lambda & 0 \\ 0 & 1-\lambda \end{bmatrix} - \frac{1}{2n} \begin{bmatrix} c(\lambda) & c(\lambda) \\ c(\lambda) & c(\lambda) \end{bmatrix}.$$

where $c(\lambda) = (1-\lambda)^{2n-2} / \left((1-\lambda)^{2n-1} - \frac{2n-2}{2n}(1-\lambda)^{2n-3} \right)$ is the upper left entry of

the inverse of the $(2n-1)\times(2n-1)$ matrix above. After some work, we obtain

$$\frac{1}{2n}c(\lambda)\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} - \left(\frac{2}{2n}c(\lambda)-(1-\lambda)\right)\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}.$$

By Lemma 3.3 in [23], $P_n^{(1)}$ is spectrally similar to $P_n^{(0)}$. The functions ϕ_0 and ϕ_1 are given by

$$\phi_0(\lambda) = \frac{1}{2n}c(\lambda), \quad \phi_1(\lambda) = \frac{2}{2n}c(\lambda) - (1-\lambda).$$

The exceptional set $\mathscr{E}$ is given by

$$\mathscr{E} = \left\{1, 1 \pm \sqrt{1-\frac{1}{n}}\right\}.$$

In particular, the set Λ_0 for which the spectral similarity relation holds is given by the complement of $\mathscr{E}$. □

Define

$$R(\lambda) := \phi_1(\lambda)/\phi_0(\lambda) = -2n\lambda^2 + 4n\lambda.$$

Let $\sigma(P_n^{(k)})$ denote the spectrum of $P_n^{(k)}$, which clearly must consist of eigenvalues. By the work in [23], we can use this quadratic l to relate the eigenvalues of $P_n^{(k)}$ to $P_n^{(k+1)}$. In particular, let

$$\mathscr{D}_k = \cup_{m=0}^{k} R^{-m}(\mathscr{E} \cup \sigma(P_n^{(0)})).$$

By part 2 of Theorem 5.8 in [23], we have

$$\sigma(P_n^{(k)}) \subseteq \mathscr{D}_k.$$

In this section, we will precisely compute the eigenvalues of $P_n^{(k)}$ and the corresponding eigenfunctions.

Lemma 5.4.2. $\sigma(P_n^{(1)}) = \{0, 1, 2\}$. *In particular, 0 and 2 are eigenvalues of multiplicity 1, and 1 is an eigenvalue of multiplicity* $2n - 1$.

Proof. One matrix representation of $P_n^{(1)}$ is given by

$$\begin{bmatrix} 1 & -\frac{1}{2n} & \cdots & -\frac{1}{2n} \\ -1 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ -1 & 0 & \cdots & 1 \end{bmatrix}.$$

Here, the first row and column correspond to the vertex 0, and the remaining $2n$ rows and columns correspond to the other vertices of degree 1. The characteristic polynomial of this matrix is given by:

$$(1-\lambda)^{2n+1} - (1-\lambda)^{2n-1}.$$

By analyzing the roots of the polynomial we obtain our result. □

Lemma 5.4.3. *The multiplicity of 1 as an eigenvalue of* $P_n^{(k)}$ *is*

$$\frac{1}{n}(2n)^k + 1$$

for $k \geq 1$.

Proof. Given a finite graph, we define a path to be a sequence of an odd number of vertices such that: (i) two consecutive vertices are connected by an edge; (ii) there

are no repeated vertices in the sequence; and (iii) the first and last vertices in the sequence are of degree one. We can associate with this path a function. In particular, to the vertices not on the path we define the eigenfunction to be identically zero, and for the vertices in the path we associate the values $1, 0, -1, 0, 1, 0, -1, \ldots$ This function may or may not be an eigenfunction of the probabilistic Laplacian, where the corresponding eigenvalue is one.

Take a basis of eigenfunctions $\mathscr{B}$ on $V_{k,n}$. There exists a path eigenfunction of path length three that is non-zero at e_1. Call this path eigenfunction f. After adding some multiple of f, without loss of generality we can suppose that the remaining basis elements attain a zero at e_1. Recall that $V_{k+1,n}$ is isomorphic to $2n$ copies of $V_{k,n}$ with the point e_1 in each copy identified as a single point (corresponding to zero). Each basis element in $\mathscr{B}$ that is not f can be made into an eigenfunction on $V_{k+1,n}$ by being placed on some copy of $V_{k,n}$ and then extended by zero. f itself can be made into an eigenfunction by placing f on every copy of $V_{k,n}$. The resulting set of eigenfunctions is not necessarily a basis on $V_{k+1,n}$, but it is clear that it generates the entire eigenspace.

Let $x_{k,n}$ denote the multiplicity of 1 as an eigenvalue of $P_n^{(k)}$. By our previous work and Theorem 2.1.6, $\operatorname{supp}(P^{(k+1)}, 1)$ is the union of $\operatorname{supp}(P^{(k)}, 1)$ on each of the $2n$ subgraphs in $V_{k+1,n}$ isomorphic to $V_{k,n}$. Since the vertex zero is being counted $2n$ times, we can deduce

$$x_{k+1,n} = 2n x_{k,n} - (2n - 1)$$

for $k \geq 1$, where $x_{1,n} = 2n - 1$. By solving this recurrence relation we obtain our result. $\square$

Lemma 5.4.4. *(Extension Algorithm) Let f be an eigenfunction of $P_n^{(k)}$ with corre-*

sponding eigenvalue λ. *Let* $\lambda_\pm = 1 \pm \sqrt{1 - \frac{\lambda}{2n}}$ *denote the preimages of* λ *under* R. *Define* $f_\pm$ *on* $V_{k+1,n}$ *by*

$$f_\pm(x) = \begin{cases} f(x) & : x \in V_{k,n} \\ \dfrac{f(a)+f(b)}{2n(1-\lambda_\pm) - (2n-2)(1-\lambda_\pm)^{-1}} & : x = \frac{a+b}{2} \text{ where } a, b \in V_{k,n} \\ \dfrac{f(a)+f(b)}{2n(1-\lambda_\pm)^2 - (2n-2)} & : x \sim c; x \notin V_{k,n}; c = \frac{a+b}{2} \text{ where } a, b \in V_{k,n} \end{cases}$$

Then $f_\pm$ *is an eigenfunction of* $P_n^{(k+1)}$ *with eigenvalue* $\lambda_\pm$.

Proof. Recall that the graph $V_{k+1,n}$ is isomorphic to $2n$ copies of $V_{k,n}$ that are all "glued" together at a single point. Alternatively, the graph $V_{k+1,n}$ can be constructed in a local manner. That is, if two vertices a and b in $V_{k,n}$ are connected by a single edge, then a vertex c is placed in the middle of the edge and $2n-2$ other vertices of degree one are connected to c. By definition, an eigenfunction g of $P_n^{(k+1)}$ must satisfy

$$(1-\lambda')g(x) = d_x^{-1} \sum_{y \sim x} g(y),$$

where λ' is the corresponding eigenvalue. This definition can be used to determine the extension algorithm for vertices not in $V_{k,n}$. By some computations, this relation holds for vertices in $V_{k,n}$. □

Define the measure δ_x on the real line to be a point mass at x. Define the measure $\mu_{k,n}$ by

$$\mu_{k,n} := \sum_{\lambda \in \sigma(P_n^{(k)})} \delta_\lambda.$$

Note that the total mass of $\mu_{k,n}$ equals the total number of eigenvalues of $P_n^{(k)}$ and that the mass at any particular value equals its multiplicity as an eigenvalue. By all

our work, we have the following.

Proposition 5.4.5. *Let $k \geq 1$. Then*

$$\mu_{k,n} = \delta_0 + \delta_2 + \sum_{j=0}^{k-1} \sum_{x \in R^{-j}(\{1\})} c_{k-j,n} \delta_x,$$

where the constant term $c_{m,n}$ is defined by

$$c_{m,n} = \frac{1}{n}(2n)^m + 1 \quad m \geq 1.$$

Proof. Recall that the number one is an eigenvalue of $P_n^{(k-1)}$ of multiplicity $c_{k-1,n}$. So by Lemma 5.4.4, the two preimages of one under R are eigenvalues of $P_n^{(k)}$ of multiplicity $c_{k-1,n}$. As an eigenvalue of $P_n^{(k-2)}$, the number one has multiplicity c_{k-2}. By applying Lemma 5.4.4 twice, the four preimages of one under R^2 are eigenvalues of $P_n^{(k)}$ of multiplicity c_{k-2}. By continuing this argument, we can conclude that

$$\mu_{k,n} \geq \delta_0 + \delta_2 + \sum_{j=0}^{k-1} \sum_{x \in R^{-j}(\{1\})} c_{k-j,n} \delta_x.$$

By a computation, the mass of the measure on the right hand side is equal to the total number of eigenvalues of $P_n^{(k)}$. Thus, the two measures are equal. □

5.5 An Example Pertaining to Fractal Tiling

The Hata tree is the unique self-similar set in the complex plane determined by the contractions $\phi_0(z) = c\bar{z}$ and $\phi_1(z) = (1 - |c|^2)\bar{z} + |c|^2$. In the special case where $c = \frac{\sqrt{5}-1}{2}i$, it is possible to construct a tiling of $\mathbb{C}$ by adding another contraction

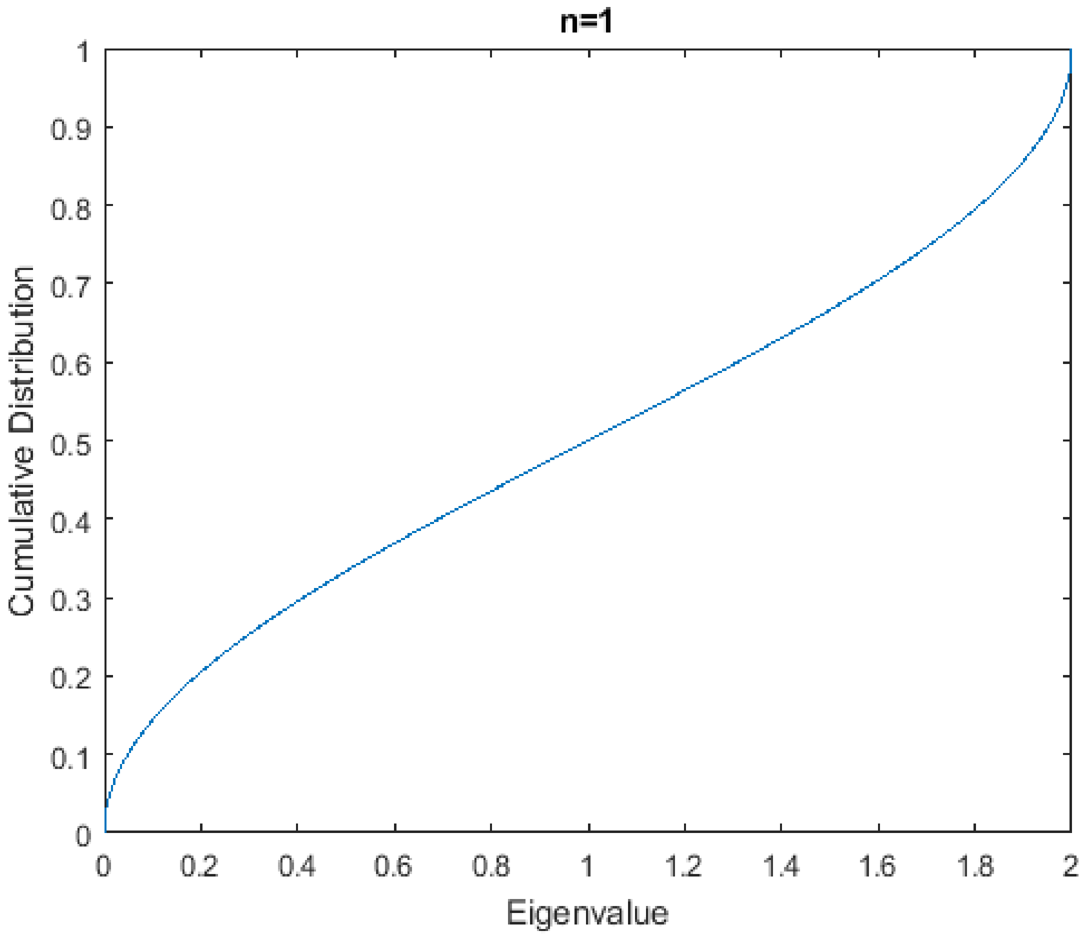


FIGURE 5.4.2: The distribution $\mu_{7,1}$

into the system. Take $\phi_2(z) = (\frac{1}{2} + \frac{1}{2}i) - |c|z$. The unique fixed point (call it A) determined by ϕ_0, ϕ_1 and ϕ_2 is the rectangle in $\mathbb{C}$ with vertices $-\frac{|c|}{2} + \frac{|c|}{2}i, \frac{1}{2} - \frac{1}{2}i, 1$ and $(\frac{1}{2} - \frac{|c|}{2}) + (\frac{1}{2} + \frac{|c|}{2})i$. By identifying $\mathbb{C}$ with $\mathbb{R}^2$, it is possible to use the methods in [1] to create a tiling of $\mathbb{C}$. Note that the iterated function system $\{\phi_0, \phi_1, \phi_2\}$ is overlapping because $\phi_0(A) \cap \phi_2(A)$ is non-empty and has a non-empty interior.

It is possible to add a different contraction. Take $\phi_3(z) = (\frac{1}{2}+\frac{1}{2}i)-\frac{1}{2}z+(|c|-\frac{1}{2})i\bar{z}$. The unique fixed point determined by ϕ_0, ϕ_1 and ϕ_3 is also A. This iterated function

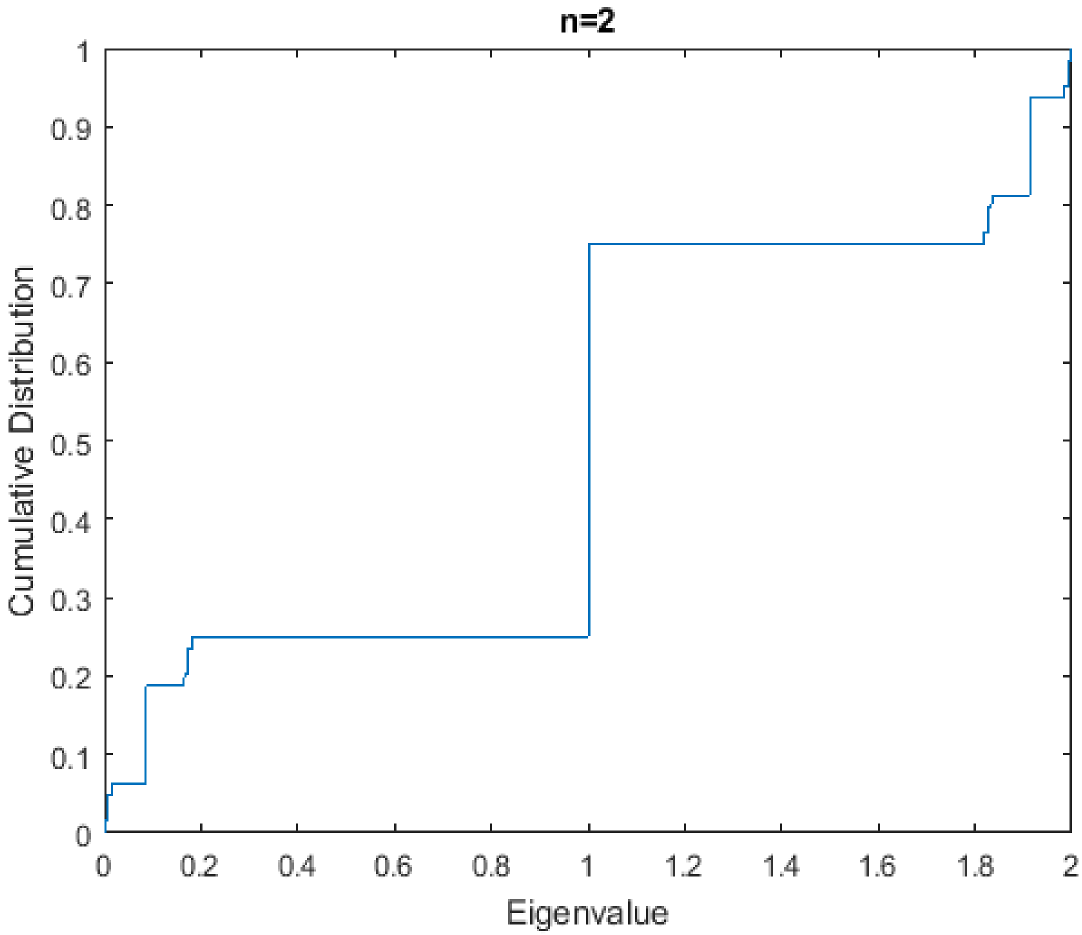


FIGURE 5.4.3: The distribution $\mu_{7,2}$

system is non-overlapping because pairwise $\phi_i(A) \cap \phi_j(A)$ are non-empty but have empty interior for $i, j \in \{0, 1, 3\}, i \neq j$.

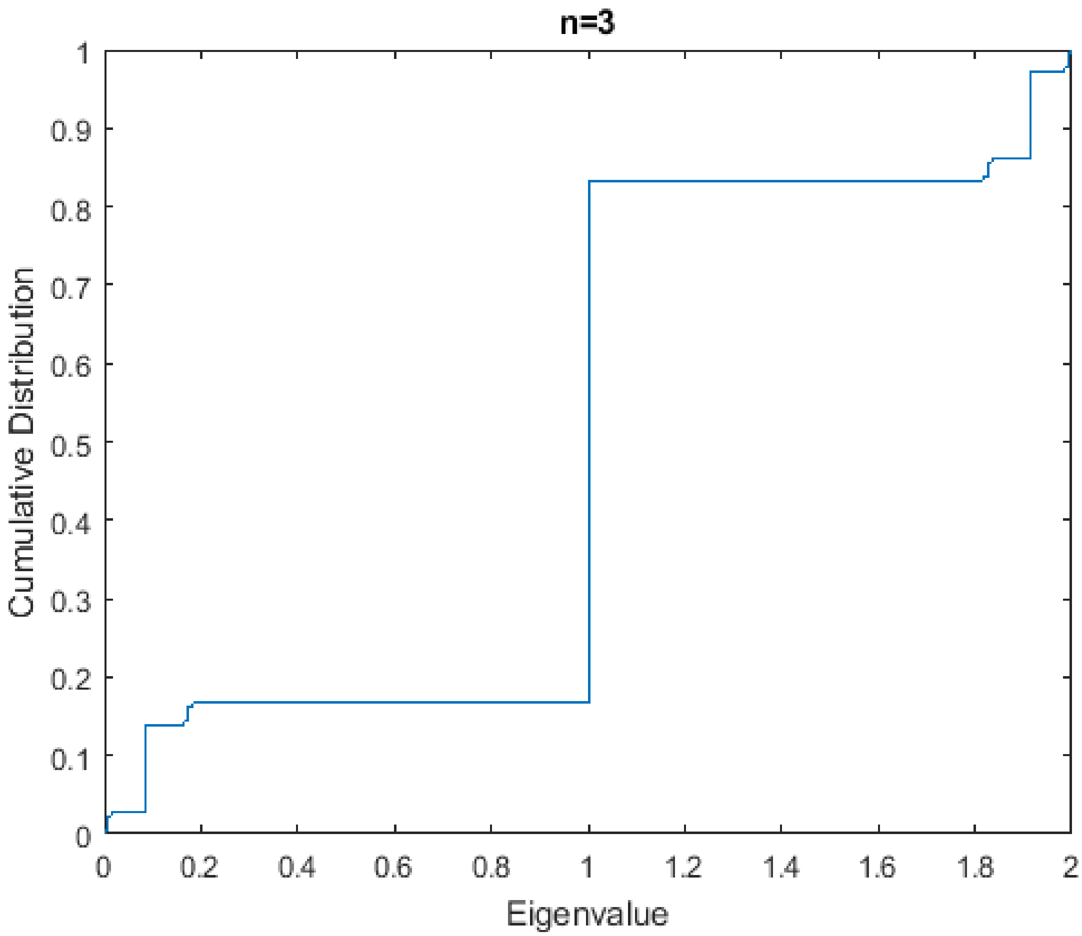


FIGURE 5.4.4: The distribution $\mu_{7,3}$

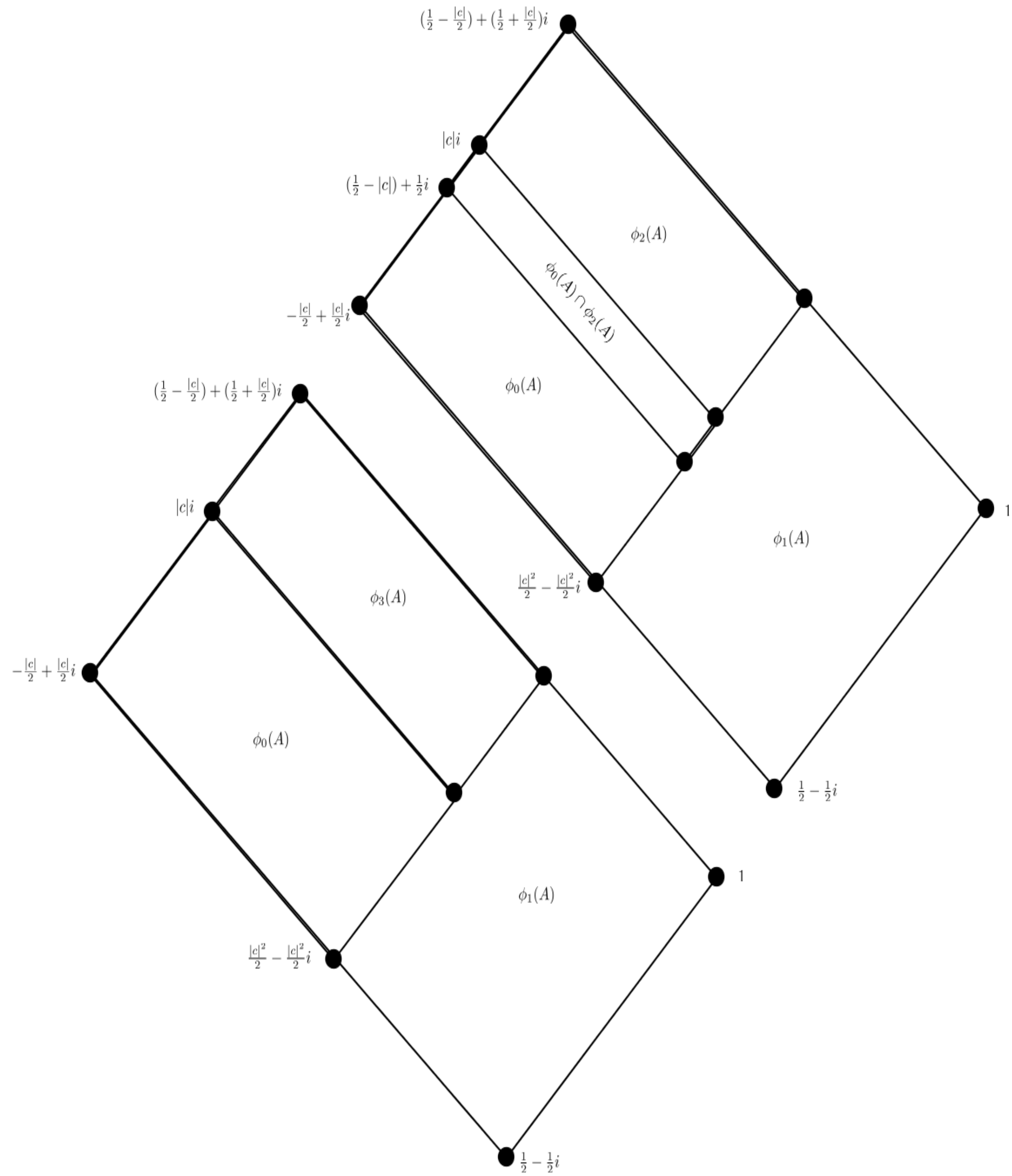


FIGURE 5.5.1: The fixed point of the IFSs $\{\phi_0, \phi_1, \phi_2\}$ and $\{\phi_0, \phi_1, \phi_3\}$

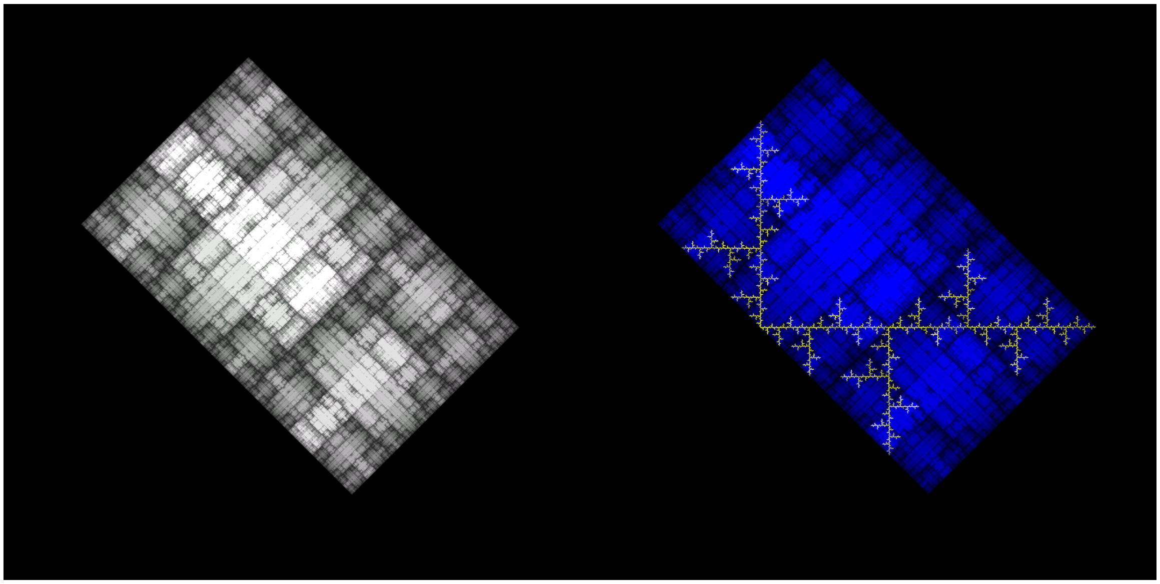


FIGURE 5.5.2: The fixed point of $\{\phi_0, \phi_1, \phi_2\}$. The Hata tree is visible in the figure on the right.

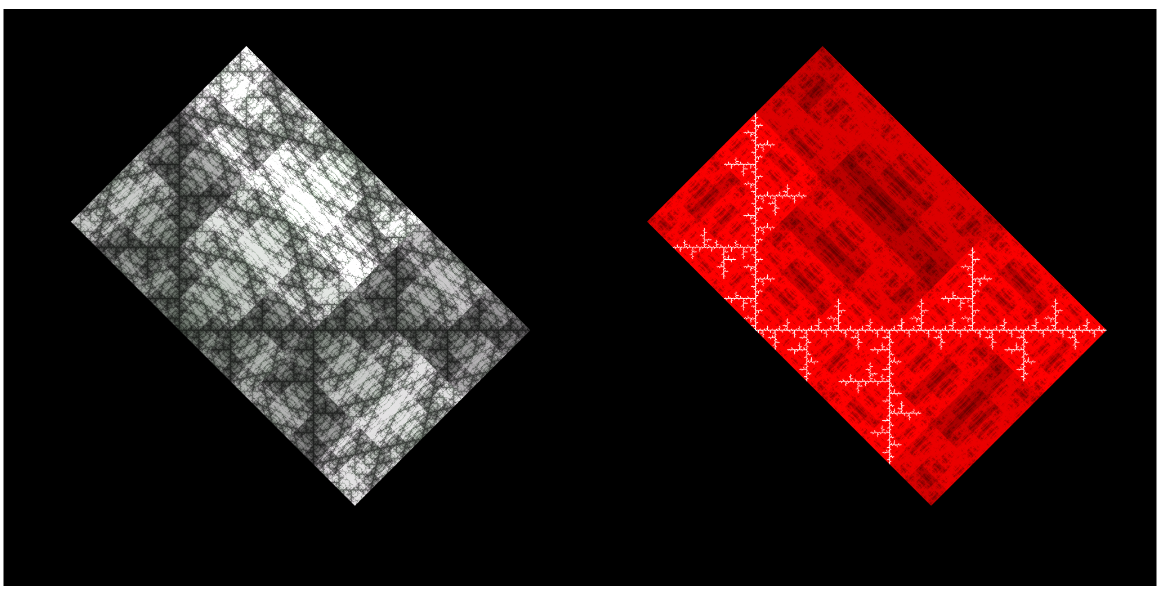


FIGURE 5.5.3: The fixed point of $\{\phi_0, \phi_1, \phi_3\}$. The Hata tree is visible in the figure on the right.

Chapter 6

Theoretical and Numerical Spectral Analysis of the Basilica Graphs

6.1 Introduction

The Basilica group is generated by a finite automaton acting on a binary tree in a self-similar fashion. It was introduced in 2002 by R. Grigorchuk and A. Zuk in [11]. They show that it does not belong to the closure of the set of groups of subexponential growth under the operations of group extension and direct limit. In [2], L. Bartholdi and B. Virag further shows that the group is amenable, making the Basilica group the first example of an amenable but not subexponentially amenable group. In [25] , V. Nekrashevych described the group as the iterated monodromy group of the polynomial $z^2 - 1$ and gave a natural way to associate it to the Basilica fractal, that is, the Julia set of $z^2 - 1$.

In [5], the authors study the finite and infinite Schreier graphs of the Basilica group acting on the binary tree in a self-similar fashion. They show that the infinite graphs have either one, two or four ends. There is only one isomorphism class of the 4-ended graphs, and uncountably many isomorphism classes of the 2-ended and 1-ended graphs.

A Schreier graph can be constructed from the action of a group on a set. Let T be a regular rooted tree. Let $G < Aut(T)$ be a finitely generated group of automorphisms of T. By fixing a set of generators S of G, one obtains a sequence $\{\Gamma_n\}_{n\geq 1}$ of finite left Schreier graphs of the action of G on T. The vertices of Γ_n are the vertices of the nth level of T, and two vertices v, v' are connected if there is a generator $s \in S$ such that $s \cdot v = v'$. The action of G on the boundary ∂T corresponds to an infinite family of orbital Schrier graphs $\{\Gamma_\xi\}_{\xi\in\partial T}$. The graphs (Γ_ξ, ξ) are the limits in the pointed Gromov-Hausdorff topology of finite Schreier graphs (Γ_n, ξ_n), with ξ_n being the prefix of ξ of length n.

The focus of this chapter will be on spectral computations on the Schreier graphs. There already exists literature on the topic. In [28], the authors construct Dirichlet forms and the corresponding Laplacians on the Basilica Julia set for which the topology in the effective resistance metric coincides with the usual topology. This is done in two different ways, by imposing a self-similar harmonic structure and by imposing a graph-directed self-similar structure on the fractal. Under the self-similar structure, it is possible to use the technique of spectral decimation to compute the spectrum of the Laplacian on approximating graphs. This is not possible under the graph-directed structure, whose graph approximations coincide with $\{\Gamma_n\}_{n\geq 1}$. In [7], the authors provide numerical techniques to approximate eigenvalues and eigenfunctions on families of Laplacians on the Julia sets of $z^2 + c$.

This chapter is divided into four parts. In the first part, we associate to $\{\Gamma_n\}_{n\geq 1}$ a sequence of graphs $\{G_n\}_{n\geq 0}$. Essentially, Γ_n can be decomposed into G_n and G_{n-1}. By studing the latter sequence, we obtain a dynamical system for the characteristic polynomial of the Laplacian. This can be used to find the characteristic polynomial of the Laplacian on Γ_n, and is given in Theorem 6.2.4. In the second part, we define a Dirichlet to Neumann map for the Laplacian on G_n. The main result in the third part is Theorem 6.4. Essentially, the Dirichlet to Neumann map is used to show that the limiting distribution of eigenvalues must have a gap. Finally, in the last part, we consider some infinite blow-ups of the graphs G_n. In particular, with the right assumptions (Assumption 1) we can deduce that the spectrum of the Laplacian on the blow-ups is pure point (Theorem 6.5.4).

6.2 Characteristic Polynomials of Finite Graphs

In the Schreier graphs Γ_n, the edges are labeled by the generators a, b of the group B and its vertices are encoded by the set $\{0,1\}^n$. In particular, the vertex 0^n is of degree four. Its removal will divide Γ_n into two subgraphs. Let us modify these subgraphs slightly by attaching to each edge incident to 0^n a vertex of degree one. Let us call the larger subgraph G_n and the smaller subgraph H_n, and let us call the attached vertices the corresponding boundary points. By the self-similar construction of the graphs Γ_n, there is a graph isomorphism between G_n and H_{n+1}. Let G_0 be H_1, the complete graph of two vertices. One can recover the graph Γ_n by identifying the boundaries of G_n and G_{n-1} as a single point.

By Proposition 3.1 in [5], we can use replacment rules to construct Γ_n recursively.

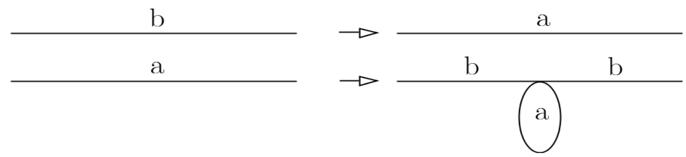


FIGURE 6.2.1: Replacement Rule

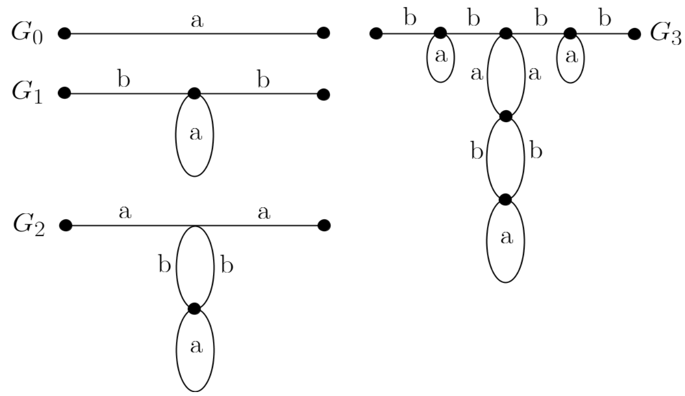


FIGURE 6.2.2: Approximating Graphs G_n, $n = 0, 1, 2, 3$

These same rules can be used to generate the graphs G_n, and are pictured in Figure 6.2.1. Let us assign the one edge in G_0 the letter b. The graph G_n can be obtained by applying the replacement rules n times to G_0. Figure 6.2.2 illustrates the first few approximating graphs of G_n.

In this chapter, we will work with a specific Laplacian. We define the graph Laplacian $L^{(n)}$ on $\ell_n^2 := \mathbb{R}^{G_n}$ by

$$L^{(n)} f(x) = \sum_{x \sim y} c_{xy}(f(x) - f(y)),$$

where c_{xy} is the number of edges between x and y. As no conditions are being imposed on the boundary, this operator is called the Neumann Laplacian. Let $L_0^{(n)}$ denote the restriction to $\mathbb{R}^{G_n \setminus \partial G_n}$. This restriction is the Dirichlet Laplacian on G_n, as a zero boundary condition is being imposed on ∂G_n.

Let $D(L^{(n)})$ and $D(L_0^{(n)})$ denote the characteristic polynomial of $L^{(n)}$ and $L_0^{(n)}$, respectively. The following theorem will be essential in constructing a dynamical system to compute these characteristic polynomials.

Theorem 6.2.1. *Let G be a finite graph. Fix a vertex u in G and let $C(u)$ be the set of cycles in G containing u. Let A denote the adjacency matrix of G, that is, $A_{x,y} = 1$ if x and y are connected by an edge, and 0 otherwise. Then the characteristic polynomial of A is*

$$D(A) = \lambda D(A_u) - \sum_{v \sim u} D(A_{uv}) - 2 \sum_{Z \in C(u)} D(A_Z),$$

where u is some fixed vertex of G, d_x is the degree of vertex x and A_Z denotes the submatrix of A with the rows and columns corresponding to the vertices in Z removed. [31]

Remark 6.2.2. For a graph Laplacian L on G_n, defined by

$$Lf(x) = \sum_{x \sim y} f(x) - f(y),$$

note that $L = D - A$, where D is a diagonal matrices containing the degrees of the vertices of G. Thus, one can immediately deduce that

$$D(L) = (\lambda - d_u) D(L_u) - \sum_{v \sim u} D(L_{uv}) - 2 \sum_{Z \in C(u)} D(L_Z).$$

For $n \geq 3$, we define six subgraphs of G_n: A_n, B_n, C_n, D_n, E_n and F_n. We set A_n to be G_n. B_n and C_n are formed by removing one or both boundary vertices, respectively. D_n and F_n are formed by removing both boundary vertices plus one

or both of the adjacent vertices, respectively. Finally, E_n is formed by removing one boundary vertex and its adjacent vertex. Figure 6.2.3 illustrates these subgraphs. Denote by a_n, b_n, c_n, d_n, e_n and f_n the characteristic polynomials of the restriction of $L^{(n)}$ to A_n, B_n, C_n, D_n, E_n and F_n, respectively. These polynomials for $n = 3$ are displayed below.

$$a_3(\lambda) = \lambda^7 - 16\lambda^6 + 93\lambda^5 - 248\lambda^4 + 309\lambda^3 - 160\lambda^2 + 28\lambda$$

$$b_3(\lambda) = \lambda^6 - 15\lambda^5 + 79\lambda^4 - 182\lambda^3 + 181\lambda^2 - 62\lambda + 4$$

$$c_3(\lambda) = \lambda^5 - 14\lambda^4 + 66\lambda^3 - 128\lambda^2 + 96\lambda - 16$$

$$d_3(\lambda) = \lambda^4 - 12\lambda^3 + 43\lambda^2 - 50\lambda + 12$$

$$e_3(\lambda) = \lambda^5 - 13\lambda^4 + 54\lambda^3 - 83\lambda^2 + 38\lambda - 4$$

$$f_3(\lambda) = \lambda^3 - 10\lambda^2 + 24\lambda - 8$$

The polynomials for $n = 4$ are displayed below.

$$a_4(\lambda) = \lambda^{12} - 32\lambda^{11} + 437\lambda^{10} - 3336\lambda^9 + 15685\lambda^8 - 47264\lambda^7 + 92248\lambda^6 - 115348\lambda^5 + 89240\lambda^4 - 39792\lambda^3 + 8928\lambda^2 - 768\lambda$$

$$b_4(\lambda) = \lambda^{11} - 31\lambda^{10} + 407\lambda^9 - 2956\lambda^8 + 13033\lambda^7 - 36094\lambda^6 + 62966\lambda^5 - 67712\lambda^4 + 42632\lambda^3 - 14160\lambda^2 + 1984\lambda - 64$$

$$c_4(\lambda) = \lambda^{10} - 30\lambda^9 + 378\lambda^8 - 2604\lambda^7 + 10708\lambda^6 - 26992\lambda^5 + 41376\lambda^4 - 37184\lambda^3 + 18176\lambda^2 - 4096\lambda + 256$$

$$d_4(\lambda) = \lambda^9 - 26\lambda^8 + 279\lambda^7 - 1606\lambda^6 + 5402\lambda^5 - 10848\lambda^4 + 12728\lambda^3 - 8112\lambda^2 + 2368\lambda - 192$$

$$e_4(\lambda) = \lambda^{10} - 27\lambda^9 + 304\lambda^8 - 1863\lambda^7 + 6812\lambda^6 - 15330\lambda^5 + 21104\lambda^4 - 17000\lambda^3 + 7216\lambda^2 - 1280\lambda + 64$$

$$f_4(\lambda) = \lambda^8 - 22\lambda^7 + 196\lambda^6 - 920\lambda^5 + 2472\lambda^4 - 3840\lambda^3 + 3264\lambda^2 - 1280\lambda + 128$$

We are now in a position to determine $D(L^{(n)})$.

Proposition 6.2.3. *For an integer n, let*

$$K(n) = \begin{cases} (n-1)/2 & : n \text{ odd} \\ (n-2)/2 & : n \text{ even} \end{cases}$$

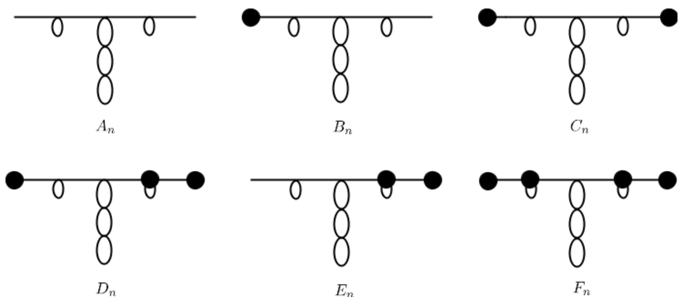


FIGURE 6.2.3: Subgraphs of G_n (removed vertices are indicated)

and

$$g_n = \prod_{j=1}^{K(n)} (c_{n-2j})^{2^{j-1}}.$$

For $n \geq 5$, let

$$a_n = (\lambda - 4)b_{n-2}^2 c_{n-1} - 2e_{n-2}b_{n-2}c_{n-1} - 2b_{n-2}^2 d_{n-1} - 2b_{n-2}^2 g_n,$$

$$b_n = (\lambda - 4)c_{n-2}b_{n-2}c_{n-1} - d_{n-2}b_{n-2}c_{n-1} - c_{n-2}e_{n-2}c_{n-1} - 2c_{n-2}b_{n-2}d_{n-1} - 2c_{n-2}b_{n-2}g_n,$$

$$c_n = (\lambda - 4)c_{n-2}^2 c_{n-1} - 2d_{n-2}c_{n-2}c_{n-1} - 2c_{n-2}^2 d_{n-1} - 2c_{n-2}^2 g_n,$$

$$d_n = (\lambda - 4)d_{n-2}c_{n-2}c_{n-1} - f_{n-2}c_{n-2}c_{n-1} - d_{n-2}d_{n-2}c_{n-1} - 2d_{n-2}c_{n-2}d_{n-1} - 2d_{n-2}c_{n-2}g_n,$$

$$e_n = (\lambda - 4)d_{n-2}b_{n-2}c_{n-1} - f_{n-2}b_{n-2}c_{n-1} - d_{n-2}e_{n-2}c_{n-1} - 2d_{n-2}b_{n-2}d_{n-1} - 2d_{n-2}b_{n-2}g_n,$$

$$f_n = (\lambda - 4)d_{n-2}^2 c_{n-1} - 2f_{n-2}d_{n-2}c_{n-1} - 2d_{n-2}^2 d_{n-1} - 2d_{n-2}^2 g_n.$$

Then

$$\begin{aligned} D(L^{(n)}) &= a_n, \\ D(L_0^{(n)}) &= c_n. \end{aligned}$$

Proof. The main point is that G_n can be constructed from two copies of G_{n-2} and one copy of G_{n-1}. This is illustrated in Figure 6.2.4. In particular, the graph is

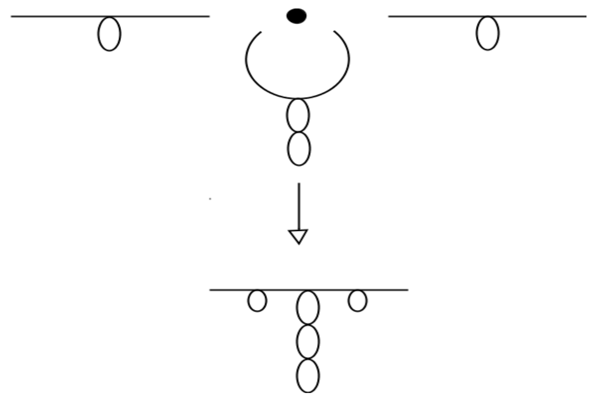


FIGURE 6.2.4: Construction of G_n from G_{n-1} and two copies of G_{n-2}

formed by identifying the two boundary vertices of G_{n-1} and boundary vertex from each copy of G_{n-2} into one vertex. One can apply Theorem 6.2.1 by decomposing the characteristic polynomial at this vertex. Note that for $n \geq 4$, our Laplacian agrees with the standard graph Laplacian at this point. Thus, our use of the theorem is valid. $\square$

Note that this dynamical system can be reduced by a half in complexity. By Theorem 6.2.1,

$$\begin{aligned} b_n &= (\lambda - 1)c_n - d_n, \\ e_n &= (\lambda - 1)d_n - f_n, \\ a_n &= (\lambda - 1)b_n - e_n = (\lambda - 1)^2 c_n - 2(\lambda - 1)d_n + f_n. \end{aligned}$$

Thus, only three of the six sequences in Proposition 6.2.3 are necessary in determining the characteristic polynomial (namely the sequences $\{c_n\}$, $\{d_n\}$ and $\{f_n\}$).

Let us define the Laplacian $L_\Gamma^{(n)}$ on Γ_n by

$$L_\Gamma^{(n)} f(x) = \sum_{x \sim y} c_{xy}(f(x) - f(y)),$$

where c_{xy} is the number of edges between x and y. Recall that the graph Γ_n can be recovered by identifying the boundary vertices of G_n and G_{n-1} into a single point. Thus, by Theorem 6.2.1, we can determine the characteristic polynomial of $L_\Gamma^{(n)}$ by decomposing the characteristic polynomial at the vertex to this identified point. This characteristic polynomial is given below.

Theorem 6.2.4. *Let $n \geq 3$. Then the characteristic polynomial of $L_\Gamma^{(n)}$ is*

$$D(L_\Gamma^{(n)}) = (\lambda - 4)c_n c_{n+1} - 2d_n c_{n+1} - 2c_n d_{n+1} - 2c_n g_{n+1} - 2g_n c_{n+1}.$$

6.3 The Dirichlet to Neumann Map

Let G be a finite graph. Let ∂G, called the boundary of G, denote a subset of vertices of G. Let L be a Laplacian operator on G. Take $f \in \ell(\partial G)$ and consider the following problem for $z \in \mathbb{C}$:

$$\begin{aligned} Lu &= zu \text{ on } G \backslash \partial G \\ u|_{\partial G} &= f \end{aligned} \tag{6.3.1}$$

Note that the case $z = 0$ is the classical Dirichlet problem.

Proposition 6.3.1. *Problem 6.3.1 has a unique solution for any initial boundary condition f if and only if z is not a Dirichlet eigenvalue of L.*

Proof. First, we can decompose L as follows

$$L = \begin{pmatrix} S & X \\ X & U \end{pmatrix},$$

where $S : \partial G \to \partial G$, $X : G\backslash\partial G \to \partial G$ and $U : G\backslash\partial G \to G\backslash\partial G$. Solving problem 6.3.1 is equivalent to finding a function $g : G\backslash\partial G \to G\backslash\partial G$ that satisfies the following matrix equation

$$\begin{pmatrix} S & X \\ X^T & U \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} f' \\ zg \end{pmatrix}.$$

Note that there is no constraint on f'. So we can deduce that

$$X^T f + Ug = zg.$$

which can be rewritten as

$$(U - z)g = -X^T f.$$

Thus, if z is not an eigenvalue of U (a Dirichlet eigenvalue of L), the matrix $U - z$ has full rank and there is a unique solution to problem 6.3.1. If z is in fact an eigenvalue, then there cannot be a solution to the problem for all f. □

For $z \in \mathbb{C}$ for which there is a unique solution to problem 6.3.1, we define a Dirichlet to Neumann map $DtN(z)$ to be an operator on $\ell(\partial G)$. In particular, for $f \in \ell(\partial G)$, we define $\big(DtN(z)f\big)(x)$ to be the normal derivative of the solution, denoted by u, at the boundary vertex x. I.e.

$$\big(DtN(z)f\big)(x) := \frac{\partial u}{\partial n}(x) = Lu(x).$$

Recall that the approximating graphs G_n to the basilica Julia set have a two point boundary. Denote these points by l_n and r_n. Let DtN_n denote the corresponding Dirichlet to Neumann map on G_n. It is a linear operator on functions on a two-point set, and thus has a two-by-two matrix representation. I.e.

$$DtN_n(z)\begin{pmatrix} f(l_n) \\ f(r_n) \end{pmatrix} = \begin{pmatrix} \alpha_n & \beta_n \\ \beta_n & \alpha_n \end{pmatrix}\begin{pmatrix} f(l_n) \\ f(r_n) \end{pmatrix},$$

for some numbers α_n and β_n. The approximating graph G_0 consists solely of the boundary vertices, and so DtN_0 must coincide with $L^{(0)}$. That is $\alpha_0 = 1$ and $\beta_0 = -1$. In general, it is possible to compute DtN_n as follows. Partition the matrix $L^{(n)}$,

$$L^{(n)} = \begin{pmatrix} S_n & X_n \\ X_n^T & U_n \end{pmatrix},$$

where $S_n : \partial G_n \to \partial G_n$, $X_n : G_n \backslash \partial G_n \to \partial G_n$ and $U_n : G_n \backslash \partial G_n \to G_n \backslash \partial G_n$. Let $f \in \ell(\partial G_n)$, and let u be the unique solution to the problem in (6.3.1). Then

$$L^{(n)}u = \begin{pmatrix} S_n & X_n \\ X_n^T & U_n \end{pmatrix}\begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} f' \\ zg \end{pmatrix}, \tag{6.3.2}$$

where the appropriate restrictions of u are implicitly made and f' is unknown. By definition, we have that $DtN_n f = f'$. By Lemma 3.5.1 in [19], the blocks S_n and U_n are invertible. If z is not an eigenvalue of U_n, then

$$g = -(U_n - z)^{-1}X_n^T f.$$

We also know that

$$S_n f + X_n g = f'.$$

Putting the two things together, we obtain

$$(S_n - X_n(U_n - z)^{-1}X_n^T)f = f'.$$

Thus,

$$DtN_n = S_n - X_n(U_n - z)^{-1}X_n^T. \tag{6.3.3}$$

We now compute DtN_1. We have

$$S_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, X_1 = \begin{pmatrix} -1 & 0 \\ -1 & 0 \end{pmatrix}, U_1 = \begin{pmatrix} 4 & -2 \\ -2 & 2 \end{pmatrix}.$$

Applying Equation 6.3.3, we have

$$DtN_1 = \left(1 + \frac{2(2-z)}{(2-z)^2 - 4}\right)^{-1} \begin{pmatrix} 1 + \frac{2-z}{(2-z)^2-4} & -\frac{2-z}{(2-z)^2-4} \\ -\frac{2-z}{(2-z)^2-4} & 1 + \frac{2-z}{(2-z)^2-4} \end{pmatrix}.$$

Thus,

$$\begin{aligned} \alpha_1 &= \left(1 + \frac{2(2-z)}{(2-z)^2 - 4}\right)^{-1} \left(1 + \frac{2-z}{(2-z)^2-4}\right), \\ \beta_1 &= -\left(1 + \frac{2(2-z)}{(2-z)^2 - 4}\right)^{-1} \frac{2-z}{(2-z)^2-4}. \end{aligned}$$

The goal in the remainder of the section is to construct a dynamical system that can be used to find DtN_n. Observe that there exists a vertex s_n in G_n such that its

removal decomposes the graph into three pieces; two of which are isomorphic to G_{n-2} and one isomorphic to G_{n-1}. Figure 6.2.4 illustrates this decomposition.

Lemma 6.3.2. *Fix n. For $z \in \mathbb{C}$, let u_z denote the solution to problem* 6.3.1 *for some fixed $f \in \ell(\partial G_n)$. Then $u_z(s_n)$ is a rational function of z with singularities at the Dirichlet eigenvalues of $L^{(n)}$.*

Proof. This follows by applying Cramer's rule to the matrix equation $(U_n - z)g = -X_n^T f$. □

Proposition 6.3.3. *Pick z so that z is not a Dirichlet eigenvalue of $L^{(n)}$, $n \geq 2$. Let u be a function on G_n satisfying $L^{(n)}u(x) = zu(x)$ for $x \in G_n \backslash \partial G_n$. Then*

$$u(s_n) = \frac{\beta_{n-2}(u(l_n) + u(r_n))}{z - 2(\alpha_{n-1} + \alpha_{n-2} + \beta_{n-1})}. \tag{6.3.4}$$

Proof. By definition, $L^{(n)}u(s_n) = \sum_{y \sim s_n}(f(s_n) - f(y))$. By decomposing the graph into three pieces, as in the explanation preceding the proposition, one can use the corresponding Neumann to Dirichlet map on that piece to compute each term in the sum. For instance, take the subgraph of G_n isomorphic to G_{n-2} containing l_n. Let x be the one neighboring vertex of s_n in this subgraph. Then by identifying l_n and s_n with l_{n-2} and r_{n-2}, respectively, we can use DtN_{n-2} to compute the term $f(s_n) - f(x)$. The remaining three terms can be handled in the same manner.

Let P_1, P_2 be operators on $\mathbb{R}^2$, where $P_1(x,y) = (x,0), P_2(x,y) = (0,y)$. Then

$$\begin{aligned} L^{(n)}u(s_n) &= P_2 DtN_{n-2}(z)\begin{pmatrix} u(l_n) \\ u(s_n) \end{pmatrix} + P_1 DtN_{n-2}(z)\begin{pmatrix} u(s_n) \\ u(r_n) \end{pmatrix} \\ &+ P_2 DtN_{n-1}(z)\begin{pmatrix} u(s_n) \\ u(s_n) \end{pmatrix} + P_1 DtN_{n-1}(z)\begin{pmatrix} u(s_n) \\ u(s_n) \end{pmatrix}. \end{aligned}$$

By our assumption on u, $L^{(n)}u(s_n) = zu(s_n)$. After a computation, we have

$$(z - 2(\alpha_{n-1} + \alpha_{n-2} + \beta_{n-1}))u(s_n) = \beta_{n-2}(u(l_n) + u(r_n)),$$

from which we can deduce the result. $\square$

Remark 6.3.4. The right hand side of equation 6.3.4 is a rational function of z. By the previous lemma, it must have singularities at the Dirichlet eigenvalues of $L^{(n)}$, but potentially has other singularities that are removable.

By applying Proposition 6.3.3, it is possible to set up a recursion to compute α_n and β_n.

Proposition 6.3.5. *For* $n \geq 3$,

$$\begin{aligned} \alpha_n &= \alpha_{n-2} + \frac{\beta_{n-2}^2}{z - 2(\alpha_{n-1} + \alpha_{n-2} + \beta_{n-1})}, \\ \beta_n &= \frac{\beta_{n-2}^2}{z - 2(\alpha_{n-1} + \alpha_{n-2} + \beta_{n-1})}. \end{aligned}$$

Proof. Fix $f \in \ell(\partial G_n)$ such that $f(l_n) \neq f(r_n)$ and $f(l_n) \neq f(-r_n)$. Let u be the

corresponding solution to problem 6.3.1. By definition,

$$DtN_n \begin{pmatrix} u(l_n) \\ u(r_n) \end{pmatrix} = \begin{pmatrix} \alpha_n & \beta_n \\ \beta_n & \alpha_n \end{pmatrix} \begin{pmatrix} u(l_n) \\ u(r_n) \end{pmatrix}.$$

However, these normal derivatives can be computed by breaking up the graph at s_n and applying the map DtN_{n-2} to the two pieces isomorphic to G_{n-2} containing l_n and r_n, respectively. In matrix language,

$$DtN_n \begin{pmatrix} u(l_n) \\ u(r_n) \end{pmatrix} = \begin{pmatrix} P_1 DtN_{n-2} \begin{pmatrix} u(l_n) \\ u(s_n) \end{pmatrix} \\ P_2 DtN_{n-2} \begin{pmatrix} u(s_n) \\ u(r_n) \end{pmatrix} \end{pmatrix}.$$

Thus, comparing these two equations, we get

$$\begin{aligned} \alpha_n u(l_n) + \beta_n u(r_n) &= \alpha_{n-2} u(l_n) + \beta_{n-2} u(s_n), \\ \beta_n u(l_n) + \alpha_n u(r_n) &= \beta_{n-2} u(s_n) + \alpha_{n-2} u(r_n). \end{aligned}$$

Thus

$$\begin{pmatrix} \alpha_n \\ \beta_n \end{pmatrix} = \begin{pmatrix} u(l_n) & u(r_n) \\ u(r_n) & u(l_n) \end{pmatrix}^{-1} \begin{pmatrix} u(l_n) & u(s_n) \\ u(r_n) & u(s_n) \end{pmatrix} \begin{pmatrix} \alpha_{n-2} \\ \beta_{n-2} \end{pmatrix}.$$

After some reduction, if $u(l_n)^2 - u(r_n)^2 \neq 0$, that is, if $f(l_n) \neq f(r_n)$ or $f(l_n) \neq$

$f(-r_n)$, we have

$$\begin{pmatrix} \alpha_n \\ \beta_n \end{pmatrix} = \begin{pmatrix} 1 & u(s_n)/(u(l_n)+u(r_n)) \\ 0 & u(s_n)/(u(l_n)+u(r_n)) \end{pmatrix} \begin{pmatrix} \alpha_{n-2} \\ \beta_{n-2} \end{pmatrix}.$$

By Proposition 6.3.3, we can rewrite $u(s_n)$ as some combination of $u(l_n)$ and $u(r_n)$. Thus,

$$\begin{aligned} \alpha_n &= \alpha_{n-2} + \frac{\beta_{n-2}^2}{z - 2(\alpha_{n-1} + \alpha_{n-2} + \beta_{n-1})}, \\ \beta_n &= \frac{\beta_{n-2}^2}{z - 2(\alpha_{n-1} + \alpha_{n-2} + \beta_{n-1})}. \quad \square \end{aligned}$$

The recursion (α_n, β_n) can be simplified somewhat. By the recursion, for $n \geq 2$ we have $\alpha_n - \alpha_{n-2} = \beta_n$. Thus,

$$\begin{aligned} \alpha_3 &= \alpha_1 + (\alpha_3 - \alpha_1) = \alpha_1 + \beta_3 \\ \alpha_4 &= \alpha_2 + (\alpha_4 - \alpha_2) = \alpha_2 + \beta_4 \\ \alpha_5 &= \alpha_3 + (\alpha_5 - \alpha_3) = (\alpha_1 + \beta_3) + \beta_5 \\ &\vdots \end{aligned}$$

Thus,

$$\alpha_n = \alpha_{n-2K(n)} + \sum_{j=0}^{K(n)-1} \beta_{n-2j}. \tag{6.3.5}$$

Therefore, it suffices to understand how the β_n's evolve. Let $C_z = z - 2(\alpha_1 + \alpha_2)$.

Then

$$\begin{aligned}
\beta_n &= \frac{\beta_{n-2}^2}{z - 2(\alpha_{n-1} + \alpha_{n-2} + \beta_{n-1})} \\
&= \frac{\beta_{n-2}^2}{z - 2(\alpha_1 + \alpha_2 + \sum_{j=3}^{n-1} \beta_j + \beta_{n-1})} \\
&= \frac{\beta_{n-2}^2}{C_z - 2\sum_{j=3}^{n-1} \beta_j - 2\beta_{n-1}}.
\end{aligned}$$

If we rearrange the last equation, we obtain

$$\sum_{j=3}^{n-1} \beta_j = -\frac{1}{2}\left(\frac{\beta_{n-2}^2}{\beta_n} - C_z + 2\beta_{n-1}\right).$$

Thus,

$$\beta_n = \sum_{j=3}^{n} \beta_j - \sum_{j=3}^{n-1} \beta_j = -\beta_n + \beta_{n-1} - \frac{1}{2}\frac{\beta_{n-1}^2}{\beta_{n+1}} + \frac{1}{2}\frac{\beta_{n-2}^2}{\beta_n}.$$

With some more rearranging,

$$\beta_n = \frac{\beta_{n-2}^2}{-4\beta_{n-1} + 2\beta_{n-2} + \frac{\beta_{n-3}^2}{\beta_{n-1}}}.$$

This equation defines a third order recursion. Let us write this recursion in the following manner:

$$(\beta_{n-3}, \beta_{n-2}, \beta_{n-1}) \mapsto \left(\beta_{n-2}, \beta_{n-1}, \frac{\beta_{n-2}^2}{-4\beta_{n-1} + 2\beta_{n-2} + \frac{\beta_{n-3}^2}{\beta_{n-1}}}\right).$$

To help us simplify the recursion even more, we define the intermediate variable

$$D_n = -4\beta_{n+2} + 2\beta_{n+1} + \frac{\beta_n^2}{\beta_{n+2}}.$$

Observe that

$$\begin{aligned}
D_{n+1} &= -4\beta_{n+3} + 2\beta_{n+2} + \frac{\beta_{n+1}^2}{\beta_{n+3}} \\
&= \frac{-4\beta_{n+1}^2}{-4\beta_{n+2} + 2\beta_{n+1} + \frac{\beta_n^2}{\beta_{n+2}}} + 2\beta_{n+2} + \frac{\beta_{n+1}^2}{\left(\frac{\beta_{n+1}^2}{-4\beta_{n+2} + 2\beta_{n+1} + \frac{\beta_n^2}{\beta_{n+2}}}\right)} \\
&= \frac{-4\beta_{n+1}^2}{D_n} + 2\beta_{n+2} + D_n.
\end{aligned}$$

The following “simplified” recursion encodes the same information.

$$(\beta_{n+1}, \beta_{n+2}, D_n) \mapsto \left(\beta_{n+2}, \frac{\beta_{n+1}^2}{D_n}, \frac{-4\beta_{n+1}^2}{D_n} + 2\beta_{n+2} + D_n\right).$$

Thus, to deduce the limiting behavior of the β_n's, it suffices to understand the behavior of

$$F(x, y, z) = \left(y, \frac{x^2}{z}, z + 2y - \frac{4x^2}{z}\right). \tag{6.3.6}$$

We conclude the section by providing a neat application of the Dirichlet to Neumann map. On the graphs G_n, let lx_n and rx_n be the unique vertices that share an edge with the boundary vertices l_n and r_n, respectively. We define T_n to be an operator that maps $\begin{pmatrix} u(l_n) \\ u(lx_n) \end{pmatrix}$ to $\begin{pmatrix} u(r_n) \\ u(rx_n) \end{pmatrix}$. This operator is called the transmission operator.

Proposition 6.3.6. *For $n \geq 1$,*

$$T_n = \frac{1}{\beta_n} \begin{pmatrix} 1 - \alpha_n & -1 \\ (\alpha_n - 1)^2 - \beta_n^2 & \alpha_n - 1 \end{pmatrix}.$$

Proof. By definition,

$$DtN_n \begin{pmatrix} u(l_n) \\ u(r_n) \end{pmatrix} = \begin{pmatrix} u(l_n) - u(lx_n) \\ u(r_n) - u(rx_n) \end{pmatrix}.$$

Thus,

$$\begin{aligned} \alpha_n u(l_n) + \beta_n u(r_n) &= u(l_n) - u(lx_n), \\ \beta_n u(l_n) + \alpha_n u(r_n) &= u(r_n) - u(rx_n), \end{aligned}$$

which can be rewritten as

$$\begin{pmatrix} u(l_n) \\ u(lx_n) \end{pmatrix} = \begin{pmatrix} \alpha_n - 1 & 1 \\ \beta_n & 0 \end{pmatrix}^{-1} \begin{pmatrix} -\beta_n & 0 \\ 1 - \alpha_n & -1 \end{pmatrix} \begin{pmatrix} u(r_n) \\ u(rx_n) \end{pmatrix}.$$

After doing a multiplication, we obtain our result. □

6.4 Gap in the Limiting Distribution of Eigenvalues

In this section, we will use the Dirichlet to Neumann maps on the graphs G_n to deduce that there is a gap in the limiting distribution of eigenvalues of $L^{(n)}$. First, we prove a useful lemma.

Lemma 6.4.1. *Let z be an eigenvalue of $L^{(n)}$. Let f be the corresponding eigenfunction. Suppose that $f(l_n) \neq 0$ or $f(r_n) \neq 0$. Then $z = \alpha_n - \beta_n$ or $z = \alpha_n + \beta_n$.*

Proof. Let z be the corresponding eigenvalue. By definition

$$\begin{pmatrix} \alpha_n & \beta_n \\ \beta_n & \alpha_n \end{pmatrix} \begin{pmatrix} f(l_n) \\ f(r_n) \end{pmatrix} = z \begin{pmatrix} f(l_n) \\ f(r_n) \end{pmatrix}.$$

Thus, z is an eigenvalue of DtN_n. This implies

$$(\alpha_n - z)^2 - \beta_n^2 = 0,$$

from which we can immediately deduce the result. $\square$

The following is the main result of the section.

Theorem 6.4.2. *In the Hausdorff metric,* $\limsup_{n\to\infty} \sigma(L^{(n)})$ *has a gap that contains the interval* $(2.5, 2.8)$.

Proof. First, we will show that $\beta_n \to 0$ for $2.5 < z < 2.8$. Note that the sequence β_n implicitly depends on z. Since $2.5 < z < 2.8$, we can deduce numerically that

$$-.16129 < \beta_2 < -.10527, \quad -.78756 < \beta_3 < -.51149, \quad -2.13921 < D_1 < -1.33148.$$

Recall from (6.3.6) that $F(\beta_{n+1}, \beta_{n+2}, D_n) = (\beta_{n+2}, \beta_{n+3}, D_{n+1})$ for $n \geq 1$. So to understand how to bound β_n and D_n in general, we need to study the map F further. Suppose that

$$a < x < b, \quad c < y < d, \quad e < z < f,$$

where a, b, c, d, e, f are all negative numbers. Let $(x', y', z') = F(x, y, z)$. Then

$$c < x' < d, \quad \frac{b^2}{f} < y' < \frac{a^2}{e}, \quad e + 2c - 4\frac{a^2}{e} < z' < f + 2d - 4\frac{b^2}{f}.$$

Note that the bounds for z' are not necessarily negative. Let us define a new function

$$G(a,b,c,d,e,f) = \left(c, d, \frac{b^2}{f}, \frac{a^2}{e}, e + 2c - 4\frac{a^2}{e}, f + 2d - 4\frac{b^2}{f}\right).$$

Let us pick tuples of negative numbers $(a_1, b_1, c_1, d_1, e_1, f_1)$ and (x_1, y_1, z_1) satisfying

$$a_1 < x_1 < b_1, \quad c_1 < y_1 < d_1, \quad e_1 < z_1 < f_1.$$

Set $(x_{n+1}, y_{n+1}, z_{n+1}) = F(x_n, y_n, z_n)$ and

$$(a_{n+1}, b_{n+1}, c_{n+1}, d_{n+1}, e_{n+1}, f_{n+1}) = G(a_n, b_n, c_n, d_n, e_n, f_n)$$

for $n \geq 1$. (In this proof, we take $(a_n, b_n, c_n, d_n, e_n, f_n)$ to be different from the sequence of polynomials defined previously.) By construction, it is clear that

$$a_2 < x_2 < b_2, \quad c_2 < y_2 < d_2, \quad e_2 < z_2 < f_2.$$

If $e_2, f_2 < 0$, then we can deduce the same set of inequalities for $n = 3$. Thus, if $a_n, b_n, c_n, d_n, e_n, f_n < 0$ for all n, then

$$a_n < x_n < b_n, \quad c_n < y_n < d_n, \quad e_n < z_n < f_n,$$

for all n. Set $(a_1, b_1, c_1, d_1, e_1, f_1) = (-.16129, -.10527, -.78756, -.51149, -2.13921, -1.33148)$. We can deduce numerically that

$$(a_n, b_n, c_n, d_n, e_n, f_n) \to (0, 0, 0, 0, M_1, M_2),$$

where $M_1 \approx -3.24073$ and $M_2 \approx -2.18943$. Furthermore, we can deduce that every term in the sequence is negative. Thus, we can conclude that $\beta_n \to 0$.

By Remark 6.3.4, no value in $(2.5, 2.8)$ can be a Dirichlet eigenvalue of $L^{(n)}$ as the dynamical system does not encounter any singularities. So if there is an eigenvalue z in the interval, the corresponding eigenvector must have a non-trivial boundary condition. By the previous lemma, we must have that $z = \alpha_n - \beta_n$ or $z = \alpha_n + \beta_n$. We will show that this cannot happen.

By Equation 6.3.5 and the fact that the β_n's are negative, for n odd

$$\alpha_n \geq \alpha_1 + \sum_{j=0}^{\infty} \beta_{3+2j}.$$

and for n even

$$\alpha_n \geq \alpha_2 + \sum_{j=0}^{\infty} \beta_{4+2j}.$$

Also recall that $\beta_{n+2} = \beta_n^2 / D_{n-1}$. By inspecting the terms e_n and f_n, we can deduce numerically that $D_n < -1.33148$. Let $C = -1.33148$. Then

$$\begin{aligned}
\sum_{j=0}^{\infty} \beta_{3+2j} &\geq \beta_3 + \frac{\beta_3^2}{C} + \frac{(\frac{\beta_3^2}{C})^2}{C} + \frac{(\frac{(\frac{\beta_3^2}{C})^2}{C})^2}{C} \cdots \\
&= \beta_3 + \frac{\beta_3^2}{C} + \frac{\beta_3^4}{C^3} + \frac{\beta_3^8}{C^7} + \cdots \\
&= C \sum_{j=0}^{n} \left(\frac{\beta_3}{C}\right)^{2^j}.
\end{aligned}$$

By the integral test from calculus,

$$\sum_{j=0}^{n} \left(\frac{\beta_3}{C}\right)^{2^j} \leq \int_0^{\infty} \left(\frac{\beta_3}{C}\right)^{2^x} dx < \int_0^{\infty} \left(\frac{78756}{133148}\right)^{2^x} dx < .76525.$$

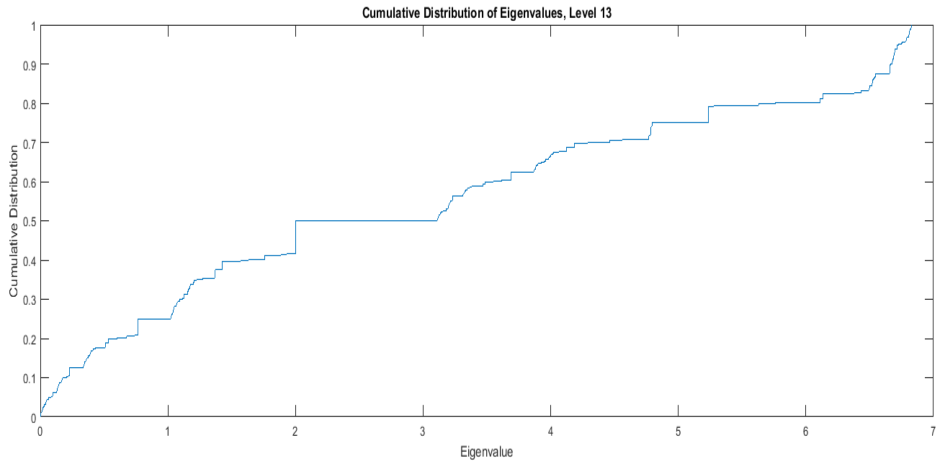


FIGURE 6.4.1: Distribution of Eigenvalues, Level 13

One can check that $2.25 < \alpha_1 < 3$. Thus for n odd, $\alpha_n > 1.23108$. Since, $.8387 < \alpha_2 < .8947$ and $-.01953 < \beta_4 < -.00518$, in a similar manner we can deduce that $\alpha_n > .83753$ for n even. So, $\alpha_n > .83753$ for all n.

We now find an upper bound for the α_n's. Since the β_n's are all negative, we can deduce that $\{\alpha_{2n}\}$ and $\{\alpha_{2n+1}\}$ are monotonically decreasing sequences. As noted before, $.8387 < \alpha_2 < .8947$. In a similar manner, we can check that $1.7385 < \alpha_3 < 2.2124$. So $\alpha_n < 2.2124$ for $n \geq 2$.

Taking the upper and lower bounds for α_n together into consideration, we have $.83753 < \alpha_n < 2.2124$ for $n \geq 2$. Since $\beta_n \to 0$ as $n \to \infty$, by taking n large enough, it is clear that $\alpha_n + \beta_n$ and $\alpha_n - \beta_n$ will not be in the interval $(2.5, 2.8)$. So for n large enough, the set $\sigma(L^{(n)})$ will have a gap containing the interval $(2.5, 2.8)$. $\square$

In the remainder of the section, we will provide evidence that there exists a gap in the limiting distribution of eigenvalues of $\Delta^{(n)} = (2\sqrt{2})^n L^{(n)}$. The choice of the scaling factor is not random. In [28], a conformally invariant resistance form and Laplacian is constructed on the Basilica Julia set. In particular, the Laplacian has

self-similar scaling and its scaling factor is $2\sqrt{2}$. The number $\sqrt{2}$ is the resistance scaling factor and 2 is the measure scaling factor).

Showing that there is a gap in the limit of $(2\sqrt{2})^n a_n^{-1}\{0\}$ is not as simple. Due to the presence of the scaling factor, we cannot use the exact same techniques of the previous section involving the Dirichlet to Neumann map. As an alternative, we will provide estimates on the order of the second smallest eigenvalue of $\Delta^{(n)}$. It is a well known fact that the graph Laplacian of a connected eigenvalue has zero as its smallest eigenvalue, and that its second smallest eigenvalue is positive. Denote by λ_{G_n} the second smallest eigenvalue of $L^{(n)}$.

We will first apply classical Cheeger's inequality (c.f. [4]) to determine an upper bound for λ_{G_n}. In the classical case, Cheeger's inequality states that the second smallest eigenvalue of a normalized Laplacian is bounded above by twice a particular constant (known as Cheeger's constant). As $L^{(n)}$ does not meet the criteria of a normalized Laplcian, we will prove a variant of the inequality holds in our case. For two vertex-disjoint subsets A and B of a graph G, let $E(A,B) = \frac{1}{2}|\{(a,b) :$ there exists an edge connecting $a \in A$ and $b \in B\}|$. In essence, $E(A,B)$ is a count of the number of connections between the vertices of A and B. For a subset $X \subset G$, define

$$h_G(X) = \frac{|E(X,X^c)|}{\min(|X|,|X^c|)},$$

where X^c denotes the complement of X in G and $|X|$ denotes the number of vertices in the set. The Cheeger constant h_G of G is defined to be

$$h_G = \min_X h_G(X).$$

Proposition 6.4.3.

$$\lambda_{G_n} \leq 4h_{G_n}.$$

Proof. We construct a function g_n on G_n based on the optimal cut which achieves h_{G_n} and separates G_n into two parts, A_n and B_n:

$$g_n(x) = \begin{cases} 1/|A_n| & : x \in A_n \\ -1/|B_n| & : x \in B_n \end{cases}$$

By the minimax principle, we have

$$\lambda_{G_n} = \min_{f \perp \mathbf{1}} \frac{\sum_{x \sim y} c_{xy}(f(x) - f(y))^2}{\sum_x f(x)^2}.$$

Subsituting f into the above equation and noting that $c_{xy} \leq 2$, we have:

$$\begin{aligned} \lambda_{G_n} &\leq 2E(A_n, B_n)(1/|A| + 1/|B|) \\ &\leq \frac{4E(A_n, B_n)}{\min(|A_n|, |B_n|)} \\ &\leq 4h_{G_n}. \quad \square \end{aligned}$$

There also exist lower bounds for the second smallest eigenvalue. In [3], these lower bounds are obtained for a general class of graph Laplacians, to which the Laplacians $L^{(n)}$ belong. Let G be a finite graph of degree n. Let us denote the vertex set $V(G) = \{1, 2, .., n\}$. Let C be an irreducible $n \times n$ matrix where $c_{ij} > 0$ if and only if $i \neq j$ and there is an edge connecting the corresponding vertices. Let $L_C(G) = \mathrm{diag}\{\delta_1, ..., \delta_n\} - C$ be our Laplacian on G. Finally, let $i_c(G) = \min\big(\sum_{i \in X, j \notin X} c_{ij}/|X|\big)$, where the minimum is taken over all non-empty subsets X of $V(G)$ satisfying $|X| \leq n/2$. This

quantity is a generalization of the classical Cheeger's constant. The corresponding inequality from [3] is stated below.

Proposition 6.4.4. *Let λ_G denote the second smallest eigenvalue of $L_C(G)$. Then*

$$\lambda_G \geq \left(\bar{\delta} - \sqrt{\bar{\delta}^2 - i_c(G)^2}\right),$$

where $\bar{\delta} = max\{\delta_1, ..., \delta_n\}$.

By Propositions 6.4.3 and 6.4.4, we can deduce the following.

Proposition 6.4.5. *For $n \geq 4$,*

$$4 - \sqrt{16 - \left(\frac{12}{5(-1)^n + 7 \cdot 2^n - 9}\right)^2} \leq \lambda_{G_n} \leq \frac{48}{5(-1)^{n-1} + 7 \cdot 2^{n-1} - 15}.$$

Proof. We start with the lower bound and apply Proposition 6.4.4. In the decomposition of $L^{(n)}$, we take the δ's to be the degrees of the vertices in G_n. So $\bar{\delta} = 4$. Note that the off-diagonal entries of $L^{(n)}$ that are non-zero are equal to either 1 or 2. Thus,

$$i_c(G_n) = \min\left(\sum_{i \in X, j \notin X} c_{ij}/|X|\right) \geq \min\left(\sum_{i \in X, j \notin X} 1/|X|\right) \geq 2/|G_n|.$$

So it suffices to find $|G_n|$. By decomposing G_n into G_{n-1} and two copies of G_{n-2}, we can deduce the following non-homogeneous recurrence relation that holds for $n \geq 4$.

$$|G_n| = |G_{n-1}| + 2|G_{n-2}| + 3.$$

The relation has the solution

$$|G_n| = \frac{1}{6}\big(5(-1)^n + 7\cdot 2^n - 9\big).$$

Putting everything together, we have

$$\lambda_{G_n} \geq 4 - \sqrt{16 - \left(\frac{12}{5(-1)^n + 7\cdot 2^n - 9}\right)^2}.$$

Now for the upper bound of λ_{G_n}, by Proposition 6.4.3 it suffices to find an upper bound for the Cheeger constant h_{G_n}. Let us pick the subgraph X_n of G_n that is isomorphic to G_{n-1}. X_n has one boundary vertex c_n that connects to two vertices in X_n^c. Thus

$$h_{G_n} \leq h_{G_n}(X_n) = \frac{2}{\min(|G_{n-1}| - 1, |G_n| - |G_{n-1}| + 1)}.$$

It can be checked that $|G_{n-1}| - 1 < |G_n| - |G_{n-1}| + 1$ for $n \geq 4$. Thus

$$\lambda_{G_n} \leq \frac{8}{|G_{n-1}| - 1} \quad = \quad \frac{48}{5(-1)^{n-1} + 7\cdot 2^{n-1} - 15}. \quad \square$$

By the previous proposition, we conclude that $\lambda_{G_n} = O(2^{-n})$ and $\lambda_{G_n} = \Omega(4^{-n})$. In fact, the numerical evidence suggests that asymptotically λ_{G_n} behaves like $(2\sqrt{2})^{-n}$. Thus, we make the following conjecture.

Conjecture 6.4.6. *The following limit exists and satisfies* $\lim_{n\to\infty} (2\sqrt{2})^n \lambda_{G_n} > 0$.

Corollary 6.4.7. *There exists* $\epsilon > 0$ *such that* $\limsup_{n\to\infty} \sigma(\Delta^{(n)}) \cap (0, \epsilon) = \emptyset$.

n	λ_{G_n}	$\lambda_{G_n}/\lambda_{G_{n+1}}$	$(2\sqrt{2})^n\lambda_{G_n}$
1	1	1	2.828427
2	1	2.618034	8
3	0.381966	1.54142	8.642904
4	0.247801	2.985018	15.85929
5	0.083015	2.448753	15.02733
6	0.033901	2.803443	17.35728
7	0.012093	2.789782	17.51197
8	0.004335	2.818787	17.75455
9	0.001538	2.822058	17.81528
10	0.000545	2.822732	17.85548
11	0.000193	2.830112	17.8915
12	0.000068		17.88085

FIGURE 6.4.2: The eigenvalue λ_{G_n}

6.5 Infinite Blow-ups

First, we define an infinite blow-up of the graphs G_n.

Definition 6.5.1. Let $\{k_n\}_{n\in\mathbb{N}} \subset \mathbb{N}$ be a strictly increasing sequence. For each n, embed G_{k_n} in some isomorphic subgraph of $G_{k_{n+1}}$. The corresponding infinite blow-up is $G_\infty := \cup_{n\geq 0} G_{k_n}$.

We define the graph Laplacian $L^{(\infty)}$ on $\ell^2 := \{f \in \mathbb{R}^{G_\infty} : ||f||_2 < \infty\}$, where $||f||_2 := \left(\sum_{x\in G_\infty} f^2(x)\right)^{1/2}$, by

$$L^{(\infty)}f(x) = \sum_{x\sim y} c_{xy}(f(x) - f(y)),$$

where c_{xy} is the number of edges between x and y.

Recall that l_n and r_n denote the left and right boundary points of G_n. Let us call the long path of G_n the minimal sequence of vertices and edges connecting l_n and

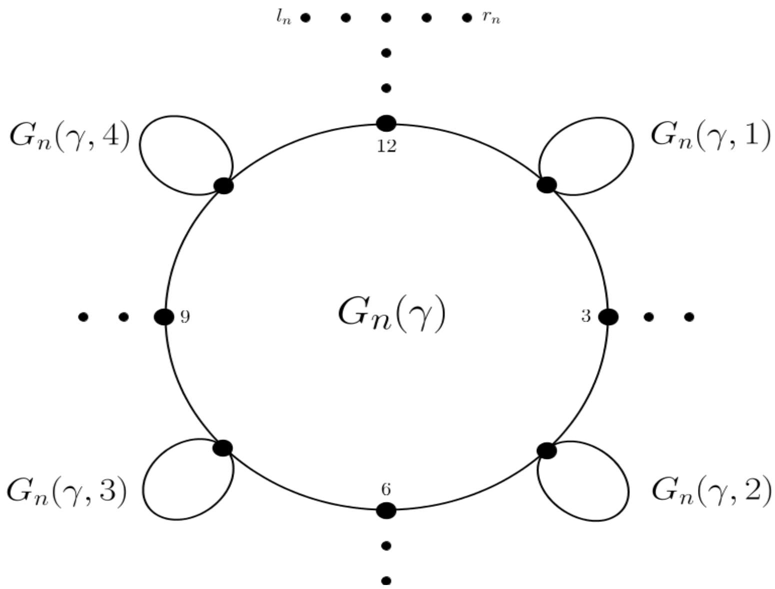


FIGURE 6.5.1: The subgraph $G_n(\gamma)$

r_n. Let us a call a loop in G_n a minimal non-trivial sequence of vertices and edges that begin and end at the same vertex. Note that all vertices of G_n, except for the boundary vertices, belong to some loop. By construction, loops in G_n will have 2^m vertices for some integer m.

Let γ be a loop in G_n with at least eight vertices. Let us call the vertex in γ whose attaching subgraph contains the boundary vertices l_n and r_n the 12 o'clock vertex. Let us call the vertices in the loop a graph distance of 2^{k-2} apart the 3, 6 and 9 o'clock vertices, with respect to clockwise orientation on the loop. Removal of the attaching subgraphs to the 3, 6, 9 and 12 o'clock vertices produces a symmetric graph which can further be divided into four subgraphs, necessarily isomorphic to each other. More specifically, they will be isomorphic to some G_m, where $m < n$. Let us refer to these subgraphs as $G_n(\gamma, i)$, $i = 1, 2, 3, 4$ and to the union of these four subgraphs and γ simply as $G_n(\gamma)$. Figure 6.5.1 illustrates this notation.

Let D_4 be the dihedral group of permutations on $\{1, 2, 3, 4\}$. For any $\pi \in D_4$, there

is a unique distance preserving bijection $\pi_{n,\gamma} : G_n(\gamma) \to G_n(\gamma)$ such that $G_n(\gamma, i)$ is mapped to $G_n(\gamma, j)$ if $\pi(i) = j$. In essence, the isometry $\pi_{n,\gamma}$ permutes the subgraphs $G_n(\gamma, i)$ in the same way π permutes the numbers $\{1, 2, 3, 4\}$ corresponding to these subgraphs. Each $\pi_{n,\gamma}$ induces an isometry $U_{\pi,n,\gamma}$ on $\ell^2_{n,\gamma} := \{f \in \ell^2 : \mathrm{supp}(f) \subset G_n(\gamma)\}$. That is, $U_{\pi,n,\gamma} f(x) := f(\pi_{n,\gamma}(x))$ for $x \in G_n(\gamma)$.

We now need to make certain assumptions on the infinite blow-ups.

Assumption 1. *The infinite blow-up G_∞ satisfies:*

- *For $n \geq 1$, the long path of $G_{k_{n-1}}$ is embedded in a loop γ_n of G_{k_n}.*
- *Apart from $l_{k_{n-1}}$ and $r_{k_{n-1}}$, no vertex of the long path can be the $3, 6, 9$ or 12 o'clock vertex of γ_n.*
- *The only vertices of G_{k_n} that connect to vertices outside G_{k_n} are the boundary vertices of G_{k_n}.*

Remark 6.5.2. There are an uncountable number of blow-ups G_∞ satisfying Assumption 1. Note that G_n can be embedded in some subgraph of G_m satisfying the conditions above if $|m - n| \geq 4$. The number of subsequences $\{k_n\}_{n \in \mathbb{N}}$ of the natural numbers satisfying $|k_{n+1} - k_n| \geq 4$ is uncountable.

Lemma 6.5.3. *Let $\ell^2_{a,k_n,\gamma_n} = \{f \in \ell^2_{k_n,\gamma_n} : U_{\pi,k_n,\gamma_n} f = (-1)^{|\pi|} f, \pi \in D_4\}$. We can consider this space as a subspace of ℓ^2. Then ℓ^2_{a,k_n,γ_n} is an invariant subspace of $L^{(\infty)}$ and of any $L^{(k_m)}$, $m \geq n$. Any eigenfunction of the restriction $L^{(k_n)}|_{\ell^2_{a,k_n,\gamma_n}}$ is an eigenfunction of $L^{(\infty)}$ and of any $L^{(k_m)}$, $m \geq n$.*

Proof. The result follows by Assumption 1, the definition of the Laplacian, and noting the following fact. Any eigenfunction of $L^{(k_n)}|_{\ell^2_{a,k_n,\gamma_n}}$ must attain a zero at the $3, 6, 9$

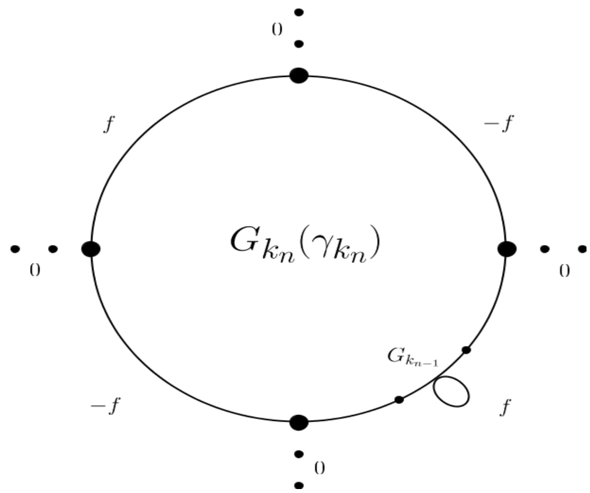


FIGURE 6.5.2: A function f in l_{a,k_n,γ_n}

and 12 o'clock vertices of γ_n. As the eigenfunction is supported in $G_{k_n}(\gamma_n)$, the sum of the differences of the eigenfunction along all edges incident to these vertices must be zero (see Figure 6.5.2). Thus, the eigenvalue equations at these vertices are satisfied and one has an eigenfunction of $L^{(\infty)}$ and of any $L^{(k_m)}$, $m \geq n$. $\square$

Recall that the graphs $G_{k_n}(\gamma_{k_n}, i)$ for $i = 1, 2, 3, 4$ will be isomorphic to some G_{j_n}, where $j_n < k_n$.

Theorem 6.5.4. *Under Assumption* 1*:*

(1) $\sigma\big(L^{(k_n)}|_{\ell^2_{a,k_n,\gamma_n}}\big) = \sigma(L_0^{(j_n)})$.

(2) To every Dirichlet eigenvalue of $L_0^{(j_n)}$ *there is a localized eigenfunction of* $L^{(\infty)}$.

(3) The spectrum of $L^{(\infty)}$ *is pure point. The set of eigenvalues of* $L^{(\infty)}$ *is*

$$\bigcup_{n\geq 0} \sigma(L_0^{(j_n)}) = \bigcup_{n\geq 0} c_{j_n}^{-1}\{0\},$$

where the polynomials c_n *are the characteristic polynomials of* $L_0^{(n)}$*, as defined in Proposition* 6.2.3.

(4) The set of finitely supported eigenfunctions of $L^{(\infty)}$ is complete in ℓ^2.

Proof. Let P_{k_n} be the orthogonal projector onto the subspace of functions with support in G_{k_n} and P_{a,k_n} be the orthogonal projector onto ℓ^2_{a,k_n}, that is $P_{k_n}f := f|_{G_{k_n}}$ and $P_{a,k_n}f := \frac{1}{8}\sum_{\pi\in D_4}(-1)^{|\pi|}U_{\pi,k_n,\gamma_n}P_{k_n}f$.

We first prove (1). Let $G_{k_n}(\gamma_n, 1)$ be identified with some G_{j_n}. If $f \in \ell^2$ is a Dirichlet eigenfunction of $L_0^{(j_n)}$ on $G_{j_n} = G_{k_n}(\gamma_n, 1)$, then $P_{a,k_n}f$ is an eigenfunction of $L^{(k_n)}|_{\ell^2_{a,k_n,\gamma_n}}$ (see Figure 6.5.2). Note that the eigenvalue equations must be satisfied at the 3,6,9 and 12 o'clock vertices. Conversely, the restriction of an eigenfunction in ℓ_{a,k_n,γ_n} to $G_{k_n}(\gamma_n, 1) = G_{j_n}$ is a Dirichlet eigenfunction of $L_0^{(j_n)}$. This is also establishes (2).

Statements (3) and (4) will follow from statement (1) and another fact. By Lemma 6.5.3 we have that $\underset{n\geq 0}{\cup}\ell^2_{a,k_n,\gamma_n}$ is contained in the space of the eigenfunctions of $L^{(\infty)}$ with finite support. Therefore it is enough to show that $\underset{n\geq 0}{\cup}\ell^2_{a,k_n,\gamma_{n+1}}$ is complete in ℓ^2.

Fix $f \in \left(\underset{n\geq 0}{\cup}\ell^2_{a,k_n,\gamma_n}\right)^{\perp}$. Note that $\langle g, U_{\pi,k_{n+1},\gamma_{n+1}}g\rangle = 0$ if the support of g is contained in G_{k_n} and π is not the identity of D_4. Therefore, $||P_{a,k_{n+1}}P_{k_n}||_2 = \frac{1}{\sqrt{8}}||P_{k_n}f||_2$. This implies

$$
\begin{aligned}
||P_{a,k_{n+1}}f||_2 &= ||P_{a,k_{n+1}}(f + P_{k_n}f - P_{k_n}f)||_2 \\
&\geq ||P_{a,k_{n+1}}P_{k_n}f||_2 - ||f - P_{k_n}f||_2 \\
&= \frac{1}{\sqrt{8}}||P_{k_n}f||_2 - ||f - P_{k_n}f||_2. \qquad (6.5.1)
\end{aligned}
$$

By Assumption 1, the computations above hold for any n. Thus (6.5.1) implies $\limsup_{n\to\infty}||P_{a,k_n}f||_2 = \frac{1}{\sqrt{8}}||f||_2$. Thus $f = 0$ and the proof is complete. $\square$

Remark 6.5.5. The proof of Theorem 6.5.4 is based on the techniques in the proof of Theorem 2 in [35].

Recall that the infinite blow-ups (Γ_ξ, ξ) are the limits in the pointed Gromov-Hausdorff topology of finite Schreier graphs (Γ_n, ξ_n), with ξ_n being the prefix of ξ of length n. The corresponding metric between two rooted graphs (Γ_1, v_1) and (Γ_2, v_2) is given by

$$\mathrm{Dist}((\Gamma_1, v_1), (\Gamma_2, v_2)) := \inf\left\{\frac{1}{r+1}; B_{\Gamma_1}(v_1, r) \text{ is isomorphic to } B_{\Gamma_2}(v_2, r)\right\}.$$

The relationship between the finite graphs G_n and Γ_n yields the following consequence.

Proposition 6.5.6. *The infinite blow-ups G_∞ that satisfy Assumption 1 are infinite blow-ups of finite Schreier graphs.*

Proof. For $n \geq 1$, the graph G_n can be written as $B_{G_n}(s_n, 2^{\lceil n/2\rceil - 1})$, where s_n is the midpoint of the long path connecting l_n and r_n. Note that the graph distance between s_n and l_n or r_n is exactly $2^{\lceil n/2\rceil - 1}$.

Next, by Assumption 1, no vertex of G_{k_n} is identified as a boundary vertex of $G_{k_{n+1}}$. Since $\Gamma_{k_{n+1}+1}$ can be constructed by identifying the boundary vertices of $G_{k_{n+1}}$ and $G_{k_{n+1}+1}$, we can identify $G_{k_n} = B_{G_{k_n}}(s_{k_n}, 2^{\lceil k_n/2\rceil - 1})$ with $B_{\Gamma_{r_n - 1}}(\xi_{r_n}, 2^{\lceil k_n/2\rceil - 1})$, where $r_n = k_{n+1} + 2$ and ξ_{r_n} is the finite word in $\{0,1\}^{r_n}$ corresponding to the vertex s_n.

Without loss of generality, let us modify the words $\{\xi_{r_n}\}_{n=0}^\infty$ by appending an infinite string of zeros to each word. By a diagonalization argument, we can find a subsequence $\{\xi_{r_{n_j}}\}_{j=0}^\infty$ that converges pointwise to some infinite word ξ_*. By the work in [5], the rooted graphs $(\Gamma_{r_{n_j}-1}, \xi_{r_{n_j}})$ converge to the infinite blow-up (Γ_{ξ_*}, ξ_*) in the

Gromov-Hausdorff metric. Thus, the corresponding subsequence of rooted graphs $(G_{k_{n_j}}, s_{k_{n_j}})$ converges to (Γ_{ξ_*}, ξ_*).

For $n \geq 1$, define $M(n) := \sup\{n_j : n \geq n_j\}$. Then the rooted graphs $(G_{k_n}, s_{k_{M(n)}})$ converge to (Γ_{ξ_*}, ξ_*) in the Gromov-Hausdorff metric. Thus, G_∞ is an infinite blow-up of finite Schreier graphs. $\square$

Appendix A

Dirichlet and Neumann Eigenvalues

Neumann Eigenvalues: Hata Tree

Level	Mult. of 1	New Eigenvalues
0	1	0,2
1	1	$1 \pm \sqrt{\frac{1}{3}}$
2	3	$1 \pm \sqrt{\frac{7}{9}}$
3	5	0.041787, 0.236647, 0.473787, 1.526212, 1.763352, 1.958212
4	9	0.012547, 0.092553, 0.130004, 0.310991, 0.452259, 0.525248, 1.474751, 1.547740, 1.689008, 1.869995, 1.907446, 1.987452
5	17	0.003973, 0.026936, 0.045802, 0.112040, 0.121881, 0.190430, 0.271716, 0.321628, 0.426888, 0.473440, 0.502936, 0.549056, 1.450943, 1.497063, 1.526559, 1.573111, 1.678371, 1.728283, 1.809569, 1.878118, 1.887959, 1.954197, 1.973063, 1.996026
6	33	0.001222, 0.008659, 0.013815, 0.034894, 0.043160, 0.059953, 0.095668, 0.115646, 0.118187, 0.129689, 0.190057, 0.190851, 0.263418, 0.279162, 0.312816, 0.332921, 0.423533, 0.443147, 0.452912, 0.473782, 0.496872, 0.515615, 0.547819, 0.549643, 1.450356, 1.452180, 1.484384, 1.503127, 1.526217, 1.547087, 1.556852, 1.576466, 1.667078, 1.687183, 1.720837, 1.736581, 1.809148, 1.809942, 1.870310, 1.881812, 1.884353, 1.904331, 1.940046, 1.956839, 1.965105, 1.986184, 1.991340, 1.998777
7	65	0.000379, 0.002650, 0.004367, 0.011047, 0.012948, 0.019114, 0.029192, 0.036193, 0.041848, 0.045782, 0.058895, 0.060881, 0.095221, 0.096189, 0.112293, 0.116347, 0.118083, 0.121815, 0.128933, 0.129859, 0.189461, 0.190160, 0.190739, 0.191776, 0.257663, 0.264979, 0.275029, 0.288531, 0.311490, 0.316185, 0.326681, 0.335369, 0.422715, 0.426863, 0.436160, 0.445177, 0.452534, 0.456250, 0.473440, 0.473787, 0.496328, 0.497498, 0.515045, 0.516532, 0.544930, 0.548491, 0.549179, 0.551856, 1.448143, 1.450820, 1.451508, 1.455069, 1.483467, 1.484954, 1.502501, 1.503671, 1.526212, 1.526559, 1.543749, 1.547465, 1.554822, 1.563839, 1.573136, 1.577284, 1.664630, 1.673318, 1.683814, 1.688509, 1.711468, 1.724970, 1.735020, 1.742336, 1.808223, 1.809260, 1.809839, 1.810538, 1.870140, 1.871066, 1.878184, 1.881916, 1.883652, 1.887706, 1.903810, 1.904778, 1.939118, 1.941104, 1.954217, 1.958151, 1.963806, 1.970807, 1.980885, 1.987051, 1.988952, 1.995632, 1.997349, 1.999620

Dirichlet Eigenvalues: Hata Tree

Level	Mult. of 1	Other Eigenvalues
1	0	$1 \pm \sqrt{\frac{1}{3}}$
2	0	$1 \pm \sqrt{\frac{1}{3}}$, 0.232408, 0.565741, 1.434258, 1.767591,
3	2	0.058081, 0.215399, 0.232408, 0.365758, 0.546082, 0.565741, 1.434258, 1.453917, 1.634241, 1.767591, 1.784600, 1.941918
4	6	0.020207, 0.058081, 0.064331, 0.167142, 0.215399, 0.215692, 0.309324, 0.365758, 0.449811, 0.541685, 0.546082, 0.553586, 1.446413, 1.453917, 1.458314, 1.550188, 1.634241, 1.690675, 1.784307, 1.784600, 1.832857, 1.935668, 1.941918, 1.979792
5	14	0.006046, 0.020207, 0.020932, 0.047837, 0.064204, 0.064331, 0.101296, 0.128211, 0.167142, 0.190395, 0.215684, 0.215692, 0.273355, 0.309324, 0.319517, 0.363223, 0.440626, 0.449811, 0.456955, 0.510420, 0.541685, 0.542398, 0.550947, 0.553586, 1.446413, 1.449052, 1.457601, 1.458314, 1.489579, 1.543044, 1.550188, 1.559373, 1.636776, 1.680482, 1.690675, 1.726644, 1.784307, 1.784315, 1.809604, 1.832857, 1.871788, 1.898703, 1.935668, 1.935795, 1.952162, 1.979067, 1.979792, 1.993953
6	30	0.001901, 0.006046, 0.006443, 0.015821, 0.020930, 0.020932, 0.032189, 0.044883, 0.047837, 0.059910, 0.064204, 0.064208, 0.095906, 0.101296, 0.112911, 0.121459, 0.128211, 0.128871, 0.167144, 0.190056, 0.190395, 0.190852, 0.215684, 0.215684, 0.263825, 0.273355, 0.278490, 0.306262, 0.318639, 0.319517, 0.330204, 0.363223, 0.425742, 0.439565, 0.440626, 0.448077, 0.456626, 0.456955, 0.473448, 0.496915, 0.510420, 0.515437, 0.542380, 0.542398, 0.546262, 0.550947, 0.550958, 0.553221, 1.446778, 1.449041, 1.449052, 1.453737, 1.457601, 1.457619, 1.484562, 1.489579, 1.503084, 1.526551, 1.543044, 1.543373, 1.551922, 1.559373, 1.560434, 1.574257, 1.636776, 1.669795, 1.680482, 1.681360, 1.693737, 1.721509, 1.726644, 1.736174, 1.784315, 1.784315, 1.809147, 1.809604, 1.809943, 1.832855, 1.871128, 1.871788, 1.878540, 1.887088, 1.898703, 1.904093, 1.935791, 1.935795, 1.940089, 1.952162, 1.955116, 1.967810, 1.979067, 1.979069, 1.984178, 1.993556, 1.993953, 1.998098
7	62	0.000584, 0.001901, 0.002007, 0.004825, 0.006439, 0.006443, 0.010042, 0.013585, 0.015821, 0.019085, 0.020930, 0.020930, 0.029308, 0.032189, 0.035685, 0.042979, 0.044883, 0.044892, 0.047943, 0.058982, 0.059910, 0.060822, 0.064208, 0.064208, 0.095220, 0.095906, 0.096196, 0.101273, 0.112437, 0.112911, 0.116099, 0.118186, 0.121459, 0.121811, 0.128324, 0.128871, 0.128884, 0.129815, 0.167144, 0.189462, 0.190056, 0.190140, 0.190433, 0.190717, 0.190852, 0.191778, 0.215684, 0.215684, 0.257610, 0.263825, 0.265619, 0.270783, 0.277639, 0.278490, 0.287722, 0.306262, 0.312040, 0.316335, 0.318639, 0.318837, 0.326852, 0.330204, 0.335362, 0.363223, 0.423436, 0.425665, 0.425742, 0.437691, 0.439565, 0.439634, 0.444986, 0.448077, 0.452668, 0.456516, 0.456626, 0.456685, 0.473446, 0.473448, 0.473782, 0.496329, 0.496915, 0.497496, 0.510434, 0.515001, 0.515437, 0.516562, 0.542380, 0.542380, 0.544999, 0.546262, 0.548245, 0.549392, 0.550958, 0.550958, 0.551827, 0.553221, 1.446778, 1.448172, 1.449041, 1.449041, 1.450607, 1.451754, 1.453737, 1.455000, 1.457619, 1.457619, 1.483437, 1.484562, 1.484998, 1.489565, 1.502503, 1.503084, 1.503670, 1.526217, 1.526551, 1.526553, 1.543314, 1.543373, 1.543483, 1.547331, 1.551922, 1.555013, 1.560365, 1.560434, 1.562308, 1.574257, 1.574334, 1.576563, 1.636776, 1.664637, 1.669795, 1.673147, 1.681162, 1.681360, 1.683664, 1.687959, 1.693737, 1.712277, 1.721509, 1.722360, 1.729216, 1.734380, 1.736174, 1.742389, 1.784315, 1.784315, 1.808221, 1.809147, 1.809282, 1.809566, 1.809859, 1.809943, 1.810537, 1.832855, 1.870184, 1.871115, 1.871128, 1.871675, 1.878188, 1.878540, 1.881813, 1.883900, 1.887088, 1.887562, 1.898726, 1.903803, 1.904093, 1.904779, 1.935791, 1.935791, 1.939177, 1.940089, 1.941017, 1.952056, 1.955107, 1.955116, 1.957020, 1.964314, 1.967810, 1.970691, 1.979069, 1.979069, 1.980914, 1.984178, 1.986414, 1.989957, 1.993556, 1.993560, 1.995174, 1.997992, 1.998098, 1.999415

Neumann Eigenvalues: Basilica Graphs G_n

Level	Eigenvalues
0	0,2
1	0,1,3
2	0, 1, 1.438447, 5.561552
3	0, 0.381966, 0.555625, 2, 2.618033, 3.781910, 6.662464
4	0, 0.247801, 0.281290, 1.018816, 1.316030, 1.326566, 2, 3.245335, 4.236410, 5.402678, 6.204916, 6.720153
5	0, 0.083015, 0.090210, 0.342548, 0.500892, 0.502534, 0.763932, 1.054474, 1.235257, 1.594158, 2, 2.293350, 2.423143, 3.188542, 3.541720, 3.739630, 3.966542, 4.782006, 5.236067, 6.516717, 6.659160, 6.679331, 6.806764
6	0, 0.033900, 0.044763, 0.134792, 0.228442, 0.256333, 0.257030, 0.356216, 0.452989, 0.648122, 0.763932, 1.022065, 1.058714, 1.081507, 1.161381, 1.321653, 1.321798, 1.370592, 1.425987, 2, 2, 2, 2, 3.125715, 3.216475, 3.239173, 3.333716, 3.689719, 3.884356, 4.027249, 4.210124, 4.503939, 4.782384, 5.236067, 5.316914, 5.726901, 6.163152, 6.441307, 6.528979, 6.655850, 6.690450, 6.717413, 6.755246, 6.814636
7	0, 0.012092, 0.012117, 0.044230, 0.087874, 0.087929, 0.096467, 0.132137, 0.154974, 0.199223, 0.228442, 0.337741, 0.354642, 0.361308, 0.392069, 0.501728, 0.501746, 0.509047, 0.530766, 0.763932, 0.763932, 0.763932, 0.763932, 1.026565, 1.046208, 1.051871, 1.070199, 1.124768, 1.156890, 1.177514, 1.218845, 1.301277, 1.370576, 1.425987, 1.493924, 1.682742, 1.888727, 2, 2, 2, 2, 2, 2, 2.360394, 2.368114, 3.119319, 3.162663, 3.184862, 3.201018, 3.233018, 3.320393, 3.377874, 3.509725, 3.657999, 3.689719, 3.717693, 3.879047, 3.931945, 3.963267, 3.995402, 4.124185, 4.186758, 4.769159, 4.781962, 4.782077, 4.793877, 5.236067, 5.236067, 5.236067, 5.236067, 6.113360, 6.133544, 6.496798, 6.516409, 6.520469, 6.543336, 6.655850, 6.657419, 6.666804, 6.678323, 6.686368, 6.695795, 6.714859, 6.787911, 6.806526, 6.809417, 6.825682

Dirichlet Eigenvalues: Basilica Graphs G_n

Level	Eigenvalues
1	1
2	0.763932, 5.236067
3	0.228442, 1.425987, 2, 3.689719, 6.655850
4	0.096467, 0.530766, 0.763932, 1.124768, 2, 3.233018, 4.186758, 5.236067, 6.113360, 6.714859
5	0.028409, 0.173202, 0.228442, 0.375404, 0.763932, 1.049280, 1.207427, 1.425987, 1.760491, 2, 2, 3.180599, 3.484031, 3.689719, 3.959702, 4.781918, 5.236067, 6.516104, 6.655850, 6.677141
6	0.012072, 0.066510, 0.096467, 0.140206, 0.228442, 0.349853, 0.416892, 0.530766, 0.677142, 0.763932, 0.763932, 1.037533, 1.094788, 1.124768, 1.165805, 1.370574, 1.425987, 2, 2, 2, 2, 3.124945, 3.212507, 3.233018, 3.332549, 3.689719, 3.883890, 4.024033, 4.186758, 4.463465, 4.782306, 5.236067, 5.236067, 5.628816, 6.113360, 6.431687, 6.528391, 6.655850, 6.690237, 6.714859, 6.751494, 6.814294
7	0.003549, 0.021606, 0.028409, 0.046702, 0.096467, 0.131024, 0.149624, 0.173202, 0.206640, 0.228442, 0.228442, 0.342261, 0.365665, 0.375404, 0.393550, 0.509038, 0.530766, 0.763932, 0.763932, 0.763932, 0.763932, 1.026268, 1.044563, 1.049280, 1.069701, 1.124768, 1.156678, 1.176126, 1.207427, 1.272874, 1.370575, 1.425987, 1.425987, 1.569852, 1.760491, 1.928431, 2, 2, 2, 2, 2, 2, 2, 3.118798, 3.157263, 3.180599, 3.200542, 3.233018, 3.319690, 3.371632, 3.484031, 3.614428, 3.689719, 3.689719, 3.878589, 3.927292, 3.959702, 3.994936, 4.124182, 4.186758, 4.769151, 4.781918, 4.782047, 4.793871, 5.236067, 5.236067, 5.236067, 5.236067, 6.113360, 6.133544, 6.496758, 6.516104, 6.520242, 6.543262, 6.655850, 6.655850, 6.664282, 6.677141, 6.686197, 6.695769, 6.714859, 6.787862, 6.806287, 6.809263, 6.825651

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