

12-15-2016

Extension of L1 Adaptive Control with Applications

jiaxing che

University of Connecticut, jiaxing.che@engineer.uconn.edu

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Extension of \mathcal{L}_1 Adaptive Control with Applications

Jiaxing Che, Ph.D.

University of Connecticut, 2016

Adaptive control is the control method used by a controller which adapt to a system with unknown or varying parameters. As a newly developed technique, \mathcal{L}_1 adaptive control has drawn increased attention in past decades. The key feature of \mathcal{L}_1 adaptive control architecture is guaranteed robustness in the presence of fast adaptation. With \mathcal{L}_1 adaptive control architecture, fast adaptation appears to be beneficial both for performance and robustness, while the trade-off between the two is resolved via the selection of the underlying filtering structure. The latter can be addressed via conventional methods from classical and robust control. Moreover, the performance bounds of \mathcal{L}_1 adaptive control architectures can be analyzed to determine the extent of the modeling of the system that is required for the given set of hardware.

The main contribution of this dissertation is to extend the framework of \mathcal{L}_1 adaptive control theory and applied in various of applications. It can be summarized with 3 different parts:

The first one is the extension of \mathcal{L}_1 adaptive to time-varying system and non-minimum phase system by using eigenvalue assignment method. This approach has been demonstrated by both theoretical models as well as high fidelity models such as flexible wing aircraft model from NASA and also the supersonic glider model

developed from the supersonic lab in Austria.

The 2nd part focuses on filter bandwidth adaptation in the \mathcal{L}_1 adaptive control architecture. The stability condition of the low-pass filter in control is relaxed by introducing an additional Lyapunov-based adaptation mechanism which results in a more systematic design with minimized tuning efforts. Adaptability for arbitrarily large nonlinear time-varying uncertainties without redesign parameters. The overall system is a non-LTI design even in the limiting case. The 3rd part introduces the concept of predictive horizon and online optimization into \mathcal{L}_1 adaptive control. This approach enables \mathcal{L}_1 adaptive control to solve the output limitation even for the non-minimum phase system.

Extension of \mathcal{L}_1 Adaptive Control with Applications

Jiaxing Che

M.S. University of Science and Technology of China, 2010

B.S. Harbin Institute of Technology, 2007

A Dissertation

Submitted in Partial Fulfillment of the

Requirements for the Degree of

Doctor of Philosophy

at the

University of Connecticut

2016

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Jiaying Che

2016

APPROVAL PAGE

Doctor of Philosophy Dissertation

Extension of \mathcal{L}_1 Adaptive Control with Applications

Presented by

Jiaxing Che, B.S., M.S.

Major Advisor

Prof. Chengyu Cao

Associate Advisor

Prof. Robert Gao

Associate Advisor

Prof. Nejat Olgac

Associate Advisor

Prof. Jiong Tang

Associate Advisor

Prof. Xu Chen

University of Connecticut

2016

ACKNOWLEDGMENTS

I would first like to acknowledge my major advisor, Dr. Chengyu Cao, for his mentorship and support over the past six years. His guidance and insights have helped me to solve challenging research problems and develop new ideas. I particularly appreciate his patience and encouragement, especially when I was in anxiety and frustration with my research. He has helped me appreciate the joy associated with doing research, despite the challenges it entails.

I would like to thank my associate advisors, Dr. Robert Gao, Dr. Nejat Olgac, Dr. Jiong Tang, and Dr. Xu Chen, for their advice, review and encouragement on my research.

I would like to extend my gratitude to our research group members: Jie Luo, Xiaotian Zou, Andrew Thompson, Jennifer Hacker, Dr. Qinmin Yang, Ali Elahidoost, John Cooper, Michael Santone, Dr. Fulai Yao, Haitham Felemban, Yuqian Liu, Robert Herman, and Roseanna Warren, with whom I share the research interests and friendship.

This dissertation is dedicated to my parents, Quanwei Che and Jie Gao, who brought me to this world and who have continuously given me their support, love, and encouragement in pursuing my dreams.

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Chapter 1

Introduction and Research Overview

1.1 Introduction of Adaptive Control

Research in adaptive control was motivated by aerospace and aviation industry. The dynamic model of the aircraft varies at a wide range of speeds and altitudes. A good autopilot needs to provide a consistent response (handling quality) to the operator at a wide range of working conditions. Hence, a constant gain controller can not satisfy this requirement. Gain scheduling design is the most common method to solve this problem in autopilot design. It change the gain (parameters) of the controller in realtime according to the airspeed and altitude reading. However, the gain scheduling method require the prior knowledge of the aircraft model. Wind tunnel test needs to be conduct at most possible operation points. Based on the wind tunnel data, the controller parameters are designed and interpolated according to the real time

airspeed reading. The gain scheduling method is very expensive to develop and verify. On the other side the adaptive control approach seems more appealing. In the early 1950s [1,2] shows an example to use adaptive control to solve the problem of changing aircraft dynamics. The typical method is model reference adaptive controller (MRAC) based on the MIT rule [3]. In the mean time Honeywell's self-oscillating adaptive controller [4] was noteworthy in the control industry.

In the 1960s, the developments in system identification and estimation theory provide new methods for adaptive control research. Different online estimation schemes as well as various design methods were combined. Two types of adaptive control emerged: the direct method, where controller parameters were estimated and directly used in the control loop, and the indirect method, where the plant model were estimated and the controller parameters were obtained indirectly using a design procedure [5].

After the evolutions of control architectures and methods, stability of the adaptive control system draws more attention. Rohr's example shows the challenge of the stability in the presence of unmodelled dynamics [6]. The adaptive parameter needs to be bounded within a certain range in order to ensure the closed loop system is stable. Several modification appears to solve this problem the σ -modification [7] and the e -modification [8]. Linear system theory and Lyapunov method provided new solutions to analysis the closed loop system. The Lyapunov method can be easily extended to a certain type of nonlinear system. Proof can be done for asymptotic stability. What type of nonlinear system can be covered has become an important topic. Backstepping method relaxed the matching conditions of a broader class of systems, such as strict-parametric feedback and feedforward systems [9–14], stability analysis these schemes in the presences of unmodelled dynamics [15–20], extensions

to output feedback with an objective to achieve global or semiglobal output feedback stabilization [21–25], extension to time-varying systems [26–29], extension to non-minimum phase systems [30,31], etc. Similar literature can be found in [5].

Beside the stability, the transient performance is another important aspect of the adaptive control. Transient performance refers to the performance of the control system during the learning process. The controller may experience larger error before the convergence of adaptive parameters. For example the self tuning regulator for robotic arm. The adaptive parameter will converge after several working cycle of the robotic arm. After the work load is changed, the arm will start a learning period again. Many papers analyze the transient performance using \mathcal{L}_∞ induced norm [32]. The bounds of the error signal could also be characterized using the tools from system theory. People want fast adaption to achieve aggressive performance. However, in the conventional MRAC, fast adaptive will lead the overall system to a high gain feedback loop. The stability margin of the system will be challenged by high-gain feedback. Fast adaption and robustness are conflicting objectives in the conventional MRAC. How to achieve better trade off between fast adaptive and robustness is worth exploring. The author had extensively researched the issue like adaptive control in the presence of unmodeled dynamics [33] as well as the negative effect of high gain observer [34] which became the main incentive of the development of \mathcal{L}_1 adaptive control theory.

1.2 Overview of \mathcal{L}_1 Adaptive Control

The \mathcal{L}_1 adaptive control theory addressed precisely the question of fast adaption without losing robustness. The first result of \mathcal{L}_1 adaptive control is presented in 2006

American control conference [35, 36]. Due to the capability of fast adaption, further theoretical results of systems with unknown time-varying uncertainties and analysis of stability margin are presented in 2007 [37, 38]. Following results are concluded in several journal papers later [39, 40]. In 2010, Dr. Cao and Dr. Hovakimyan further summarized the theory to a book [5]. In 2012, a paper of \mathcal{L}_1 adaptive on safety critical system presents the theory in and more intuitive and education focused manner [41].

1.2.1 Main Theoretical Result of \mathcal{L}_1 Adaptive Control

The basic \mathcal{L}_1 adaptive control architecture could be explained by the following content. Consider the following SISO system

$$\dot{x}(t) = A_x(t)x(t) + b(u(t) + \theta^T x(t)), \quad y(t) = c^T x(t), \quad x(0) = x_0, \quad (1.2.1)$$

where $x(t) \in \mathbb{R}^n$ is the system state vector (measurable), $u(t) \in \mathbb{R}$ is the control input, $y(t) \in \mathbb{R}$ is the system output, A is a known $n \times n$ Hurwitz matrix, $b, c \in \mathbb{R}^n$ are known constant vectors with the zeros of $c^T(s\mathbb{I} - A)b$ in the open left-half s plane, $\theta(t) \in \mathbb{R}^n$ is a vector of bounded unknown disturbances, and x_0 is the initial value of $x(t)$.

We consider the following state predictor

$$\dot{\hat{x}}(t) = A\hat{x}(t) + b(\hat{\theta}(t)x(t) + u(t)), \quad \hat{y}(t) = c^T \hat{x}(t), \quad \hat{x}(0) = x_0, \quad (1.2.2)$$

where $\hat{x}(t) \in \mathbb{R}^n$ is the predicted state, $\hat{y}(t) \in \mathbb{R}$ is the predicted output, $\hat{\theta}(t) \in \mathbb{R}^n$ is the vector of adaptive parameters, serve as an estimate of the parameter θ governed by the following projection-type adaptive law:

Letting $\tilde{x}(t) = \hat{x}(t) - x(t)$, the update law for $\hat{\sigma}(t)$ is given by

$$\dot{\hat{\theta}}(t) = \Gamma \text{Proj}(\hat{\theta}(t), -\tilde{x}Pbx(t)), \quad \hat{\theta}(0) = \hat{\theta}_0 \in \Theta \quad (1.2.3)$$

where $\hat{x}(t) = \hat{x}(t) - x(t)$ is the prediction error, $\Gamma \in \mathbb{R}$ is the adaptation gain, and $P = P^T > 0$ solves the algebraic Lyapunov equation $A^T P + P A = -Q$ for arbitrary symmetric $Q = Q^T > 0$. The projection is confined to the set Θ . The laplace transform of the adaptive control signal is defined as

$$u(s) = -C(s)(\hat{\eta} - k_g r(s)) \quad (1.2.4)$$

where $r(s)$ and $\hat{\eta}(s)$ are the laplace transforms of $r(t)$ and $\hat{\eta} \triangleq \hat{\theta}(t)x(t)$ respectively. $k_g \triangleq -1/(c^T A^{-1}b)$, and $C(s)$ is a BIBO-stable and strictly proper transfer function with DC gain $C(0) = 1$ and its state-space realization assumes zero initialization.

The over all control architecture can be demonstrated in the following figure

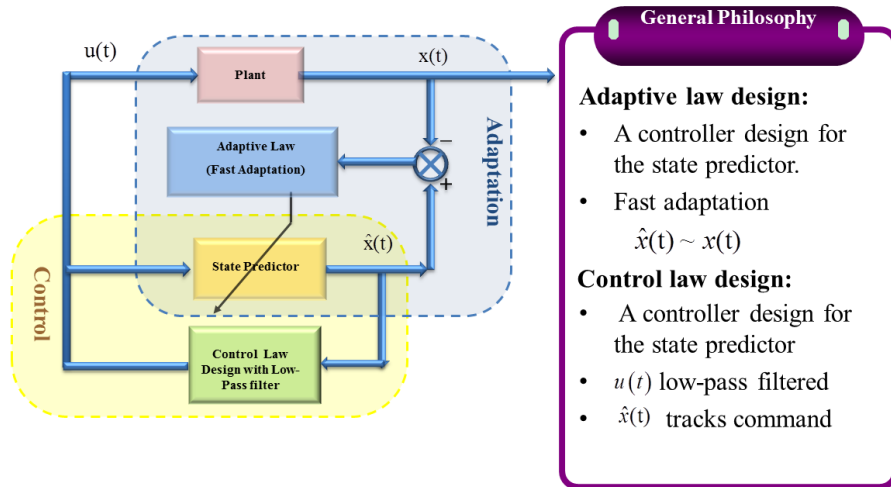


FIGURE 1.2.1: The \mathcal{L}_1 adaptive control structure.

The overall system consists of two loops: the adaptation loop and control loop. The objective of the adaptation loop is getting the estimation of the unknown disturbances. The control loop cancels the effects of the disturbances and uncertainties in the system output. Fast adaptation is in the adaptation loop. Since the adaptation loop is purely software loop. So there is no time delay in this loop. Hence, high adaptation gain is desired to achieve fast adaptation. The adaptation gain is only limited to sampling rate of the sensors and the available CPU computing power. In the control loop, the low pass filter mechanism is introduced to remove the noise in the sensor and recovered the time delay margins of the control system. The fast adaptation and low-pass filter will not be helpful if they work alone independently. For example, the fast adaptation alone without the filter will lead the system to high gain feedback and reduce the time delay margin of the system. Also, low pass filter alone without high gain will further deteriorate the system performance and leads to sluggish response. And even sometimes reduced the phase margin of the system because it introduced more control loop delay. But when these two elements work together, the situation become different. The adaptation gain can be tuned much higher if the low pass filter is in presence, and also some of the negative effect of low pass filter could be compensated by fast adaption. The overall system performance could be improved. Fast adaptation could be achieved in the presents of robustness.

More detailed results of \mathcal{L}_1 adaptive control theory can be found in the book and the papers [5, 40].

1.2.2 Extension of \mathcal{L}_1 Adaptive Control

After introducing the fundamental theory of \mathcal{L}_1 adaptive control. Further results was developend to expend the system coverage. [42, 43] extend the results of the \mathcal{L}_1 adaptive control theory from affine system to non-affine system and further discussed in [44, 45]. These papers use the similar technique—transforming nonaffine-in-control systems into equivalent linear time-varying (LTV) systems with uncertainties and designing the \mathcal{L}_1 adaptive controller for the transformed systems. A more general \mathcal{L}_1 adaptive control design for non-affine multi-input multi-output nonlinear systems with system dynamics in the normal form was presented in [46]. The design [47] further extends the uncertain nonaffine-in-control nonlinear systems by introducing a more general filtering structure and relaxing the assumptions in [42–45].

For the output feedback case, there are very few \mathcal{L}_1 adaptive control designs [46, 48–50]. The \mathcal{L}_1 adaptive control design in [48, 49] is limited to first order linear time invariant systems. The \mathcal{L}_1 adaptive controller in [50] extends to systems with unknown state-dependent and time-varying nonlinearities and has a control law with two low-pass filters for matched and unmatched adaptive estimates.

In the \mathcal{L}_1 adaptive control architecture, adaptive law design is an important component, which is used to update the adaptive parameters in the predictor. Projection-type adaptive law [51] and piece-wise constant adaptive law [49] are two typical adaptive law design. The piece-wise constant adaptive law can be extended and applied to other areas, such as adaptive estimates for linear time-varying systems [52] and cooperative control for multi-agent systems [53–56]. An alternative adaptive law design using sliding mode control is provided in [57], which gives a good estimation independent of matched uncertainties.

Unknown time-varying parameters and disturbances in the presence of non-zero trajectory initialization error is address in [58]. The \mathcal{L}_1 adaptive is further extended to a strict parameter feedback form in [37]. Optimization of the time-delay margin of \mathcal{L}_1 adaptive controller via the design of the underlying filter is introduced in [59]. Performance analysis in the presents of unmodelled actuator dynamics is shown in [60–63]. The results of output feedback system is presented in [64–66]. Later the \mathcal{L}_1 adaptive control architecture has been further extended to a larger coverage of system. Output feedback controller is designed for systems with time-varying unknown parameters and bounded disturbances [67]. \mathcal{L}_1 adaptive controller for multi-input multi-output systems is introduced in [68–70]. System coverage is extended to larger class of nonlinearities in [71–75]. Novel control loop design with new feedback control technique is also a important branch. \mathcal{L}_1 adaptive control combing with eigenvalue control law is presented in [76], \mathcal{L}_1 adaptive control combing artificial nero-network is analysed in [77, 78]. \mathcal{L}_1 adaptive control coupled with sliding mode technique is introduced in [79]. Other extensions includes: LTV reference system [80], additional uncertainty bias estimation [81], control output limits [82–84], input saturation [85, 86], Decentralized \mathcal{L}_1 adaptive control for large-scale systems [87], systems with hysteresis uncertainties [88]. The \mathcal{L}_1 control architecture also inspired adaptive cooperative control for flocking of mobile agents [89, 90].

1.2.3 Application of \mathcal{L}_1 Adaptive Control

The application of \mathcal{L}_1 adaptive theory has begun before the theory is developed. During that time, the author has conduct research on various control problems such as path following algorithm, aerial refueling and flight control system. The trade off

between transient performance and robustness has drawn the author's attention and becomes the main incentive to develop the theory. More applications are conducted along with the development. The \mathcal{L}_1 adaptive controller has not only been successfully implemented and tested in a large number of flight tests and flight simulation environments [69, 91, 92], but also been applied and simulated in other areas such as boiler-turbine control [93, 94], main steam temperature process control [95], manipulation of micro-nano objects [96] as well as well drilling systems [97]. In flight control area, \mathcal{L}_1 adaptive controller has been successfully applied to pitch controller for miniature air vehicles [98], wing rock [99], autonomous rotorcraft [100], missile longitudinal autopilot design [101], flight envelope limiting [102], pilot-induced oscillation suppression [103, 104], flexible wing aircraft [105–107]. It has also been applied to some novel aircraft such as air-breathing hypersonic vehicle model in the presence of unmodeled dynamics [108], tailless unstable aircraft [109] in the presence of unknown actuator failures [110], NASA AirSTAR flight test vehicle [111, 112], The theories are also developed accordingly like flight validation of metrics driven adaptive control [113, 114], multi-criteria analysis of an \mathcal{L}_1 adaptive flight control system [115]. The guaranteed transient performance of \mathcal{L}_1 adaptive controller make it an ideal tool for performance critical application such as aerial refueling [116–122] and path following algorithm [123–128]. Similar application on tracking a moving object is presented in [129–131]. Other applications include: magnetic-based cube-sat attitude control [132], satellite orbit stabilization [133], automated manipulation of micro-nano objects with spherical parallel manipulator [134], control of a nonlinear pressure-regulating engine bleed valve in aircraft air management systems [135], attitude control of quadrotors [136], autonomous underwater vehicle design and control strategy [137].

1.3 Contributions of Dissertation

The dissertation focuses on three parts: (a) Extension of \mathcal{L}_1 adaptive control to NMP and time-varying systems. (b) Extension of \mathcal{L}_1 adaptive control to semi-global uncertainties. (c) Maintaining output limits in uncertain systems. Part (a) modified the dynamic inversion control law for the unmatched uncertainty such that the controller is capable to deal with non-minimum phase reference system with a relaxed stability condition. This approach is successfully tested in the flexible wing aircraft control and supersonic glider system control. Moreover, this approach can also be extended to linear time-varying (LTV) system. The second part of the dissertation focuses on the design of the low-pass filter. A Lyapunov-based adaptation mechanism is introduced in the filter design which leads to a more systematic design with minimized tuning efforts as well as the adaptability for arbitrarily large nonlinear time-varying uncertainties without redesign parameters. In the third part of the proposed dissertation, an online optimization method is proposed to enable the adaptive controller achieve the control objective while maintaining the output constraint condition is satisfied.

1.4 Thesis Outline

The remainder of the dissertation is organized as follows.

In Chapter 2, we discuss an adaptive approach for disturbance rejection in the presence of unmatched time-varying disturbances. This unknown disturbance is estimated by introducing an \mathcal{L}_1 piece-wise constant adaptive law from \mathcal{L}_1 adaptive control architecture. With the estimated disturbance, a novel disturbance rejection control law design inspired by the eigenvalue assignment method is proposed. This control

law design does not require the dynamic inversion of the plant (desired reference system). The disturbance will be compensated following a performance determined by the eigenvalues assigned to the controller. Properly chosen eigenvalues will result in reasonable performance for disturbance rejection. This approach provides closed-loop stability when dealing with systems having unstable zeros in the presence of unmatched disturbances, with a relaxed stability condition. This approach can stabilize a extended type of system especially in the flexible structures.

Chapter 3 is motivated by the challenging control problem in a light, high-aspect ratio, flexible aircraft configuration that exhibits strong rigid body/flexible mode coupling. An multiple-input multiple-output (MIMO) \mathcal{L}_1 adaptive output feedback controller is presented for a semi-span wind tunnel model capable of motion. A modification to the \mathcal{L}_1 output feedback controller is proposed to make it more suitable for flexible structures. This controller is evaluated in the presence of gust load which can be represented as a general unmatched uncertainty. A linear-quadratic-Gaussian (LQG) controller is employed here to stabilize the system and provide damping for the flexible mode. The \mathcal{L}_1 adaptive output feedback controller is designed to control the rigid body mode and reject disturbances while not exciting the frequencies of other flexible mode. An \mathcal{L}_1 adaptive output feedback controller is designed for a single test point and is then applied to all the test cases. The simulation results show that the \mathcal{L}_1 augmented controller can stabilize and meet the performance requirements for all 10 test conditions ranging from 30 psf to 130 psf of dynamic pressure.

Chapter 4 presents an adaptive approach for disturbance rejection in the presence of unmatched unknown time-varying disturbances for a class of linear time-varying (LTV) systems. The unknown disturbance is estimated by a piece-wise constant adaptive law which uses information of the system at discrete times. Eigenvalue as-

signment architecture is employed to stabilize the system and to transform the time-varying system $A(t), B(t), C(t)$, into a control canonical form. This form involves time-invariant A, B matrices and a time-varying $C(t)$ matrix. A control law is designed for disturbance rejection and tracking such that the output of LTV systems with disturbances can practically track the reference signal. The stability is analyzed and tracking performance is characterized in the main theorem.

In Chapter 5, the design of the low-pass filter in \mathcal{L}_1 control is relaxed by introducing an additional Lyapunov-based adaptation mechanism. This chapter includes three main contributions. First is a more systematic design with minimized tuning efforts. Second is the adaptability for arbitrarily large nonlinear time-varying uncertainties without redesign parameters. Third is a non-LTI design even in the limiting case, which clarifies its difference with linear designs. State feedback is adopted for the simplicity to illustrate this new adaptation mechanism. However, the same technique can be easily applied to output feedback controller design. Simulation examples verify the theoretical findings.

In Chapter 6, a new controller framework incorporating \mathcal{L}_1 adaptive control and an online optimization scheme is presented. This framework provides a method for maintaining output constraints in the presence of non-linear time-varying matched uncertainties by predicting future output trajectories within a finite time horizon. The stability conditions of the system are derived and the prediction error is characterized. The online optimization is formulated into a linear program problem for which many efficient algorithms exist. Simulation results demonstrate the effectiveness of the controller framework.

In Chapter 7, we summarize the dissertation and discuss the future research directions.

Chapter 2

Extension of \mathcal{L}_1 Adaptive Control to NMP and Time-Varying systems

2.1 Introduction

Tracking and disturbance rejection are the main purpose for the control system design. In many practical problems, we cannot measure all of the disturbances, or we may choose not to measure some of them due to technical or economic reasons [138]. Therefore, it is important to develop disturbance estimation techniques. After receiving an estimated disturbance, a controller needs to be designed to reject the disturbance with a reasonable performance. There are two types of disturbance estimation problems. The first type is *estimation of unknown disturbances with a known structures* : Disturbances with known structures refer to the disturbance signal which can be described by differential equations. The most common assumption is that the disturbance is constant or sinusoidal signal with a fixed frequency. The structured

disturbance estimation problem can be transformed to a state estimation problem by representing the disturbance parameters as extended states. [139] presents the method for the unknown constant disturbance. [140] demonstrates the adaptive disturbance rejection method for sinusoidal disturbances with unknown frequency. The second type of disturbance estimation is *general time-varying disturbance estimation*. Disturbance observer [141] can estimate the disturbance by inverting the plant dynamic. Extended State Observer (ESO) [142] defines the disturbance as an extended state and designs a state observer. By choosing high observer gain, ESO can track the time-varying disturbance signal with arbitrary desired eigenvalues. Generalized Extended State Observer (GESO) [143] extends the estimated state to the n -th order derivative. GESO can track and converge to the disturbance with an n -th order constant derivative.

\mathcal{L}_1 adaptive control architecture is a recently developed method for unknown time-varying uncertain systems. Through fast and robust adaptation, complex nonlinear uncertain systems can be controlled with improved performance without enforcing persistent excitation, applying gain-scheduling, or resorting to high-gain feedback. These benefits highly improve application of the \mathcal{L}_1 adaptive control in a broad range of systems experiencing unknown uncertainties including aerospace, flight control, and industrial systems [144–148].

The disturbance estimation method in \mathcal{L}_1 adaptive control architecture uses a different methodology. The \mathcal{L}_1 adaptive law is the controller for the state predictor designed to make the output of the state predictor converge to the output of the plant. Any aggressive controller design can serve as the adaptive law in \mathcal{L}_1 estimation. The advantage of the \mathcal{L}_1 estimation is that it can provide a characterized performance bound of the tracking error. This will also be demonstrated in Lemma 2.4.1 of this

chapter.

Disturbance rejection is another important goal for the control system design. The internal model principle based control design is summarized in [149,150], but the internal model principle based methods require prior knowledge of the disturbance type. Usually it only works for constant and sinusoidal disturbances with known frequency. For systems with general time-varying disturbances, an intuitive method of disturbance rejection involves creating a control signal $u(t)$ which causes in an equal, opposing output to cancel the effects of the disturbance. Internal model control method is another popular tool for disturbance rejection [151]. In a \mathcal{L}_1 adaptive controller, the disturbance is first decomposed into matched and unmatched components [112]. The matched disturbance can be directly canceled in the control channel. For the unmatched disturbance, the controller requires a dynamic inversion of the desired plant in order to compensate the effects of disturbances in the output. Similar method could be found in [144–148].

However, these methods mentioned above all require a dynamic inversion of the plant, therefore they cannot work for the non-minimum phase plants. Output tracking control of non-minimum phase systems is a highly challenging problem encountered in the control of flexible manipulators and space structures [152]. Performance limitations of non-minimum phase systems are discussed in [153]. The dynamic inversion will transform the unstable zeros in the plant to unstable poles in the control law and yield an unbounded control signal. Moreover, the closed-loop stability condition is difficult to satisfy for non-minimum phase plants.

To solve this problem, this chapter presents a disturbance rejection method based on eigenvalue assignment inspired by [154]. This method was used to assign eigenvalues with desired trajectories and to achieve closed-loop stability. In order to com-

compensate for disturbances, the control law transforms the state predictor into a control canonical form. A tracking method is designed to cancel the disturbance with a reasonable performance. The disturbance is compensated following a rate determined by the eigenvalues assigned to the controller. Properly chosen eigenvalues yield a reasonable response to cancel the disturbances. The most beneficial feature of this approach is the closed-loop stability which occurs even in the non-minimum phase nominal plant.

The early work of this chapter is presented in [155]. In addition [155], the stability analysis is introduced in Theorem 2.4.5 with the consideration of the system state. Moreover, bound of tracking error from the desired system is analyzed in Theorem 2.4.6.

2.2 Problem Formulation

Consider the following Single-Input Single-Output (SISO) system:

$$\dot{x}(t) = Ax(t) + bu(t) + \sigma(t), \quad y(t) = c^T x(t), \quad (2.2.1)$$

where $x \in \mathbb{R}^n$ is the system state vector (measurable), $u \in \mathbb{R}$ is the control signal. b and $c \in \mathbb{R}^n$ are known constant vectors, A is a known $n \times n$ Hurwitz matrix, (A, b) is controllable, and $\sigma(t)$ is the unknown time-varying bounded disturbance subject to the following assumption.

Assumption 2.2.1. There exist constants $b_\sigma > 0$ and $b_{d\sigma} > 0$ such that

$$\|\sigma(t)\| \leq b_\sigma, \quad \|\dot{\sigma}(t)\| \leq b_{d\sigma} \quad (2.2.2)$$

holds uniformly in $t \geq 0$.

The control objective is to design an adaptive state feedback control signal $u(t)$ such that the system output $y(t)$ tracks reference input $r(t)$ and compensates the disturbance with a reasonable performance. This performance can be specified by the eigenvalues assigned to the control law. Without loss of generality, we suppose there exist constants B_r and B_{dr} such that:

$$\|r(t)\| \leq B_r, \quad \|\dot{r}(t)\| \leq B_{dr}. \quad (2.2.3)$$

2.3 \mathcal{L}_1 Adaptive Controller

In this section, a novel control law design is introduced into the \mathcal{L}_1 adaptive architecture. The entire controller consists of state predictor, adaptive law and control law.

State Predictor:

The state predictor together with the adaptive law are designed for fast estimation of the unknown disturbance $\sigma(t)$. The following state predictor is considered.

$$\begin{aligned} \dot{\hat{x}}(t) &= A\hat{x}(t) + bu(t) + \hat{\sigma}(t), \\ \hat{y}(t) &= c^T \hat{x}(t), \hat{x}(0) = x_0 \end{aligned} \quad (2.3.1)$$

where $\hat{\sigma}(t)$ is defined in the following adaptive laws:

Adaptive Laws:

The piece-wise constant adaptive laws are used in this chapter in order to make $\hat{x}(t)$ track $x(t)$. The update law for $\hat{\sigma}(t)$ is given by

$$\hat{\sigma}(t) = \hat{\sigma}(iT), \quad t \in [iT, (i+1)T) \quad (2.3.2)$$

$$\hat{\sigma}(iT) = \Phi^{-1}(T)e^{AT}\tilde{x}(iT), \quad i = 0, 1, 2, 3\ldots \quad (2.3.3)$$

where $\tilde{x}(t) = \hat{x}(t) - x(t)$ is the estimation error vector of the state predictor and $\Phi(T)$ is defined as: $\Phi(T) = \int_0^T e^{A(T-\tau)} d\tau$.

Control Laws:

The control law is a controller of the state predictor and is designed to make $\hat{y}(t)$ track $r(t)$ following a reasonable performance and in the meantime the disturbance $\hat{\sigma}(t)$ will be inhibited following the desired eigenvalues.

Define

$$\dot{\hat{x}}_1(t) = A\hat{x}_1(t) + bu_1(t), \quad y_1(t) = c^T \hat{x}_1(t). \quad (2.3.4)$$

Letting

$$\begin{aligned} u_2(t) &= u(t) - u_1(t), \\ \hat{x}_2(t) &= \hat{x}(t) - \hat{x}_1(t), \\ \hat{y}_2(t) &= \hat{y}(t) - \hat{y}_1(t), \end{aligned} \quad (2.3.5)$$

substituting (2.3.4) and (2.3.26) in (2.3.1), one derives:

$$\dot{\hat{x}}_2(t) = A\hat{x}_2(t) + bu_2(t) + \hat{\sigma}(t), \quad \hat{y}_2(t) = c^T \hat{x}_2. \quad (2.3.6)$$

The following section introduces the design strategies for u_1 and u_2 . In order for the output of system $c^T \hat{x}(t)$ to tracks the reference signal $r(t)$, the control signal $u_1(t)$ and $u_2(t)$ are designed such that $c^T \hat{x}_1(t)$ tracks $r(t)$ and $c^T(t)\hat{x}_2(t)$ tracks 0.

$u_1(t)$ design: In order to make $c^T \hat{x}_1(t)$ track $r(t)$, $u_1(t)$ is defined as

$$u_1(t) = k_g r(t), \quad k_g = -\frac{1}{c^T A^{-1} b}. \quad (2.3.7)$$

$u_2(t)$ design : The eigenvalue assignment method in [154] is used to compensate the disturbance following the rate of the designed eigenvalues. Before the control law, the following definition is required:

$$C = \begin{bmatrix} b & Ab & \cdots & A^{n-1}b \end{bmatrix}, \quad (2.3.8)$$

$$D = C^{-1} = \begin{bmatrix} D_{n-1} \\ D_{n-2} \\ \vdots \\ D_0 \end{bmatrix}, \quad (2.3.9)$$

$$(2.3.10)$$

where D_0 is the last row of D . Then, we define

$$\bar{D} = \begin{bmatrix} D_0 \\ D_0 A \\ \vdots \\ D_0 A^{n-1} \end{bmatrix} = \begin{bmatrix} \bar{D}_0 \\ \bar{D}_1 \\ \vdots \\ \bar{D}_{n-1} \end{bmatrix}, \quad (2.3.11)$$

$$\begin{bmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_n \end{bmatrix} = D_0 A^n \bar{D}^{-1}, \quad (2.3.12)$$

$$k = \left(\begin{array}{c} - \begin{bmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_n \end{bmatrix} \\ + \begin{bmatrix} d_1 & d_2 & \cdots & d_n \end{bmatrix} \end{array} \right) \bar{D}, \quad (2.3.13)$$

where $\begin{bmatrix} d_1 & d_2 & \cdots & d_n \end{bmatrix}$ are assigned according to the desired PD-eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ by the following equation

$$\begin{aligned} s^n + d_n s^{n-1} + d_{n-2} s^{n-2} + \dots + d_1 = \\ (s + \lambda_1)(s + \lambda_2) \dots (s + \lambda_n). \end{aligned} \quad (2.3.14)$$

We first define

$$W = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -d_1 & -d_2 & \cdots & \cdots & -d_n \end{bmatrix}, \quad (2.3.15)$$

where $\begin{bmatrix} d_1 & d_2 & \cdots & d_n \end{bmatrix}$ are defined in (2.3.14), and

$$C(s) = \begin{bmatrix} F(s) & 0 & 0 & \cdots & 0 \\ 0 & F(s) & 0 & \cdots & 0 \\ 0 & 0 & F(s) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & F(s) \end{bmatrix}, \quad (2.3.16)$$

where $F(s)$ is a strict proper stable low-pass filter array with relative degree n .

Let

$$\hat{\sigma}_r(s) = C(s)\hat{\sigma}(s), \quad (2.3.17)$$

and

$$\Lambda(t) = \begin{bmatrix} \Lambda_0(t) \\ \Lambda_1(t) \\ \vdots \\ \Lambda_{n-1}(t) \end{bmatrix} = \bar{D}\hat{\sigma}_r(t), \quad (2.3.18)$$

where \bar{D} is defined in (2.3.10). We note that $\hat{\sigma}_r$ is smooth and have an n order bounded derivative. At last, we define

$$R(\Lambda(t)) = \begin{bmatrix} 0 & \Lambda_0^{(0)}(t) & \cdots & \sum_{m=0}^{n-2} \Lambda_m^{(n-2-m)}(t) \end{bmatrix}^T \quad (2.3.19)$$

and

$$g(\hat{\sigma}_r(t), t) = - \begin{bmatrix} d_1 & d_2 & \cdots & d_n \end{bmatrix} R(\Lambda(t)) - \sum_{m=0}^{n-1} \Lambda_m^{(n-1-m)}(t). \quad (2.3.20)$$

To further introduce the control law, we define:

$$\dot{\hat{x}}_3(t) = A\hat{x}_3(t) + \hat{\sigma}(t) - \hat{\sigma}_r(t). \quad (2.3.21)$$

Let $\hat{x}_4(t) = \hat{x}_2(t) - \hat{x}_3(t)$. Using the definition of (2.3.6), we have:

$$\dot{\hat{x}}_4(t) = A\hat{x}_4(t) + bu_2(t) + \hat{\sigma}_r(t), \quad (2.3.22)$$

where $\hat{\sigma}_r(t)$ is the filtered $\hat{\sigma}(t)$ defined in (2.3.17). The control signal $u_2(t)$ is defined as:

$$u_2(t) = u_{21}(t) + u_{22}(t). \quad (2.3.23)$$

and

$$u_{21}(t) = -k\hat{x}_4(t) + g(\hat{\sigma}_r(t), t), \quad (2.3.24)$$

where k and $g(\hat{\sigma}_r(t), t)$ are defined in (2.3.13) and (2.3.20).

$$u_{22}(t) = \frac{c^T \bar{D}^{-1} R(\Lambda(t))}{c^T \bar{D}^{-1} W^{-1} \begin{bmatrix} 0 & 0 & \dots & 1 \end{bmatrix}^T}, \quad (2.3.25)$$

where \bar{D} , $R(\Lambda(t))$ and W are defined in (2.3.10), (2.3.19) and (2.3.15).

The overall control law is

$$u(t) = u_1(t) + u_2(t), \quad (2.3.26)$$

where $u_1(t)$ and $u_2(t)$ are defined in (2.3.7) and (2.3.23).

Remark 2.3.1. The above transformation transforms a system with full state time-varying disturbance into a canonical form. Its purpose will be clarified later in the proof.

2.4 Analysis of \mathcal{L}_1 Adaptive Controller

Let:

$$\begin{aligned} \gamma_0(T) &= \int_0^T \sqrt{\lambda_{\max}((e^{A(t-\tau)})^\top (e^{A(t-\tau)}))} b_\sigma d\tau, \\ t &\in [0, T) \end{aligned} \quad (2.4.1)$$

$$\gamma_1(T) = \sqrt{\lambda_{\max}(A^\top A)}\gamma_0(T) + b_{d\sigma}T, \quad (2.4.2)$$

$$\gamma_2(T) = \sqrt{n}\|(sI - A)^{-1}\|_{\mathcal{L}_1}\gamma_1(T). \quad (2.4.3)$$

The following lemma states that the adaptive parameter, $\hat{\sigma}(t)$, serves as a good estimate of the unknown disturbance $\sigma(t)$.

Lemma 2.4.1. *Considering the system described in (2.2.1) together with the state predictor (2.3.1), adaptive law (2.3.2) and control law (2.3.26), we have*

$$\|(\hat{\sigma}(t) - \sigma(t))\| \leq \gamma_1(T), \quad \|\tilde{x}\| \leq \gamma_2(T). \quad (2.4.4)$$

Proof: Subtracting (2.3.1) from (2.2.1), we get

$$\dot{\tilde{x}}(t) = A\tilde{x}(t) + \hat{\sigma}(t) - \sigma(t). \quad (2.4.5)$$

The solution of (2.4.5) in $[iT, iT + T), t \in [0, T)$ is

$$\begin{aligned} \tilde{x}(iT + t) &= e^{At}\tilde{x}(iT) + \int_{iT}^{iT+t} e^{A(iT+t-\tau)}\hat{\sigma}(iT)d\tau \\ &\quad - \int_{iT}^{iT+t} e^{A(iT+t-\tau)}\sigma(\tau)d\tau \\ &= e^{At}\tilde{x}(iT) + \int_0^t e^{A(t-\tau)}\hat{\sigma}(iT)d\tau \\ &\quad - \int_{iT}^{iT+t} e^{A(iT+t-\tau)}\sigma(\tau)d\tau. \end{aligned} \quad (2.4.6)$$

When $t = T$, it follows from (2.4.6) that

$$\begin{aligned}\tilde{x}(iT + T) &= e^{AT} \tilde{x}(iT) + \int_{iT}^{iT+T} e^{A(iT+t-\tau)} \hat{\sigma}(iT) d\tau \\ &\quad - \int_{iT}^{iT+T} e^{A(iT+t-\tau)} \sigma(\tau) d\tau.\end{aligned}\tag{2.4.7}$$

According to the choice of adaptive law in (2.3.2), we have

$$e^{AT} \tilde{x}(iT) + \int_{iT}^{(i+1)T} e^{A((i+1)T-\tau)} \hat{\sigma}(iT) d\tau = 0.\tag{2.4.8}$$

It follows from (2.4.7) that

$$\tilde{x}(iT + T) = - \int_{iT}^{iT+T} e^{A(iT+T-\tau)} \sigma(\tau) d\tau.\tag{2.4.9}$$

Using the condition in assumption 2.2.1 and definition of $\gamma_0(T)$ in (2.4.1), we get

$$\tilde{x}(iT) < \gamma_0(T).\tag{2.4.10}$$

In what follows, we derive the upper-bound of $\tilde{\sigma}(t)$ where $\tilde{\sigma}(t)$ is defined as

$$\tilde{\sigma}(t) = \hat{\sigma}(t) - \sigma(t).\tag{2.4.11}$$

It follows from (2.4.9) that

$$\tilde{x}(iT) = - \int_{(i-1)T}^{iT} e^{A(iT-\tau)} \sigma(\tau) d\tau.\tag{2.4.12}$$

It follows from (2.4.8) that

$$\begin{aligned}
\tilde{x}(iT) &= (I - e^{AT})\tilde{x}(iT) \\
&\quad - \int_{iT}^{(i+1)T} e^{A((i+1)T-\tau)} \hat{\sigma}(iT) d\tau \\
&= - \int_{iT}^{(i+1)T} e^{A((i+1)T-\tau)} A\tilde{x}(iT) d\tau \\
&\quad - \int_{iT}^{(i+1)T} e^{A((i+1)T-\tau)} \hat{\sigma}(iT) d\tau \\
&= - \int_{iT}^{(i+1)T} e^{A((i+1)T-\tau)} (\hat{\sigma}(iT) + A\tilde{x}(iT)) d\tau.
\end{aligned} \tag{2.4.13}$$

Hence, (2.4.12) and (2.4.13) imply that

$$\begin{aligned}
&\int_{(i-1)T}^{iT} e^{A(iT-\tau)} \sigma(\tau) d\tau = \\
&\int_{iT}^{(i+1)T} e^{A((i+1)T-\tau)} (\hat{\sigma}(iT) + A\tilde{x}(iT)) d\tau,
\end{aligned} \tag{2.4.14}$$

and hence there exists $t_p \in [(i-1)T, iT]$ such that

$$\hat{\sigma}(iT) + A\tilde{x}(iT) = \sigma(t_p). \tag{2.4.15}$$

For any t , there exists t_p such that $|t - t_p| \leq T$ which satisfies (2.4.15) and therefore

implies that

$$\begin{aligned}
\|\hat{\sigma}(t) - \sigma(t)\| &\leq \|\hat{\sigma}(t) - \sigma(t_p)\| + \|\sigma(t) - \sigma(t_p)\| \\
&\leq \|\hat{\sigma}(iT) - \sigma(t_p)\| + \|\sigma(t) - \sigma(t_p)\| \\
&\leq A\|\tilde{x}(iT)\| + \int_{t_p}^t \|\dot{\sigma}(\tau)\| d\tau.
\end{aligned} \tag{2.4.16}$$

It follows from Assumption 2.2.1 that $\dot{\sigma}(t)$ is bounded such that

$$\|\hat{\sigma}(t) - \sigma(t)\| \leq \sqrt{\lambda_{\max}(A^\top A)} \gamma_0(T) + b_{d\sigma} T. \tag{2.4.17}$$

Using the dynamic in (2.4.5), we have

$$\tilde{x}(s) = (sI - A)^{-1}(\hat{\sigma}(s) - \sigma(s)). \tag{2.4.18}$$

Hence, we have

$$\begin{aligned}
\|\tilde{x}_t\|_{\mathcal{L}_\infty} &\leq \|(sI - A)^{-1}\|_{\mathcal{L}_1} \|\hat{\sigma}(t) - \sigma(t)\|_{\mathcal{L}_\infty} \\
&\leq \|(sI - A)^{-1}\|_{\mathcal{L}_1} \gamma_1(T),
\end{aligned} \tag{2.4.19}$$

for any $t \geq 0$. It follows that

$$\|\tilde{x}\| \leq \sqrt{n} \|(sI - A)^{-1}\|_{\mathcal{L}_1} \gamma_1(T), \tag{2.4.20}$$

which completes the proof. \square

Lemma 2.4.2.

$$\lim_{T \rightarrow 0} \gamma_0(T) \rightarrow 0, \quad \lim_{T \rightarrow 0} \gamma_1(T) \rightarrow 0, \quad \lim_{T \rightarrow 0} \gamma_2(T) \rightarrow 0. \quad (2.4.21)$$

Proof : Recall the definition of $\gamma_0(T)$ in (2.4.1), where everything inside the integration is bounded values and bounded functions. Therefore, $T \rightarrow 0$ yields $\gamma_0(T) \rightarrow 0$. Similarly, the limits of $\gamma_1(T)$ and $\gamma_2(T)$ go to zero which completes the proof. \square

Remark 2.4.3. Lemma 2.4.1 and lemma 2.4.2 show the performance of the state predictor. The estimated state variable $\hat{x}(t)$ has a bounded error $\gamma_1(T)$ with the real system state variable $x(t)$, and T can be chosen to make this bound of error arbitrarily small. Similarly, the real disturbance $\sigma(t)$ and estimated disturbance $\hat{\sigma}(t)$ also have a bounded error $\gamma_2(T)$ which can also be arbitrarily small.

Define

$$E_1 = \begin{bmatrix} 0 & 0 & \dots & 1 \end{bmatrix}^T \quad (2.4.22)$$

Define a reference system

$$\begin{aligned}
x_{ref}(s) &= \frac{(sI - A)^{-1}b}{-c^T A^{-1}b} r(s) + \\
& (sI - A)^{-1}(I - C(s))\sigma(s) + \\
& \bar{D}^{-1} \left(\frac{(sI - W)^{-1} E_1 c^T \bar{D}^{-1} R(\Lambda(t))}{c^T \bar{D}^{-1} W^{-1} E_1} \right) - \\
& \bar{D}^{-1} R(\Lambda(t))
\end{aligned} \tag{2.4.23}$$

$$\begin{aligned}
y_{ref}(s) &= c^T x_{ref}(s) = \frac{c^T (sI - A)^{-1}b}{-c^T A^{-1}b} r(s) + \\
& c^T (sI - A)^{-1}(I - C(s))\sigma(s) + \\
& \left(\frac{c \bar{D}^{-1} (sI - W)^{-1} E_1}{c \bar{D}^{-1} W^{-1} E_1} - 1 \right) c \bar{D}^{-1} R(\Lambda(s)).
\end{aligned} \tag{2.4.24}$$

The errors between the real system and the reference system are

$$x_e(s) = x(s) - x_{ref}(s), \quad y_e(s) = y(s) - y_{ref}(s). \tag{2.4.25}$$

Lemma 2.4.4. *The reference system in (2.4.23) is stable and*

$$\begin{aligned}
\|x_{ref}\|_{\mathcal{L}_\infty} &\leq \left\| \frac{(sI - A)^{-1}b}{-c^T A^{-1}b} \right\|_{\mathcal{L}_1} B_r + \\
& \left\| (sI - A)^{-1}(I - C(s)) \right\|_{\mathcal{L}_1} b_\sigma + \\
& \left\| \frac{\bar{D}^{-1}((sI - W)^{-1} E_1 c^T \bar{D}^{-1})}{c^T \bar{D}^{-1} W^{-1} E_1} - \bar{D}^{-1} \right\|_{\mathcal{L}_1} \sqrt{n} B_{R\lambda}
\end{aligned} \tag{2.4.26}$$

where $B_{R\lambda}$ is defined as

$$\begin{aligned}
B_{R\lambda} &= \left\| H_\Lambda(s) \bar{D} C(s) \right\|_{\mathcal{L}_1} \\
& (b_\sigma + \sqrt{\lambda_{\max}(A^\top A)} \gamma_0(T) + b_{d\sigma} T)
\end{aligned} \tag{2.4.27}$$

Proof: In (2.4.23), we noted that $r(t)$ and $\sigma(t)$ are bounded by (2.2.3) and (2.2.2); $R(\Lambda(t))$ in (2.4.23) is defined in (2.3.19), $\Lambda(t)$ in (2.3.19) is defined in (2.3.18), $\hat{\sigma}_r(s)$ in (2.3.18) is defined in (2.3.17). Consider the $\hat{\sigma}(t)$ in (2.3.17)

$$\begin{aligned}\|\hat{\sigma}(t)\| &= \|\sigma(t) + (\hat{\sigma}(t) - \sigma(t))\| \\ &\leq \|\sigma(t)\| + \|\hat{\sigma}(t) - \sigma(t)\| .\end{aligned}\tag{2.4.28}$$

Substituting the bounds from (2.2.2) and (2.4.17) into (2.4.28)

$$\|\hat{\sigma}(t)\| \leq b_\sigma + \sqrt{\lambda_{\max}(A^\top A)}\gamma_0(T) + b_{d\sigma}T .\tag{2.4.29}$$

Define

$$H_\Lambda(s) = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ s & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ s^{n-2} & s^{n-3} & \cdots & 1 & 0 \end{bmatrix} ,\tag{2.4.30}$$

Eqn (2.3.19) can be represented by

$$R(\Lambda(s)) = H_\Lambda(s)\Lambda(s) .\tag{2.4.31}$$

It follows from (2.3.17), (2.3.18) and (2.4.31),

$$R(\Lambda(s)) = H_\Lambda(s)\bar{D}C(s)\hat{\sigma}(s) ,\tag{2.4.32}$$

where $H_\Lambda(s)$, \bar{D} and $C(s)$ are defined in (2.4.30), (2.3.9), (2.3.16). Thus

$$\|R(\Lambda(t))\|_\infty \leq \|H_\Lambda(s)\bar{D}C(s)\|_{\mathcal{L}_1} \|\hat{\sigma}\|_\infty. \quad (2.4.33)$$

Plugging in (2.4.29) into (2.4.33) we have

$$\begin{aligned} \|R(\Lambda(t))\|_\infty &\leq \|H_\Lambda(s)\bar{D}C(s)\|_{\mathcal{L}_1} \\ &(b_\sigma + \sqrt{\lambda_{\max}(A^\top A)}\gamma_0(T) + b_{d\sigma}T) \end{aligned} \quad (2.4.34)$$

Eqn (2.4.34) is also the definition of $B_{R\lambda}$ defined in (2.4.27) Substituting $\|R(\Lambda(t))\|_\infty$, $r(s)$ and $\sigma(t)$ in (2.4.34), (2.2.3) and (2.2.2) into (2.4.23), we can readily obtain (2.4.26). \square

Theorem 2.4.5. *Given the system in (2.2.1) and the \mathcal{L}_1 adaptive controller in (2.3.1), (2.3.2) and (2.3.26), we have*

$$\begin{aligned} \|x_e\|_{\mathcal{L}_\infty} &\leq \|(sI - A)^{-1}(I - C(s))\|_{\mathcal{L}_1} \gamma_2(T) + \\ &\gamma_1(T). \end{aligned} \quad (2.4.35)$$

$$\begin{aligned} \|y_e\|_{\mathcal{L}_\infty} &\leq \|c^T(sI - A)^{-1}(I - C(s))\|_{\mathcal{L}_1} \gamma_2(T) + \\ &\|c^T\|_{\mathcal{L}_1} \gamma_1(T). \end{aligned} \quad (2.4.36)$$

$$\begin{aligned}
\|x\|_{\mathcal{L}_\infty} \leq & \left\| \frac{(sI - A)^{-1}b}{-c^T A^{-1}b} \right\|_{\mathcal{L}_1} B_r + \\
& \|(sI - A)^{-1}(I - C(s))\|_{\mathcal{L}_1} b_\sigma + \\
& \left\| \frac{\bar{D}^{-1}((sI - W)^{-1}E_1 c^T \bar{D}^{-1})}{c^T \bar{D}^{-1} W^{-1} E_1} - \bar{D}^{-1} \right\|_{\mathcal{L}_1} \sqrt{n} B_{R\lambda} + \\
& \|(sI - A)^{-1}(I - C(s))\|_{\mathcal{L}_1} \gamma_2(T) + \gamma_1(T).
\end{aligned} \tag{2.4.37}$$

where $x_e(t)$, $y_e(t)$, $x_{ref}(t)$, and $y_{ref}(t)$ are defined in (2.4.25), (2.4.23), (2.4.24).

Proof: First, we analyze the behavior of the state predictor (2.3.1) together with the control law described in (2.3.26). Following the definition of $\hat{x}(t)$, $\hat{x}_1(t)$, $\hat{x}_2(t)$, $\hat{x}_3(t)$, $\hat{x}_4(t)$ in (2.3.1), (2.3.4), (2.3.6), (2.3.21), (2.3.22), we have

$$\hat{x}(t) = \hat{x}_1(t) + \hat{x}_2(t), \tag{2.4.38}$$

$$\hat{x}_2(t) = \hat{x}_3(t) + \hat{x}_4(t). \tag{2.4.39}$$

The overall performance of the state predictor in (2.3.1) can be concluded as follows

$$\begin{aligned}
\hat{x}(t) &= \hat{x}_1(t) + \hat{x}_3(t) + \hat{x}_4(t), \\
\hat{y}(t) &= c^T(\hat{x}_1(t) + \hat{x}_3(t) + \hat{x}_4(t)),
\end{aligned} \tag{2.4.40}$$

$\hat{x}_1(t)$, $\hat{x}_3(t)$, $\hat{x}_4(t)$ are analyzed individually as shown next.

Plugging in the control law (2.3.7) in (2.3.4), we have

$$\hat{x}_1(s) = \frac{(sI - A)^{-1}b}{-c^T A^{-1}b} r(s). \tag{2.4.41}$$

Substituting $\hat{\sigma}_r(t)$ defined in (2.3.17) into (2.3.21), we have

$$\dot{\hat{x}}_3(t) = A\hat{x}_3 + (1 - C(s))\hat{\sigma}(t), \quad (2.4.42)$$

and hence

$$\hat{x}_3(s) = (sI - A)^{-1}(I - C(s))\hat{\sigma}(s). \quad (2.4.43)$$

Substituting $\hat{\sigma}(t) = \sigma(t) + (\hat{\sigma}(t) - \sigma(t))$ into (2.4.43), we have

$$\hat{x}_3(s) = (sI - A)^{-1}(I - C(s))\sigma(s) + e_1(s). \quad (2.4.44)$$

where

$$e_1(s) = (sI - A)^{-1}(I - C(s))(\hat{\sigma}(s) - \sigma(s)). \quad (2.4.45)$$

Substituting the boundary of $\hat{\sigma}(t) - \sigma(t)$ in Lemma 2.4.1 into (2.4.45), we have

$$\|e_1\|_{\mathcal{L}_\infty} \leq \|(sI - A)^{-1}(I - C(s))\|_{\mathcal{L}_1} \gamma_2(T). \quad (2.4.46)$$

$$\|c^T e_1\|_{\mathcal{L}_\infty} \leq \|c^T (sI - A)^{-1}(I - C(s))\|_{\mathcal{L}_1} \gamma_2(T). \quad (2.4.47)$$

It follows from (2.4.44) that

$$\hat{y}_3(s) = c^T (sI - A)^{-1}(I - C(s))\sigma(s) + c^T e_1(s). \quad (2.4.48)$$

For $\hat{x}_4(t)$, Substituting the control law $u_{21}(t)$ defined in (2.3.24), we can transform

the system in (2.3.22) into a nice control canonical form [76] as follows.

$$\begin{aligned} \dot{Z}(t) = & \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -d_1 & -d_2 & \cdots & \cdots & -d_n \end{bmatrix} Z(t) \\ & + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \vdots \\ 1 \end{bmatrix} u_{22}(t). \end{aligned} \quad (2.4.49)$$

where $Z(t)$ is defined as:

$$Z(t) = \bar{D}\hat{x}_4(t) + R(\Lambda(t)). \quad (2.4.50)$$

Using the definition of W in (2.3.15), (2.4.49) can be rewritten as

$$\dot{Z}(t) = WZ(t) + E_1 u_{22}(t). \quad (2.4.51)$$

The Laplace transformation of (2.4.51) is

$$Z(s) = (sI - W)^{-1} E_1 u_{22}(t). \quad (2.4.52)$$

It follows from (2.4.50) that $\hat{x}_4(t)$ and $\hat{y}_4(t)$ can be expressed by $Z(t)$ and $R(\Lambda(t))$ as

$$\hat{x}_4(t) = \bar{D}^{-1}Z(t) - \bar{D}^{-1}R(\Lambda(t)), \quad (2.4.53)$$

$$\hat{y}_4(t) = c^T \hat{x}_4(t) = c\bar{D}^{-1}Z(t) - c\bar{D}^{-1}R(\Lambda(t)). \quad (2.4.54)$$

Substituting the dynamics of $Z(s)$ in (2.4.52) and the control law in (2.3.25) into (2.4.53) (2.4.54), we have :

$$\begin{aligned} \hat{x}_4(s) &= \frac{c\bar{D}^{-1}(sI - W)^{-1}E_1c\bar{D}^{-1}R(\Lambda(s))}{cD^{-1}W^{-1}E_1} - \\ &\quad \bar{D}^{-1}R(\Lambda(s)) \end{aligned} \quad (2.4.55)$$

$$\begin{aligned} c^T \hat{x}_4(s) &= \left(\frac{c\bar{D}^{-1}(sI - W)^{-1}E_1}{cD^{-1}W^{-1}E_1} - 1 \right) \\ &\quad c\bar{D}^{-1}R(\Lambda(s)). \end{aligned} \quad (2.4.56)$$

Substituting the result in (2.4.41), (2.4.44) and (2.4.55) into (2.4.40), we have

$$\begin{aligned} \hat{x}(s) &= \frac{(sI - A)^{-1}b}{-c^T A^{-1}b} r(s) + \\ &\quad (sI - A)^{-1}(I - C(s))\sigma(s) + e_1(s) + \\ &\quad \bar{D}^{-1} \left(\frac{(sI - W)^{-1}E_1c^T \bar{D}^{-1}R(\Lambda(t))}{c^T \bar{D}^{-1}W^{-1}E_1} \right) - \bar{D}^{-1}R(\Lambda(t)) \end{aligned} \quad (2.4.57)$$

where $e_1(s)$ is defined in (2.4.45).

Substituting the result in (2.4.41), (2.4.48) and (2.4.56) into (2.4.40), we have

$$\begin{aligned}\hat{y}(s) &= \frac{c^T(sI - A)^{-1}b}{-c^T A^{-1}b}r(s) + \\ &\quad c^T(sI - A)^{-1}(I - C(s))\sigma(s) + c^T e_1(s) + \\ &\quad \left(\frac{c\bar{D}^{-1}(sI - W)^{-1}E_1}{cD^{-1}W^{-1}E_1} - 1\right)c\bar{D}^{-1}R(\Lambda(s)).\end{aligned}\quad (2.4.58)$$

Next, we consider the real system state $x(t)$ and output $y(t)$ of system in (2.2.1)

$$x(s) = \hat{x}(s) + \tilde{x}(s), y(s) = c^T(\hat{x}(s) + \tilde{x}(s)). \quad (2.4.59)$$

Substituting the result in (2.4.57), (2.4.58) into (2.4.59) we have:

$$\begin{aligned}x(s) &= \hat{x}(s) + \tilde{x}(s) \\ &= \frac{(sI - A)^{-1}b}{-c^T A^{-1}b}r(s) + (sI - A)^{-1}(I - C(s))\sigma(s) + \\ &\quad e_1(s) + \bar{D}^{-1}\left(\frac{(sI - W)^{-1}E_1 c^T \bar{D}^{-1}R(\Lambda(t))}{c^T \bar{D}^{-1}W^{-1}E_1}\right) - \\ &\quad \bar{D}^{-1}R(\Lambda(t)) + \tilde{x}(s),\end{aligned}\quad (2.4.60)$$

$$\begin{aligned}y(s) &= \frac{c^T(sI - A)^{-1}b}{-c^T A^{-1}b}r(s) + \\ &\quad c^T(sI - A)^{-1}(I - C(s))\sigma(s) + c^T e_1(s) + \\ &\quad \left(\frac{c\bar{D}^{-1}(sI - W)^{-1}E_1}{cD^{-1}W^{-1}E_1} - 1\right)c\bar{D}^{-1}R(\Lambda(s)) + c^T \tilde{x}(s).\end{aligned}\quad (2.4.61)$$

Using the definition of $x_e(s)$, $y_e(s)$ in (2.4.25) and the definition of $x_{ref}(s)$, $y_{ref}(s)$ in

(2.4.23) and (2.4.24), we have:

$$x_e(s) = e_1(s) + \tilde{x}(s), \quad y_e(s) = c^T e_1(s) + c^T \tilde{x}(s). \quad (2.4.62)$$

It follows from (2.4.23), (2.4.62), (2.4.46), (2.4.47) and Lemma 2.4.1 that

$$\begin{aligned} \|x_e\| &\leq \|(sI - A)^{-1}(I - C(s))\|_{\mathcal{L}_1} \gamma_2(T) + \gamma_1(T). \\ \|y_e\| &\leq \|c^T(sI - A)^{-1}(I - C(s))\|_{\mathcal{L}_1} \gamma_2(T) + \\ &\quad \|c^T\|_{\mathcal{L}_1} \gamma_1(T). \end{aligned} \quad (2.4.63)$$

It follows from the definition of $x_e(s)$ and $y_e(s)$ in (2.4.25) that,

$$x(s) = x_{ref}(s) + x_e(s), \quad y(s) = y_{ref}(s) + y_e(s). \quad (2.4.64)$$

Hence,

$$\|x\|_{\mathcal{L}_\infty} \leq \|x_{ref}\|_{\mathcal{L}_\infty} + \|x_e\|_{\mathcal{L}_\infty}. \quad (2.4.65)$$

Using the results in Lemma 2.4.4 and (2.4.26) and (2.4.35), we can readily obtain (2.4.37) which completes the proof. Thus completing the proof. \square

Define

$$e(t) = y_{ref}(t) - y_{des}(t) \quad (2.4.66)$$

where $y_{des}(s) = \frac{c^T(sI-A)^{-1}b}{-c^T A b} r(s)$ is the desired system.

Theorem 2.4.6. *Given the system in (2.2.1) and the \mathcal{L}_1 adaptive controller in*

(2.3.1), (2.3.2) and (2.3.26), the tracking error $e(t)$ in (2.4.66) is given as:

$$\begin{aligned} \|e(t)\| &\leq \|c^T(sI - A)^{-1}(I - C(s))\|_{\mathcal{L}_1} b_{d\sigma} \\ &+ \left\| \frac{c\bar{D}^{-1}}{cD^{-1}W^{-1}E_1} \right\|_{\mathcal{L}_1} \frac{2b_{d\xi}\lambda_{\max}(P)}{\lambda_{\min}(Q)}. \end{aligned} \quad (2.4.67)$$

Proof: Compare (2.4.66) and (2.4.24), we have

$$e(s) = g_1(s)\sigma(s) + g_2(s)c\bar{D}^{-1}R(\Lambda(s)), \quad (2.4.68)$$

where $g_1(s)$ and $g_2(s)$ are defined as

$$g_1(s) = c^T(sI - A)^{-1}(I - C(s)), \quad (2.4.69)$$

$$g_2(s) = \frac{c\bar{D}^{-1}(sI - W)^{-1}E_1}{cD^{-1}W^{-1}E_1} - 1. \quad (2.4.70)$$

Define

$$e_1(s) = g_1(s)\sigma(s), \quad e_2(s) = g_2(s)c\bar{D}^{-1}R(\Lambda(s)). \quad (2.4.71)$$

To further analysis the bound of signal $e_2(t)$, the following definition is needed

$$\begin{aligned} \dot{\xi}(t) &= W\xi(t) + E_1\bar{r}(t), \quad \xi_{ss}(t) = -WE_1\bar{r}(t) \\ \tilde{\xi}(t) &= \xi(t) - \xi_{ss}(t), \quad \bar{r}(s) = c\bar{D}^{-1}R(\Lambda(s)). \end{aligned} \quad (2.4.72)$$

Then $e_2(t)$ can be rewritten as

$$e_2(t) = -\frac{c\bar{D}^{-1}\tilde{\xi}}{cD^{-1}W^{-1}E_1} . \quad (2.4.73)$$

Dynamic of $\tilde{\xi}(t)$ can be expressed as

$$\begin{aligned} \dot{\tilde{\xi}}(t) &= W\tilde{\xi}(t) + E_1\bar{r}(t) + \\ &W\xi_{ss}(t) + WE_1\dot{\bar{r}}(t) - \dot{\xi}_{ss}(t) . \end{aligned} \quad (2.4.74)$$

Given any positive-defined matrix Q and solving P by the following lyapunov equation

$$W^T P + PW = -Q . \quad (2.4.75)$$

Consider the lyapunov function candidate $V(t) = \tilde{\xi}(t)^T P \tilde{\xi}(t)$ we have.

$$\begin{aligned} \dot{V}(t) &= \dot{\tilde{\xi}}(t)^T P \tilde{\xi}(t) + \tilde{\xi}(t)^T P \dot{\tilde{\xi}}(t) \\ &= -\tilde{\xi}(t)^T Q \tilde{\xi}(t) + 2\tilde{\xi}(t)^T P E_1 \bar{r}(t) + \\ &2\tilde{\xi}(t)^T P W E_1 \dot{\bar{r}}(t) + 2\tilde{\xi}(t)^T P W \xi_{ss}(t) - 2\tilde{\xi}(t)^T P \dot{\xi}_{ss}(t) . \end{aligned}$$

Using the definition of $\xi_{ss}(t)$ we have

$$\dot{V}(t) = -\tilde{\xi}(t)^T Q \tilde{\xi}(t) - 2\tilde{\xi}(t)^T P \dot{\xi}_{ss}(t) . \quad (2.4.76)$$

Follows by the assumption 2.2.1, there exist a bound $b_{d\xi}$ such that

$$\left\| \dot{\xi}_{ss}(t) \right\| < b_{d\xi} . \quad (2.4.77)$$

For any $t_1 > 0$, if

$$V(t_1) > \frac{4b_{d\xi}^2 \lambda_{\max}^3(P)}{\lambda_{\min}^2(Q)}, \quad (2.4.78)$$

then

$$\|\tilde{\xi}(t_1)\| \geq \sqrt{\frac{V(t_1)}{\lambda_{\max}(P)}} > \frac{2b_{d\xi} \lambda_{\max}(P)}{\lambda_{\min}(Q)}. \quad (2.4.79)$$

It follows from (2.4.76) that

$$\begin{aligned} \dot{V}(t_1) &= -\tilde{\xi}(t)^T Q \tilde{\xi}(t) - 2\tilde{\xi}(t)^T P \dot{\xi}_{ss}(t) \\ &\leq \lambda_{\min}(Q) \|\tilde{\xi}(t_1)\|^2 + \\ &\quad 2b_{d\xi} \lambda_{\max}(P) \|\tilde{\xi}(t_1)\| < 0. \end{aligned} \quad (2.4.80)$$

Eqn (2.4.78) to (2.4.80) means that if $V(0) > \frac{4b_{d\xi}^2 \lambda_{\max}^3(P)}{\lambda_{\min}^2(Q)}$, $V(t)$ will keep decreasing until $V(t) \leq \frac{4b_{d\xi}^2 \lambda_{\max}^3(P)}{\lambda_{\min}^2(Q)}$. If $V(0) \leq \frac{4b_{d\xi}^2 \lambda_{\max}^3(P)}{\lambda_{\min}^2(Q)}$, then $V(t) \leq \frac{4b_{d\xi}^2 \lambda_{\max}^3(P)}{\lambda_{\min}^2(Q)}$. As a result

$$\|\tilde{\xi}(t)\| \leq \sqrt{\frac{V(t)}{\lambda_{\max}(P)}} < \frac{2b_{d\xi} \lambda_{\max}(P)}{\lambda_{\min}(Q)}. \quad (2.4.81)$$

Thus,

$$\|e_2(t)\| \leq \left\| \frac{c\bar{D}^{-1}}{cD^{-1}W^{-1}E_1} \right\|_{\mathcal{L}_1} \frac{2b_{d\xi} \lambda_{\max}(P)}{\lambda_{\min}(Q)}. \quad (2.4.82)$$

As a result

$$\begin{aligned}
\|e(t)\| &\leq \|e_1(t)\| + \|e_2(t)\| \\
&\leq \|c^T(sI - A)^{-1}(I - C(s))\|_{\mathcal{L}_1} b_{d\sigma} \\
&\quad + \left\| \frac{c\bar{D}^{-1}}{cD^{-1}W^{-1}E_1} \right\|_{\mathcal{L}_1} \frac{2b_{d\xi}\lambda_{\max}(P)}{\lambda_{\min}(Q)}. \tag{2.4.83}
\end{aligned}$$

□

Remark 2.4.7. Theorem 2.4.6 shows the relationship of the tracking error $e(t)$, filter bandwidth $C(s)$, variation rate of disturbances $b_{d\xi}$, and $\lambda_{\max}(P)$, $\lambda_{\min}(Q)$. Moreover, the assigned eigenvalues of matrix W will affect the $\lambda_{\max}(P)$ and $\lambda_{\min}(Q)$ governed by eqn (2.4.75).

2.5 Simulation

We consider the system in (2.2.1) with

$$A = \begin{bmatrix} 3 & 0 \\ 3 & -1 \end{bmatrix}, b = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, c = \begin{bmatrix} 0 & -1 \end{bmatrix}. \tag{2.5.1}$$

We notice that $c^T(sI - A)^{-1}b = \frac{s-3}{s^2+4s+3}$ has a zero in the right half complex plane.

We consider the \mathcal{L}_1 adaptive state feedback controller defined via (2.3.1), (2.3.2) and (2.3.26), where $C(s) = \frac{1}{200s+1}$, $T = 10^{-2}$. The reference input $r(t) = 1$. To further

demonstrate the effects of the disturbance, we decouple the $\sigma(t)$ as follows:

$$\sigma(t) = \sigma_m(t) + \sigma_{um}(t) \quad (2.5.2)$$

where $\sigma_m(t)$ is the matched disturbance component and $\sigma_{um}(t)$ is the unmatched disturbance.

i) In the presence of a constant disturbance

Define unit step function $H(t, \tau) = \begin{cases} 0 & \text{if } t \leq \tau \\ 1 & \text{else} \end{cases}$, reference input $r(t) = 0$. Two sets of eigenvalues with $\lambda_{min} = 1$ and $\lambda_{min} = 10$ are simulated. We first apply the matched disturbance: $\sigma(t) = \sigma_m(t) = H(t, 3) \begin{bmatrix} 2 & -1 \end{bmatrix}^T$. The disturbance rejection performance for this case is shown in figure 2.5.1. The output $y(t)$ after adding the unmatched disturbance: $\sigma(t) = \sigma_{um}(t) = H(t, 3) \begin{bmatrix} 1 & 2 \end{bmatrix}^T$ is shown in figure 2.5.1. We can see from the figure that the controller with larger eigenvalue will lead to a faster disturbance compensation. Other than the eigenvalues, the sampling time T is also related with the performance. Reducing sampling time also helps to improve performance.

ii) In the presence of a time-varying disturbance

We first defined the matched disturbance as $\sigma_m(t) = 0.2\sin(1/2\pi t) \begin{bmatrix} 2 & -1 \end{bmatrix}^T$ and the unmatched disturbance: $\sigma_{um}(t) = 0.3\sin(1/3\pi t) \begin{bmatrix} 1 & 2 \end{bmatrix}^T$. The reference input $r(t) = 0$, the output $y(t)$ and the control signal $u(t)$ are shown in figures 2.5.3 and 2.5.4. Comparing the results of $\lambda_{min} = 1$ and $\lambda_{min} = 10$. The disturbance is compensated better when λ_{min} is larger. The simulation verified the property of the transfer function $g_2(s)$ discussed in remark 2.4.7. Figure 2.5.5 and 2.5.6 shows the tracking performance and system state with unmatched sinusoidal function.

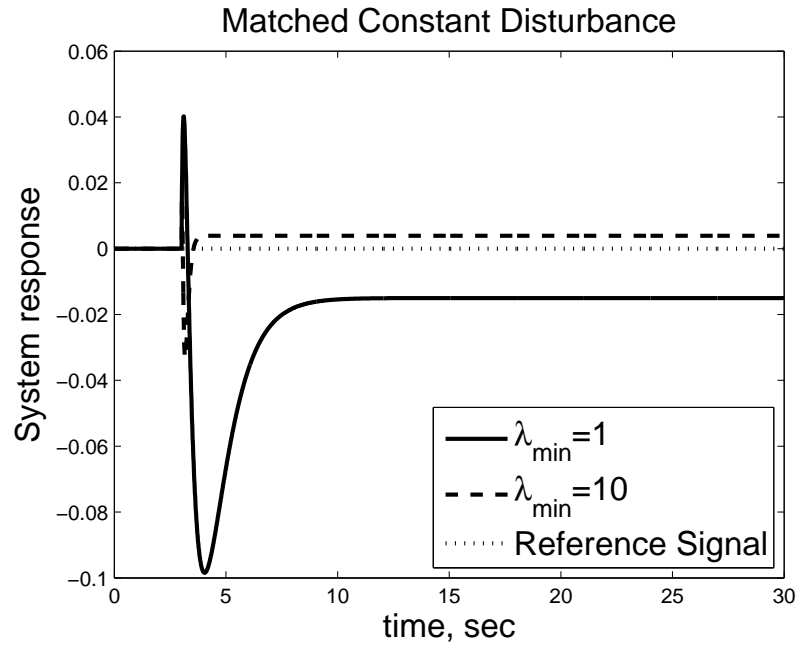


FIGURE 2.5.1: Performance for matched constant disturbance , $\lambda_{\min} = 1$ and $\lambda_{\min} = 10$.

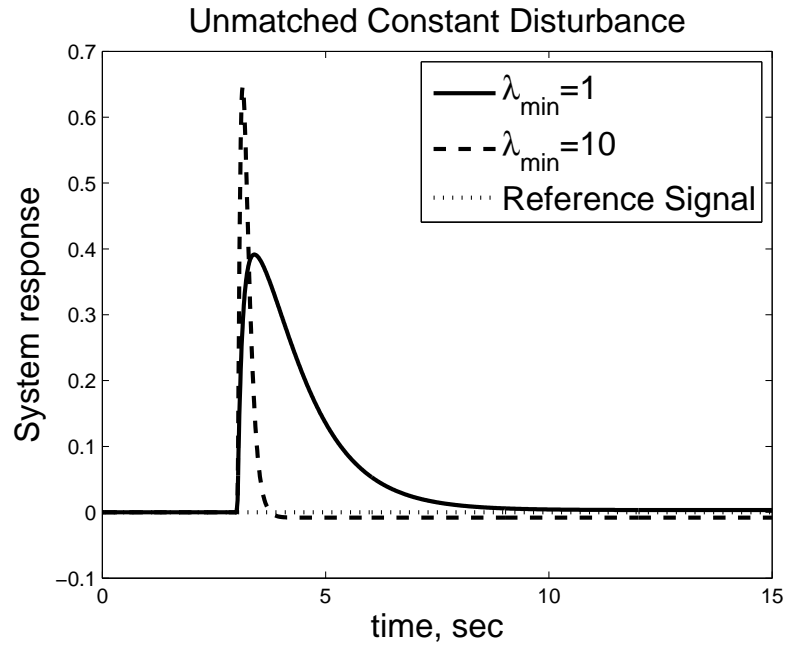


FIGURE 2.5.2: Performance for unmatched constant disturbance , $\lambda_{\min} = 1$ and $\lambda_{\min} = 10$.

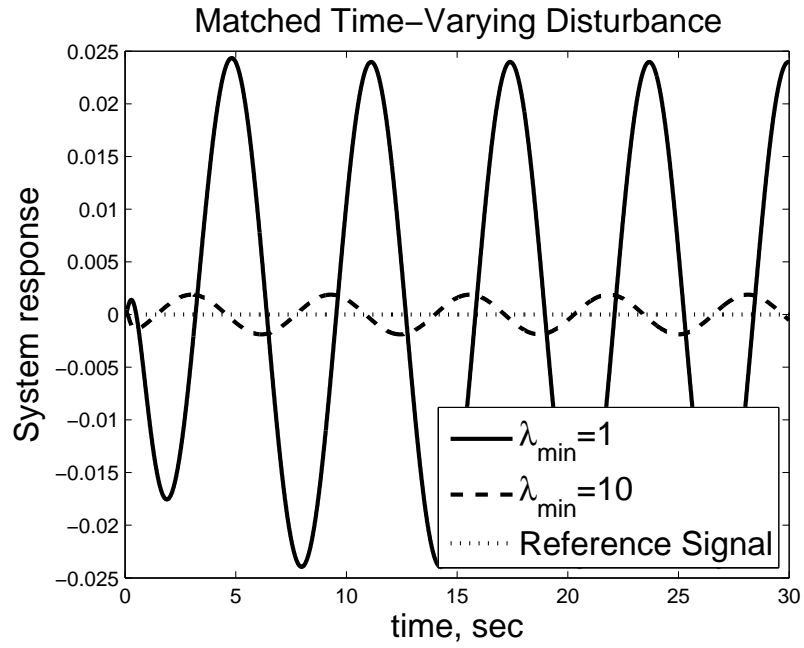


FIGURE 2.5.3: Performance for matched time-varying disturbance, $\lambda_{\min} = 1$ and $\lambda_{\min} = 10$.

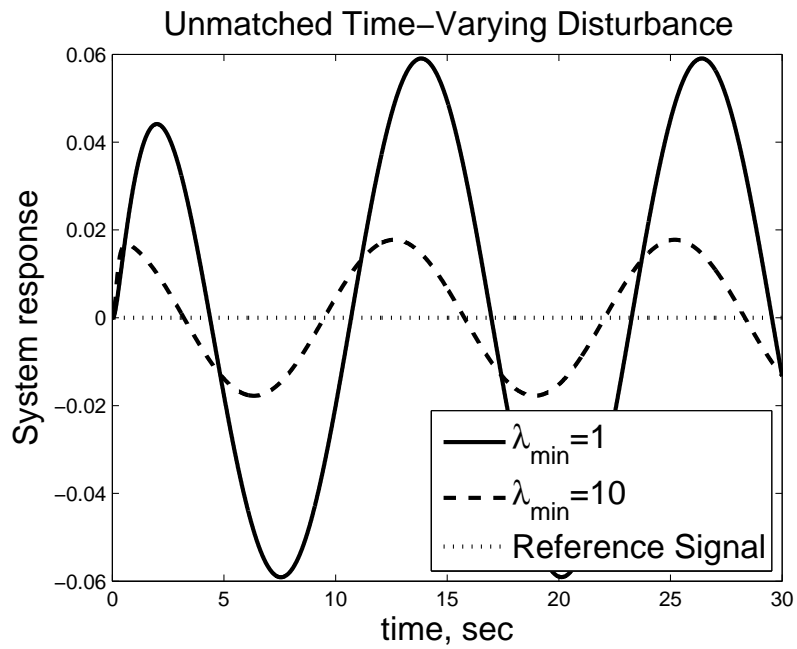


FIGURE 2.5.4: Performance for unmatched Time-varying disturbance, when $\lambda_{\min} = 1$ and $\lambda_{\min} = 10$.

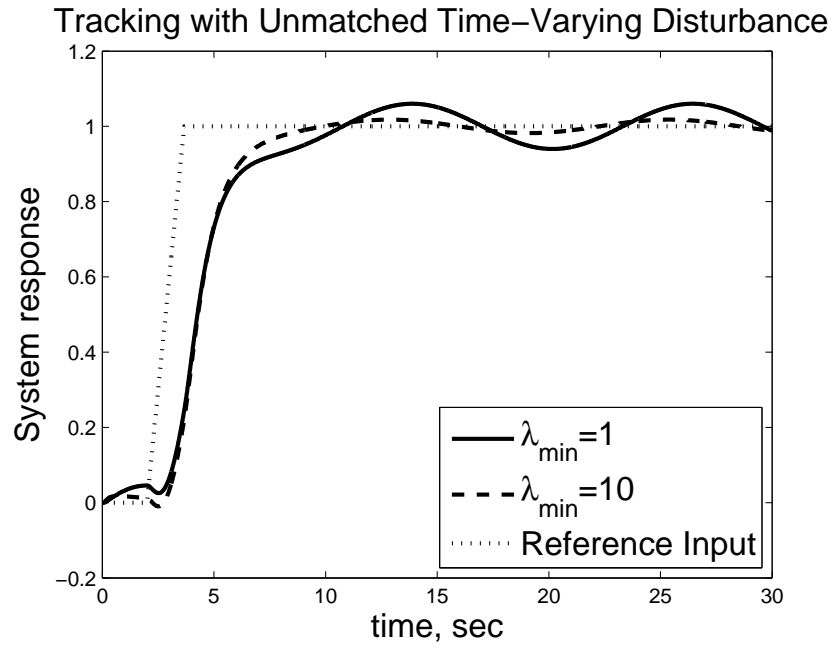


FIGURE 2.5.5: Tracking performance for unmatched time-varying disturbance, when $\lambda_{\min} = 1$ and $\lambda_{\min} = 10$.

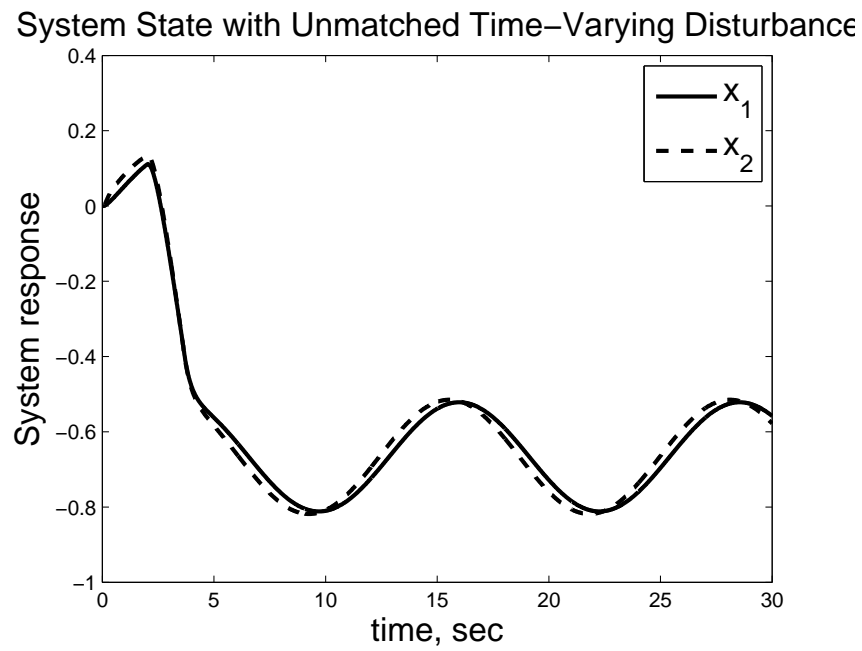


FIGURE 2.5.6: System response for unmatched time-varying disturbance, when $\lambda_{\min} = 10$.

2.6 Summary

A novel control law for the \mathcal{L}_1 control architecture is provided in the presence of unmatched time-varying disturbances. This control law avoids the dynamic inversion of the desired system. Consequently, it can deal with the non-minimum phase plant with closed loop stability. Simulation results demonstrate the viability and performance of this proposed control method. Future work includes the extension of this approach for linear time-varying (LTV) systems and Multiple Input Multiple Output (MIMO) systems.

Chapter 3

Application of \mathcal{L}_1 Adaptive Output Feedback Control Design for Flexible Wing Aircraft

3.1 Introduction

The next generation of efficient subsonic aircraft will need to be both high aspect ratio and light weight, and the associated structural flexibility presents both challenges and opportunities. An example of advanced vehicles under consideration is illustrated in **figure 3.1.1**. Robust active aeroelastic control and Gust Load Alleviation (GLA) have a potential to substantially reduce weight by relaxing structural strength requirements for the wings. However, current tools such as traditional linear aeroelastic and aeroservoelastic analysis methods are inadequate to reliably predict aeroelastic stability and assess active aeroelastic control effectiveness for this type of vehicle. This type of vehicle represents a nonlinear stability and control problem involving complex in-



FIGURE 3.1.1: New generation of efficient subsonic aircraft.

teractions among the flexible structure, unsteady aerodynamics, flight control system, propulsion system, environmental conditions, and vehicle flight dynamics. Furthermore, because of the inherent flexibility of the aircraft, the lower order structural mode frequencies are of the same order as the rigid-body mode frequencies. The close proximity of flexible and rigid-body dynamics does not allow for the more traditional designs based on the separation between these two set of dynamics. Hence, this trend in aeronautics and high level of uncertainty associated with available aircraft design tools present an important challenge for adaptive control theory. Some early research on the flexible wing program is summarized in [156, 157]. In [158, 159] the dynamic inversion control law and its modification is proposed for flexible wing controller. An output feedback adaptive controller for flexible aircraft is introduced in [107]. Based on the same simulation model, the integrated flight/structural mode control for Very flexible aircraft is developed in [105]. The fundamental theory of \mathcal{L}_1 adaptive control is introduced in [37, 39, 61, 160]. State feedback \mathcal{L}_1 adaptive control has been used in a number of challenging applications; the theory has been verified and the design

process has matured against real world challenges. For example, the \mathcal{L}_1 adaptive control has been recently used to enable real-time dynamic modeling of the edges of a flight envelope of a dynamically scaled model of a generic transport aircraft with a conventional configuration [112]. On the other hand, the output feedback \mathcal{L}_1 adaptive control is in the early stages of application to new challenges such as very flexible aircraft. The \mathcal{L}_1 output feedback control design is developed in [161–164]. This paper presents an extension of a recent \mathcal{L}_1 adaptive output feedback controller in [164] to an advanced highly flexible aircraft configuration. Firstly, the theoretical analysis in [164] is extended to a MIMO system setup, which fits the flexible aircraft control problem. In [165], the disturbance and uncertainties are assumed to depend on the output of the system, in this paper, this assumption is relaxed that the uncertainties are depend on system state. Secondly, the control law design in [164] is modified to only compensate the unmatched disturbance and uncertainties in steady state. In other words, Dc gain inversion is used instead of dynamic inversion. Hence, it has little influence on the flexible modes which is highly uncertain and unpredictable in different operating points. An independent LQG controller is designed to stabilize these flexible modes. Thirdly, the control law in [105] is modified slight to have a independent state predictor for the adaptive law. In [105] the LQG controller and \mathcal{L}_1 output feedback adaptive law share the same state predictor which causes some restriction in parameter tuning. To demonstrate the potential of the proposed \mathcal{L}_1 output feedback approach, a semi-span high aspect ratio flexible aircraft wind tunnel model from Ref. [10] is revisited. The numerical simulation results for a very flexible aircraft configuration illustrate the algorithm’s performance and quantify the achievable performance bound as a function of the control computer CPU for this class of vehicles.

3.2 Problem Formulation

Consider the following multiple-input multiple-output (MIMO) system

$$\begin{aligned}\dot{x}(t) &= A_m x(t) + B_m \left(f(x, t) + u(t) \right) + \sigma(t), \\ y(t) &= C_m^\top x(t), \quad y(0) = y_0,\end{aligned}\tag{3.2.1}$$

where $x(t) \in \mathbb{R}^n$ is the system state vector (unmeasurable), $u(t) \in \mathbb{R}^p$ is the control input, $y(t) \in \mathbb{R}^q$ is the system output, A_m is a known $n \times n$ Hurwitz matrix, $B_m \in \mathbb{R}^{p \times n}$, $C_m \in \mathbb{R}^{n \times q}$ are known constant matrices with ranks p and q , zeros of $C_m^\top(s\mathbb{I} - A_m)B_m$ lie in the open left-half s plane, $f : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^p$ is an unknown nonlinear function, and $\sigma(t) \in \mathbb{R}^n$ are unknown disturbances.

Assumption 3.2.1. [Semiglobal Lipschitz condition on x] For any $\delta > 0$, there exist $L(\delta) > 0$ and $B > 0$ such that

$$\|f(x, t) - f(\bar{x}, t)\| \leq L(\delta)\|x - \bar{x}\|_\infty, \quad \|f(0, t)\| \leq B,$$

for all $\|x\|_\infty \leq \delta$ and $\|\bar{x}\|_\infty \leq \delta$ uniformly in u and t .

Assumption 3.2.2. There exist $B_\sigma > 0$ such that

$$\|\sigma(t)\| \leq B_\sigma$$

for all $t \geq 0$, where the numbers B_σ can be arbitrarily large.

The control objective is to design an adaptive output feedback controller $u(t)$ such

that the system output $y(t)$ tracks the reference system output $y_{des}(t)$ described by

$$\begin{aligned}\dot{x}_{des}(t) &= A_m x_{des}(t) + B_d \bar{k}_g r(t), \\ y_{des}(t) &= C_d^\top x_{des}(t),\end{aligned}\tag{3.2.2}$$

where B_d and C_d are the selected input and output channels for control. In this paper we mainly focus on control of vertical displacement. One control input channel B_d and one control output channel C_d is selected for tracking (first column of B_m and first row of C_m). The other column of control input matrix B_m is used for disturbance rejection and stability augmentation. The other rows of C_m are used for output feedback observer and adaptive law design. $\bar{k}_g = -(C_d^\top A_m^{-1} B_d)^{-1}$, $r(t)$ is a given bounded reference input signal with $|r(t)| \leq \|r\|_{\mathcal{L}_\infty}$.

3.3 \mathcal{L}_1 Adaptive Output Feedback Controller

We consider the following output predictor

$$\begin{aligned}\dot{\hat{x}}(t) &= A_m \hat{x}(t) + B_m u(t) + \hat{\sigma}(t), \\ \hat{y}(t) &= C_m^\top \hat{x}(t), \quad \hat{y}(0) = y_0,\end{aligned}\tag{3.3.1}$$

where $\hat{\sigma}(t) \in \mathbb{R}^n$ is the vector of adaptive parameters. We can find matrix $B_{um} \in \mathbb{R}^{n \times (n-p)}$ such that $B_m^\top B_{um} = \mathbf{0}^{p \times (n-1)}$ and $\text{rank}([B_m \ B_{um}]) = n$. Then, equation

(3.3.1) can be written as

$$\begin{aligned}\dot{\hat{x}}(t) &= A_m \hat{x}(t) + B_m(u(t) + \hat{\sigma}_1(t)) + B_{um} \hat{\sigma}_2(t), \\ \hat{y}(t) &= C_m^\top \hat{x}(t), \quad \hat{y}(0) = y_0,\end{aligned}\tag{3.3.2}$$

where $\hat{\sigma}_1(t)$ represents the matched component of the uncertainties $\hat{\sigma}(t)$, and $\hat{\sigma}_2(t)$ represents the unmatched component.

Letting $\tilde{y}(t) = \hat{y}(t) - y(t)$, the update law for $\hat{\sigma}(t)$ is given by

$$\begin{aligned}\hat{\sigma}(t) &= \hat{\sigma}(iT), \quad t \in [iT, (i+1)T), \\ \hat{\sigma}(iT) &= -\Phi^{-1}(T)\mu(iT), \quad i = 0, 1, 2, \dots,\end{aligned}\tag{3.3.3}$$

where $\Phi(T) = \int_0^T e^{\Lambda A_m \Lambda^{-1}(T-\tau)} \Lambda d\tau$ and

$$\mu(iT) = e^{\Lambda A_m \Lambda^{-1}T} \mathbf{1}_1 \tilde{y}(iT), \quad i = 0, 1, 2, 3, \dots\tag{3.3.4}$$

where $\mathbf{1}_1$ is defined as

$$\mathbf{1}_1 = \begin{bmatrix} 1^{q \times q} \\ 0^{(n-q) \times q} \end{bmatrix}\tag{3.3.5}$$

The control signal is defined as follows

$$\begin{bmatrix} \hat{\sigma}_1(t) \\ \hat{\sigma}_2(t) \end{bmatrix} = \begin{bmatrix} B_m & B_{um} \end{bmatrix}^{-1} \hat{\sigma}(t), \quad (3.3.6)$$

$$\begin{aligned} u(s) &= \mathbf{1}_2 k_g r(s) - C_1(s) \hat{\sigma}_1(s) \\ &\quad - C_2(s) M \hat{\sigma}_2(s), \end{aligned} \quad (3.3.7)$$

where $\mathbf{1}_2 = \begin{bmatrix} 1 \\ 0^{(p-1) \times 1} \end{bmatrix}$, $r(s)$ is the Laplace transformation of the reference signal $r(t)$, $k_g = -(c_d^\top A_m^{-1} B_d)^{-1}$, $M = \frac{C_d^\top A_m^{-1} B_{um}}{C_d^\top A_m^{-1} B_d}$, both $C_1(s)$ and $C_2(s)$ are low pass filters with unit DC gain, and $\hat{\sigma}_1(s)$ and $\hat{\sigma}_2(s)$ are Laplace transformations of matched uncertainties $\hat{\sigma}_1(t)$ and unmatched uncertainties $\hat{\sigma}_2(t)$ respectively. The \mathcal{L}_1 adaptive controller consists of (3.3.1), (3.3.3) and (3.3.7).

Remark 3.3.1. In [112], the control law in equation (17) is designed based on dynamic inversion and is designed to cancel the effects of the output $y(t)$. An important modification in this paper is that the DC gain matrix of the system is used instead of dynamic inversion. By doing this, we are not pursuing a perfect cancellation through dynamic inversion even during the transient phase. Instead, only the effects of steady state components are cancelled. This will allow the high frequency components to settle naturally. It is noted that the intention to cancel high frequency components during the transient phase may lead to a fast varying control signal, which may excite additional unmodeled high frequency components. This modification is necessary because the flexible vehicle model structure has a high system dimension and high frequency lightly damped modes. Under nominal conditions, dynamic inversion will improve the tracking performance, but the control signal will easily excite the high

frequency mode shapes, especially when dealing with high system dimensions with large parameter deviations due to a wide range of encompassed flight conditions. The frequencies of the flexible modes are uncertain, and the first and second flexible modes have frequencies close to rigid body dynamics and are close to each other, which makes cancellation by inversion completely impractical. In addition, the rank of the controllability matrix is low compared to the real system dimensions, which means many of the high frequency mode shapes are nearly uncontrollable. To avoid exciting the uncertain flexible mode frequencies, the dynamic inversion is changed to a static inversion. Only compensating the static error which makes more sense for the flexible structure. This results in a different stability condition for the output feedback controller.

3.4 Preliminaries for the Main Result

Since A_m is Hurwitz, there exists a positive-definite matrix $P = P^\top > 0$ that satisfies the following Lyapunov equation

$$A_m^\top P + P A_m = -Q, \quad Q > 0.$$

From the properties of P , there exists a non-singular matrix \sqrt{P} such that

$$P = (\sqrt{P})^\top \sqrt{P}.$$

Given the vector $C_m^\top(\sqrt{P})^{-1}$, let D be a $(n - q) \times n$ matrix that contains the null space of $C_m^\top(\sqrt{P})^{-1}$, i.e.,

$$D(C_m^\top(\sqrt{P})^{-1})^\top = 0. \quad (3.4.1)$$

Then we define

$$\Lambda = \begin{bmatrix} C_m^\top \\ D\sqrt{P} \end{bmatrix}. \quad (3.4.2)$$

Lemma 3.4.1. *For any $\xi = \begin{bmatrix} y \\ z \end{bmatrix} \in \mathbb{R}^n$, where $y \in \mathbb{R}^p$ and $z \in \mathbb{R}^{n-q}$, there exist and positive definite $P_1 \in \mathbb{R}^{q \times q}$, $P_2 \in \mathbb{R}^{(n-q) \times (n-q)}$ such that*

$$\xi^\top (\Lambda^{-1})^\top P \Lambda^{-1} \xi = y^\top P_1 y + z^\top P_2 z.$$

Proof. Using $P = (\sqrt{P})^\top \sqrt{P}$, one can write

$$\xi^\top (\Lambda^{-1})^\top P \Lambda^{-1} \xi = \xi^\top (\sqrt{P} \Lambda^{-1})^\top (\sqrt{P} \Lambda^{-1}) \xi.$$

We notice that

$$\Lambda(\sqrt{P})^{-1} = \begin{bmatrix} c_m^\top(\sqrt{P})^{-1} \\ D \end{bmatrix}.$$

Let

$$Q_1 = (c_m^\top(\sqrt{P})^{-1})(c_m^\top(\sqrt{P})^{-1})^\top, \quad Q_2 = DD^\top.$$

From (3.4.1) we have

$$(\Lambda(\sqrt{P})^{-1})(\Lambda(\sqrt{P})^{-1})^\top = \begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix}.$$

Non-singularity of Λ and \sqrt{P} implies that $(\Lambda(\sqrt{P})^{-1})(\Lambda(\sqrt{P})^{-1})^\top$ is non-singular, and therefore Q_2 is also non-singular. Hence,

$$(\sqrt{P}\Lambda^{-1})^\top(\sqrt{P}\Lambda^{-1}) = (\Lambda(\sqrt{P})^{-1})(\Lambda(\sqrt{P})^{-1})^\top)^{-1} = (\Lambda(\sqrt{P})^{-1})^{-\top}(\sqrt{P}\Lambda^{-1}) = \begin{bmatrix} Q_1^{-1} & 0 \\ 0 & Q_2^{-1} \end{bmatrix}.$$

Denoting $P_1 = Q_1^{-1}$ and $P_2 = Q_2^{-1}$, completes the proof. \square

Remark 3.4.2. Lemma 3.4.1 will be used in the stability analysis. The lyapunov function defined in ξ space has a nice decoupling form which allows the adaptive law design to drive the output of the state predictor (companion model) to the real system while maintaining the internal stability of the predictor. Because of the decoupling between y and z , the z dynamic will not be driven to increase. Instead, z will be self stabilized according to lyapunov stability theory.

Let

$$\alpha = \lambda_{\max}(\Lambda^{-\top}P\Lambda^{-1})\Delta^2 \tag{3.4.3}$$

where $\Delta = \frac{2\|\Lambda^{-\top}PB_m\|(L(\gamma_x)\gamma_x+B)}{\lambda_{\min}(\Lambda^{-\top}Q\Lambda^{-1})} + \frac{2\|\Lambda^{-\top}P\|B_\sigma}{\lambda_{\min}(\Lambda^{-\top}Q\Lambda^{-1})}$, γ_x is a positive constant, and $L(\gamma_x)$

is a Lipschitz constant. Consider the inverse of Λ as

$$\Lambda^{-1} = \begin{bmatrix} \varrho_1 & \varrho_2 \end{bmatrix}, \quad (3.4.4)$$

where ϱ_1 represents the first column of Λ^{-1} , and ϱ_2 represents the remaining columns.

Further, let

$$\begin{aligned} \varrho_3(s) &= C_1(s) \mathbf{1}_1^\top \begin{bmatrix} B_m & B_{um} \end{bmatrix}^{-1} \\ &\quad + C_2(s) M \mathbf{1}_3 \begin{bmatrix} B_m & B_{um} \end{bmatrix}^{-1} \end{aligned} \quad (3.4.5)$$

where and $\mathbf{1}_3 = \begin{bmatrix} 0^{(n-p) \times p} & \mathbf{I}^{(n-p) \times (n-p)} \end{bmatrix}$.

The norm of ϱ_4 is given by

$$\begin{aligned} \|\varrho_4\| &= \|(s\mathbb{I} - A_m)^{-1} B_m\|_{\mathcal{L}_1} \|\varrho_3\|_{\mathcal{L}_1} \\ &\quad + \|(s\mathbb{I} - A_m)^{-1}\|_{\mathcal{L}_1}. \end{aligned} \quad (3.4.6)$$

Letting

$$\mathbf{1}_1^\top e^{\Lambda A_m \Lambda^{-1} t} = \begin{bmatrix} \eta_{y_0}(t) & \bar{\eta}_{y_0}^\top(t) \end{bmatrix}, \quad (3.4.7)$$

$$\mathbf{1}_3 e^{\Lambda A_m \Lambda^{-1} t} = \begin{bmatrix} \eta_{z_0}(t) & \bar{\eta}_{z_0}(t) \end{bmatrix}, \quad (3.4.8)$$

where $\eta_{y_0}(t) \in \mathbb{R}^q$ and $\bar{\eta}_{y_0}^\top \in \mathbb{R}^{n-q}$ contain the first q and $q+1$ to n elements of the row vector $\mathbf{1}_1^\top e^{\Lambda A_m \Lambda^{-1} t}$ respectively, $\eta_{z_0}(t) \in \mathbb{R}^{(n-q) \times 1}$ and $\bar{\eta}_{z_0} \in \mathbb{R}^{(n-q) \times (n-q)}$ contain the first q and $q+1$ to n columns of the matrix $\mathbf{1}_3 e^{\Lambda A_m \Lambda^{-1} t}$ respectively. We further

introduce the following functions

$$\beta_{y_0}(T) = \max_{t \in [0, T]} |\eta_{y_0}(t)|, \quad \bar{\beta}_{y_0}(T) = \max_{t \in [0, T]} \|\bar{\eta}_{y_0}(t)\|, \quad (3.4.9)$$

$$\beta_{z_0}(T) = \max_{t \in [0, T]} \|\eta_{z_0}(t)\|, \quad \bar{\beta}_{z_0}(T) = \max_{t \in [0, T]} \|\bar{\eta}_{z_0}(t)\|. \quad (3.4.10)$$

Let

$$\eta_1(T) = \int_0^T \|\mathbf{1}_1^\top \varsigma(T - \tau)\| d\tau, \quad (3.4.11)$$

$$\eta_2(T) = \int_0^T |\mathbf{1}_1^\top \varsigma(T - \tau) B_m| d\tau, \quad (3.4.12)$$

where T is any positive constant and $\varsigma(T - \tau) = e^{\Lambda A_m \Lambda^{-1}(T - \tau)} \Lambda$, and further define

$$\begin{aligned} \nu(T) = & \|\phi(T)\| \sqrt{\frac{\alpha}{\lambda_{\max}(P_2)}} + \eta_1(T) B_\sigma \\ & + \eta_2(T) (L(\gamma_x) \gamma_x + B), \end{aligned} \quad (3.4.13)$$

where $\phi(T) \in \mathbb{R}^{n-1}$ is a vector, which consists of 2 to n elements of $\mathbf{1}_1^\top e^{\Lambda A_m \Lambda^{-1} T}$, and P_2 is positive definite.

Let

$$\beta_1(T) = \max_{t \in [0, T]} \int_0^t \|\mathbf{1}_1^\top \varsigma(t - \tau)\| d\tau, \quad (3.4.14)$$

$$\beta_2(T) = \max_{t \in [0, T]} \int_0^t |\mathbf{1}_1^\top \varsigma(t - \tau) B_m| d\tau, \quad (3.4.15)$$

$$\beta_3(T) = \max_{t \in [0, T]} \int_0^t \|S\varsigma(t - \tau)\| d\tau, \quad (3.4.16)$$

$$\beta_4(T) = \max_{t \in [0, T]} \int_0^t \|S\varsigma(t - \tau) B_m\| d\tau, \quad (3.4.17)$$

$$\beta_5(T) = \max_{t \in [0, T]} \int_0^t |\mathbf{1}_1^\top \varsigma(t - \tau) \Phi(T) \varphi(T) \mathbf{1}_1| d\tau, \quad (3.4.18)$$

$$\beta_6(T) = \max_{t \in [0, T]} \int_0^t \|S\varsigma(t - \tau) \Phi(T) \varphi(T) \mathbf{1}_1\| d\tau, \quad (3.4.19)$$

where $\varphi(T) = e^{\Lambda A_m \Lambda^{-1} T}$, and further define

$$\begin{aligned} \gamma_{\tilde{y}} &= \beta_{y_0}(T) \nu(T) + \bar{\beta}_{y_0}(T) \sqrt{\frac{\alpha}{\lambda_{max}(P_2)}} \\ &\quad + \beta_5(T) \nu(T) + \beta_1(T) B_\sigma \\ &\quad + \beta_2(T) (L(\gamma_x) \gamma_x + B), \end{aligned} \quad (3.4.20)$$

$$\begin{aligned} \gamma_{\tilde{z}} &= \beta_{z_0}(T) \nu(T) + \bar{\beta}_{z_0}(T) \sqrt{\frac{\alpha}{\lambda_{max}(P_2)}} \\ &\quad + \beta_6(T) \nu(T) + \beta_3(T) B_\sigma \\ &\quad + \beta_4(T) (L(\gamma_x) \gamma_x + B). \end{aligned} \quad (3.4.21)$$

For the proof of stability and uniform performance bounds, the choices of $C_1(s)$, $C_2(s)$, and integration step T together with system dynamics need to ensure that

there exists γ_x such that

$$\begin{aligned}
& \|\varrho_1\|\gamma_{\tilde{y}}(T) + \|\varrho_2\|\gamma_{\tilde{z}} + \|\varrho_4\|\|A_m\Lambda^{-1}\mathbf{1}_1\|\nu(T) \\
& + \|\varrho_4\|(\|B_m\|(L(\gamma_x)\gamma_x + B) + B_\sigma) \\
& + \|\varrho_4\|\|A_m\Lambda^{-1}\|\sqrt{\frac{\alpha}{\lambda_{max}(P_2)}} + \|\bar{r}_{t'}\|_{\mathcal{L}_\infty} < \gamma_x
\end{aligned} \tag{3.4.22}$$

3.5 Analysis of \mathcal{L}_1 Adaptive Controller

In this section, we analyze the performance bounds of the \mathcal{L}_1 adaptive controller. Let $\tilde{x}(t) = \hat{x}(t) - x(t)$. The error dynamics between (3.2.1) and (3.3.1) are

$$\dot{\tilde{x}}(t) = A_m\tilde{x}(t) + \hat{\sigma}(t) - B_m f(x, t) - \sigma(t), \tag{3.5.1}$$

$$\tilde{y}(t) = C_m^\top \tilde{x}(t), \quad \tilde{y}(0) = 0. \tag{3.5.2}$$

Considering the following state transformation

$$\tilde{\xi} = \Lambda \tilde{x}, \tag{3.5.3}$$

it follows from (3.5.2) that

$$\begin{aligned}
\dot{\tilde{\xi}}(t) &= \Lambda A_m \Lambda^{-1} \tilde{\xi}(t) + \Lambda \hat{\sigma}(t) - \Lambda B_m f(x, t) \\
&\quad - \Lambda \sigma(t),
\end{aligned} \tag{3.5.4}$$

$$\tilde{y}(t) = \tilde{\xi}_1(t), \tag{3.5.5}$$

where $\tilde{\xi}_1(t)$ is the first q element of $\tilde{\xi}(t)$.

Theorem 3.5.1. *Given the system in (3.2.1) and the \mathcal{L}_1 adaptive controller in (3.3.1), (3.3.3) and (3.3.7) subject to (3.4.22), if $x(0) < \gamma_x$, and $\hat{x}(0)$ in the output predictor is chosen such that $\tilde{z}^\top(0)P_2\tilde{z}(0) \leq \alpha$, then*

$$\|\tilde{y}\|_{\mathcal{L}_\infty} \leq \gamma_{\tilde{y}}(T), \quad (3.5.6)$$

$$\|\tilde{z}\|_{\mathcal{L}_\infty} \leq \gamma_{\tilde{z}}, \quad (3.5.7)$$

$$\|x\|_{\mathcal{L}_\infty} < \gamma_x, \quad (3.5.8)$$

$$\|u\|_{\mathcal{L}_\infty} < \gamma_u, \quad (3.5.9)$$

$\bar{\gamma}_{\tilde{y}}(T)$ and $\gamma_{\tilde{z}}$ are introduced in (3.4.20) and (3.4.21) respectively, γ_x is a positive constant, and

$$\begin{aligned} \gamma_u &= \|\varrho_3\|_{\mathcal{L}_1}\|A_m\Lambda^{-1}\mathbf{1}_1\|\nu(T) + \|k_g r_{t'}\|_{\mathcal{L}_\infty} \\ &\quad + \|\varrho_3\|_{\mathcal{L}_1}\|A_m\Lambda^{-1}\|\sqrt{\frac{\alpha}{\lambda_{\max}(P_2)}} \\ &\quad + \|\varrho_3\|_{\mathcal{L}_1}\|B_m\|(L(\gamma_x)\gamma_x + B) + \|\varrho_3\|_{\mathcal{L}_1}B_\sigma \end{aligned} \quad (3.5.10)$$

Proof. Since $x(0) < \gamma_x$ and $x(t)$ is continuous, then assuming the opposite implies that there exists t' such that

$$x(t') = \gamma_x, \quad (3.5.11)$$

while

$$\|x_{t'}\|_{\mathcal{L}_\infty} \leq \gamma_x. \quad (3.5.12)$$

At first, we prove that for all $iT < t'$, we have

$$\|\tilde{y}(iT)\| \leq \nu(T), \quad (3.5.13)$$

$$\tilde{z}^\top(iT)P_2\tilde{z}(iT) \leq \alpha. \quad (3.5.14)$$

We prove the bounds in (3.5.13) and (3.5.14) by induction. At the beginning, when $t = 0$, we have

$$\|\tilde{y}(0)\| = 0 \leq \nu(T), \quad (3.5.15)$$

$$\tilde{z}^\top(0)P_2\tilde{z}(0) \leq \alpha. \quad (3.5.16)$$

where P_2 is positive definite. In the next step, we will prove that if (3.5.13) and (3.5.14) hold at time jT , then they also hold at time $(j+1)T$.

It follows from (3.5.4) that

$$\begin{aligned} \tilde{\xi}((j+1)T) &= e^{\Lambda A_m \Lambda^{-1}T} \tilde{\xi}(jT) \\ &\quad + \int_0^T \varsigma(T-\tau) \hat{\sigma}(jT) d\tau \\ &\quad - \int_0^T \varsigma(T-\tau) B_m f(x, jT + \tau) d\tau \\ &\quad - \int_0^T \varsigma(T-\tau) \sigma(jT + \tau) d\tau. \end{aligned} \quad (3.5.17)$$

Since

$$\tilde{\xi}(jT) = \begin{bmatrix} \tilde{y}(jT) \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \tilde{z}(jT) \end{bmatrix}, \quad (3.5.18)$$

equation (3.5.17) can be written as

$$\tilde{\xi}((j+1)T) = \chi((j+1)T) + \zeta((j+1)T), \quad (3.5.19)$$

where

$$\begin{aligned} \chi((j+1)T) &= e^{\Lambda A_m \Lambda^{-1} T} \begin{bmatrix} \tilde{y}(jT) \\ 0 \end{bmatrix} \\ &+ \int_0^T \varsigma(T-\tau) \hat{\sigma}(jT) d\tau, \end{aligned} \quad (3.5.20)$$

$$\begin{aligned} \zeta((j+1)T) &= e^{\Lambda A_m \Lambda^{-1} T} \begin{bmatrix} 0 \\ \tilde{z}(jT) \end{bmatrix} \\ &- \int_0^T \varsigma(T-\tau) \sigma(jT+\tau) d\tau \\ &- \int_0^T \varsigma(T-\tau) B_m f(x, jT+\tau) d\tau. \end{aligned} \quad (3.5.21)$$

Substitution of the adaptive law (3.3.3) into (3.5.20) results in

$$\chi((j+1)T) = 0. \quad (3.5.22)$$

Following from (3.5.21), consider $\zeta(t)$ as the solution of the following dynamics

$$\dot{\zeta}(t) = \Lambda A_m \Lambda^{-1} \zeta(t) - \Lambda B_m f(x, t) - \Lambda \sigma(t), \quad (3.5.23)$$

$$\zeta(jT) = \begin{bmatrix} 0 \\ \tilde{z}(jT) \end{bmatrix}, \quad t \in [jT, (j+1)T]. \quad (3.5.24)$$

Consider the following function

$$V(t) = \zeta^\top(t) \Lambda^{-\top} P \Lambda^{-1} \zeta(t) \quad (3.5.25)$$

over $t \in [jT, (j+1)T]$. Note that Λ is non-singular and P is positive definite, $\Lambda^{-\top} P \Lambda^{-1}$ is positive definite, and therefore $V(t)$ is a positive definite function. It follows from Lemma 3.9.3 and (3.5.24) that

$$V(\zeta(jT)) = \tilde{z}^\top(jT) P_2 \tilde{z}(jT) \leq \alpha. \quad (3.5.26)$$

Following from (3.5.23) over $t \in [jT, (j+1)T]$, we obtain the derivative of $V(t)$

$$\begin{aligned} \dot{V}(t) &= -\zeta^\top(t) \Lambda^{-\top} Q \Lambda^{-1} \zeta(t) \\ &\quad - 2\zeta^\top(t) \Lambda^{-\top} P B_m f(x, t) \\ &\quad - 2\zeta^\top(t) \Lambda^{-\top} P \sigma(t). \end{aligned} \quad (3.5.27)$$

It follows from Assumption 3.2.1 and (3.5.12) that

$$|f(x, t)| \leq L(\gamma_x) \gamma_x + B. \quad (3.5.28)$$

From Assumption 3.2.2 and (3.5.28), we can further derive the upper bound of $\dot{V}(t)$ over $t \in [jT, (j+1)T]$

$$\begin{aligned} \dot{V}(t) &\leq -\lambda_{\min}(\Lambda^{-\top} Q \Lambda^{-1}) \|\zeta(t)\|^2 \\ &\quad + 2 \|\zeta(t)\| \|\Lambda^{-\top} P B_m\| (L(\gamma_x) \gamma_x + B) \\ &\quad + 2 \|\zeta(t)\| \|\Lambda^{-\top} P\| B_\sigma. \end{aligned} \quad (3.5.29)$$

If

$$V(t) \geq \alpha, \quad (3.5.30)$$

then, from (3.5.25) and the definition of α we have

$$\begin{aligned} \|\zeta(t)\| &\geq \sqrt{\frac{\alpha}{\lambda_{\max}(\Lambda^{-\top} P \Lambda^{-1})}} \\ &\geq \frac{2\|\Lambda^{-\top} P B_m\|(L(\gamma_x)\gamma_x + B)}{\lambda_{\min}(\Lambda^{-\top} Q \Lambda^{-1})} \\ &\quad + \frac{2\|\Lambda^{-\top} P\|B_\sigma}{\lambda_{\min}(\Lambda^{-\top} Q \Lambda^{-1})}, \end{aligned} \quad (3.5.31)$$

which together with (3.5.29) yields

$$\dot{V}(t) \leq 0. \quad (3.5.32)$$

It follows from (3.5.26), (3.5.30) and (3.5.32) that

$$V(t) \leq \alpha, \quad \forall t \in [jT, (j+1)T]. \quad (3.5.33)$$

Using the result of Lemma 3.9.3 in together with (3.5.33), we can derive that

$$\tilde{z}^\top((j+1)T)P_2\tilde{z}((j+1)T) \leq \alpha, \quad (3.5.34)$$

which implies that (3.5.14) holds for $(j+1)T$.

It follows from (3.5.18), (3.5.19), (3.5.21) and (3.5.22) that

$$\begin{aligned}
\tilde{y}((j+1)T) &= \mathbf{1}_1^\top \zeta((j+1)T) \\
&= \mathbf{1}_1^\top e^{\Lambda A_m \Lambda^{-1} T} \begin{bmatrix} 0 \\ \tilde{z}(jT) \end{bmatrix} \\
&\quad - \mathbf{1}_1^\top \int_0^T \varsigma(T-\tau) B_m f(x, jT+\tau) d\tau \\
&\quad - \mathbf{1}_1^\top \int_0^T \varsigma(T-\tau) \sigma(jT+\tau) d\tau, \tag{3.5.35}
\end{aligned}$$

By using definitions in (3.4.11), (3.4.12) and (3.4.13), we arrive at the following upper bound

$$\begin{aligned}
\|\tilde{y}((j+1)T)\| &\leq \|\phi(T)\| \|\tilde{z}(jT)\| + \eta_1(T) B_\sigma \\
&\quad + \eta_2(T) (L(\gamma_x) \gamma_x + B) \\
&\leq \nu(T), \tag{3.5.36}
\end{aligned}$$

This confirms the upper bound in (3.5.13) holds for $(j+1)T$. Hence, (3.5.13) and (3.5.14) hold for all $iT \leq t'$.

For all $iT + t \leq t'$, where $0 \leq t \leq T$, it follows from (3.5.4) that

$$\begin{aligned}
\tilde{y}(iT + t) &= \mathbf{1}_1^\top e^{\Lambda A_m \Lambda^{-1} t} \tilde{\xi}(iT) \\
&\quad + \mathbf{1}_1^\top \int_0^t \varsigma(t - \tau) \hat{\sigma}(iT) d\tau \\
&\quad - \mathbf{1}_1^\top \int_0^t \varsigma(t - \tau) \sigma(iT + \tau) d\tau \\
&\quad - \mathbf{1}_1^\top \int_0^t \varsigma(t - \tau) B_m f(x, iT + \tau) d\tau, \tag{3.5.37}
\end{aligned}$$

$$\begin{aligned}
\tilde{z}(iT + t) &= \mathbf{1}_3 e^{\Lambda A_m \Lambda^{-1} t} \tilde{\xi}(iT) + \mathbf{1}_3 \int_0^t \varsigma(t - \tau) \hat{\sigma}(iT) d\tau \\
&\quad - \mathbf{1}_3 \int_0^t \varsigma(t - \tau) \sigma(iT + \tau) d\tau \\
&\quad - \mathbf{1}_3 \int_0^t \varsigma(t - \tau) B_m f(x, iT + \tau) d\tau. \tag{3.5.38}
\end{aligned}$$

Considering (3.5.13)-(3.5.14) and recalling the definitions of $\beta_{y_0}(T)$, $\bar{\beta}_{y_0}(T)$, $\beta_1(T)$, $\beta_2(T)$, and $\beta_5(T)$ in (3.4.9), (3.4.14), (3.4.15) and (3.4.18), we arrive at the following upper bound

$$\begin{aligned}
\|\tilde{y}(iT + t)\| &\leq \beta_{y_0}(T) \nu(T) + \bar{\beta}_{y_0}(T) \sqrt{\frac{\alpha}{\lambda_{max}(P_2)}} \\
&\quad + \beta_5(T) \nu(T) + \beta_1(T) B_\sigma \\
&\quad + \beta_2(T) (L(\gamma_x) \gamma_x + B). \tag{3.5.39}
\end{aligned}$$

Similarly, by introducing (3.4.10), (3.4.16), (3.4.17), and (3.4.19), we have

$$\begin{aligned}
\|\tilde{z}(iT + t)\| &\leq \beta_{z_0}(T) \nu(T) + \bar{\beta}_{z_0}(T) \sqrt{\frac{\alpha}{\lambda_{max}(P_2)}} \\
&\quad + \beta_6(T) \nu(T) + \beta_3(T) B_\sigma \\
&\quad + \beta_4(T) (L(\gamma_x) \gamma_x + B). \tag{3.5.40}
\end{aligned}$$

Then, for all $t \in [0, t']$, it follows from (3.5.39) - (3.5.40) and definitions of $\gamma_{\tilde{y}}(T)$ in (3.4.20) and $\gamma_{\tilde{z}}$ in (3.4.21) that

$$\|\tilde{y}(t)\| \leq \gamma_{\tilde{y}}(T), \quad (3.5.41)$$

$$\|\tilde{z}(t)\| \leq \gamma_{\tilde{z}}. \quad (3.5.42)$$

Since $x(t) = \hat{x}(t) - \tilde{x}(t)$, we have

$$\|x(t)\| \leq \|\hat{x}(t)\| + \|\tilde{x}(t)\|. \quad (3.5.43)$$

It follows from (3.5.3) that

$$\tilde{x}(t) = \Lambda^{-1} \tilde{\xi}(t) = \Lambda^{-1} \begin{bmatrix} \tilde{y}(t) \\ \tilde{z}(t) \end{bmatrix}. \quad (3.5.44)$$

Then, the upper bound of $\tilde{x}(t)$ is written as

$$\|\tilde{x}(t)\| \leq \|\varrho_1\| |\tilde{y}(t)| + \|\varrho_2\| \|\tilde{z}(t)\|, \quad (3.5.45)$$

where ϱ_1 and ϱ_2 are introduced in (3.4.4).

Furthermore, it follows from (3.3.1) that

$$\begin{aligned} \hat{x}(s) &= (s\mathbb{I} - A_m)^{-1} B_m u(s) + (s\mathbb{I} - A_m)^{-1} \hat{\sigma}(s) \\ &\quad + (s\mathbb{I} - A_m)^{-1} x_0. \end{aligned} \quad (3.5.46)$$

Then, we arrive at following upper bound of

$$\begin{aligned}\|\hat{x}_{t'}\|_{\mathcal{L}_\infty} &\leq \|(s\mathbb{I} - A_m)^{-1}B_m a\|_{\mathcal{L}_1}\|u_{t'}\|_{\mathcal{L}_\infty} \\ &\quad + \|(s\mathbb{I} - A_m)^{-1}\|_{\mathcal{L}_1}\|\hat{\sigma}_{t'}\|_{\mathcal{L}_\infty} + \|r_0\|_{\mathcal{L}_\infty}\end{aligned}\tag{3.5.47}$$

over $t \in [0, t']$, where $r_0(s) = (s\mathbb{I} - A_m)^{-1}x_0$.

From (3.3.6) and (3.3.7), we arrive at the following upper bound

$$\|u_{t'}\|_{\mathcal{L}_\infty} \leq \|\varrho_3\|_{\mathcal{L}_1} \|\hat{\sigma}_{t'}\|_{\mathcal{L}_\infty} + \|k_g r_{t'}\|_{\mathcal{L}_\infty}, \tag{3.5.48}$$

where ϱ_3 is defined in (3.4.5). Substitution of (3.5.48) into (3.5.47) yields

$$\|\hat{x}_{t'}\|_{\mathcal{L}_\infty} \leq \|\varrho_4\| \|\hat{\sigma}_{t'}\|_{\mathcal{L}_\infty} + \|\bar{r}_{t'}\|_{\mathcal{L}_\infty}, \tag{3.5.49}$$

where $\|\varrho_4\|$ is defined in (3.4.6), and $\|\bar{r}_t\|_{\mathcal{L}_\infty} = \|(s\mathbb{I} - A_m)^{-1}B_m a k_g\|_{\mathcal{L}_1}\|r_t\|_{\mathcal{L}_\infty} + \|r_0\|_{\mathcal{L}_\infty}$.

From the adaptive law in (3.3.3) and (3.3.4), we have

$$\int_0^T \varsigma(T - \tau) \hat{\sigma}(iT) d\tau + e^{\Lambda A_m \Lambda^{-1} T} \mathbf{1}_1 \tilde{y}(iT) = 0. \tag{3.5.50}$$

We further obtain that

$$\begin{aligned}\tilde{y}(iT) &= (1 - \mathbf{1}_1^\top e^{\Lambda A_m \Lambda^{-1} T} \mathbf{1}_1) \tilde{y}(iT) \\ &\quad - \int_0^T \mathbf{1}_1^\top \varsigma(T - \tau) \hat{\sigma}(iT) d\tau \\ &= - \int_0^T \mathbf{1}_1^\top \varsigma(T - \tau) A_m \Lambda^{-1} \mathbf{1}_1 \tilde{y}(iT) d\tau \\ &\quad - \int_0^T \mathbf{1}_1^\top \varsigma(T - \tau) \hat{\sigma}(iT) d\tau.\end{aligned}\tag{3.5.51}$$

It follows from (3.5.35) that

$$\begin{aligned}
\tilde{y}(iT) &= \int_0^T \mathbf{1}_1^\top \varsigma(T - \tau) A_m \Lambda^{-1} \begin{bmatrix} 0 \\ \tilde{z}((i-1)T) \end{bmatrix} d\tau \\
&\quad - \int_0^T \mathbf{1}_1^\top \varsigma(T - \tau) B_m f(x, (i-1)T + \tau) d\tau \\
&\quad - \int_0^T \mathbf{1}_1^\top \varsigma(T - \tau) \sigma((i-1)T + \tau) d\tau. \tag{3.5.52}
\end{aligned}$$

Following from the relation of (3.5.51) and (3.5.52), (3.5.28) and Assumption 5.2.3, we arrive at the following upper bound

$$\begin{aligned}
\|\hat{\sigma}(iT)\| &\leq \|A_m \Lambda^{-1} \mathbf{1}_1\| \|\tilde{y}(iT)\| \\
&\quad + \|A_m \Lambda^{-1}\| \|\tilde{z}((i-1)T)\| \\
&\quad + \|B_m\| (L(\gamma_x) \gamma_x + B) + B_\sigma. \tag{3.5.53}
\end{aligned}$$

Note that $\hat{\sigma}(t)$ is piece-wise continuous. Following from (3.5.13) and (3.5.14), we obtain

$$\begin{aligned}
\|\hat{\sigma}_{t'}\|_{\mathcal{L}_\infty} &\leq \|A_m \Lambda^{-1} \mathbf{1}_1\| \nu(T) \\
&\quad + \|A_m \Lambda^{-1}\| \sqrt{\frac{\alpha}{\lambda_{\max}(P_2)}} \\
&\quad + \|B_m\| (L(\gamma_x) \gamma_x + B) + B_\sigma. \tag{3.5.54}
\end{aligned}$$

Substitution of (3.5.54) into (3.5.49) yields

$$\begin{aligned}
\|\hat{x}_{t'}\|_{\mathcal{L}_\infty} &\leq \|\varrho_4\| \|A_m \Lambda^{-1} \mathbf{1}_1\| \nu(T) + \|\bar{r}_{t'}\|_{\mathcal{L}_\infty} \\
&\quad + \|\varrho_4\| \|A_m \Lambda^{-1}\| \sqrt{\frac{\alpha}{\lambda_{\max}(P_2)}} \\
&\quad + \|\varrho_4\| (\|B_m\| (L(\gamma_x) \gamma_x + B) + B_\sigma). \tag{3.5.55}
\end{aligned}$$

Finally, following from (3.5.43), (3.5.45) and (3.5.55), we obtain the upper bound of $x(t)$

$$\begin{aligned}
\|x_{t'}\|_{\mathcal{L}_\infty} &\leq \|\varrho_1\| \gamma_{\bar{y}}(T) + \|\varrho_2\| \gamma_{\bar{z}} + \|\bar{r}_{t'}\|_{\mathcal{L}_\infty} \\
&\quad + \|\varrho_4\| \|A_m \Lambda^{-1} \mathbf{1}_1\| \nu(T) \\
&\quad + \|\varrho_4\| \|A_m \Lambda^{-1}\| \sqrt{\frac{\alpha}{\lambda_{\max}(P_2)}} \\
&\quad + \|\varrho_4\| (\|B_m\| (L(\gamma_x) \gamma_x + B) + B_\sigma). \tag{3.5.56}
\end{aligned}$$

By considering stability condition, (3.5.56) becomes

$$\|x_{t'}\|_{\mathcal{L}_\infty} < \gamma_x, \tag{3.5.57}$$

which contradicts (3.5.12) and proves (3.5.8). Following from (3.5.8), (3.5.41) and (3.5.42), we further obtain results (3.5.6) and (3.5.7). It follows from (3.5.8), (3.5.48)

and (3.5.54) that

$$\begin{aligned}
\|u\|_{\mathcal{L}_\infty} &< \|\varrho_3\|_{\mathcal{L}_1}\|A_m\Lambda^{-1}\mathbf{1}_1\|\nu(T) + \|k_g r_{t'}\|_{\mathcal{L}_\infty} \\
&+ \|\varrho_3\|_{\mathcal{L}_1}\|A_m\Lambda^{-1}\|\sqrt{\frac{\alpha}{\lambda_{\max}(P_2)}} \\
&+ \|\varrho_3\|_{\mathcal{L}_1}\|B_m\|(L(\gamma_x)\gamma_x + B) + \|\varrho_3\|_{\mathcal{L}_1}B_\sigma \\
&< \gamma_u
\end{aligned} \tag{3.5.58}$$

which proves (3.5.9) and concludes the proof. \square

3.6 Simulation

The dearth of publicly available control-centric nonlinear models for very flexible aircraft led the authors to revisit the model from Ref. [107]. The details of the physical wind tunnel model can be found in Ref. [166]. The range of vertical motion is constrained to ± 12 inches, hard stop to hard stop, and the model angle of attack is limited by loading considerations to single digits in degrees. The wind tunnel model is instrumented with accelerometers along the spar, strain gauges at the root and mid-spar, a rate gyro at the wing tip, and a rate gyro and accelerometers at the tunnel attachment point.

The mathematical model is linear and includes rigid body translational and rotational displacements and velocities (z, θ, w, q) , as well as twelve flexible modes. The flexible modes are represented by generalized displacements, η_i , and velocities, $\dot{\eta}_i$. There were 10 test points for the flexible vehicle wind tunnel model, each cor-

responding to a different dynamic pressure ranging from 30 psf to 130 psf shown in Table 1, and an associated linear model described above. The simulation consists of linear models with third order actuator dynamics, typical of aeroservoelastic models for each of the control surfaces in the simulation.

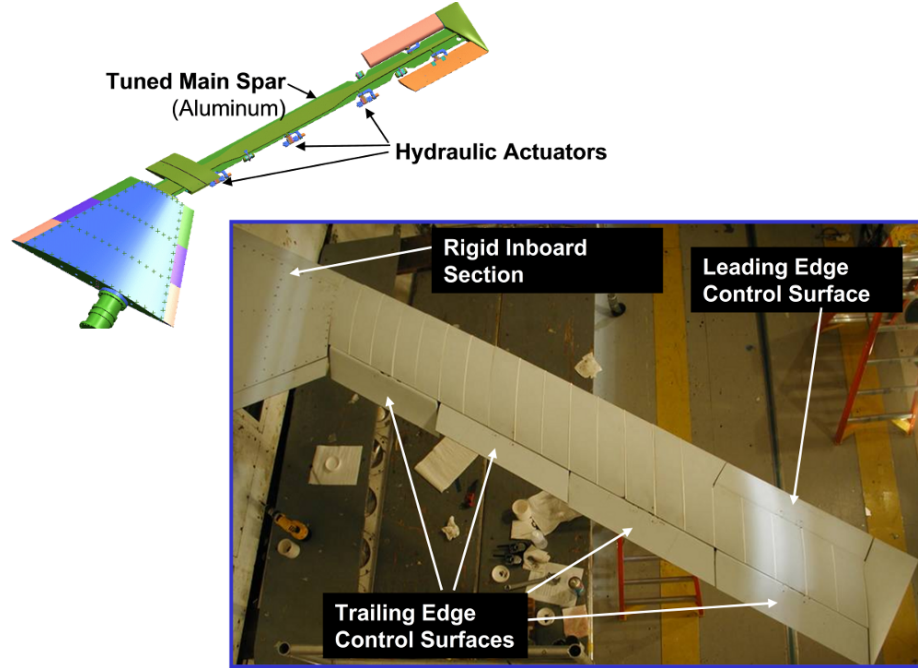


FIGURE 3.6.1: Wind tunnel model

Table 1. Dynamic pressure for different test points.

Test Point Index	1	2	3	4	5	6	7	8	9	10
Dynamic Pressure (psf)	30	35	40	45	50	60	70	80	90	130

The general structure of the flexible model is given by

$$\begin{bmatrix} \dot{x}_r \\ \dot{x}_e \\ \dot{x}_{lag} \\ \dot{x}_\delta \end{bmatrix} = \begin{bmatrix} A_r & A_r^e & A_r^{lag} & A_r^\delta \\ A_e^r & A_e & A_e^{lag} & A_e^\delta \\ A_{lag}^r & A_{lag}^e & A_{lag} & A_{lag}^\delta \\ 0 & 0 & 0 & A_\delta \end{bmatrix} \begin{bmatrix} x_r \\ x_e \\ x_{lag} \\ x_\delta \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ B_{cmd} \end{bmatrix} \delta_{cmd} \quad (3.6.1)$$

where x_r represents rigid body position and rates, x_e represents elastic mode deflections and rates, x_{lag} represents aerodynamic lag states, and x_δ represents actuator states. For control design purposes, the model is residualized to eliminate lag states and then is further reduced by eliminating higher frequency flexible modes. Furthermore, the actuator dynamics are neglected, and as a result, the control design model is reduced from 112 to 12 state variables consisting of 2 rigid (half-span) modes and 4 flexible modes. Thus, the model used for design has the format

$$\begin{bmatrix} \dot{\bar{x}}_r \\ \dot{\bar{x}}_e \end{bmatrix} = \begin{bmatrix} A_r & A_r^e \\ A_e^r & A_e \end{bmatrix} \begin{bmatrix} \bar{x}_r \\ \bar{x}_e \end{bmatrix} + \begin{bmatrix} A_\delta^r \\ A_\delta^e \end{bmatrix} \delta$$

$$\bar{y} = \begin{bmatrix} C_r & C_e \end{bmatrix} \begin{bmatrix} \bar{x}_r \\ \bar{x}_e \end{bmatrix} + D\delta \quad (3.6.2)$$

where $\delta = [\delta_{LE}, \delta_{TE1}, \delta_{TE2}, \delta_{TE3}, \delta_{TE4}, \delta_D]^T$ are the control surface command. The last element δ_D is the gust load input command. y is the output of sensors described in Ref [166]. In order to improve stability and add damping into the system, a LQG control structure is chosen for the baseline controller. In order to create a model structure compatible with \mathcal{L}_1 output feedback several of the sensor measurements, combined in a way that isolates flexible wing modes, are added to the system as in Eq. (3.6.2) as states. In addition, the integrator on position is augmented to the

system in Eq. (3.6.2), resulting in a new system structure given by

$$\begin{aligned}\dot{x} &= A_m x + B_m u \\ y &= C_m x\end{aligned}\tag{3.6.3}$$

where $u = \delta$ is a vector of control inputs (one leading edge and four trailing edges), y is a vector of sensor and integrator outputs, and $A_m \in R^{16 \times 16}$, $B_m \in R^{16 \times 5}$, $C_m \in R^{8 \times 16}$ are matrices with appropriate dimensions. A baseline LQG controller, using the information A_m, B_m, C_m at one testing point is designed to make the wrapped system stable and provide damping to the flexible mode. Figure 3.6.3 shows the performance of \mathcal{L}_1 adaptive controller designed and tested at 130 psf. Then the testing pressure is changed from 130 psf to 30 psf and still use the design parameter in 130 psf. The system remains stable as shown in figure 3.6.4. Figure 3.6.5 and 3.6.6 show the performance when gust load channel is activated.

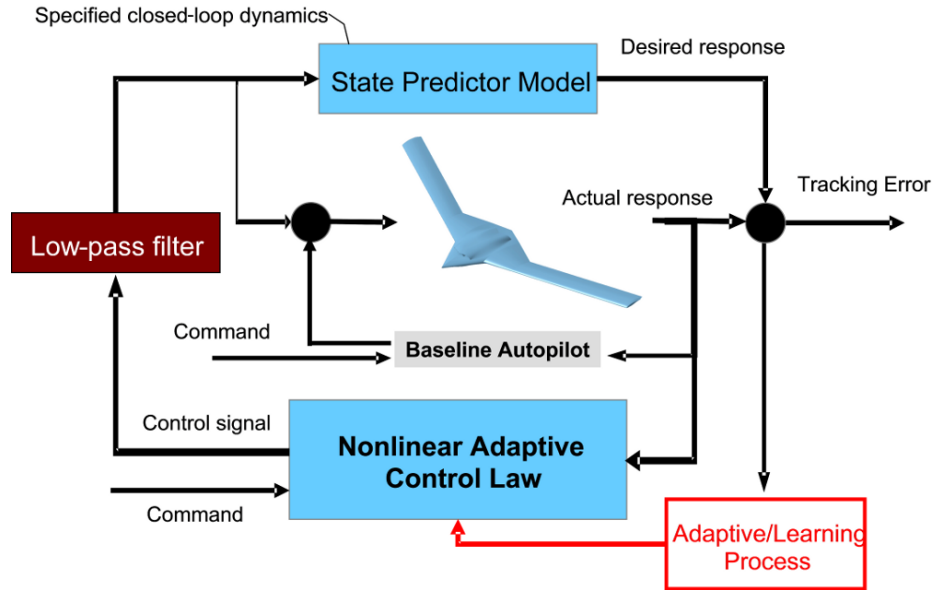
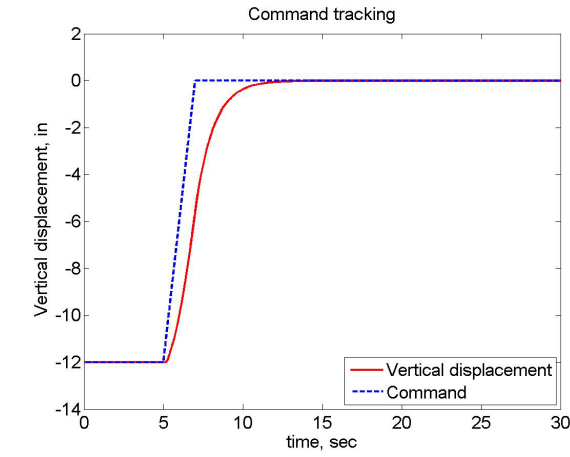
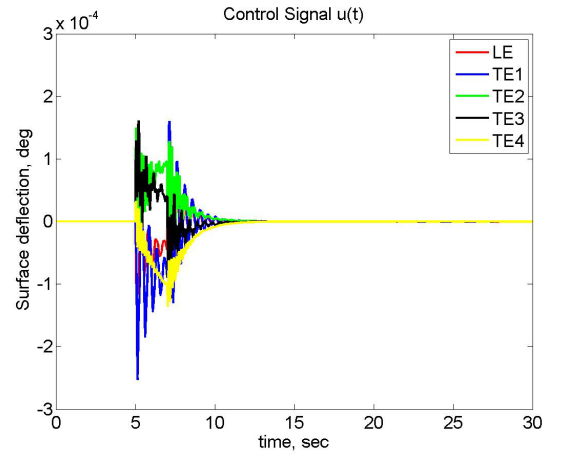


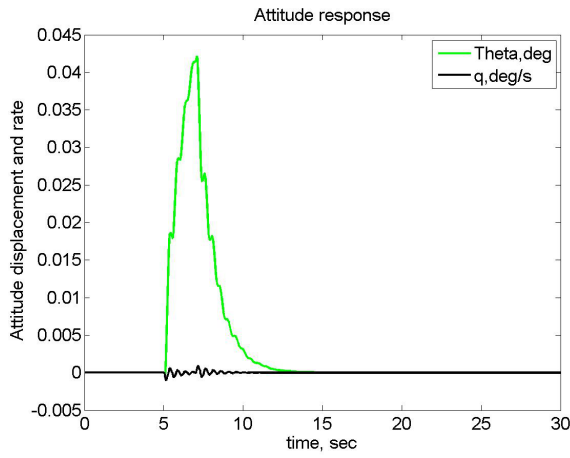
FIGURE 3.6.2: \mathcal{L}_1 adaptive control architecture for stability augmentation



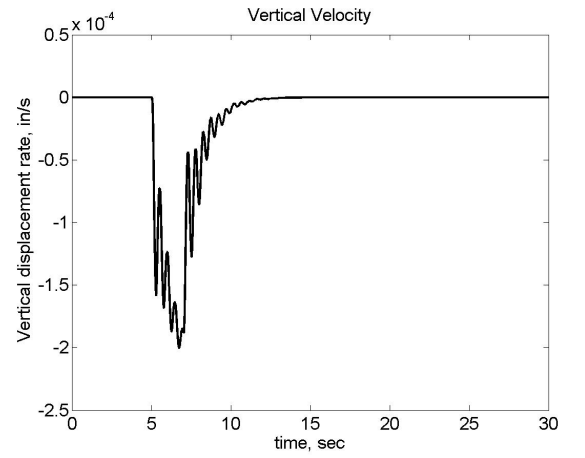
(A) Vertical tunnel position (in)



(B) Control deflection (deg)

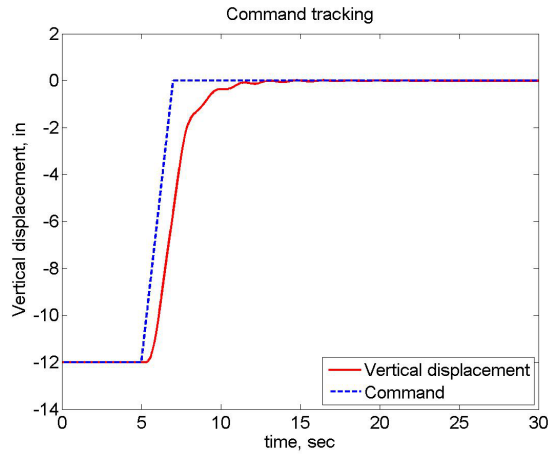


(C) Model attitude θ (deg) and q (deg/s)

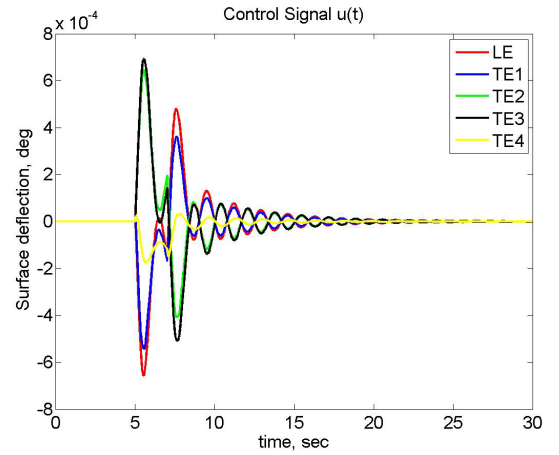


(D) Vertical velocity V_z (in/s)

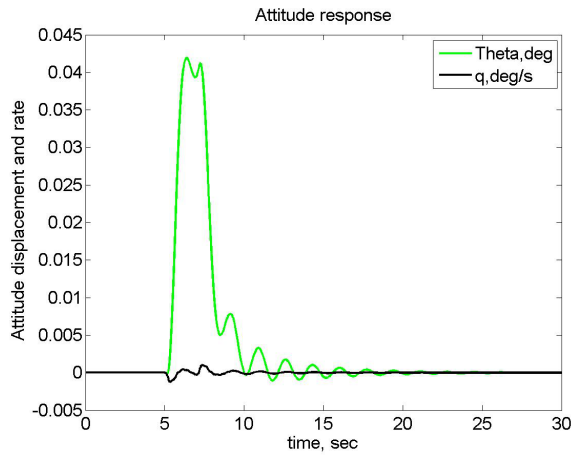
FIGURE 3.6.3: Measured output responses for the \mathcal{L}_1 adaptive controller (130 psf) evaluated at 130 psf (ir=11)



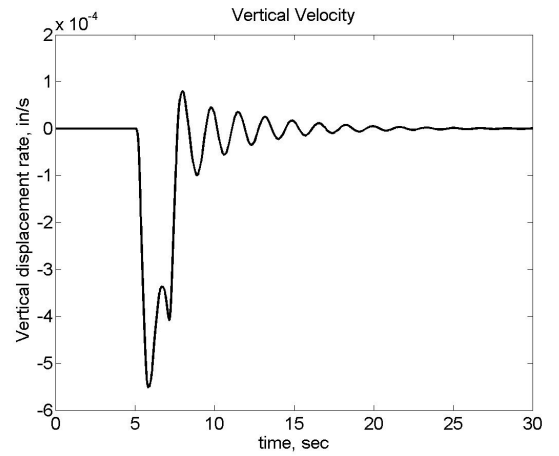
(A) Vertical tunnel position (in)



(B) Control deflection (deg)

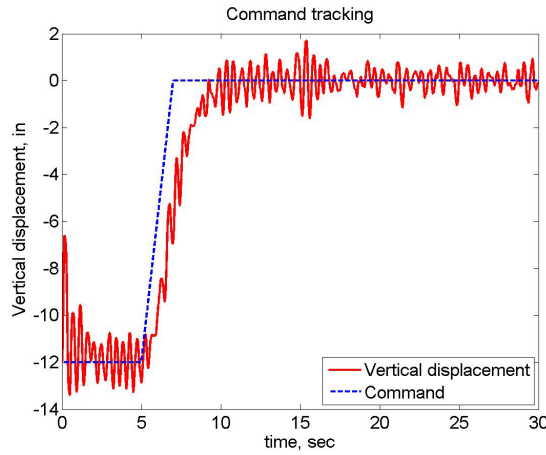


(C) Model attitude θ (deg) and q (deg/s)

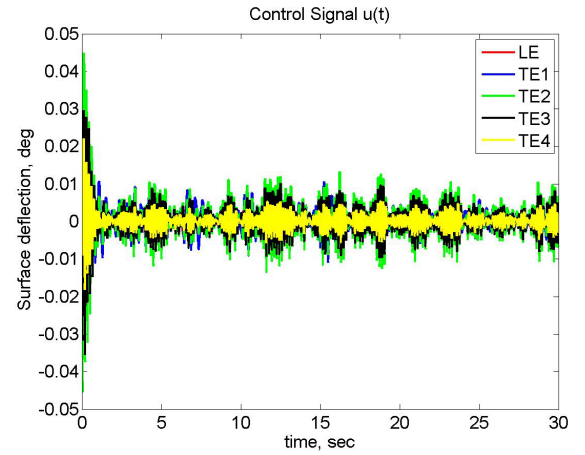


(D) Vertical velocity V_z (in/s)

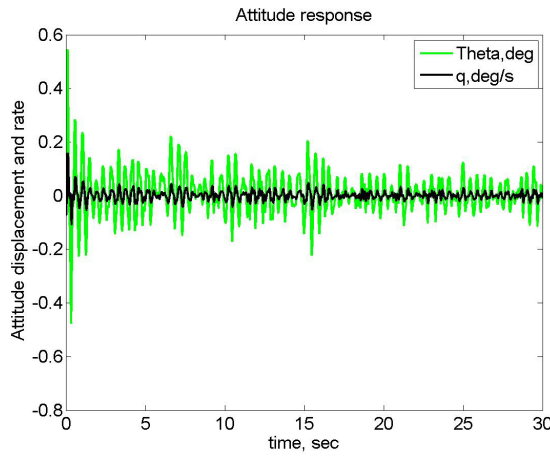
FIGURE 3.6.4: Measured output responses for the \mathcal{L}_1 adaptive controller (130 psf) evaluated at 30 psf (ir=2)



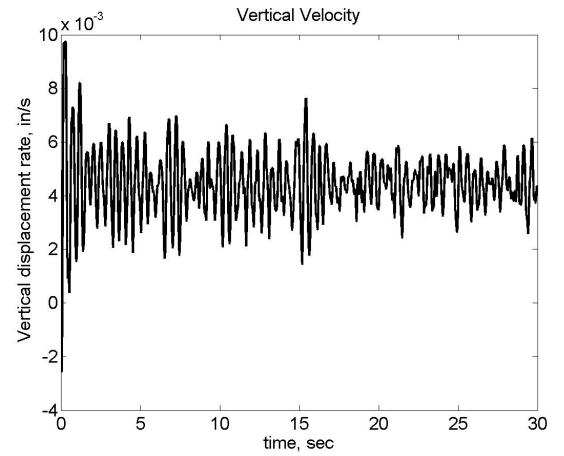
(A) Vertical tunnel position (in)



(B) Control deflection (deg)

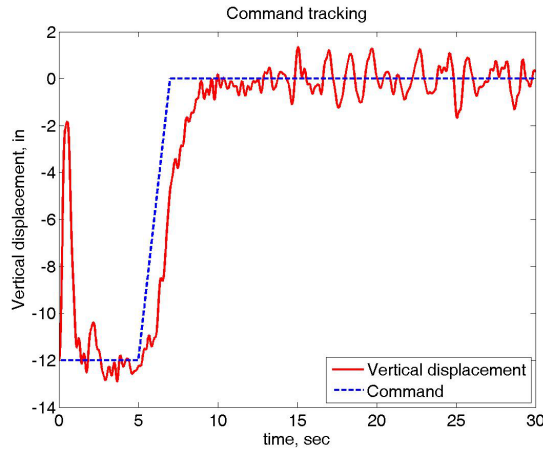


(C) Model attitude θ (deg) and q (deg/s)

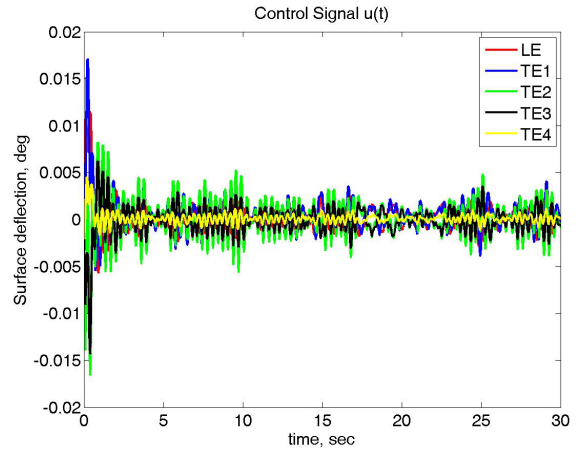


(D) Vertical velocity V_z (in/s)

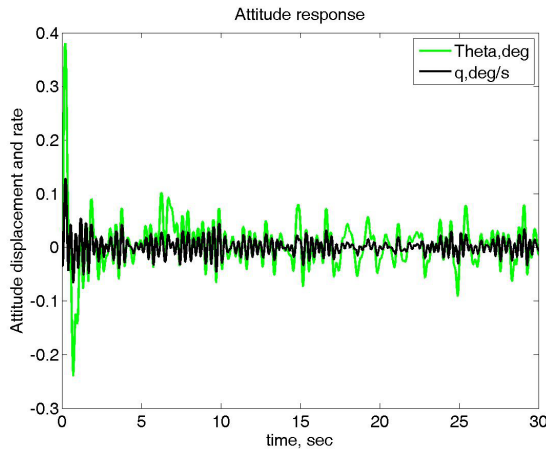
FIGURE 3.6.5: Measured output responses for the \mathcal{L}_1 adaptive controller (130 psf) evaluated at 130 psf (ir=11) with gust load



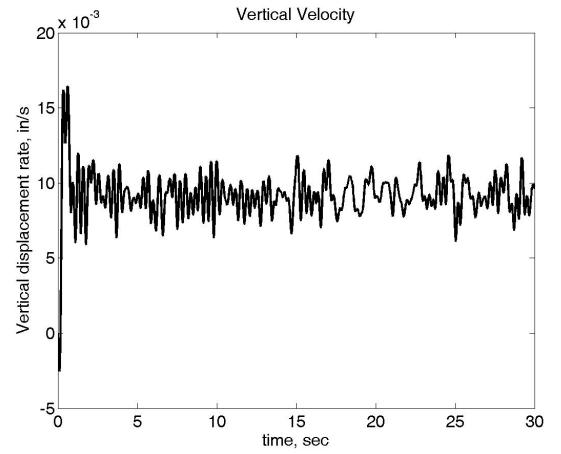
(A) Vertical tunnel position (in)



(B) Control deflection (deg)



(C) Model attitude θ (deg) and q (deg/s)



(D) Vertical velocity V_z (in/s)

FIGURE 3.6.6: Measured output responses for the \mathcal{L}_1 adaptive controller (130 psf) evaluated at 30 psf ($ir=2$) with gust load

3.7 Conclusions

This paper presents an extension of a recent \mathcal{L}_1 adaptive output feedback controller in [164] to an advanced, highly flexible aircraft configuration. Firstly, the theoretical analysis in [164] is extended to a MIMO setup, which fits the flexible aircraft control problem. Secondly, the control law design in [164] is modified to only compensate the unmatched disturbance and uncertainties in steady state. In other words, here the Dc gain inversion is used instead of dynamic inversion. Hence, it has little influence on the flexible modes, which are highly uncertain and unpredictable in different operating points. An independent LQG controller is designed to stabilize these flexible modes. Thirdly, the control law in [105] is modified slightly to have an independent state predictor for the adaptive law and to remove some parameter tuning restrictions. The numerical simulation results for a very flexible aircraft configuration illustrate the algorithm's performance and quantify the achievable performance bound as a function of the control computer CPU for this class of vehicles.

3.8 Acknowledgement

The authors are thankful to Franton Lin sharing the valuable advice.

3.9 Appendix

Definition 3.9.1. [138] For a signal $\xi(t)$, $t \geq 0$, $\xi \in \mathbb{R}^n$, its \mathcal{L}_∞ and truncated \mathcal{L}_∞ norms are $\|\xi\|_{\mathcal{L}_\infty} = \max_{i=1,\dots,n} (\sup_{\tau \geq 0} |\xi_i(\tau)|)$, $\|\xi_t\|_{\mathcal{L}_\infty} = \max_{i=1,\dots,n} (\sup_{0 \leq \tau \leq t} |\xi_i(\tau)|)$, where ξ_i is the i^{th} component of ξ .

Definition 3.9.2. [138] The \mathcal{L}_1 -norm of a stable proper SISO system is defined $\|H(s)\|_{\mathcal{L}_1} = \int_0^\infty |h(t)|dt$, where $h(t)$ is the impulse response of $H(s)$.

Lemma 3.9.3. [138] *For a stable proper multi-input multi-output (MIMO) system $H(s)$ with input $r(t) \in \mathbb{R}^m$ and output $x(t) \in \mathbb{R}^n$, we have $\|x_t\|_{\mathcal{L}_\infty} \leq \|H(s)\|_{\mathcal{L}_1} \|r_t\|_{\mathcal{L}_\infty}, \forall t \geq 0$.*

Chapter 4

Adaptive Controller for Unknown Disturbance Rejection in Linear Time-Varying Systems

4.1 Introduction

Many practical control problems, such as the flight control system can be modeled as control of Linear Time-Varying (LTV) system in the presence of unmatched disturbances. In many practical problems we cannot measure all of the disturbances, or we may choose not to measure some of them due to technical or economic reasons [138]. Therefore, it is important to develop disturbance estimation techniques. [139] presents the method for the unknown constant disturbance. [140] demonstrates the adaptive disturbance rejection method for sinusoidal disturbances with unknown frequency. Disturbance observer [141] can estimate the disturbance by inverting the plant dynamic. Extended State Observer (ESO) [142] defines the disturbance as an extended state and designs a state observer. By choosing high observer gain, ESO can track the time-varying disturbance signal with

arbitrary desired eigenvalues. Kalman-based extended state observer can be extended to time-varying systems.

\mathcal{L}_1 adaptive control architecture is a recently developed method for unknown time-varying uncertain systems [39]. Through fast and robust adaptation, complex nonlinear uncertain systems can be controlled with verifiable performance without enforcing persistent excitation, applying gain-scheduling, or resorting to high-gain feedback. \mathcal{L}_1 adaptive control has applications in a wide range of systems in aerospace, flight control and industrial areas with unknown uncertainties.

The disturbance estimation method in \mathcal{L}_1 adaptive control architecture uses a different methodology. The \mathcal{L}_1 adaptive law is the controller for the state predictor and is designed to make the output of the state predictor converge to the output of the plant. Generally, any aggressive controller design can serve as the adaptive law in \mathcal{L}_1 estimation. By utilizing the results of control theory, the estimation error bound of \mathcal{L}_1 estimation can be characterized.

Stability is another challenge for the control of time-varying system, [167], [168], [169]. Eigenvalue assignment is a powerful tool used to stabilize the LTV system. [170] proposed an arbitrary eigenvalue assignment for linear time-varying multivariable control systems. [171] proposed a method which can assign time-varying eigenvalue trajectories to the LTV systems, which is based on PD-spectral theory for multivariable linear time-varying systems [172]. [76] take the effects of disturbance into consideration.

Control law design for tracking and disturbance rejection is also a challenge of the LTV system. Gain scheduling is a prevailing approach for practical LTV systems [173]. There are similar concepts like "non-minimum phase" in LTV systems, which introduces some performance limitation [174].

The incentive of this chapter is to adapt the eigenvalue assignment method into the control law design of the adaptive controller. This method is applied to LTI system in [175] and further extended to LTV system. in [76]. However, the analysis in [76] does not include

the adaptive law, sampling time and over all stability condition.

The work of this chapter can be summarized in the following 3 parts: 1. The piecewise adaptive law for disturbance estimation in \mathcal{L}_1 adaptive controller is incorporated and extended to linear time-varying systems. By only using the current information $A(iT)$ at discrete time, the disturbance is estimated with a bounded error which has been analyzed in lemma 1 of this chapter. 2. The eigenvalue assignment approach for linear time-varying systems is adopted in the control law design in order to stabilize the system. This control law design firstly transforms the linear-time-varying system into an control conical space with time-invariant A and B matrices and a time-varying $C(t)$ matrix. The estimated disturbances are decoupled into the matched and unmatched disturbances. The matched disturbances are directly compensated in the control channel. The effects of unmatched disturbances on system output are compensated by the control law to achieve a reasonable performance. 3. The tracking performance is analyzed using a lyapunov-based method. The overall control algorithm in this chapter only uses the current system information (causal), unlike other optimal control methods which use future information [176].

The paper is organized as follows. Section 4.2 gives the problem formulation. In Section 4.3, the adaptive control architecture is introduced. Stability and uniform performance bounds are presented in Section 4.4. In Section 4.5, simulation results are presented, while Section 4.7 concludes the chapter.

4.2 Problem Formulation

Consider the following time-varying single-input single-output (SISO) system:

$$\dot{x}(t) = A(t)x(t) + b(t)u(t) + \sigma(t), \quad y(t) = c(t)^T x(t), \quad (4.2.1)$$

where $x \in \mathbb{R}^n$ is the system state vector (measurable), $u \in \mathbb{R}$ is the control signal. $b(t)$ and $c(t) \in \mathbb{R}^n$ are known vectors, $A(t), B(t)$ is known time-varying matrices with $(n+1)$ th order derivative, $(A(t), b(t))$ is uniformly controllable and $\sigma(t)$ is the unknown time-varying bounded disturbance subject to the following assumption:

Assumption 4.2.1. There exist constants $b_\sigma > 0$ and $b_{d\sigma} > 0$

$$\|\sigma(t)\| \leq b_\sigma, \quad \|\dot{\sigma}(t)\| \leq b_{d\sigma} \quad (4.2.2)$$

hold uniformly in $t \geq 0$.

Assumption 4.2.2. There exist constants $b_A > 0, b_{dA} > 0, b_b > 0, b_{db} > 0, b_c > 0, b_{dc} > 0$

$$\|A(t)\| \leq b_A, \quad \left\| \frac{d^k A(t)}{dt^k} \right\| \leq b_{dA}, \quad (4.2.3)$$

$$\|b(t)\| \leq b_b, \quad \left\| \frac{d^k b(t)}{dt^k} \right\| \leq b_{db}, \quad (4.2.4)$$

$$\|c(t)\| \leq b_c, \quad \left\| \frac{d^k c(t)}{dt^k} \right\| \leq b_{dc}, \quad (4.2.5)$$

hold uniformly in $t \geq 0$ and $k = 1, 2, \dots, n$.

Assumption 4.2.3. $A(t)$ is nonsingular uniformly in $t \geq 0$.

The control objective is to design an adaptive state feedback control signal $u(t)$ such that the system output $y(t)$ tracks reference input $r(t)$ and compensates the disturbance with a reasonable performance, which can be tuned by the bandwidth of the low-pass filter as well as the eigenvalues assigned to the control law. Without loss of generality, we suppose

there exist constants B_r and B_{dr} such that:

$$\|r\|_{\mathcal{L}_\infty} < B_r, \quad \|\dot{r}\|_{\mathcal{L}_\infty} < B_{dr}. \quad (4.2.6)$$

4.3 Adaptive Controller

In this section, a novel control law design is introduced into \mathcal{L}_1 adaptive architecture. The entire controller consists of state predictor, adaptive law and control law.

State Predictor: The state predictor, together with the adaptive law, is designed for fast estimation of the unknown disturbance $\sigma(t)$. The following state predictor is considered:

$$\dot{\hat{x}}(t) = \hat{A}(t)\hat{x}(t) + \hat{b}(t)u(t) + \hat{\sigma}(t), \quad \hat{y}(t) = c(t)^T \hat{x}(t), \quad \hat{x}(0) = x_0, \quad (4.3.1)$$

where $\hat{A}(t)$, $\hat{b}(t)$, $\hat{\sigma}(t)$ are piece-wise constant parameters updated at time iT as following.

$$\hat{A}(t) = \hat{A}(iT) = A(iT), \quad \hat{b}(t) = \hat{b}(iT) = b(iT), \quad \hat{\sigma}(t) = \hat{\sigma}(iT), \quad t \in [iT, (i+1)T), \quad (4.3.2)$$

where T is the sampling interval and $\hat{\sigma}(iT)$ is defined in following adaptive laws:

Adaptive Laws:

The piece-wise constant adaptive laws are used in here in order to make $\hat{x}(t)$ track $x(t)$.

The update law for $\hat{\sigma}(iT)$ is given by

$$\begin{aligned} \hat{\sigma}(t) &= \hat{\sigma}(iT), t \in [iT, (i+1)T), \\ \hat{\sigma}(iT) &= \left[\int_0^T e^{A(iT)(T-\tau)} d\tau \right]^{-1} e^{A(iT)T} \tilde{x}(iT), i = 0, 1, 2, 3... \end{aligned} \quad (4.3.3)$$

where $\tilde{x}(t) = \hat{x}(t) - x(t)$ is the estimation error vector of the state predictor.

Control Laws:

The control law is a controller of the state predictor designed to make $\hat{y}(t)$ track $r(t)$, during which time the disturbance $\hat{\sigma}(t)$ will be inhibited following the desired eigenvalues.

The eigenvalue assignment method [154] is used to compensate the disturbance while following the rate of the eigenvalue we assigned. Before the control law, the following definition is required:

From the definition in [177], there exists a nonsingular controllability matrix for a uniformly controllable system

$$\tilde{\mathcal{C}}(t) = \begin{bmatrix} \delta_1(t) & \delta_2(t) & \cdots & \delta_n(t) \end{bmatrix} \quad (4.3.4)$$

where $\delta_{i+1}(t) = A(t)\delta_i(t) - \dot{\delta}_i(t)$ with $\delta_1(t) = b(t)$. There also exists an inverse matrix $D(t) = \tilde{\mathcal{C}}^{-1}(t)$ such that

$$D(t)\tilde{\mathcal{C}}(t) = \begin{bmatrix} D_{n-1}(t) \\ D_{n-2}(t) \\ \vdots \\ D_0(t) \end{bmatrix} \begin{bmatrix} \delta_1(t) & \delta_2(t) & \cdots & \delta_n(t) \end{bmatrix} = \mathbb{I}, \quad (4.3.5)$$

where $D_i(t)$ is the $(i+1)$ th row vector of $D(t)$. Define

$$\bar{D}(t) = \begin{bmatrix} \bar{D}_{n-1}(t) \\ \bar{D}_{n-2}(t) \\ \vdots \\ \bar{D}_0(t) \end{bmatrix}, \quad (4.3.6)$$

with $\bar{D}_0(t) = D_0(t)$, and

$$\bar{D}_p(t) = \dot{\bar{D}}_{p-1}(t) + \bar{D}_{p-1}(t)A(t), \quad (4.3.7)$$

for each $p = 1, 2, \dots, n$. [154] shows that there exists set of linear combination coefficients $\alpha_1(t), \alpha_2(t), \dots, \alpha_n(t)$ such that

$$\bar{D}_n(t) = \begin{bmatrix} \alpha_1(t) & \alpha_2(t) & \dots & \alpha_n(t) \end{bmatrix} \begin{bmatrix} \bar{D}_0(t) \\ \bar{D}_1(t) \\ \vdots \\ \bar{D}_{n-1}(t) \end{bmatrix} \quad (4.3.8)$$

and it is not hard to verify that

$$\dot{\bar{D}}(t) + \bar{D}(t)A(t) = \begin{bmatrix} \bar{D}_1(t) \\ \bar{D}_2(t) \\ \vdots \\ \bar{D}_n(t) \end{bmatrix}. \quad (4.3.9)$$

(4.3.9) will be used in lemma (4.4.4) to derive the control canonical form.

Define

$$k(t) \triangleq \left(- \begin{bmatrix} \alpha_1(t) & \alpha_2(t) & \dots & \alpha_n(t) \end{bmatrix} + \begin{bmatrix} d_1 & d_2 & \dots & d_n \end{bmatrix} \right) \begin{bmatrix} \bar{D}_0(t) \\ \bar{D}_1(t) \\ \vdots \\ \bar{D}_{n-1}(t) \end{bmatrix}, \quad (4.3.10)$$

where $\begin{bmatrix} d_1 & d_2 & \dots & d_n \end{bmatrix}$ are calculated according to the desired eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ by the following equation

$$s^n + d_n s^{n-1} + d_{n-2} s^{n-2} + \dots + d_1 = (s + \lambda_1)(s + \lambda_2) \dots (s + \lambda_n). \quad (4.3.11)$$

The control signal $u_1(t)$ is defined as follows:

$$u_1(t) = -k(t)x(t), \quad (4.3.12)$$

$$W \triangleq \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -d_1 & -d_2 & \cdots & \cdots & -d_n \end{bmatrix}, \quad (4.3.13)$$

where $\begin{bmatrix} d_1 & d_2 & \cdots & d_n \end{bmatrix}$ are defined in (4.3.11), $\hat{\sigma}_r(t)$ is defined as follows:

$$\hat{\sigma}_r(s) \triangleq C(s)\hat{\sigma}(s), \quad (4.3.14)$$

where the low-pass filter array, $C(s)$, is designed to make the disturbance estimation $\hat{\sigma}(t)$ smooth and also to recover the time-delay margin of the closed loop system [38].

Define

$$C(s) \triangleq \begin{bmatrix} tf(s) & 0 & 0 & \cdots & 0 \\ 0 & tf(s) & 0 & \cdots & 0 \\ 0 & 0 & tf(s) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & tf(s) \end{bmatrix} \quad (4.3.15)$$

where $tf(s)$ is a strict proper stable low pass filter to make $\hat{\sigma}_r$ smooth and have bounded

derivative. Define

$$\Lambda(t) = \begin{bmatrix} \Lambda_0(t) \\ \Lambda_1(t) \\ \vdots \\ \Lambda_{n-1}(t) \end{bmatrix} = \bar{D}(t)\sigma(t), \quad \hat{\Lambda}(t) = \begin{bmatrix} \Lambda_0(t) \\ \Lambda_1(t) \\ \vdots \\ \Lambda_{n-1}(t) \end{bmatrix} = \bar{D}(t)\hat{\sigma}_r(t). \quad (4.3.16)$$

If followed from assumption 4.2.1 that there exists constant $b_\Lambda > 0$ such that

$$\|\Lambda(t)\| \leq b_\Lambda \quad (4.3.17)$$

where b_Λ can be defined as

$$b_\Lambda = \max_{t>0} \|\bar{D}(t)\| b_\sigma \quad (4.3.18)$$

where $\Lambda(t)$ and $\hat{\Lambda}(t)$ can be further decomposed into the matched channel $\Lambda_m(t)$, $\hat{\Lambda}_m(t)$ and unmatched channel $\Lambda_{um}(t)$, $\hat{\Lambda}_{um}(t)$ as follows:

$$\Lambda_m(t) = \begin{bmatrix} 0 & 0 & \cdots & \Lambda_n(t) \end{bmatrix}^T, \quad \Lambda_{um}(t) = \begin{bmatrix} \Lambda_0(t) & \Lambda_1(t) & \cdots & 0 \end{bmatrix}^T \quad (4.3.19)$$

$$\hat{\Lambda}_m(t) = \begin{bmatrix} 0 & 0 & \cdots & \hat{\Lambda}_n(t) \end{bmatrix}^T, \quad \hat{\Lambda}_{um}(t) = \begin{bmatrix} \hat{\Lambda}_0(t) & \hat{\Lambda}_1(t) & \cdots & 0 \end{bmatrix}^T. \quad (4.3.20)$$

Let

$$u_2(t) = k_g(t)(r(t) + c(t)D^{-1}(t)W^{-1}\hat{\Lambda}_{um}(t)) - \hat{\Lambda}_n(t), \quad (4.3.21)$$

where \bar{D} , W and $\Lambda(t)$ are defined in (4.3.6), (4.3.16) and (4.3.13). $k_g(t)$ is designed as

$$k_g(t) = \frac{-1}{c(t)D^{-1}(t)W^{-1}\begin{bmatrix} 0 & \dots & 0 & 1 \end{bmatrix}^T}. \quad (4.3.22)$$

The overall control law is

$$u(t) = u_1(t) + u_2(t). \quad (4.3.23)$$

4.4 Analysis of Adaptive Controller

Lemma 4.4.1. *Given*

$$\int_0^T e^{A_m(T-\tau)} \zeta(\tau) d\tau = \varepsilon, \quad (4.4.1)$$

where $\zeta(\tau) \in \mathbb{R}^n$ is a continuous bounded signal $\|\zeta(\tau)\| \leq b_\zeta$ and has bounded derivative $\|\dot{\zeta}(\tau)\| \leq b_{d\zeta}$, $\varepsilon \in \mathbb{R}^n$ is an arbitrary vector. Then

$$\|\zeta(\tau)\| \leq \chi_2(T) (\|\varepsilon\|/T + \sqrt{n}Td_q), \quad (4.4.2)$$

where $\chi_0(T)$, $\chi_1(T)$ and d_q are defined as

$$\chi_0(T) = \max_{\tau \in [0, T)} \|e^{A_m T} A_m e^{-A_m \tau}\|, \quad (4.4.3)$$

$$\chi_1(T) = \max_{\tau \in [0, T)} \|e^{A_m T} e^{A_m \tau}\|, \quad (4.4.4)$$

$$\chi_2(T) = \max_{\tau \in [0, T)} \|e^{-A_m T} e^{A_m \tau}\|, \quad (4.4.5)$$

$$d_q = \chi_0(T)b_\zeta + \chi_1(T)b_{d\zeta}. \quad (4.4.6)$$

Proof. Let

$$q(\tau) = e^{A_m(T-\tau)}\zeta(\tau). \quad (4.4.7)$$

The boundary of $\dot{q}(\tau)$ can be derived as following:

$$\begin{aligned} \dot{q}(\tau) &= \frac{d}{d\tau} \left(e^{A_m(T-\tau)}\zeta(\tau) \right) \\ &= e^{A_m T} \frac{d}{d\tau} \left(e^{-A_m \tau} \right) \zeta(\tau) + e^{A_m T} e^{-A_m \tau} \frac{d}{d\tau} (\zeta(\tau)) \\ &= -e^{A_m T} A_m e^{-A_m \tau} \zeta(\tau) + e^{A_m T} e^{-A_m \tau} \dot{\zeta}(\tau). \end{aligned} \quad (4.4.8)$$

Using (4.4.8) and the definition of $\chi_0(T)$ together with the bounds of $\zeta(\tau)$ and $\dot{\zeta}(\tau)$, we have

$$\|\dot{q}(\tau)\| \leq \chi_0(T)b_\zeta + \chi_1(T)b_{d\zeta}. \quad (4.4.9)$$

It follows from the definition of d_q in (4.4.6) that

$$\|\dot{q}(\tau)\| \leq d_q \quad (4.4.10)$$

Eqn (4.4.1) can be written as

$$\int_0^T q(\tau) d\tau = \varepsilon. \quad (4.4.11)$$

Consider the i th element of equation in (4.4.11), we have

$$\int_0^T q_i(\tau) d\tau = \varepsilon_i. \quad (4.4.12)$$

Since $q(\tau)$ is continuous, for any T , there exists $t_i \in [0, T]$ such that

$$q_i(t_i) = \varepsilon_i/T, \quad (4.4.13)$$

the variable $q_i(\tau)$ can be rewritten as follows

$$q_i(\tau) = q_i(t_i) + \int_{t_i}^{\tau} \dot{q}_i(\alpha) d\alpha \quad (4.4.14)$$

Since $\tau - t_i \leq T$. Applying the results in (4.4.10) we have

$$q_i(\tau) < \varepsilon_i/T + Td_q \quad (4.4.15)$$

As a result

$$\|q(\tau)\| \leq \|\varepsilon\|/T + \sqrt{n}Td_q \quad (4.4.16)$$

It follows from the definition in (4.4.7)

$$\begin{aligned} \zeta(\tau) &= \left[e^{A_m(T-\tau)} \right]^{-1} q(\tau) \\ &= e^{-A_m T} e^{A_m \tau} q(\tau) \end{aligned} \quad (4.4.17)$$

The bound of (4.4.17) can be further derived as.

$$\begin{aligned} \|\zeta(\tau)\| &\leq \chi_2(T) \|q(\tau)\| \\ &\leq \chi_2(T) (\|\varepsilon\|/T + \sqrt{n}Td_q) \end{aligned} \quad (4.4.18)$$

□

For any positive constants ρ_x , ρ_u and T , we define the following constants.

$$\eta(T) = \int_0^T \sqrt{\lambda_{\max}((e^{A_m(T-\tau)})^\top e^{A_m(T-\tau)})} d\tau, \quad (4.4.19)$$

$$\gamma_0(T, \rho_x, \rho_u) = \eta(T)(b_{dA}T\sqrt{n}\rho_x + b_{db}T\sqrt{n}\rho_u + b_\sigma), \quad (4.4.20)$$

$$\eta_1(T) = \left[e^{A(iT)T} - \mathbb{I} \right]^{-1} A(iT)e^{A(iT)T}, \quad (4.4.21)$$

$$\eta_2(T, \rho_x, \rho_u) = \sqrt{\lambda_{\max}(\eta_1(T)^\top \eta_1(T))} \gamma_0(T, \rho_x, \rho_u), \quad (4.4.22)$$

$$\begin{aligned} \gamma_1(T, \rho_x, \rho_u) = & Tb_{d\sigma} + \sqrt{n}T(\chi_0(T)b_\zeta(T, \rho_x, \rho_u) + \chi_1(T)b_{d\sigma}) + \\ & \chi_2(T)((\eta(T)/T)(b_A\gamma_0(T, \rho_x, \rho_u) + Tb_{dA}\sqrt{n}\rho_x + Tb_{db}\sqrt{n}\rho_u)) \end{aligned} \quad (4.4.23)$$

$$b_\zeta(T, \rho_x, \rho_u) = b_\sigma + \eta_2(T, \rho_x, \rho_u), \quad (4.4.24)$$

$$\begin{aligned} b_{d\tilde{x}}(T, \rho_x, \rho_u) = & \sqrt{\lambda_{\max}(A(iT)^\top A(iT))} \gamma_0(T, \rho_x, \rho_u) + b_{dA}T\rho_x + \\ & b_{db}T\rho_u + \gamma_0(T, \rho_x, \rho_u), \end{aligned} \quad (4.4.25)$$

$$\gamma_2(T, \rho_x, \rho_u) = \gamma_0(T, \rho_x, \rho_u) + Tb_{d\tilde{x}}(T, \rho_x, \rho_u). \quad (4.4.26)$$

Define

$$\tilde{\sigma}(t) = \hat{\sigma}(t) - \sigma(t). \quad (4.4.27)$$

Lemma 4.4.2. *Considering the SISO system described in (4.2.1) together with the state predictor (4.3.1), adaptive law (4.3.3) and control law (4.3.23), if the truncated \mathcal{L}_∞ norm $\|x_{t_1}\|_{\mathcal{L}_\infty} \leq \rho_x$, $\|u_{t_1}\|_{\mathcal{L}_\infty} \leq \rho_u$ for any time $t_1 \geq 0$, then*

$$\|\tilde{\sigma}_{t_1}\| \leq \gamma_1(T, \rho_x, \rho_u), \quad \|\tilde{x}_{t_1}\| \leq \gamma_2(T, \rho_x, \rho_u).$$

Proof. In time interval $[iT, (i+1)T)$, $t \in [0, T)$, the system matrices $A(iT+t)$ and $b(iT+t)$

can be written as:

$$\begin{aligned} A(iT+t) &= A(iT) + \int_{iT}^{iT+t} \dot{A}(\alpha) d\alpha, \\ b(iT+t) &= b(iT) + \int_{iT}^{iT+t} \dot{b}(\alpha) d\alpha. \end{aligned} \quad (4.4.28)$$

Substituting (4.4.28) into (4.2.1), we have

$$\dot{x}(iT+t) = A(iT)x + \int_{iT}^{iT+t} \dot{A}(\alpha) d\alpha x + b(iT)u + \int_{iT}^{iT+t} \dot{b}(\alpha) d\alpha u + \sigma(iT+t). \quad (4.4.29)$$

Compare the dynamics of the system in (4.4.29) and (4.3.1) at time instant $iT+t$, $t \in [0, T)$ as follows

$$\dot{\tilde{x}}(iT+t) = A(iT)\tilde{x}(iT) + \hat{\sigma}(iT) - \int_{iT}^{iT+t} \dot{A}(\alpha) d\alpha x - \int_{iT}^{iT+t} \dot{b}(\alpha) d\alpha u - \sigma(iT+t). \quad (4.4.30)$$

Let

$$\sigma_0(iT+t) = \int_{iT}^{iT+t} \dot{A}(\alpha) d\alpha x(iT+t) + \int_{iT}^{iT+t} \dot{b}(\alpha) d\alpha u(iT+t) + \sigma(iT+t). \quad (4.4.31)$$

It follows from (4.4.30) that

$$\begin{aligned} \tilde{x}(iT+t) &= e^{A(iT)t} \tilde{x}(iT) + \int_0^t e^{A(iT)(t-\tau)} \hat{\sigma}(iT) d\tau \\ &\quad - \int_0^t e^{A(iT)(t-\tau)} \sigma_0(iT+\tau) d\tau, \end{aligned} \quad (4.4.32)$$

which implies that

$$\begin{aligned}\tilde{x}((i+1)T) &= e^{A(iT)T}\tilde{x}(iT) + \int_0^T e^{A(iT)(T-\tau)}\hat{\sigma}(iT)d\tau - \\ &\quad \int_0^T e^{A(iT)(T-\tau)}\sigma_0(iT+\tau)d\tau.\end{aligned}\tag{4.4.33}$$

Substitution of the adaptive law (4.3.3) into (4.4.33) results in

$$\tilde{x}((i+1)T) = - \int_0^T e^{A(iT)(T-\tau)}\sigma_0(iT+\tau)d\tau.\tag{4.4.34}$$

It follows from Assumptions 4.2.1, 4.2.2 and definition in (4.4.31) that

$$\|\sigma_0(t)\| \leq b_{dA}T\sqrt{n}\rho_x + b_{db}T\sqrt{n}\rho_u + b_\sigma,\tag{4.4.35}$$

for $t \in [0, t']$. By using the definitions in (4.4.19) and (4.4.20), it follows from (4.4.34) and (4.4.35) that the following upper bound holds:

$$\begin{aligned}\|\tilde{x}((i+1)T)\| &\leq \eta(T)(b_{dA}T\sqrt{n}\rho_x + b_{db}T\sqrt{n}\rho_u + b_\sigma) \\ &\leq \gamma_0(T, \rho_x, \rho_u),\end{aligned}\tag{4.4.36}$$

for $(i+1)T \in [0, t']$.

Eqn (4.4.36) also implies

$$\|\tilde{x}(iT)\| \leq \gamma_0(T, \rho_x, \rho_u)\tag{4.4.37}$$

for all $iT \in [0, t']$. According to the choice of adaptive law (4.3.3), we have

$$e^{A(iT)T}\tilde{x}(iT) + \int_0^T e^{A(iT)(T-\tau)}\hat{\sigma}(iT)d\tau = 0.\tag{4.4.38}$$

Following from (4.4.38), we further obtain that

$$\begin{aligned}\tilde{x}(iT) &= (\mathbb{I} - e^{A(iT)T})\tilde{x}(iT) - \int_0^T e^{A(iT)(T-\tau)}\hat{\sigma}(iT)d\tau \\ &= - \int_0^T e^{A(iT)(T-\tau)}(\hat{\sigma}(iT) + A(iT)\tilde{x}(iT))d\tau.\end{aligned}\quad (4.4.39)$$

It follows from (4.4.34) that

$$\tilde{x}(iT) = - \int_0^T e^{A(iT)(T-\tau)}\sigma_0((i-1)T + \tau)d\tau. \quad (4.4.40)$$

Hence, (4.4.39) and (4.4.40) imply

$$\int_0^T e^{A(iT)(T-\tau)}\sigma_0((i-1)T + \tau)d\tau = \int_0^T e^{A(iT)(T-\tau)}(\hat{\sigma}(iT) + A(iT)\tilde{x}(iT))d\tau, \quad (4.4.41)$$

which further implies that

$$\int_0^T e^{A(iT)(T-\tau)}(\sigma_0((i-1)T + \tau) - \hat{\sigma}(iT) - A(iT)\tilde{x}(iT))d\tau = 0. \quad (4.4.42)$$

Derivation of the bound on $\hat{\sigma}(iT)$:

Using assumption 4.2.3, eqn (4.3.3) can be written as

$$\hat{\sigma}(iT) = \left[e^{A(iT)T} - \mathbb{I} \right]^{-1} A(iT)e^{A(iT)T}\tilde{x}(iT), i = 0, 1, 2, 3... \quad (4.4.43)$$

It follows from the definition of $\eta_1(T)$, $\eta_2(T)$ in (4.4.21), (4.4.22) and (4.4.36) that

$$\|\hat{\sigma}(iT)\| \leq \eta_2(T, \rho_x, \rho_u). \quad (4.4.44)$$

Let

$$\zeta(\tau) = \sigma((i-1)T + \tau) - \hat{\sigma}(iT). \quad (4.4.45)$$

It follows from assumption 4.2.1, (4.4.44) and the definition in (4.4.24) that

$$\begin{aligned} \|\zeta(\tau)\| &\leq b_\sigma + \eta_2(T, \rho_x, \rho_u) \\ &\leq b_\zeta(T, \rho_x, \rho_u). \end{aligned} \quad (4.4.46)$$

It follows from assumption 4.2.1 and definition in (4.4.45) that

$$\|\dot{\zeta}(\tau)\| \leq b_{d\sigma}. \quad (4.4.47)$$

Let

$$\varepsilon = \int_0^T e^{A_m(T-\tau)} \left(A(iT) \tilde{x}(iT) - \int_{iT}^{iT+t} \dot{A}(\alpha) d\alpha x(iT+t) - \int_{iT}^{iT+t} \dot{b}(\alpha) d\alpha u(iT+t) \right) d\tau. \quad (4.4.48)$$

We have

$$\|\varepsilon\| \leq \eta(T) (b_A \gamma_0(T, \rho_x, \rho_u) + T b_{dA} \sqrt{n} \rho_x + T b_{db} \sqrt{n} \rho_u). \quad (4.4.49)$$

Using (4.4.48), (4.4.45), eqn (4.4.42) can be rewritten as

$$\int_0^T e^{A_m(T-\tau)} \zeta(\tau) d\tau = \varepsilon \quad (4.4.50)$$

Using the results in (4.4.46), (4.4.47), (4.4.50) and it follows from lemma 4.4.1 that

$$\|\zeta(\tau)\| \leq \chi_2(T) (\|\varepsilon\| / T + \sqrt{n} T d_q). \quad (4.4.51)$$

Substituting (4.4.49) into (4.4.51), we have

$$\begin{aligned} \|\sigma((i-1)T + \tau) - \hat{\sigma}(iT)\| &\leq \sqrt{n}T (\chi_0(T)b_\zeta(T, \rho_x, \rho_u) + \chi_1(T)b_{d\sigma}) + \\ &\chi_2(T) \left((\eta(T)/T) (b_A\gamma_0(T, \rho_x, \rho_u) + Tb_{dA}\sqrt{n}\rho_x + Tb_{db}\sqrt{n}\rho_u) \right). \end{aligned} \quad (4.4.52)$$

Since $\hat{\sigma}(iT + \tau) = \hat{\sigma}(iT)$ over the interval $[iT, (i+1)T)$,

$$\begin{aligned} \tilde{\sigma}(iT + \tau) &= \sigma(iT + \tau) - \hat{\sigma}(iT + \tau) \\ &= \sigma(iT + \tau) - \sigma((i-1)T + \tau) + \sigma((i-1)T + \tau) - \hat{\sigma}(iT) \\ &= \int_{(i-1)T+\tau}^{iT+\tau} \dot{\sigma}(\alpha) d\alpha + \sigma((i-1)T + \tau) - \hat{\sigma}(iT) \end{aligned} \quad (4.4.53)$$

Using the (4.4.52) and the bound on $\dot{\sigma}(\alpha)$ in assumption 4.2.1, for all time $iT + \tau \leq t_1$, the bounds of (4.4.53) can be derived as

$$\begin{aligned} \|\tilde{\sigma}(iT + \tau)\| &\leq Tb_{d\sigma} + \sqrt{n}T (\chi_0(T)b_\zeta(T, \rho_x, \rho_u) + \chi_1(T)b_{d\sigma}) \\ &\chi_2(T) \left((\eta(T)/T) (b_A\gamma_0(T, \rho_x, \rho_u) + Tb_{dA}\sqrt{n}\rho_x + Tb_{db}\sqrt{n}\rho_u) \right) \end{aligned} \quad (4.4.54)$$

which is

$$\|\tilde{\sigma}_{t_1}\| \leq \gamma_1(T, \rho_x, \rho_u). \quad (4.4.55)$$

Eqn (4.4.30) can be written as

$$\dot{\tilde{x}}(iT + t) = A(iT)\tilde{x}(iT) - \int_{iT}^{iT+t} \dot{A}(\alpha) d\alpha x - \int_{iT}^{iT+t} \dot{b}(\alpha) d\alpha u + \tilde{\sigma}(iT + t). \quad (4.4.56)$$

Combing (4.4.55) and 4.4.56 together with the definition in (4.4.25), we have

$$\begin{aligned}
\|\dot{\tilde{x}}(iT + t)\| &\leq \sqrt{\lambda_{max}(A(iT)^\top A(iT))} \gamma_0(T, \rho_x, \rho_u) + b_{dA}T\rho_x + b_{dB}T\rho_u + \gamma_0(T, \rho_x, \rho_u) \\
&\leq b_{d\tilde{x}}(T, \rho_x, \rho_u).
\end{aligned} \tag{4.4.57}$$

For all $iT + t \leq t_1$.

$$\tilde{x}(iT + t) = \tilde{x}(iT) + \int_{iT}^{iT+t} \dot{\tilde{x}}(\alpha) d\alpha. \tag{4.4.58}$$

The bound of (4.4.58) can be further derived as

$$\|\tilde{x}(iT + t)\| \leq \gamma_0(T, \rho_x, \rho_u) + Tb_{d\tilde{x}}(T, \rho_x, \rho_u). \tag{4.4.59}$$

Using the definition in (4.4.26), we have

$$\|\tilde{x}_{t_1}\| \leq \gamma_2(T, \rho_x, \rho_u), \tag{4.4.60}$$

which completes the proof. \square

Lemma 4.4.3. *For any given bound ρ_x and ρ_u*

$$\begin{aligned}
\lim_{T \rightarrow 0} \gamma_0(T, \rho_x, \rho_u) &\rightarrow 0, \\
\lim_{T \rightarrow 0} \gamma_1(T, \rho_x, \rho_u) &\rightarrow 0, \\
\lim_{T \rightarrow 0} \gamma_2(T, \rho_x, \rho_u) &\rightarrow 0.
\end{aligned} \tag{4.4.61}$$

Proof. We can readily obtain that all these variables can be reduced to some bounded value multiplied by a time step T . When $T \rightarrow 0$, All $\gamma_0(T, \rho_x, \rho_u)$, $\gamma_1(T, \rho_x, \rho_u)$ and $\gamma_2(T, \rho_x, \rho_u)$ go to zero. \square

Define

$$L_H = \left\| \left(\frac{c^T \bar{D}(t)^{-1} \begin{bmatrix} \Lambda_0(t) & \Lambda_1(t) & \cdots & 0 \end{bmatrix}^T}{c^T \bar{D}(t)^{-1} W^{-1} \begin{bmatrix} 0 & 0 & \cdots & 1 \end{bmatrix}^T} - \begin{bmatrix} 0 & 0 & \cdots & 1 \end{bmatrix}^T \right) C(s) \right\|_{\mathcal{L}_1}. \quad (4.4.62)$$

Lemma 4.4.4. *Considering the system described in (4.2.1) together with the state predictor (4.3.1), adaptive law (4.3.3) and control law (4.3.23), if $\|x_{t_1}\|_{\mathcal{L}_\infty} \leq \rho_x$ and $\|u_{t_1}\|_{\mathcal{L}_\infty} \leq \rho_u$ for any time $t_1 \geq 0$, then*

$$\|u_{t_1}\|_{\mathcal{L}_\infty} \leq \|k_g\|_{\mathcal{L}_1} B_r + \|k\|_{\mathcal{L}_1} \rho_x + L_H (b_\sigma + \gamma_1(T, \rho_x, \rho_u)) \quad (4.4.63)$$

and

$$\begin{aligned} \|x_{t_1}\|_{\mathcal{L}_\infty} &\leq \|\bar{D}^{-1}\|_{\mathcal{L}_1} \left\| (s\mathbb{I} - W)^{-1} \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \end{bmatrix}^T \right\|_{\mathcal{L}_1} \\ &\quad \left(\|k_g\|_{\mathcal{L}_1} B_r + L_H (b_\sigma + \gamma_1(T, \rho_x, \rho_u)) \right) \\ &\quad + \|\bar{D}^{-1}\|_{\mathcal{L}_1} \left\| (s\mathbb{I} - W)^{-1} \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \end{bmatrix}^T \right\|_{\mathcal{L}_1} b_\Lambda. \end{aligned} \quad (4.4.64)$$

Proof:

$$\begin{aligned} u(t) &= k_g r(t) - k(t)x(t) - \Lambda_{n-1}(t) + \frac{c(t)^T \bar{D}(t)^{-1} \begin{bmatrix} \Lambda_0(t) & \Lambda_1(t) & \cdots & 0 \end{bmatrix}^T}{c(t)^T \bar{D}(t)^{-1} W^{-1} \begin{bmatrix} 0 & 0 & \cdots & 1 \end{bmatrix}^T} \\ &= k_g r(t) - k(t)x(t) \\ &\quad + \left(\frac{c(t)^T \bar{D}(t)^{-1} \begin{bmatrix} \Lambda_0(t) & \Lambda_1(t) & \cdots & 0 \end{bmatrix}^T}{c(t)^T \bar{D}(t)^{-1} W^{-1} \begin{bmatrix} 0 & 0 & \cdots & 1 \end{bmatrix}^T} - \begin{bmatrix} 0 & 0 & \cdots & 1 \end{bmatrix}^T \right) C(s) \end{aligned} \quad (4.4.65)$$

Hence,

$$\begin{aligned} \|u_{t_1}\|_{\mathcal{L}_\infty} &\leq \|k_g\|_{\mathcal{L}_1} B_r + \|k\|_{\mathcal{L}_1} \rho_x \\ &+ \left\| \left(\frac{c(t)^T \bar{D}(t)^{-1} \begin{bmatrix} \Lambda_0(t) & \Lambda_1(t) & \cdots & 0 \end{bmatrix}^T}{c(t)^T \bar{D}(t)^{-1} W^{-1} \begin{bmatrix} 0 & 0 & \cdots & 1 \end{bmatrix}^T} - \begin{bmatrix} 0 & 0 & \cdots & 1 \end{bmatrix}^T \right) C(s) \right\|_{\mathcal{L}_1} \|\hat{\sigma}_{t_1}\|_{\mathcal{L}_\infty} \end{aligned} \quad (4.4.66)$$

Eqn (4.4.66) can be rewritten as

$$\begin{aligned} \|u_{t_1}\|_{\mathcal{L}_\infty} &\leq \|k_g\|_{\mathcal{L}_1} B_r + \|k\|_{\mathcal{L}_1} \rho_x + L_H \|\hat{\sigma}\|_{\mathcal{L}_\infty} \\ &\leq \|k_g\|_{\mathcal{L}_1} B_r + \|k\|_{\mathcal{L}_1} \rho_x + L_H (b_\sigma + \gamma_1(T, \rho_x, \rho_u)) \end{aligned} \quad (4.4.67)$$

which completes the first part of this lemma. In the following part, we will derive the upper bound on $\|x_{t_1}\|_{\mathcal{L}_\infty}$. Substituting the control law $u(t)$ defined in (4.3.12) and $u_1(t)$ in (4.3.23) into system equation (4.2.1)

$$\dot{x}(t) = (A(t) - b(t)k(t))x(t) + b(t)u_2(t) + \sigma(t). \quad (4.4.68)$$

Define

$$z(t) = \bar{D}(t)x(t) \quad (4.4.69)$$

where the $\bar{D}(t)$ are defined in eqn (4.3.6) we have

$$x(t) = \bar{D}^{-1}(t)z(t). \quad (4.4.70)$$

Plugging eqn (4.4.70) into eqn (4.4.68) and notice that $\dot{z}(t) = \dot{\bar{D}}(t)x(t) + \bar{D}(t)\dot{x}(t)$. Use the

system equation (4.2.1) for $\dot{x}(t)$

$$\dot{z}(t) = \dot{\bar{D}}(t)x(t) + \bar{D}(t) (A(t)x(t) + b(t)u(t) + \sigma(t)) \quad (4.4.71)$$

which can be further simplified as

$$\dot{z}(t) = \left(\dot{\bar{D}}(t) + \bar{D}(t)A(t) \right) x(t) + \bar{D}(t)b(t)u(t) + \bar{D}(t)\sigma(t). \quad (4.4.72)$$

Substituting (4.4.70) into (4.4.72), we have

$$\dot{z}(t) = \left(\dot{\bar{D}}(t) + \bar{D}(t)A(t) \right) \bar{D}^{-1}(t)z(t) + \bar{D}(t)b(t)u(t) + \bar{D}(t)\sigma(t). \quad (4.4.73)$$

Then we further simplify $\left(\dot{\bar{D}}(t) + \bar{D}(t)A(t) \right) \bar{D}^{-1}(t)$. Use the results in (4.3.9)

$$\left(\dot{\bar{D}}(t) + \bar{D}(t)A(t) \right) \bar{D}^{-1}(t) = \begin{bmatrix} \bar{D}_1(t) \\ \bar{D}_2(t) \\ \vdots \\ \bar{D}_n(t) \end{bmatrix} \bar{D}^{-1}(t). \quad (4.4.74)$$

Use the results in (4.3.8) we have

$$\left(\dot{\bar{D}}(t) + \bar{D}(t)A(t) \right) \bar{D}^{-1}(t) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -\alpha_1(t) & -\alpha_2(t) & \cdots & \cdots & -\alpha_n(t) \end{bmatrix}. \quad (4.4.75)$$

Plugging (4.4.75) into (4.4.73) and using the property of $\bar{D}(t)b(t) = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \end{bmatrix}^T$

[154] we have

$$\dot{z}(t) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -\alpha_1(t) & -\alpha_2(t) & \cdots & \cdots & -\alpha_n(t) \end{bmatrix} z(t) + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u(t) + \bar{D}(t)\sigma(t). \quad (4.4.76)$$

Substitute control law $u_1(t)$ in (4.3.12) into (4.4.73)

$$\dot{z}(t) = Wz(t) + \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \end{bmatrix}^T u_2(t) + \Lambda(t) \quad (4.4.77)$$

where W and $\Lambda(t)$ are defined in (4.3.13) and (4.3.16), similar to (4.4.67). The bound of $u_2(t)$ can be derived as

$$\|u_{t_1}\|_{\mathcal{L}_\infty} \leq \|k_g\|_{\mathcal{L}_1} B_r + L_H (b_\sigma + \gamma_1(T, \rho_x, \rho_u)). \quad (4.4.78)$$

$$\begin{aligned} \|z_{t_1}\|_{\mathcal{L}_\infty} &\leq \left\| (s\mathbb{I} - W)^{-1} \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \end{bmatrix}^T \right\|_{\mathcal{L}_1} \| (u_2)_{t_1} \|_{\mathcal{L}_\infty} \\ &\quad + \left\| (s\mathbb{I} - W)^{-1} \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \end{bmatrix}^T \right\|_{\mathcal{L}_1} b_\Lambda \end{aligned} \quad (4.4.79)$$

where b_Λ is defined in (4.3.18). Plugging (4.4.78) into (4.4.79) we have

$$\begin{aligned} \|z_{t_1}\|_{\mathcal{L}_\infty} &\leq \left\| (s\mathbb{I} - W)^{-1} \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \end{bmatrix}^T \right\|_{\mathcal{L}_1} \left(\|k_g\|_{\mathcal{L}_1} B_r + L_H (b_\sigma + \gamma_1(T, \rho_x, \rho_u)) \right) \\ &\quad + \left\| (s\mathbb{I} - W)^{-1} \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \end{bmatrix}^T \right\|_{\mathcal{L}_1} b_\Lambda. \end{aligned} \quad (4.4.80)$$

Due to (4.4.70), we have

$$\begin{aligned}
\|x_{t_1}\|_{\mathcal{L}_\infty} &\leq \|\bar{D}^{-1}\|_{\mathcal{L}_1} \left\| (s\mathbb{I} - W)^{-1} \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \end{bmatrix}^T \right\|_{\mathcal{L}_1} \\
&\quad \left(\|k_g\|_{\mathcal{L}_1} B_r + L_H (b_\sigma + \gamma_1(T, \rho_x, \rho_u)) \right) \\
&\quad + \|\bar{D}^{-1}\|_{\mathcal{L}_1} \left\| (s\mathbb{I} - W)^{-1} \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \end{bmatrix}^T \right\|_{\mathcal{L}_1} b_\Lambda \quad (4.4.81)
\end{aligned}$$

which completes the proof. \square

Lemma 4.4.5. *There exist $\rho_x > 0$, $\rho_u > 0$ and $T > 0$ such that*

$$\begin{aligned}
\rho_x &> \|\bar{D}^{-1}\|_{\mathcal{L}_1} \\
&\quad \left\| (s\mathbb{I} - W)^{-1} \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \end{bmatrix}^T \right\|_{\mathcal{L}_1} \left(\|k_g\|_{\mathcal{L}_1} B_r + L_H (b_\sigma + \gamma_1(T, \rho_x, \rho_u)) \right) \\
&\quad + \|\bar{D}^{-1}\|_{\mathcal{L}_1} \left\| (s\mathbb{I} - W)^{-1} \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \end{bmatrix}^T \right\|_{\mathcal{L}_1} b_\Lambda \quad (4.4.82)
\end{aligned}$$

and

$$\rho_u > \|k_g\|_{\mathcal{L}_1} B_r + \|k\|_{\mathcal{L}_1} \rho_x + L_H (b_\sigma + \gamma_1(T, \rho_x, \rho_u)) . \quad (4.4.83)$$

Proof: Let $\Delta > 0$ be any constant.

$$\begin{aligned}
\rho_x &= \|\bar{D}^{-1}\|_{\mathcal{L}_1} \left\| (s\mathbb{I} - W)^{-1} \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \end{bmatrix}^T \right\|_{\mathcal{L}_1} \left(\|k_g\|_{\mathcal{L}_1} B_r + L_H (b_\sigma + \Delta) \right) \\
&\quad + \|\bar{D}^{-1}\|_{\mathcal{L}_1} \left\| (s\mathbb{I} - W)^{-1} \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \end{bmatrix}^T \right\|_{\mathcal{L}_1} b_\Lambda \quad (4.4.84)
\end{aligned}$$

and

$$\rho_u = \|k_g\|_{\mathcal{L}_1} B_r + \|k\|_{\mathcal{L}_1} \rho_x + L_H (b_\sigma + \Delta) . \quad (4.4.85)$$

It follows from lemma 4.4.3 that there exists a T such that $\gamma_1(T, \rho_x, \rho_u) < \Delta$. Substituting this into (4.4.84) and (4.4.85) one finds that both (4.4.82) and (4.4.83) holds which completes the proof. \square

Theorem 4.4.6. *Consider the system in (4.2.1) together with the \mathcal{L}_1 adaptive controller in (4.3.1), (4.3.3) and (4.3.23), choosing T as in lemma 5 to make (4.4.82) and (4.4.83) hold, then*

$$\|x\|_{\mathcal{L}_\infty} < \rho_x, \quad \|u\|_{\mathcal{L}_\infty} < \rho_u . \quad (4.4.86)$$

Proof The proof will be done by contradiction. From (4.2.1) and (4.3.1), $x(0) = 0$ and $\hat{x}(0) = x_0$. It follows from the definition of ρ_x and ρ_u in (4.4.84) and (4.4.85) that

$$x(0) < \rho_x, \quad u(0) < \rho_u . \quad (4.4.87)$$

Assume (4.4.86) is not true, since $x(t)$ and $u(t)$ are continuous . There exist $t' \leq 0$ such that

$$\|x(t')\|_\infty = \rho_x \quad \text{or} \quad \|u(t')\|_\infty = \rho_u , \quad (4.4.88)$$

while

$$\|x_{t'}\|_{\mathcal{L}_\infty} \leq \rho_x, \quad \|u_{t'}\|_{\mathcal{L}_\infty} \leq \rho_u . \quad (4.4.89)$$

Letting $t_1 = t'$ it follows from lemma 4.4.4 that

$$\begin{aligned}
\|x_{t'}\|_{\mathcal{L}_\infty} &\leq \|\bar{D}^{-1}\|_{\mathcal{L}_1} \left\| (s\mathbb{I} - W)^{-1} \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \end{bmatrix}^T \right\|_{\mathcal{L}_1} \\
&\quad \left(\|k_g\|_{\mathcal{L}_1} B_r + L_H (b_\sigma + \gamma_1(T, \rho_x, \rho_u)) \right) \\
&\quad + \|\bar{D}^{-1}\|_{\mathcal{L}_1} \left\| (s\mathbb{I} - W)^{-1} \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \end{bmatrix}^T \right\|_{\mathcal{L}_1} b_\Lambda \quad (4.4.90) \\
&< \rho_x
\end{aligned}$$

and

$$\begin{aligned}
\|u_{t'}\|_{\mathcal{L}_\infty} &\leq \|k_g\|_{\mathcal{L}_1} B_r + \|k\|_{\mathcal{L}_1} \rho_x + L_H (b_\sigma + \gamma_1(T, \rho_x, \rho_u)) \\
&< \rho_u. \quad (4.4.91)
\end{aligned}$$

As this finding (4.4.91) clearly contradicts the statements in (4.4.82) and (4.4.83), t' does not exist. As a result $\|x_t\|_{\mathcal{L}_\infty} < \rho_x$ and $\|u_t\|_{\mathcal{L}_\infty} < \rho_u$ holds for all $t \leq 0$, thus proving theorem 4.4.86. \square

To further simplify the derivation, we define

$$E_1 = \begin{bmatrix} 0 & 0 & \cdots & 1 \end{bmatrix}^T, \quad I_1 = \begin{bmatrix} \mathbb{I}_{(n-1) \times (n-1)} & 0 \\ 0 & 0 \end{bmatrix}. \quad (4.4.92)$$

$$\begin{aligned}
b_e &= \left\| (s\mathbb{I} - W)^{-1} E_1 k_g(t) c(t) D^{-1}(t) W^{-1} I_1 \bar{D}(t) (\mathbb{I} - C(s)) \right\|_{\mathcal{L}_1} b_\sigma \\
&\quad + \left\| (s\mathbb{I} - W)^{-1} E_1 k_g(t) c(t) D^{-1}(t) W^{-1} I_1 \bar{D}(t) C(s) \right\|_{\mathcal{L}_1} \gamma_2(T, \rho_x, \rho_u) \\
&\quad + \left\| (s\mathbb{I} - W)^{-1} \bar{D}_0(t) (\mathbb{I} - C(s)) \right\|_{\mathcal{L}_1} b_\sigma \\
&\quad + \left\| (s\mathbb{I} - W)^{-1} \bar{D}_0(t) C(s) \right\|_{\mathcal{L}_1} \gamma_2(T, \rho_x, \rho_u). \quad (4.4.93)
\end{aligned}$$

$$\begin{aligned}
z_{ss}(t) &= -W^{-1}E_1k_g(t)(r(t) + c(t)D^{-1}(t)W^{-1}\Lambda_{um}(t)) - W^{-1}\Lambda_{um}(t) \\
&= -W^{-1}E_1\hat{u}_2(t) - W^{-1}\Lambda_{um}(t).
\end{aligned} \tag{4.4.94}$$

$z_{ss}(t)$ is designed to have the following property

$$\begin{aligned}
y_{ss}(t) &\triangleq c(t)\bar{D}^{-1}(t)z_{ss}(t) \\
&= -c(t)\bar{D}^{-1}(t)W^{-1}E_1k_g(t)(r(t) + c(t)D^{-1}(t)W^{-1}\Lambda(t)_{um}) - c(t)\bar{D}^{-1}(t)W^{-1}\Lambda_{um}(t) \\
&= r(t)
\end{aligned} \tag{4.4.95}$$

(4.4.95) means that if the system state, $z(t)$ in (4.4.77) can track $z_{ss}(t)$, then the output $y(t)$ can track the reference input $r(t)$.

Define

$$\tilde{y}(t) = r(t) - y(t). \tag{4.4.96}$$

Theorem 4.4.7. *The asymptotic tracking performance of the system is given by*

$$\|\tilde{y}(t)\| \leq \|c\bar{D}^{-1}\|_{\mathcal{L}_1} \left(b_e + \frac{2b_{dz}\lambda_{\max}(P)}{\lambda_{\min}(Q)} \right). \tag{4.4.97}$$

Proof: The first part of the proof (eqn(4.4.98) to (4.4.110)) prepares the equations needed for the performance bound characterization.

After applying the control law $u_1(t)$, through which the system dynamics were transformed into (4.4.77) for the convinence of analysis, we use $E_1 = \begin{bmatrix} 0 & 0 & \dots & 0 & 1 \end{bmatrix}^T$. The dynamics of (4.2.1) can be expressed as

$$\begin{aligned}
\dot{z}(t) &= Wz(t) + E_1u_2(t) + \Lambda(t) \\
y(t) &= c(t)x(t) = c(t)\bar{D}^{-1}(t)z(t).
\end{aligned} \tag{4.4.98}$$

Define:

$$\tilde{\Lambda}(t) = \hat{\Lambda}(t) - \Lambda(t), \quad \tilde{\Lambda}_n(t) = \hat{\Lambda}_n(t) - \Lambda_n(t), \quad \tilde{\Lambda}_{um}(t) = \hat{\Lambda}_{um}(t) - \Lambda_{um}(t). \quad (4.4.99)$$

Substituting $u_2(t)$ defined in (4.3.21) into (4.4.98), we have

$$\begin{aligned} \dot{z}(t) &= Wz(t) + E_1k_g(t)(r(t) + c(t)D^{-1}(t)W^{-1}\hat{\Lambda}_{um}(t)) - E_1\hat{\Lambda}_n(t) + \Lambda(t) \\ &= Wz(t) + E_1k_g(t)(r(t) + c(t)D^{-1}(t)W^{-1}\hat{\Lambda}_{um}(t)) - \hat{\Lambda}_m(t) + \Lambda_m(t) + \Lambda_{um}(t) \\ &= Wz(t) + E_1k_g(t)(r(t) + c(t)D^{-1}(t)W^{-1}\hat{\Lambda}_{um}(t)) - \tilde{\Lambda}_m(t) + \Lambda_{um}(t) \\ &= Wz(t) + E_1k_g(t)(r(t) + c(t)D^{-1}(t)W^{-1}\Lambda_{um}(t)) + \Lambda_{um}(t) \\ &\quad + E_1k_g(t)c(t)D^{-1}(t)W^{-1}\tilde{\Lambda}_{um}(t) - \tilde{\Lambda}_m(t). \end{aligned} \quad (4.4.100)$$

Decomposing the $z(t)$ into two parts, $z_1(t)$ and $e(t)$, yields

$$z(t) = z_1(t) + e(t) \quad (4.4.101)$$

where the error signal $e(t)$ is defined as

$$\dot{e}(t) = We(t) + E_1k_g(t)c(t)D^{-1}(t)W^{-1}\tilde{\Lambda}_{um}(t) - \tilde{\Lambda}_m(t). \quad (4.4.102)$$

Comparing the dynamics of $e(t)$ in (4.4.102) and $z(t)$ in (4.4.100) we have.

$$\dot{z}_1(t) = Wz_1(t) + E_1k_g(t)(r(t) + c(t)D^{-1}(t)W^{-1}\Lambda_{um}(t)) + \Lambda_{um}(t). \quad (4.4.103)$$

Define

$$\hat{u}_2(t) = k_g(t)(r(t) + c(t)D^{-1}(t)W^{-1}\Lambda_{um}(t)). \quad (4.4.104)$$

Substituting (4.4.104) into (4.4.103), we have

$$\dot{z}_1(t) = Wz_1(t) + E_1\hat{u}_2(t) + \Lambda_{um}(t) . \quad (4.4.105)$$

Define

$$\tilde{z}(t) = z_1(t) - z(t)_{ss} . \quad (4.4.106)$$

So

$$\begin{aligned} \dot{\tilde{z}}(t) &= \dot{z}_1(t) - \dot{z}_{ss}(t) \\ &= W\tilde{z}(t) + E_1\hat{u}_2(t) + \Lambda_{um}(t) + Wz_{ss}(t) - \dot{z}_{ss}(t) \end{aligned} \quad (4.4.107)$$

Given any positive-defined matrix Q and solving P by the following lyapunov equation

$$W^TP + PW = -Q \quad (4.4.108)$$

$$\begin{aligned}
\dot{V}(t) &= \dot{\tilde{z}}(t)^T P \tilde{z}(t) + \tilde{z}(t)^T P \dot{\tilde{z}}(t) \\
&= \left(\tilde{z}(t)^T W^T + \hat{u}_2(t)^T E_1^T + \Lambda_{um}(t)^T + z_{ss}(t)^T W^T - \dot{z}_{ss}(t)^T \right) P \tilde{z}(t) + \\
&\quad \tilde{z}(t)^T P (W \tilde{z}(t) + E_1 \hat{u}_2(t) + \Lambda_{um}(t) + W z_{ss}(t) - \dot{z}_{ss}(t)) \\
&= \tilde{z}(t)^T W^T P \tilde{z}(t) + \hat{u}_2(t)^T E_1^T P \tilde{z}(t) + \Lambda_{um}(t)^T P \tilde{z}(t) + z_{ss}(t)^T W^T P \tilde{z}(t) \\
&\quad - \dot{z}_{ss}(t)^T P \tilde{z}(t) + \tilde{z}(t)^T P W \tilde{z}(t) + \tilde{z}(t)^T P E_1 \hat{u}_2(t) + \\
&\quad \tilde{z}(t)^T P \Lambda_{um}(t) + \tilde{z}(t)^T P W z_{ss}(t) - \tilde{z}(t)^T P \dot{z}_{ss}(t) \\
&= \tilde{z}(t)^T (W^T P + P W) \tilde{z}(t) + 2 \tilde{z}(t)^T P E_1 \hat{u}_2(t) + \\
&\quad 2 \tilde{z}(t)^T P \Lambda_{um}(t) + 2 \tilde{z}(t)^T P W z_{ss}(t) - 2 \tilde{z}(t)^T P \dot{z}_{ss}(t) \\
&= -\tilde{z}(t)^T Q \tilde{z}(t) + 2 \tilde{z}(t)^T P E_1 \hat{u}_2(t) + \\
&\quad 2 \tilde{z}(t)^T P \Lambda_{um}(t) + 2 \tilde{z}(t)^T P W z_{ss}(t) - 2 \tilde{z}(t)^T P \dot{z}_{ss}(t). \tag{4.4.109}
\end{aligned}$$

Using the definition of $z_{ss}(t)$ in (4.4.94), the equation above can be rewritten as

$$\begin{aligned}
\dot{V}(t) &= -\tilde{z}(t)^T Q \tilde{z}(t) + 2 \tilde{z}(t)^T P E_1 \hat{u}_2(t) + 2 \tilde{z}(t)^T P \Lambda_{um}(t) + \\
&\quad 2 \tilde{z}(t)^T P W (-W^{-1} E_1 \hat{u}_2(t) - W^{-1} \Lambda_{um}(t)) - 2 \tilde{z}(t)^T P \dot{z}_{ss}(t) \\
&= -\tilde{z}(t)^T Q \tilde{z}(t) - 2 \tilde{z}(t)^T P \dot{z}_{ss}(t). \tag{4.4.110}
\end{aligned}$$

In the following part eqn (4.4.102) and (4.4.110) are mainly used to derive the performance bound of tracking. In eqn(4.4.102), the variable $\tilde{\Lambda}_m(t)$ can be expressed as

$$\begin{aligned}
\tilde{\Lambda}_m(s) &= \Lambda_m(s) - \hat{\Lambda}_m(s) \\
&= \bar{D}_0(t) (\sigma(s) - C(s) \hat{\sigma}(s)) \\
&= \bar{D}_0(t) (\sigma(s) - C(s) (\sigma(s) - \tilde{\sigma}(s))) \\
&= \bar{D}_0(t) ((\mathbb{I} - C(s)) \sigma(s) + C(s) \tilde{\sigma}(s)) \\
&= \bar{D}_0(t) (\mathbb{I} - C(s)) \sigma(s) + \bar{D}_0(t) C(s) \tilde{\sigma}(s). \tag{4.4.111}
\end{aligned}$$

Similarly

$$\tilde{\Lambda}_{um}(s) = I_1 \bar{D}(t) (\mathbb{I} - C(s)) \sigma(s) + I_1 \bar{D}(t) C(s) \tilde{\sigma}(s). \quad (4.4.112)$$

Substituting (4.4.111) and (4.4.112) into (4.4.102), we have

$$\begin{aligned} \dot{e}(t) &= We(t) + E_1 k_g(t) c(t) D^{-1}(t) W^{-1} \tilde{\Lambda}_{um}(t) - \tilde{\Lambda}_m(t) \\ &= We(t) + E_1 k_g(t) c(t) D^{-1}(t) W^{-1} I_1 \bar{D}(t) (\mathbb{I} - C(s)) \sigma(s) \\ &\quad + E_1 k_g(t) c(t) D^{-1}(t) W^{-1} I_1 \bar{D}(t) C(s) \tilde{\sigma}(s) \\ &\quad + \bar{D}_0(t) (\mathbb{I} - C(s)) \sigma(s) + \bar{D}_0(t) C(s) \tilde{\sigma}(s). \end{aligned} \quad (4.4.113)$$

Use the definition of I_1 in (4.4.92). We derive the upper bound of the error signal $e(t)$ as

$$\begin{aligned} \|e(t)\|_\infty &\leq \left\| (s\mathbb{I} - W)^{-1} E_1 k_g(t) c(t) D^{-1}(t) W^{-1} I_1 \bar{D}(t) (\mathbb{I} - C(s)) \right\|_{\mathcal{L}_1} \|\sigma\|_{\mathcal{L}_\infty} \\ &\quad + \left\| (s\mathbb{I} - W)^{-1} E_1 k_g(t) c(t) D^{-1}(t) W^{-1} I_1 \bar{D}(t) C(s) \right\|_{\mathcal{L}_1} \|\tilde{\sigma}\|_{\mathcal{L}_\infty} \\ &\quad + \left\| (s\mathbb{I} - W)^{-1} \bar{D}_0(t) (\mathbb{I} - C(s)) \right\|_{\mathcal{L}_1} \|\sigma\|_{\mathcal{L}_\infty} \\ &\quad + \left\| (s\mathbb{I} - W)^{-1} \bar{D}_0(t) C(s) \right\|_{\mathcal{L}_1} \|\tilde{\sigma}\|_{\mathcal{L}_\infty}. \end{aligned} \quad (4.4.114)$$

Using the bound in lemma 4.4.1 and assumption 4.2.1

$$\begin{aligned} \|e(t)\|_\infty &\leq \left\| (s\mathbb{I} - W)^{-1} E_1 k_g(t) c(t) D^{-1}(t) W^{-1} I_1 \bar{D}(t) (\mathbb{I} - C(s)) \right\|_{\mathcal{L}_1} b_\sigma \\ &\quad + \left\| (s\mathbb{I} - W)^{-1} E_1 k_g(t) c(t) D^{-1}(t) W^{-1} I_1 \bar{D}(t) C(s) \right\|_{\mathcal{L}_1} \gamma_2(T, \rho_x, \rho_u) \\ &\quad + \left\| (s\mathbb{I} - W)^{-1} \bar{D}_0(t) (\mathbb{I} - C(s)) \right\|_{\mathcal{L}_1} b_\sigma \\ &\quad + \left\| (s\mathbb{I} - W)^{-1} \bar{D}_0(t) C(s) \right\|_{\mathcal{L}_1} \gamma_2(T, \rho_x, \rho_u). \end{aligned} \quad (4.4.115)$$

Using the definition of b_e in (4.4.93) we have

$$\|e(t)\|_\infty \leq b_e. \quad (4.4.116)$$

Eqn (4.4.115) summarized the error of $e(t)$, the following analyzes the tracking error from eqn (4.4.110) are analysed. Follows by the assumption, there exist a bound b_{dz} such that

$$\|\dot{z}_{ss}(t)\| < b_{dz}. \quad (4.4.117)$$

For any $t_1 > 0$, if

$$V(t_1) > \frac{4b_{dz}^2 \lambda_{\max}^3(P)}{\lambda_{\min}^2(Q)}, \quad (4.4.118)$$

then

$$\|\tilde{z}(t_1)\| \geq \sqrt{\frac{V(t_1)}{\lambda_{\max}(P)}} > \frac{2b_{dz} \lambda_{\max}(P)}{\lambda_{\min}(Q)}. \quad (4.4.119)$$

It follows from (4.4.110) that

$$\begin{aligned} \dot{V}(t_1) &= -\tilde{z}(t)^T Q \tilde{z}(t) - 2\tilde{z}(t)^T P \dot{z}_{ss}(t) \\ &\leq \lambda_{\min}(Q) \|\tilde{z}(t_1)\|^2 + 2b_{dz} \lambda_{\max}(P) \|\tilde{z}(t_1)\| < 0. \end{aligned} \quad (4.4.120)$$

Eqn (4.4.118) to (4.4.120) means that if $V(0) > \frac{4b_{dz}^2 \lambda_{\max}^3(P)}{\lambda_{\min}^2(Q)}$, $V(t)$ will keep decreasing untill $V(t) \leq \frac{4b_{dz}^2 \lambda_{\max}^3(P)}{\lambda_{\min}^2(Q)}$. If $V(0) \leq \frac{4b_{dz}^2 \lambda_{\max}^3(P)}{\lambda_{\min}^2(Q)}$, then $V(t) \leq \frac{4b_{dz}^2 \lambda_{\max}^3(P)}{\lambda_{\min}^2(Q)}$

$$\|\tilde{z}(t)\| \leq \sqrt{\frac{V(t)}{\lambda_{\max}(P)}} < \frac{2b_{dz} \lambda_{\max}(P)}{\lambda_{\min}(Q)}. \quad (4.4.121)$$

The tracking error can be written as:

$$\begin{aligned}
\tilde{y}(t) &= r(t) - y(t) \\
&= c(t)\bar{D}^{-1}(t)z_{ss}(t) - c(t)\bar{D}^{-1}(t)z(t) \\
&= -c(t)\bar{D}^{-1}(t)(z(t) - z_{ss}(t)) \\
&= -c(t)\bar{D}^{-1}(t)((z(t) - z_1(t)) + (z_1(t) - z_{ss}(t))) \\
&= -c(t)\bar{D}^{-1}(t)(e(t) + \tilde{z}(t)) .
\end{aligned} \tag{4.4.122}$$

Using the results in (4.4.116) and (4.4.121), we have.

$$\|\tilde{y}(t)\| \leq \|c\bar{D}^{-1}\|_{\mathcal{L}_1} \left(b_e + \frac{2b_{dz}\lambda_{\max}(P)}{\lambda_{\min}(Q)} \right) \tag{4.4.123}$$

which completes the proof. \square

Remark 4.4.8. By tuning both the bandwidth of the low pass filter $C(s)$, as well as the sampling time, one can make the b_e term in theorem 4.4.7 arbitrarily small. The term $\frac{2b_{dz}\lambda_{\max}(P)}{\lambda_{\min}(Q)}$ can be affected by the assigned eigenvalues to the W matrix.

4.5 Simulation

We will demonstrate the performance of the eigenvalue assignment control method with a numerical example. Consider an certain SISO LTV system with unmodeled dynamics and

disturbances

$$A(t) = \begin{bmatrix} 0 & 0 & -\sin(t) \\ 0 & 0 & \cos(t) \\ \cos(t) & \sin(t) & 0 \end{bmatrix}, \quad B(t) = \begin{bmatrix} \cos(t) \\ \sin(t) \\ 0 \end{bmatrix}, \quad C(t) = \begin{bmatrix} -\sin(t) \\ 1/(10e^t) + \cos(t) \\ 1 \end{bmatrix}.$$

The sampling time of the system $T = 0.01s$. The tracking error is given by the following.

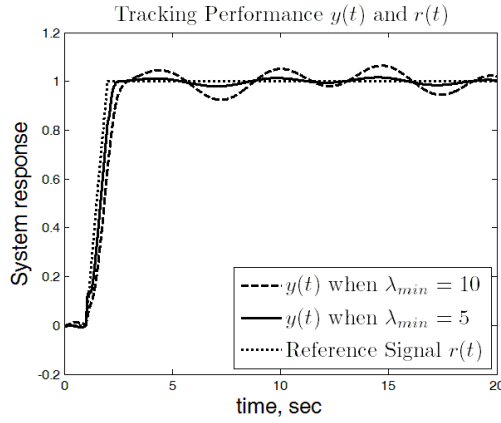


FIGURE 4.5.1: Performance for Tracking ,
 $\lambda_{min} = 5$ and $\lambda_{min} = 10$

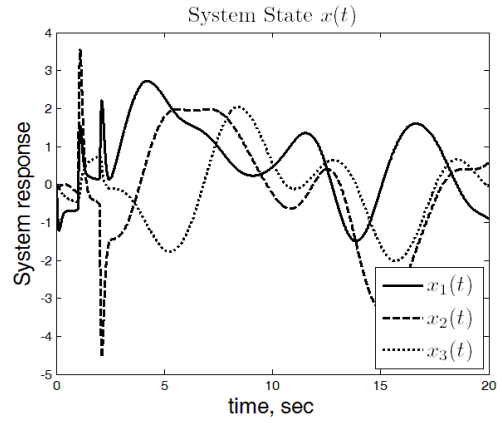


FIGURE 4.5.2: Performance of System
State , $\lambda_{min} = 10$

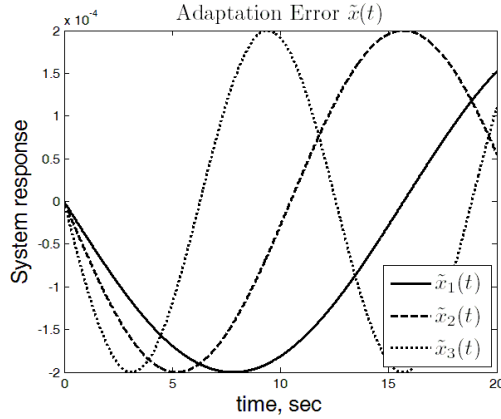


FIGURE 4.5.3: Adaptation performance
 $\tilde{x}(t)$

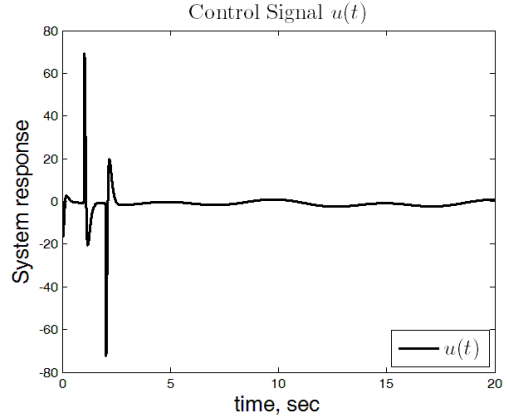


FIGURE 4.5.4: Control signal $u(t)$

We can tell from figure 4.5.1 that as we change the eigenvalues of the W matrix, the tracking performance can be configured reasonably. This is consistence to the results in Remark 4.4.8.

4.6 Application on Supersonic Glider

In [178], the similar approach are designed for the control of supersonic glider. Figure 4.6.1 shows a example of supersonic glider.

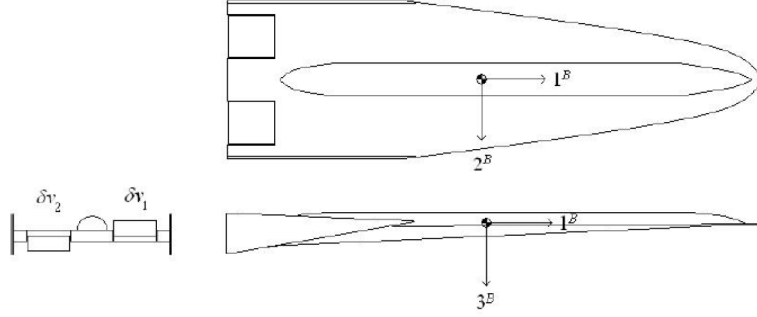


FIGURE 4.6.1: Generic Hypersonic Glider Model

System equation of the glider is given as follows

$$\begin{bmatrix} \dot{q} \\ \dot{\alpha} \\ \dot{V} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} M_q & M_\alpha & M_V & 0 \\ 1 & \frac{N_\alpha}{V} & \frac{N_V}{V} & 0 \\ 0 & X_\alpha & X_V & -g \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} q \\ \alpha \\ V \\ \theta \end{bmatrix} + \begin{bmatrix} M_{\delta e} & M_{\delta t} \\ \frac{N_{\delta e}}{V} & \frac{N_{\delta t}}{V} \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \delta e \\ \delta t \end{bmatrix}. \quad (4.6.1)$$

The system setup is a nonminimum phase Linear Time Varying (LTV) state feedback system. The eigen-value assignment \mathcal{L}_1 adaptive controller (namely, the \mathcal{L}_1 augmented pole placement controller in the literature [178]) is shown to have robustness in the presence of time invariant and time varying errors in the aerodynamic coefficients, control surface and gravimetric uncertainties. The proposed controller improves the performance of the baseline controller in the presence of these uncertainties. This is concluded with a reduction in the tracking error norm and the control surface norm, which is the energy of the deviation between the reference and real control signal.

The simulation results in figure 4.6.2 shows the performance of the \mathcal{L}_1 adaptive controller with a reduction in the tracking error norm and the control surface norm.

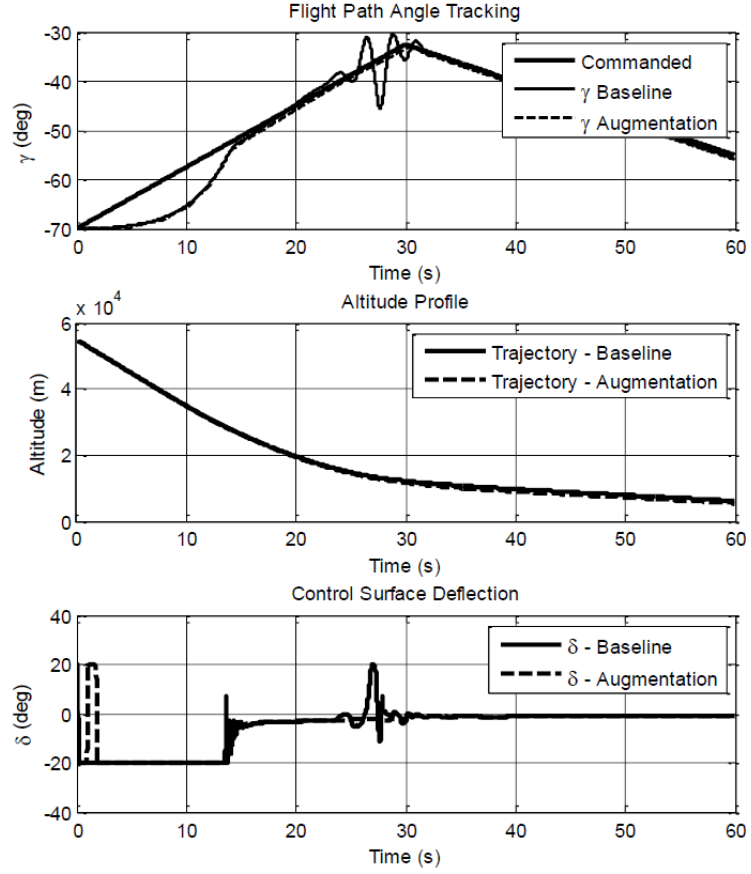


FIGURE 4.6.2: Simulation results of the proposed LTV eigenvalue assignment controller

Problem formulation, controller design, simulation and more information could be found in [178].

4.7 Summary

This chapter presents an adaptive approach for disturbance rejection in the presence of unmatched unknown time-varying disturbances for a class of linear time-varying (LTV)

systems. The piece-wise constant adaptive law from \mathcal{L}_1 architecture is employed here to estimate the unknown disturbances by using partial information of the system. The adaption and control algorithm in this chapter is causal. Eigenvalue assignment architecture transforms the time-varying system $A(t), B(t), C(t)$ into a control canonical form including the time-invariant A, B matrix system with desired eigenvalues . A control law is designed for disturbance rejection and tracking such that the output of LTV systems with disturbances can practically track the reference signal. The stability is analyzed. Estimation error and tracking error upper bounds are characterized in the main theorem. Simulation results as well as the simulation in [178] validate the approach of this chapter.

Acknowledgements

This work is supported by NASA contract and NSF Award IIS 1208499. Thanks for the helpful input from Roseanna Warren.

Chapter 5

Filter Bandwidth Adaptation in the \mathcal{L}_1 Control

5.1 Introduction

Adaptive control has been used and developed for several decades. Many books and papers have been and continue to be written in this area. Model Reference Adaptive Control (MRAC) is a special type of adaptive control that has been widely used since its introduction [179–183]. Artificial neural networks (ANNs) [184, 185] were later introduced to adaptive control to be able to handle unknown nonlinear functions. The \mathcal{L}_1 adaptive control is a modification of MRAC that uses fast adaption and low-pass filtering to handle time-varying uncertainties [49, 186–192]. However, the filter bandwidth of the original \mathcal{L}_1 controller must be tuned case by case so that it can handle disturbances with various frequencies and achieve the desired performance without becoming unstable. If the bandwidth of the filter in the \mathcal{L}_1 adaptive controller is reduced, its capability to deal with high frequency disturbances decreases, and its response becomes slower. This will in turn adversely

affect its performance. If the bandwidth is increased significantly, its ability to track the reference trajectory improves, but the time delay margin of the system is reduced, and the system can easily become unstable. According to the stability condition of the controller, increasing the bandwidth is necessary to stabilize the system. However, the larger the Lipschitz constant of the uncertainties is, the larger the bandwidth must be. For real systems with time-delay, this is obviously a problem, as the Lipschitz constant for the uncertainties must not exceed a certain upper limit.

In this chapter, we have proposed an additional gain adaptation augmented to the original \mathcal{L}_1 control system to compensate for the limitation in low pass filter bandwidth. This adaptation adjusts the bandwidth of the controller based on changes in disturbance estimation over the time which allows for a much more systematic design approach for the \mathcal{L}_1 controller and reduces the need for tuning. This provides a major improvement for applications where the system's time-delay margin or robustness is a concern. By using the bandwidth adaptation method proposed here, the \mathcal{L}_1 controller can be applied to systems with arbitrarily large Lipschitz constants for uncertainties. With this method, the controller's stability condition can be easily satisfied by choosing an arbitrarily large fixed bandwidth. The method adds a separate filter to ensure smooth control signals and maintains robustness. The bandwidth of this new filter adapts to be higher when the uncertainties change faster and lower when they change more slowly. Furthermore, this new design results in a non-LTI system that will help clarify the distinction between \mathcal{L}_1 control and linear controllers. For simplicity, this chapter considers a problem involving a single-input-single-output (SISO) system with full state feedback, yet the method discussed here is a fundamental technique that can easily be applied to the \mathcal{L}_1 controllers developed for other classes of systems such as multiple-input-multiple-output (MIMO) [192] and output feedback [193,194]. The definitions and properties of the norms used in this chapter can be found in [189]. All the proofs are in the appendix.

5.2 Problem Formulation

Consider the following SISO system

$$\dot{x}(t) = A_m x(t) + b_m(u(t) + f(x, t)) + \sigma(t), \quad y(t) = c_m^\top x(t), \quad x(0) = x_0, \quad (5.2.1)$$

where $x(t) \in \mathbb{R}^n$ is the system state vector (measurable), $u(t) \in \mathbb{R}$ is the control input, $y(t) \in \mathbb{R}$ is the system output, A_m is a known $n \times n$ Hurwitz matrix, $b_m, c_m \in \mathbb{R}^n$ are known constant vectors with the zeros of $c_m^\top(s\mathbb{I} - A_m)b_m$ in the open left-half s plane, $f : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}$ is an unknown nonlinear function, $\sigma(t) \in \mathbb{R}^n$ is a vector of bounded unknown disturbances, and x_0 is the initial value of $x(t)$.

Assumption 5.2.1. [Semiglobal Lipschitz condition in x] For any $\delta > 0$, there exists $L(\delta) > 0$ such that $|f(x, t) - f(\bar{x}, t)| \leq L(\delta)\|x - \bar{x}\|_\infty$ for all $\|x\|_\infty \leq \delta$ and $\|\bar{x}\|_\infty \leq \delta$.

Assumption 5.2.2. For any $t \geq 0$, $|f(0, t)| \leq B_1, \|\sigma(t)\|_\infty \leq B_2$.

Assumption 5.2.3. For any $\delta > 0$, there exist $d_{f_x}(\delta) > 0$, $d_{f_t}(\delta) > 0$, and $d_\sigma > 0$ such that for any $\|x\|_\infty \leq \delta$ the partial derivatives of $f(x, t)$ with respect to x and t are piecewise continuous and bounded, and the derivative of $\sigma(t)$ is also piecewise continuous and bounded, i.e.,

$$\left\| \frac{\partial f(x, t)}{\partial x} \right\| \leq d_{f_x}(\delta), \quad \left| \frac{\partial f(x, t)}{\partial t} \right| \leq d_{f_t}(\delta), \quad \left\| \frac{d\sigma(t)}{dt} \right\|_\infty \leq d_\sigma.$$

5.3 \mathcal{L}_1 Adaptive Feedback Controller

We consider the following state predictor

$$\dot{\hat{x}}(t) = A_m \hat{x}(t) + b_m u(t) + \hat{\sigma}(t), \quad \hat{y}(t) = c_m^\top \hat{x}(t), \quad \hat{x}(0) = x_0, \quad (5.3.1)$$

where $\hat{x}(t) \in \mathbb{R}^n$ is the predicted state, $\hat{y}(t) \in \mathbb{R}$ is the predicted output, $\hat{\sigma}(t) \in \mathbb{R}^n$ is the vector of adaptive parameters.

Letting $\tilde{x}(t) = \hat{x}(t) - x(t)$, the update law for $\hat{\sigma}(t)$ is given by

$$\hat{\sigma}(t) = \hat{\sigma}(iT), \quad t \in [iT, (i+1)T), \hat{\sigma}(iT) = - \left[\int_0^T e^{A_m(T-\tau)} d\tau \right]^{-1} e^{A_m T} \tilde{x}(iT), \quad (5.3.2)$$

where T is a positive constant, $i = 0, 1, 2, \dots$ and τ is a dummy variable. We can find matrix $B_{um} \in \mathbb{R}^{n \times (n-1)}$ such that $b_m^\top B_{um} = 0$ and $\text{rank}([b_m \ B_{um}]) = n$. Then $\hat{\sigma}(t)$ can be decoupled into matched and unmatched components via the transformation $\begin{bmatrix} \hat{\sigma}_1(t) & \hat{\sigma}_2(t) \end{bmatrix}^T = \begin{bmatrix} b_m & B_{um} \end{bmatrix}^{-1} \hat{\sigma}(t)$, where $\hat{\sigma}_1(t)$ represents the matched component of the uncertainty estimate, $\hat{\sigma}(t)$, and $\hat{\sigma}_2(t)$ represents the unmatched component. Then, equation (5.3.1) can be written as

$$\dot{\hat{x}}(t) = A_m \hat{x}(t) + b_m(u(t) + \hat{\sigma}_1) + B_{um} \hat{\sigma}_2, \quad \hat{y}(t) = c_m^\top \hat{x}(t), \quad \hat{x}(0) = x_0. \quad (5.3.3)$$

The control signal is defined as follows

$$u(s) = k_g r(s) - u_f(s), \quad (5.3.4)$$

where $r(s)$ is the Laplace transformation of the reference signal $r(t)$ and $u_f(t)$ is generated via a novel time-varying filter:

$$\dot{u}_f(t) = -\omega(t) (u_f(t) - \sigma_f(t)), \quad u_f(0) = 0, \quad (5.3.5)$$

with its bandwidth $\omega(t)$ adapted according to

$$\omega(t) = \omega_0 + \mu |\dot{\sigma}_f(t)|, \quad (5.3.6)$$

where ω_0 is a positive constant, μ is a constant with a large positive value and $\sigma_f(t)$ is a signal with the Laplace transformation:

$$\sigma_f(s) = C_1(s)\hat{\sigma}_1(s) + C_2(s)M(s)\hat{\sigma}_2(s) \quad (5.3.7)$$

where $M(s) = \frac{c_m^\top(s\mathbb{I}-A_m)^{-1}B_{um}}{c_m^\top(s\mathbb{I}-A_m)^{-1}b_m}$, both $C_1(s)$ and $C_2(s)$ are low pass filters with unit DC gain, and $C_2(s)$ needs to ensure that $\frac{C_2(s)c_m^\top(s\mathbb{I}-A_m)^{-1}B_{um}}{c_m^\top(s\mathbb{I}-A_m)^{-1}b_m}$ is a strictly proper transfer function. Here $\hat{\sigma}_1(s)$ and $\hat{\sigma}_2(s)$ are Laplace transformations of matched uncertainties $\hat{\sigma}_1(t)$ and $\hat{\sigma}_2(t)$ respectively. The internal states of $C_1(s)$ and $C_2(s)$ are initialized at zero and thus $\sigma_f(0) = 0$. Since $C_1(s)$ and $C_2(s)$ are low-pass filters, σ_f in (5.3.6) differentiable and can be obtained with proper transfer functions. The entire \mathcal{L}_1 adaptive controller with bandwidth adaptation consists of (5.3.1-5.3.2) and (5.3.4-5.3.6).

5.4 Analysis of \mathcal{L}_1 Adaptive Controller with Filter Bandwidth Adaptation

In the following lemma, an important property of the filter bandwidth adaptation is derived which establishes the bounds of $u_f - \sigma_f$.

Lemma 5.4.1. *It follows from Eq. (5.3.5-5.3.6) that $|u_f(t) - \sigma_f(t)| \leq 1/\mu, \forall t \geq 0$.*

It is noted that any adaptive filter instead of (5.3.5-5.3.6) can be adopted if it ensures Lemma 5.4.1. The following lemma establishes the effect of the design constant T on certain error bounds.

Lemma 5.4.2. *Given*

$$\int_0^T e^{A_m(T-\tau)} \zeta(\tau) d\tau = \varepsilon, \quad (5.4.1)$$

where $\zeta(\tau) \in \mathbb{R}^n$ is a continuous bounded signal $\|\zeta(\tau)\| \leq b_\zeta$ and has bounded derivative

$\|\dot{\zeta}(\tau)\| \leq b_{d\zeta}$, $\varepsilon \in \mathbb{R}^n$ is an arbitrary vector, then $\|\zeta(\tau)\| \leq \chi_2(T)(\|\varepsilon\|/T + \sqrt{n}Td_q)$, where τ is a dummy variable and $\chi_0(T)$, $\chi_1(T)$ and d_q are defined as

$$\begin{aligned}\chi_0(T) &= \max_{\tau \in [0, T)} \|e^{A_m T} A_m e^{-A_m \tau}\|, & \chi_1(T) &= \max_{\tau \in [0, T)} \|e^{A_m T} e^{A_m \tau}\|, \\ \chi_2(T) &= \max_{\tau \in [0, T)} \|e^{-A_m T} e^{A_m \tau}\|, & d_q &= \chi_0(T)b_\zeta + \chi_1(T)b_{d\zeta}.\end{aligned}\quad (5.4.2)$$

For any positive constants γ_x and T , we define the following constants.

$$\eta(T) = \int_0^T \sqrt{\lambda_{\max}((e^{A_m(T-\tau)})^\top e^{A_m(T-\tau)})} d\tau, \quad (5.4.3)$$

$$\nu(\gamma_x, T) = \eta(T)(\|b_m\|(L(\gamma_x)\gamma_x + B_1) + B_2). \quad (5.4.4)$$

where L , B_1 and B_2 are introduced in Assumptions 5.2.1 and 5.2.2.

For any matrix $Q_{n \times n}$, we define the induced 1-norm, $\|Q\|_1 = \max_{j=1, \dots, n} (\sum_{i=1}^n |Q_{ij}|)$, which has the property $\|Qv\|_\infty \leq \|Q\|_1 \|v\|_\infty$ for any $v \in \mathbb{R}^n$. Define

$$\eta_1(T) = [e^{A_m T} - \mathbb{I}]^{-1} A_m e^{A_m T}, \quad (5.4.5)$$

$$\eta_2(T, \gamma_x) = \sqrt{\lambda_{\max}(\eta_1(T)^\top \eta_1(T))} \nu(\gamma_x, T), \quad (5.4.6)$$

$$b_\zeta(T, \gamma_x) = \|b_m\|(L(\gamma_x)\gamma_x + B_1) + B_2 + \eta_2(T, \gamma_x). \quad (5.4.7)$$

For any γ_x , we define the following constants:

$$\gamma_1 = \|A_m\|_1 \gamma_x + \|b_m\|_\infty (\gamma_u + L(\gamma_x)\gamma_x + B_1) + B_2, \quad (5.4.8)$$

$$\gamma_2 = \|b_m\|_\infty \gamma_u + d_\sigma, \quad (5.4.9)$$

$$\eta_4(T) = \chi_2(T) \left(\left\| \int_0^T e^{A_m(T-\tau)} A_m \tilde{x}(iT) d\tau \right\| / T + \sqrt{n}Td_q \right). \quad (5.4.10)$$

$$\eta_5(T) = \|[b_m \ B_{um}]^{-1}\|_1 \eta_4(T). \quad (5.4.11)$$

where

$$\begin{aligned}\gamma_u(T) &= \|k_g\| \|r\|_{\mathcal{L}_\infty} + 1/\mu + \|C_1(s)\|_{\mathcal{L}_1} (L(\gamma_x)\gamma_x + B_1 + \|[b_m \ B_{um}]^{-1}\|_1 B_2 + \eta_5(T)) \\ &\quad + \|C_2(s)M(s)\|_{\mathcal{L}_1} (B_2 + \eta_5(T)) + \epsilon_0 ,\end{aligned}\tag{5.4.12}$$

and ϵ_0 is arbitrary positive constant. Define

$$G(s) = (s\mathbb{I} - A_m)^{-1} b_m (1 - C_1(s)) ,\tag{5.4.13}$$

$$\gamma_e(T) = \|(s\mathbb{I} - A_m)^{-1}\|_{\mathcal{L}_1} \eta_4(T) ,\tag{5.4.14}$$

$$\begin{aligned}\gamma_0(T) &= \|(s\mathbb{I} - A_m)^{-1} k_g\|_{\mathcal{L}_1} \|r\|_\infty + \|(s\mathbb{I} - A_m)^{-1} b_m\|_{\mathcal{L}_1} 1/\mu \\ &\quad + \|(s\mathbb{I} - A_m)^{-1} b_m C_2(s)M(s) + (s\mathbb{I} - A_m)^{-1} B_{um}\|_{\mathcal{L}_1} (B_2 + \eta_5(T))\end{aligned}\tag{5.4.15}$$

The stability condition of the \mathcal{L}_1 adaptive controller with bandwidth adaptation is

$$\|G(s)\|_{\mathcal{L}_1} (L(\gamma_x)\gamma_x + B_1 + \|[b_m \ B_{um}]^{-1}\|_1 B_2 + \eta_5(T)) + \gamma_0 + \gamma_e(T) < \gamma_x .\tag{5.4.16}$$

Subtracting (5.2.1) from (5.3.1) gives us the error dynamics,

$$\dot{\tilde{x}}(t) = A_m \tilde{x}(t) + \hat{\sigma}(t) - b_m f(x, t) - \sigma(t) , \quad \tilde{y}(t) = c_m^\top \tilde{x}(t), \quad \tilde{x}(0) = 0 .\tag{5.4.17}$$

In the first Theorem, we analyze the performance bounds of the \mathcal{L}_1 adaptive controller.

Theorem 5.4.3. *Choose T and $C_1(s)$ such that there exists γ_x satisfying (5.4.16). Given the system in (5.2.1) and the \mathcal{L}_1 adaptive controller in (5.3.1), (5.3.2) and (5.3.4), if*

$x(0) < \gamma_x$ and $u(0) < \gamma_u$, then

$$\|\tilde{x}\|_{\mathcal{L}_\infty} \leq \gamma_e(T), \quad (5.4.18)$$

$$\|x\|_{\mathcal{L}_\infty} < \gamma_x, \quad (5.4.19)$$

$$\|u\|_{\mathcal{L}_\infty} < \gamma_u, \quad (5.4.20)$$

where $\gamma_e(T)$ is defined in (5.4.14), γ_x is introduced in (5.4.16), and γ_u is introduced in (5.4.12).

In the following Theorem, we analyze the circumstances under which the stability condition in (5.4.16) will hold. For simplicity, we consider a first order low-pass filter, i.e.

$$C_1(s) = \omega_1 / (s + \omega_1). \quad (5.4.21)$$

Theorem 5.4.4. *There exist a small T and large ω_1 that satisfy Eq. (5.4.16).*

We note that $G(s)$ is composed of a low-pass system, $(s\mathbb{I} - A_m)^{-1}b_m$, cascaded with a high-pass system, $(1 - C_1(s))$. If the bandwidth of $C_1(s)$ is higher than that of $(s\mathbb{I} - A_m)^{-1}b_m$, then $G(s)$ will attenuate all input frequencies, which leads to (5.7.42). This result also holds if $C_1(s)$ is a higher order low-pass filter.

We note that T and $C_1(s)$ are design parameters for the controller. By reducing T and increasing the bandwidth of $C_1(s)$, ω_1 , the closed-loop system can always be made stable regardless of the magnitude of the uncertainty bounds in Assumptions 5.2.1 and 5.2.2. The low-pass filtering mechanism is used because the adaptive law uses fast adaptation (small T), which results in a high-gain feedback loop. Without the filter, high-gain feedback can directly lead to aggressive control signals resulting in reduced robustness and time-delay margin. For this reason, the low-pass filter bandwidth cannot be made arbitrarily large in previous versions the \mathcal{L}_1 controller, and thus the stability condition requires an upper bound

on the the constants L , B_1 , and B_2 . However, in the method presented in this chapter, $C_1(s)$ simply serves to ensure that σ_f is differentiable. The adaptive filter in (5.3.5) takes on the role of ensuring smooth control signals to maintain robustness and time-delay margin. Therefore, the bandwidth of $C_1(s)$ can be made arbitrarily large, and the constants L , B_1 , and B_2 need only be finite for us to find a suitable ω_1 for (5.4.16) to hold.

It is possible for the time-varying bandwidth, $\omega(t)$, to vary with a high frequency, but this will not lead to overly aggressive control signals since $\omega(t)$ is merely the bandwidth of a filter. It is also noted that for high frequency uncertainties, $\omega(t)$ will have a large value. This is because $\omega(t)$ is proportional to the rate of change of the uncertainties, as a larger bandwidth or faster control signal is necessary to handle high frequency uncertainties.

5.5 Simulations

Case I: Filter Performance of Timer-Varying disturbances

Firstly, a test is conducted to show the performance of the filter gain adaptation mechanism in Eq. (5.3.5-5.3.6) and further verify the theoretical finding in Lemma 1 by comparing the input and output signal of the newly induced filter. A chirp signal is used to show the performance response of the new filter in figure 5.5.1, 5.5.2. It is shown that with the regular low-pass filter in figure 5.5.1, the tracking error will increase when the signal frequency increase. When experiencing high frequency disturbance or sharp signal introduced by large uncertainty. The newly introduced filter in 5.5.2 will adapt it's bandwidth according to the rate of the signal and keep the tracking error with a bound defined by μ following Lemma 1. The disturbance rejection performance comparison is shown in figure 5.5.3.

The configuration of the closed loop simulation is as following: Consider the system

in (5.2.1) with $A_m = \begin{bmatrix} -2 & 0.9 \\ -6 & -5 \end{bmatrix}$, $b_m = \begin{bmatrix} 1.2 \\ 10 \end{bmatrix}$, $c_m = \begin{bmatrix} 1 & 0 \end{bmatrix}$, and $\sigma(t) = \begin{bmatrix} 0.3 \sin(0.2t) \\ 0.1 \cos(0.1t) \end{bmatrix}$. $r(t) = 5 \sin(0.2t)$. In the implementation of the \mathcal{L}_1 adaptive controller, the integration step is $T = 10^{-4}$ s, $\omega_0 = 250$ rad/s, $\mu = 0.1$, and the two low pass filters in (5.3.7) are chosen with very large bandwidths as $C_1(s) = \frac{5000}{s+5000}$ and $C_2(s) = \frac{5000}{s+5000}$.

In the following, the performance and time-delay margin of the controller are demonstrated.

Case II: System Performance of Uncertainties

System Performance of Small Uncertainty: For the system with the uncertainty $f(x, t) = (0.05x_1)^2 + e^{0.05x_2} + 5x_1 + 5x_2 + \cos(t)$, the simulation results of the \mathcal{L}_1 adaptive controller with filter bandwidth adaptation are shown in Fig. 5.5.4. Fig. 5.5.4a shows the tracking performance of the system output. It can be seen that the tracking error remains at a small value. Even though the magnitude of $\omega(t)$ varies, the control input, $u(t)$, is still smooth, as shown in Fig. 5.5.4b.

System Performance of Large Uncertainty: The simulation results for the same controller with the larger uncertainty $f(x, t) = (0.2x_1)^2 + e^{0.2x_2} + 20x_1 + 20x_2 + \cos(t)$ are shown in Fig. 5.5.5. As shown in Fig. 5.5.5a, the controller still obtains desirable tracking performance with a small tracking error.

Both Fig. 5.5.4 and 5.5.5 demonstrate that the bandwidth adaptation mechanism is able to handle uncertainties of various magnitudes and rates.

Time-Delay Margin: As stated in [188], the low-pass filter design can maintain robustness while introducing fast adaptation to the system. The time-delay margin can be tuned by selection of the low-pass filters $C_1(s)$ and $C_2(s)$. The choice of these fixed low-pass filters directly affects the time-delay margin of the system. Bandwidth adaptation can reduce the tuning process of choosing the proper bandwidth to maintain robustness. The

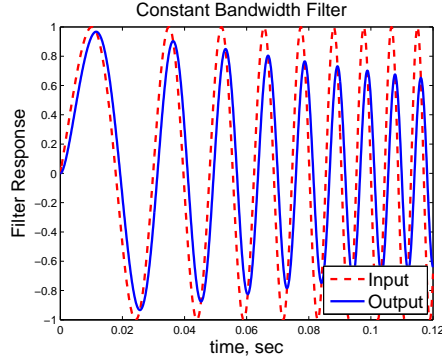


FIGURE 5.5.1: Simulation results of a conventional low-pass filter with chirp reference signal.

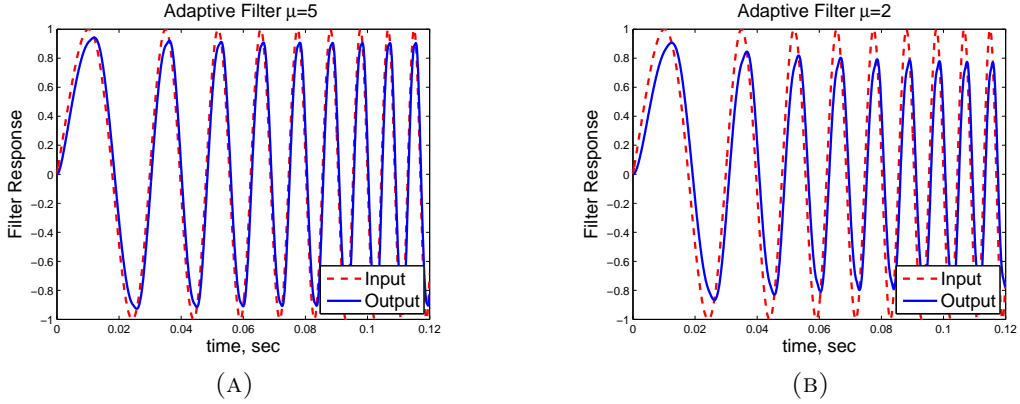


FIGURE 5.5.2: Simulation results the newly induce filter bandwidth adaptation mechanism with chirp signal.

time-delay margins are 0.0047 seconds for the system with small uncertainty and 0.0031 seconds for the system with large uncertainty.

For the \mathcal{L}_1 adaptive controller with fixed low-pass filter bandwidth, the control law design in (5.3.4) is changed to $u(s) = k_g r(s) - \sigma_f(s)$ with fixed bandwidth low-pass filters, and the low pass filters have been chosen as $C_1(s) = \frac{500}{s+500}$ and $C_2(s) = \frac{500}{s+500}$. For the \mathcal{L}_1 adaptive controller with fixed low-pass filter bandwidth, the time-delay margin of the closed-loop system with the small uncertainty is 0.0023 seconds, while the time-delay margin of the closed-loop system with the large uncertainty is 0.0029 seconds. Without tuning

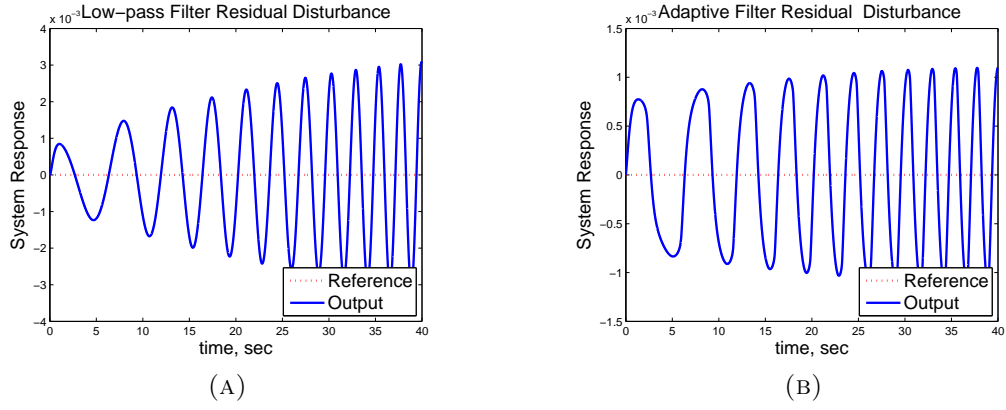


FIGURE 5.5.3: Simulation results the newly induce filter bandwidth adaptation mechanism with chirp signal.

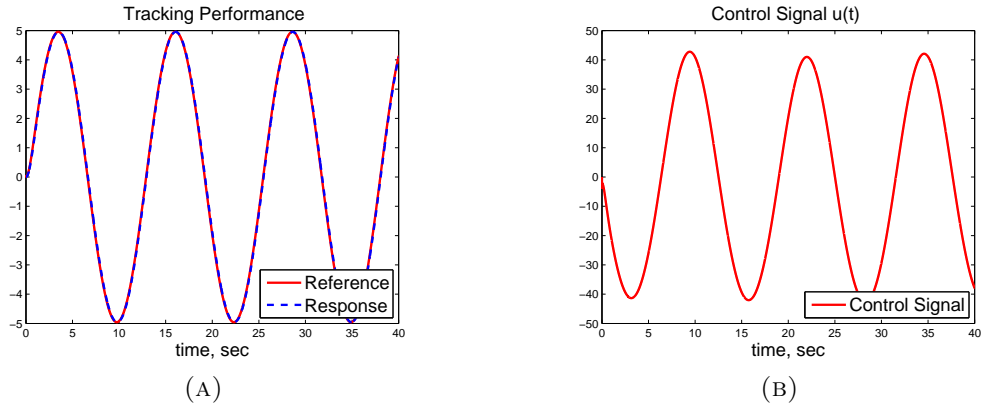


FIGURE 5.5.4: Simulation results of controller with filter bandwidth adaptation for the systems with the small uncertainty.

the bandwidth of the low-pass filters, the \mathcal{L}_1 adaptive controller with filter bandwidth adaptation can provide a better time-delay margin for the system.

5.6 Summary

This chapter presents an \mathcal{L}_1 adaptive controller that uses a new bandwidth adaptation method to increase the range of uncertainties that can be handled by a single controller

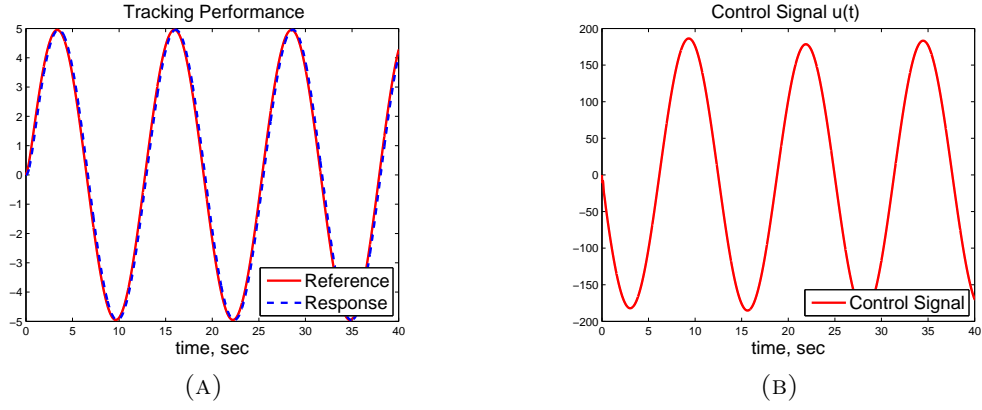


FIGURE 5.5.5: Simulation results of controller with filter bandwidth adaptation for the system with the large uncertainty.

design. This is an important improvement for applications because the Lipschitz constant on uncertainties may be too high for the low-pass filter bandwidth to handle without seriously reducing the system's robustness and time-delay margin. Filter bandwidth adaptation is a new fundamental technique that can be applied to any other existing \mathcal{L}_1 controller, such as those designed for MIMO [192], output feedback [193, 194]. The theoretical analysis given here shows that the system is stable, and simulation results verify that tracking performance is maintained for systems with different uncertainties. Additionally, simulation results show an increase in the time-delay margin of the system when bandwidth adaptation is used.

5.7 Appendix

Proof of Lemma 5.4.1: Since $\sigma_f(0) = 0$ and $u_f(0) = 0$, we have $u_f(0) - \sigma_f(0) = 0$. Considering the following Lyapunov candidate function $V(t) = (u_f(t) - \sigma_f(t))^2/2$, we obtain $\dot{V} = (\dot{u}_f(t) - \dot{\sigma}_f(t))(u_f(t) - \sigma_f(t))$. Substituting in (5.3.5) and (5.3.6) yields $\dot{V} = -\omega(t)(u_f(t) - \sigma_f(t))^2 - \dot{\sigma}_f(t)(u_f(t) - \sigma_f(t))$

$= -(\omega_0 + \mu|\dot{\sigma}_f(t)|)(u_f(t) - \sigma_f(t))^2 - \dot{\sigma}_f(t)(u_f(t) - \sigma_f(t))$. Then, we can further obtain

$$\begin{aligned}\dot{V} &< -\mu|\dot{\sigma}_f(t)|(u_f(t) - \sigma_f(t))^2 - \dot{\sigma}_f(t)(u_f(t) - \sigma_f(t)) \\ &< -|\dot{\sigma}_f(t)|\left(\mu(u_f(t) - \sigma_f(t))^2 + \text{sgn}(\dot{\sigma}_f(t))(u_f(t) - \sigma_f(t))\right).\end{aligned}\quad (5.7.1)$$

For any t such that

$$V(t) \geq 1/(2\mu^2), \quad (5.7.2)$$

we have $|u_f(t) - \sigma_f(t)| \geq 1/\mu$. Multiplying both sides by $\sqrt{\mu}$ and then subtracting $1/(2\sqrt{\mu})$ from both sides gives us

$$\sqrt{\mu}|u_f(t) - \sigma_f(t)| - 1/(2\sqrt{\mu}) \geq 1/(2\sqrt{\mu}). \quad (5.7.3)$$

The left-hand side of (5.7.3) can be rewritten as $|\sqrt{\mu}(u_f(t) - \sigma_f(t))| - |-\text{sgn}(\dot{\sigma}_f(t))/(2\sqrt{\mu})|$. Then using the properties of the absolute value, we can obtain $|\sqrt{\mu}(u_f(t) - \sigma_f(t)) + \text{sgn}(\dot{\sigma}_f(t))/(2\sqrt{\mu})| \geq 1/(2\sqrt{\mu})$. Squaring both sides and subtracting $1/(4\mu)$ yields $\left(\sqrt{\mu}(u_f(t) - \sigma_f(t)) + \text{sgn}(\dot{\sigma}_f(t))/(2\sqrt{\mu})\right)^2 - 1/(4\mu) \geq 0$. Multiplying out the squared term gives us

$$\mu(u_f(t) - \sigma_f(t))^2 + \text{sgn}(\dot{\sigma}_f(t))(u_f(t) - \sigma_f(t)) \geq 0. \quad (5.7.4)$$

Combining (5.7.1) and (5.7.4), we note that if (5.7.2) holds, then $\dot{V}(t) \leq 0$. Since $V(0) = 0$, we have $V(t) \leq 1/(2\mu^2), \forall t \geq 0$, and thus $|u_f(t) - \sigma_f(t)| \leq 1/\mu, \forall t \geq 0$, which proves lemma 5.4.1. \square

Proof of Lemma 5.4.2: Let

$$q(\tau) = e^{A_m(T-\tau)}\zeta(\tau). \quad (5.7.5)$$

The boundary of $\dot{q}(\tau)$ can be derived as following:

$$\begin{aligned}\dot{q}(\tau) &= \frac{d}{d\tau} \left(e^{A_m(T-\tau)} \zeta(\tau) \right) = e^{A_m T} \frac{d}{d\tau} \left(e^{-A_m \tau} \right) \zeta(\tau) + e^{A_m T} e^{-A_m \tau} \frac{d}{d\tau} (\zeta(\tau)) \\ &= -e^{A_m T} A_m e^{-A_m \tau} \zeta(\tau) + e^{A_m T} e^{-A_m \tau} \dot{\zeta}(\tau).\end{aligned}\quad (5.7.6)$$

Using (5.7.6) and the definition of $\chi_0(T)$ together with the bounds of $\zeta(\tau)$ and $\dot{\zeta}(\tau)$, we have $\|\dot{q}(\tau)\| \leq \chi_0(T)b_\zeta + \chi_1(T)b_{d_\zeta}$. It follows from the definition of d_q in (5.4.2) that

$$\|\dot{q}(\tau)\| \leq d_q \quad (5.7.7)$$

Eqn (5.4.1) can be written as

$$\int_0^T q(\tau) d\tau = \varepsilon. \quad (5.7.8)$$

Consider the i th element of equation in (5.7.8), we have

$$\int_0^T q_i(\tau) d\tau = \varepsilon_i. \quad (5.7.9)$$

Since $q(\tau)$ is continuous, for any T , there exists $t_i \in [0, T]$ such that $q_i(t_i) = \varepsilon_i/T$, the variable $q_i(\tau)$ can be rewritten as follows $q_i(\tau) = q_i(t_i) + \int_{t_i}^{\tau} \dot{q}_i(\alpha) d\alpha$. Since $\tau - t_i \leq T$. Applying the results in (5.7.7) we have $q_i(\tau) \leq \varepsilon_i/T + Td_q$. As a result $\|q(\tau)\| \leq \|\varepsilon\|/T + \sqrt{n}Td_q$. It follows from the definition in (5.7.5) that $\zeta(\tau) = [e^{A_m(T-\tau)}]^{-1} q(\tau) = e^{-A_m T} e^{A_m \tau} q(\tau)$. The bound can be further derived as

$$\|\zeta(\tau)\| \leq \chi_2(T) \|q(\tau)\| \leq \chi_2(T) (\|\varepsilon\|/T + \sqrt{n}Td_q), \text{ which completes the proof.} \quad \square$$

Proof of Theorem 5.4.3: The proof will be done by contradiction. Assume that (5.4.19) and (5.4.20) are not true. Since $x(0) < \gamma_x, u(0) < \gamma_u$, and $x(t), u(t)$ are continuous, then there exists t' such that

$$x(t') = \gamma_x \quad \text{or} \quad u(t') = \gamma_u, \quad (5.7.10)$$

while

$$\|x_{t'}\|_{\mathcal{L}_\infty} \leq \gamma_x, \quad \|u_{t'}\|_{\mathcal{L}_\infty} \leq \gamma_u. \quad (5.7.11)$$

Define $\sigma_0(t) = b_m f(x(t), t) + \sigma(t)$. It follows from (5.4.17) that $\tilde{x}(iT + t) = e^{A_m t} \tilde{x}(iT) + \int_0^t e^{A_m(t-\tau)} \hat{\sigma}(iT) d\tau - \int_0^t e^{A_m(t-\tau)} \sigma_0(iT + \tau) d\tau$, which implies that

$$\begin{aligned} \tilde{x}((i+1)T) = & \quad (5.7.12) \\ e^{A_m T} \tilde{x}(iT) + \int_0^T e^{A_m(T-\tau)} \hat{\sigma}(iT) d\tau - \int_0^T e^{A_m(T-\tau)} \sigma_0(iT + \tau) d\tau. \end{aligned}$$

Substitution of the adaptive law (5.3.2) into (5.7.13) results in

$$\tilde{x}((i+1)T) = - \int_0^T e^{A_m(T-\tau)} \sigma_0(iT + \tau) d\tau. \quad (5.7.13)$$

It follows from Assumptions 5.2.1 and 5.2.2 and (5.7.11) that

$$|f(x(t), t)| \leq L(\gamma_x) \gamma_x + B_1, \quad (5.7.14)$$

and thus

$$\|\sigma_0(t)\| \leq \|b_m\| (L(\gamma_x) \gamma_x + B_1) + B_2, \quad (5.7.15)$$

for $t \in [0, t']$. By using the definitions in (5.4.3) and (5.4.4), it follows from (5.7.13) and (5.7.15) that the following upper bound holds:

$$\|\tilde{x}((i+1)T)\| \leq \eta(T) (\|b_m\| (L(\gamma_x) \gamma_x + B_1) + B_2) \leq \nu(\gamma_x, T), \quad (5.7.16)$$

for $(i+1)T \in [0, t']$.

In what follows, we derive the bound of $\hat{\sigma}(iT)$. Since A_m is a hurwitz matrix, eqn (5.3.2) can be written as $\hat{\sigma}(iT) = [e^{A_m T} - \mathbb{I}]^{-1} A_m e^{A_m T} \tilde{x}(iT)$. It follows from the definition of

$\eta_1(T)$, $\eta_2(T)$ in (5.4.5), (5.4.6) and eqn (5.7.16) that

$$\|\hat{\sigma}(iT)\| \leq \eta_2(T, \gamma_x). \quad (5.7.17)$$

In what follows, we characterize the bounds of $\|\hat{\sigma}(iT + \tau) - \sigma_0(iT + \tau)\|_\infty$. First, we derive the bounds of $\dot{\sigma}_0(t)$. For any $t \in [0, t']$, it follows from (5.7.11) that $\|\dot{x}(t)\|_\infty \leq \gamma_1$. where γ_1 is defined in (5.4.8). Assumption 5.2.3 further implies $\left| \frac{df(x(t), t)}{dt} \right| \leq d_{f_x}(\gamma_x)\gamma_1 + d_{f_t}(\gamma_x)$, and thus

$$\|\dot{\sigma}_0(t)\|_\infty \leq \gamma_2, \quad (5.7.18)$$

Let

$$\zeta(\tau) = \sigma_0(iT + \tau) - \hat{\sigma}(iT). \quad (5.7.19)$$

It follows from (5.7.15), (5.7.17) and the definition in (5.4.7) that

$$\|\zeta(\tau)\| \leq \|b_m\|(L(\gamma_x)\gamma_x + B_1) + B_2 + \eta_2(T, \gamma_x) = b_\zeta(T, \gamma_x). \quad (5.7.20)$$

It follows from (5.7.18) and definition in (5.7.19) that

$$\left\| \dot{\zeta}(\tau) \right\| \leq \gamma_2. \quad (5.7.21)$$

According to the choice of adaptive law (5.3.2), we have $e^{A_m T} \tilde{x}(iT) + \int_0^T e^{A_m(T-\tau)} \hat{\sigma}(iT) d\tau = 0$. we further obtain that

$$\begin{aligned} \tilde{x}(iT) &= (\mathbb{I} - e^{A_m T}) \tilde{x}(iT) - \int_0^T e^{A_m(T-\tau)} \hat{\sigma}(iT) d\tau \\ &= - \int_0^T e^{A_m(T-\tau)} (\hat{\sigma}(iT) + A_m \tilde{x}(iT)) d\tau. \end{aligned} \quad (5.7.22)$$

It follows from (5.7.13) that

$$\tilde{x}(iT) = - \int_0^T e^{A_m(T-\tau)} \sigma_0((i-1)T + \tau) d\tau. \quad (5.7.23)$$

Hence, (5.7.22) and (5.7.23) imply $\int_0^T e^{A_m(T-\tau)} \sigma_0((i-1)T + \tau) d\tau = \int_0^T e^{A_m(T-\tau)} (\hat{\sigma}(iT) + A_m \tilde{x}(iT)) d\tau$, which further implies that $\int_0^T e^{A_m(T-\tau)} (\hat{\sigma}(iT) - \sigma_0((i-1)T + \tau)) d\tau = - \int_0^T e^{A_m(T-\tau)} A_m \tilde{x}(iT) d\tau$. Let $\varepsilon = - \int_0^T e^{A_m(T-\tau)} A_m \tilde{x}(iT) d\tau$. Combing (5.7.20), (5.7.21), it follows from Lemma 5.4.2, we have

$\|\zeta(\tau)\| \leq \chi_2(T) (\|\varepsilon\| / T + \sqrt{n} T d_q) = \chi_2(T) (\|\int_0^T e^{A_m(T-\tau)} A_m \tilde{x}(iT) d\tau\| / T + \sqrt{n} T d_q)$. Since (5.7.20) and (5.7.21) holds for any i such that $(i+1)T \leq t'$, we have

$$\|\hat{\sigma}(t) - \sigma_0(t)\|_\infty \leq \eta_4(T), t \in [0, t'], \quad (5.7.24)$$

where $\eta_4(T)$ is defined in (5.4.10). Define

$$\begin{bmatrix} \sigma_1(t) & \sigma_2(t) \end{bmatrix}^T = \begin{bmatrix} f(x(t), t) & 0 \end{bmatrix}^T + [b_m \ B_{um}]^{-1} \sigma(t). \quad (5.7.25)$$

It can be verified that $\sigma_0(t) = [b_m \ B_{um}] \begin{bmatrix} \sigma_1(t) & \sigma_2(t) \end{bmatrix}^T$. Since $\hat{\sigma}(t) = [b_m \ B_{um}] \begin{bmatrix} \sigma_1(t) & \sigma_2(t) \end{bmatrix}^T$, we have

$$\begin{bmatrix} \hat{\sigma}_1(t) & \hat{\sigma}_2(t) \end{bmatrix}^T - \begin{bmatrix} \sigma_1(t) & \sigma_2(t) \end{bmatrix}^T = [b_m \ B_{um}]^{-1} (\hat{\sigma}(t) - \sigma_0(t)). \quad (5.7.26)$$

It follows from (5.7.24) and (5.7.26) that

$$\left\| \begin{bmatrix} \hat{\sigma}_1(t) & \hat{\sigma}_2(t) \end{bmatrix}^T - \begin{bmatrix} \sigma_1(t) & \sigma_2(t) \end{bmatrix}^T \right\|_\infty \leq \eta_5(T), \forall t \in [0, t'], \quad (5.7.27)$$

where $\eta_5(T)$ is defined in (5.4.11). Following the definition of ∞ norm, Eq. (5.7.27) implies

$$\|(\hat{\sigma}_1 - \sigma_1)_{t'}\|_{\mathcal{L}_\infty} \leq \eta_5(T), \|(\hat{\sigma}_2 - \sigma_2)_{t'}\|_{\mathcal{L}_\infty} \leq \eta_5(T). \quad (5.7.28)$$

The control law in (5.3.4) can be written as

$$u(t) = k_g r(t) - (u_f(t) - \sigma_f(t)) - \sigma_f(t), \quad (5.7.29)$$

Substituting (5.7.29) into (5.3.3) yields

$$\begin{aligned} \dot{\hat{x}}(t) &= A_m \hat{x}(t) + b_m k_g r(t) - b_m (u_f(t) - \sigma_f(t)) + \\ &\quad b_m (-\sigma_f(t) + \hat{\sigma}_1) + B_{um} \hat{\sigma}_2. \end{aligned} \quad (5.7.30)$$

It follows from (5.3.7) that

$$-\sigma_f(s) + \hat{\sigma}_1(s) = (1 - C_1(s))\hat{\sigma}_1(s) - C_2(s)M(s)\hat{\sigma}_2(s). \quad (5.7.31)$$

Substituting (5.7.31) into (5.7.30) yields

$$\begin{aligned} \hat{x}(s) &= (s\mathbb{I} - A_m)^{-1} b_m k_g r(s) + (s\mathbb{I} - A_m)^{-1} b_m (1 - C_1(s))\hat{\sigma}_1(s) \\ &\quad - (s\mathbb{I} - A_m)^{-1} b_m (u_f(s) - \sigma_f(s)) - (s\mathbb{I} - A_m)^{-1} b_m C_2(s)M(s)\hat{\sigma}_2(s) + (s\mathbb{I} - A_m)^{-1} B_{um} \hat{\sigma}_2(s), \end{aligned}$$

and thus

$$\|\hat{x}_{t'}\|_{\mathcal{L}_\infty} \leq \|G(s)\|_{\mathcal{L}_1} \|\hat{\sigma}_{1_{t'}}\|_{\mathcal{L}_\infty} + \|\zeta_{t'}\|_{\mathcal{L}_\infty}, \quad (5.7.32)$$

where $G(s)$ is defined in (5.4.13) and $\zeta(t)$ is signal with its Laplace transformation $(s\mathbb{I} - A_m)^{-1} b_m k_g r(s) - (s\mathbb{I} - A_m)^{-1} b_m C_2(s)M(s)\hat{\sigma}_2(s) - (s\mathbb{I} - A_m)^{-1} b_m (u_f(t) - \sigma_f(t)) + (s\mathbb{I} - A_m)^{-1} B_{um} \hat{\sigma}_2(s)$. It follows from (5.7.14) and (5.7.25) that

$$\|\sigma_{1_{t'}}\|_{\mathcal{L}_\infty} \leq L(\gamma_x)\gamma_x + B_1 + \|[b_m \ B_{um}]^{-1}\|_{\mathcal{L}_1} B_2. \quad (5.7.33)$$

Since $\|\hat{\sigma}_{1_{t'}}\|_{\mathcal{L}_\infty} \leq \|\sigma_{1_{t'}}\|_{\mathcal{L}_\infty} + \|(\hat{\sigma}_1 - \sigma_1)_{t'}\|_{\mathcal{L}_\infty}$, It follows from (5.7.33) and (5.7.28) that

$$\|\hat{\sigma}_{1_{t'}}\|_{\mathcal{L}_\infty} \leq L(\gamma_x)\gamma_x + B_1 + \|[b_m \ B_{um}]^{-1}\|_1 B_2 + \eta_5(T), \quad \|\hat{\sigma}_{2_{t'}}\|_{\mathcal{L}_\infty} \leq B_2 + \eta_5(T). \quad (5.7.34)$$

It follows from Lemma 5.4.1, (5.4.15) and (5.7.34) that $\|\zeta_{t'}\|_{\mathcal{L}_\infty} \leq \gamma_0$, and thus (5.7.32)

implies

$$\|\hat{x}_{t'}\|_{\mathcal{L}_\infty} \leq \|G(s)\|_{\mathcal{L}_1} (L(\gamma_x)\gamma_x + B_1 + \|[b_m \ B_{um}]^{-1}\|_{\mathcal{L}_1} B_2 + \eta_5(T)) + \gamma_0. \quad (5.7.35)$$

Following (5.4.16) and (5.7.35), we obtain

$$\|\hat{x}_{t'}\|_{\mathcal{L}_\infty} < \gamma_x - \gamma_e(T). \quad (5.7.36)$$

It follows from (5.4.17) and (5.7.24) that

$$\|\tilde{x}_{t'}\|_{\mathcal{L}_\infty} \leq \gamma_e(T), \quad (5.7.37)$$

where $\gamma_e(T)$ is defined in (5.4.14). Since $x(t) = \hat{x}(t) - \tilde{x}(t)$, we have $\|x_{t'}\|_{\mathcal{L}_\infty} \leq \|\hat{x}_{t'}\|_{\mathcal{L}_\infty} + \|\tilde{x}_{t'}\|_{\mathcal{L}_\infty}$, which combined with (5.7.36) and (5.7.37) leads to

$$\|x_{t'}\|_{\mathcal{L}_\infty} < \gamma_x. \quad (5.7.38)$$

It follows from Lemma 5.4.1 and (5.7.29) that

$$\|u_{t'}\|_{\mathcal{L}_\infty} \leq |k_g| \|r\|_{\mathcal{L}_\infty} + 1/\mu + \|\sigma_{f_{t'}}\|_{\mathcal{L}_\infty}, \quad (5.7.39)$$

and it follows from (5.3.7) that

$$\begin{aligned} \|\sigma_{f_{t'}}\|_{\mathcal{L}_\infty} &\leq \|C_1(s)\|_{\mathcal{L}_1} (L(\gamma_x)\gamma_x + B_1 + \|[b_m \ B_{um}]^{-1}\|_{\mathcal{L}_1} B_2 + \eta_5(T)) + \\ &\quad \|C_2(s)M(s)\|_{\mathcal{L}_1} (B_2 + \eta_5(T)). \end{aligned} \quad (5.7.40)$$

Combining (5.7.39) and (5.7.40) yields

$$\|u_{t'}\|_{\mathcal{L}_\infty} < \gamma_u, \quad (5.7.41)$$

where γ_u is defined in (5.4.12).

Note that (5.7.38) and (5.7.41) contradict (5.7.11) and thus (5.4.19) and (5.4.20) must be true. Furthermore, (5.4.18) follows from (5.7.37) directly since (5.4.19-5.4.20) hold. This concludes the proof. \square

Proof of Theorem 5.4.4: For the first order low-pass filter in (5.4.21), it follows from “Verifying the \mathcal{L}_1 -Norm Bound” part in [187] that

$$\lim_{\omega \rightarrow \infty} \|G(s)\|_{\mathcal{L}_1} = 0. \quad (5.7.42)$$

Following the definitions of $\eta(T)$ in (5.4.3), $\nu(\gamma_x, T)$ in (5.4.4) and $\eta_4(T)$ in (5.4.10), it can be easily verified that $\lim_{T \rightarrow 0} \eta(T) = 0$, $\lim_{T \rightarrow 0} \nu(\gamma_x, T) = 0$ and $\lim_{T \rightarrow 0} \eta_4(T) = 0$. Then, we can further obtain that

$$\lim_{T \rightarrow 0} \eta_5(T) = 0, \quad (5.7.43)$$

$$\begin{aligned} \lim_{T \rightarrow 0} \gamma_0 &= \|(s\mathbb{I} - A_m)^{-1}k_g\|_{\mathcal{L}_1}\|r\|_{\infty} + \|(s\mathbb{I} - A_m)^{-1}b_m\|_{\mathcal{L}_1}1/\mu \\ &\quad + \|(s\mathbb{I} - A_m)^{-1}b_mC_2(s)M(s) + (s\mathbb{I} - A_m)^{-1}B_{um}\|_{\mathcal{L}_1}B_2, \end{aligned} \quad (5.7.44)$$

$$\lim_{T \rightarrow 0} \gamma_e(T) = 0, \quad (5.7.45)$$

For any γ_x such that

$$\gamma_x > \gamma_0 + \gamma_e(T), \quad (5.7.46)$$

if

$$\|G(s)\|_{\mathcal{L}_1} < \frac{\gamma_x - \gamma_0 - \gamma_e(T)}{L(\gamma_x)\gamma_x + B_1 + \|[b_m \ B_{um}]^{-1}\|_1 B_2 + \eta_5(T)}, \quad (5.7.47)$$

(5.4.16) can always be satisfied. Reducing T , it follows from (5.7.45) that there exists a finite positive γ_x satisfying (5.7.46) and hence $\frac{\gamma_x - \gamma_0 - \gamma_e(T)}{L(\gamma_x)\gamma_x + B_1 + \|[b_m \ B_{um}]^{-1}\|_1 B_2 + \eta_5(T)}$ is a finite positive number. Hence, it follows from (5.7.42) and (5.7.47) that (5.4.16) can always be satisfied by increasing ω and reducing T . Therefore, it follows from Theorem 5.4.3 that the entire closed-loop system can always be made stable with a small enough value of T and a large enough value of ω_1 . \square

Chapter 6

Adaptive Control for Systems with Output Constraints Using an Online Optimization Method

6.1 Introduction

In practical applications of control engineering, systems are commonly subject to constraints. These include, input constraints due to physical actuator limits and state or output constraints resulting from material or structural limits of the plant. A good example is flight envelope protection [195–197], if the controller is unable to maintain the aircraft within its flight envelope it could yield catastrophic results. This chapter analyzes linear time-invariant systems in the presence of non-linear time-varying matched uncertainties subject to output constraints. The results can also be extended to systems including non-minimum phase output constraints which present serious challenges to control engineers in industry today.

Due to industry demand, much research has been conducted on control problems with the presence of input, state, or output constraints. For constrained systems, Model Pre-

dictive Control (MPC) [198, 199] has gained significant popularity in both academia and industry. This is because the MPC method makes it convenient to introduce constraints into the problem formulation. Although, the presence of active constraints may yield closed-loop system instability. Some earlier analysis on the stability and performance of MPC algorithms in the presence of system constraints include [200, 201] a more detailed historical review of MPC stability analysis can be found in [202]. Constraint softening schemes have been established to address closed loop instability issues [203–205]. Contractive model predictive control [206] is also shown to improve closed loop stability by adding an additional state constraint. However, when systems demonstrate non-minimum phase characteristics constraint softening methods may have reduced performance. An extension of constraint softening which also employs time dependent weightings on the objective functions to improve controller performance is found in [207]. Though, these methods do not sufficiently consider the effects of unknown disturbances and/or uncertainties which are often present in complex systems.

Introducing disturbances or uncertainty into the system can destabilize predictive control schemes. As model predictive control theory matured research activities began to investigate robust model predictive control where disturbances and/or uncertainties are included in the problem formulation. In [208] a modified min-max problem for guaranteeing robust stability for a set of finite impulse response (FIR) models was developed. This work provides some of the first analysis to explicitly deal with system uncertainty though is applicable to a very specific class of systems. A less restrictive framework which implements linear matrix inequalities (LMI) for systems with polytopic uncertainty was investigated in [209]. A more general set of polytopic uncertain systems was addressed through an improved LMI based approach by employing multiple Lyapunov functions in [210]. Similar to system uncertainties, disturbances can also present challenges in regards to stability. A predictive control scheme in the presence of persistent input disturbances through constraint restrictions is presented in [211]. Though, plants are usually subject to both input and output disturbances i.e. measurement noise. A scheme using a combination of a stable state estimator (Luenberger observer) and a tube based model predictive controller to control constrained systems in the presence of unknown input and output disturbances was devel-

oped in [212] which was later extended to include a time-varying state estimator presented in [213]. Though, many of these methods have large on-line computational demands, off-line schemes have been investigated to reduce the computational burden in [214, 215].

While significant progress has been made in dealing with constrained systems subject to uncertainties, they do not address systems subject to time-varying non-linear uncertainties. The controller proposed in this chapter consists of two major components. The first component is the L1 controller [5]. In this architecture, fast adaptation and satisfactory transient response are delivered with stable tracking performance in the presence of unknown high-frequency gain, time-varying unknown parameters and time-varying bounded disturbances [37, 148]. The L1 control law consists of two components, the reference tracking component and the uncertainty cancellation component. In our proposed scheme, the uncertainty cancellation component is always active whereas the tracking component is subject to a switching logic. The second component employs a finite prediction horizon similar to that found in other predictive control schemes. The control signal space for the prediction horizon optimization is limited to first order hold trajectories. At each sampling time the finite prediction horizon optimization finds a feasible solution set which can maintain minimum-phase and non-minimum phase output constraints. By tuning the time horizon it is possible to prevent output violations for not only low order systems but also high order systems with slower dynamics. The controller will remain in the tracking mode (L1 reference tracking component) as long as the feasible solution set space is a finite volume (i.e. a non-unique non-empty solution set). Though, if the feasible solution set space reduces to zero the control law switches from the tracking mode to the first order hold dictated by the finite prediction horizon optimization. The control sequence is applied until the optimization returns to a non-empty solution set. The online optimization can be formulated into a linear program (LP) for which many efficient solver strategies have been developed.

The chapter is organized as follows. Section 6.2 provides the problem formulation. Section 6.3 presents the adaptive law and control law. Section 6.4 provides analysis for the L1 adaptive controller. Section 6.5 includes simulation results for the proposed controller. Finally, Section 6.6 discusses a summary of the theoretical framework presented and results.

6.2 Problem Formulation

Consider the following n th order multiple-input multiple-output (MIMO) system with full state feedback:

$$\begin{aligned}\dot{x}(t) &= Ax(t) + B(u(t) + \sigma(t)) \\ y(t) &= C^T x(t) \\ y_c(t) &= C_c x(t) \quad x(0) = 0,\end{aligned}\tag{6.2.1}$$

where $x \in \mathbb{R}^n$ is the system state vector (measurable), $u \in \mathbb{R}^p$ is the control signal. $B \in \mathbb{R}^{n \times p}$ is a known constant matrix, $y_r \in \mathbb{R}^{n \times r}$ is the tracking output, $y_c \in \mathbb{R}^{n \times c}$ is the constrained output. A is a known $n \times n$ Hurwitz matrix, (A, B) is controllable, $\sigma(t) = f(t, x(t)) \in \mathbb{R}^p$ is the unknown uncertainty subject to the following assumption:

Assumption 6.2.1. There exist constants $L > 0$ and $L_0 > 0$ such that the following inequalities hold uniformly in $t \geq 0$:

$$\|f(t, x_1) - f(t, x_2)\|_\infty \leq L\|x_1 - x_2\|_\infty + L_0.\tag{6.2.2}$$

Assumption 6.2.2. There exist constants $L_1 > 0$, $L_2 > 0$ and $L_3 > 0$ such that the following inequalities hold uniformly in $t \geq 0$:

$$\|\dot{\sigma}(t)\|_\infty \leq L_1\|\dot{x}(t)\|_\infty + L_2\|x(t)\|_\infty + L_3.\tag{6.2.3}$$

Remark 6.2.3. The Assumptions 6.2.1 and 6.2.2 are the global Lipschitz condition [138] page93. These two assumptions ensure the solution of an ODE system exists. In this chapter, Assumption 6.2.1 and 6.2.2 particularly ensure the signals $\sigma(t)$ and $\dot{\sigma}(t)$ are bounded by the state variable $x(t)$ and derivative of the state variable $\dot{x}(t)$.

The control objective is to design an adaptive state feedback control signal $u(t)$ such that the system output $y_r(t)$ tracks reference input $r(t)$ following a desired reference system

$y_r(s) \approx c^T(sI - A)^{-1}B$ and maintain the output $y_c(t)$ within certain limits

$P_{min}(i) \leq y_{ci}(t) \leq P_{max}(i)$. $y_{ci}(t)$ indicates the i^{th} element in the vector y_c , and $P_{min}(i)$, $P_{max}(i)$ are the lower and upper bounds for $y_{ci}(t)$, $\forall t \geq 0$. Without loss of generality, we suppose there exist constants B_r and B_{dr} that:

$$\|r(t)\|_\infty < B_r, \|\dot{r}(t)\|_\infty < B_{dr}. \quad (6.2.4)$$

6.3 \mathcal{L}_1 Adaptive Controller

In this section, we introduce a novel control law design into the \mathcal{L}_1 adaptive architecture to handle the output constraints problem. The entire controller consists of the state predictor, adaptive law and control law. In what follows, we introduce each controller component separately.

6.3.1 State Predictor

The state predictor together with the adaptive law are designed for fast estimation of the unknown disturbance $\sigma(t)$. We consider the following state predictor

$$\begin{aligned} \dot{\hat{x}}(t) &= A\hat{x}(t) + Bu(t) + \hat{\sigma}(t), \\ \hat{y}(t) &= C^T\hat{x}(t), \quad \hat{x}(0) = x_0 \end{aligned} \quad (6.3.1)$$

where $\hat{\sigma}(t)$ is defined in following adaptive law:

6.3.2 Adaptive Law

The piece-wise constant adaptive law is used in this chapter in order to make $\hat{x}(t)$ track $x(t)$. The update law for $\hat{\sigma}(t)$ is given by

$$\begin{aligned}\hat{\sigma}(t) &= \hat{\sigma}(iT), \quad t \in [iT, (i+1)T] \\ \hat{\sigma}(iT) &= -\Phi^{-1}(T)e^{AT}\tilde{x}(iT), i = 0, 1, 2, 3\ldots\end{aligned}\tag{6.3.2}$$

where $\tilde{x}(t) = x(t) - \hat{x}(t)$ is the estimation error vector of the state predictor and $\Phi(T)$ is defined as: $\Phi(T) = \int_0^T e^{A(T-\tau)}d\tau$. The adaptive law drives $\tilde{x}(t)$ to be small and as a result, $\hat{\sigma}(t)$ will approach $Bf(t, x(t))$ (Details in lemma 6.4.2). The information of $\hat{\sigma}(t)$ will be used in the control law design.

6.3.3 Control Law

The overall control law design consists of two components as shown in Figure 6.3.1. The

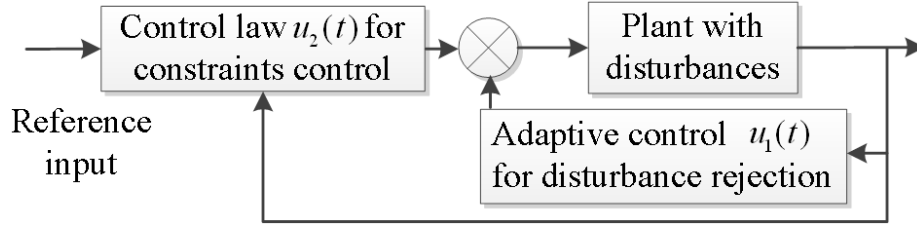


FIGURE 6.3.1: Control architecture

first component aims at correcting the plant with unknown disturbances and nonlinearities present to a nominal plant, namely, disturbance rejection control law. The second component is the output constraint violation avoidance control law, when an output is in danger of violating a constraint in the prediction horizon, the control objective changes from the tracking objective to the output constraint violation avoidance objective. The overall

control law is defined as follows:

$$u(t) = u_1(t) + u_2(t). \quad (6.3.3)$$

For the convenience of the analysis, the state predictor described in (6.3.1) can be separated into the following two subsystems $\hat{x}_1(t)$ and $\hat{x}_2(t)$ in (6.3.4) and (6.3.6)

$$\begin{aligned} \dot{\hat{x}}_1(t) &= A\hat{x}_1(t) + Bu_1(t) + \hat{\sigma}(t), \\ \hat{y}_1(t) &= C^T \hat{x}_1(t). \end{aligned} \quad (6.3.4)$$

Letting

$$u_2(t) := u(t) - u_1(t), \quad \hat{x}_2(t) := \hat{x}(t) - \hat{x}_1(t), \quad \hat{y}_2(t) := \hat{y}(t) - \hat{y}_1(t). \quad (6.3.5)$$

substituting (6.3.4) and (6.3.5) into (6.3.1), yields:

$$\begin{aligned} \dot{\hat{x}}_2(t) &= A\hat{x}_2(t) + Bu_2(t), \\ \hat{y}_2(t) &= C^T \hat{x}_2(t). \end{aligned} \quad (6.3.6)$$

Let $x_1(t) := x(t) - \hat{x}_2(t)$ using eqn (6.2.1), we have

$$\begin{aligned} \dot{x}_1(t) &= Ax_1(t) + B(u_1(t) + \sigma(t)), \\ y_1(t) &= C^T x_1(t). \end{aligned} \quad (6.3.7)$$

Disturbance Rejection Control Law

The control law $u_1(t)$ is a controller of the state predictor designed to cancel the effects of the disturbance $\sigma(t)$ in (6.3.7):

$$u_1(t) = -C(s)(B^T B)^{-1} B^T \hat{\sigma}(t), \quad (6.3.8)$$

where $C(s)$ is a low-pass filter array to filter the control signal.

Remark 6.3.1. The objective of adaptive law is to make the adaptive parameter $\hat{\sigma}(t)$ approach the real disturbance $\sigma(t)$. The effects of the control law $u_1(t)$ cancels the effects of $\sigma(t)$ and as a result $\hat{x}_1(t)$ in (6.3.4) and $x_1(t)$ in (6.3.7) will become very small. Since $x(t) = x_1(t) + \hat{x}_2(t)$, the system response $x(t)$ in (6.2.1) is transformed very similar to $\hat{x}_2(t)$ in (6.3.6). The bound of $x_1(t)$ will be discussed later.

Constraint Violation Avoidance Control Law

The control law $u_2(t)$ is designed for the system with constraints. Since the control law $u_1(t)$ has already canceled the unknown disturbances and transformed the system dynamic close to (6.3.6) which has been discussed in Remark 3.1. The constraint violation avoidance control law is designed based on system (6.3.6) with no uncertainties.

The control objective is to make $y_r(t)$ track a desired reference signal $r(t)$ with reasonable performance. While maintaining output constraints $P_{min}(i) \leq y_{ci}(t) \leq P_{max}(i)$, $i = 1, 2, 3, \dots, n$, $\forall t \geq 0$, where y_c is defined in (6.2.1).

There are two working modes for the $u_2(t)$, namely tracking mode and Constraint Violation Avoidance Mode (CVAM). The system normally works in the tracking mode if there is no risk to violate the constraints in the prediction horizon. The CVAM mode is designed to avoid the constraints violation. The switching logic between two modes is based on a online optimization program. The details of the control law u_2 is introduced in the following procedures:

At first, we need to design a predicting diagram for the constrained output. For the issue of output prediction, the most critical part is the trajectory of the control signal. We confine the CVAM control signal space to a first-order hold trajectory, which is

$$u_c(t_0, t_0 + \tau) = u(t_0) + \beta\tau, \quad (6.3.9)$$

where $\beta \in \mathbb{R}^p$ is an unknown vector to be determined. Note that upon switching, the control signal remains continuous. In the following, the selection of parameters β and switching time are discussed.

With the deterministic known system obtained in (6.3.6), the output at each time could be calculated based on the control signal which will be defined later in (6.3.27). Consider N time points ahead of current time instant t (including the current time instant), we can derive the output at each time point as follows. Without loss of generality, we could assume u_0 as the current control signal $u(t_0) = u_0$ and $x(t_0)$ as the current state value. In the next section, the time domain analysis will be introduced to derive the control law.

To introduce the control law $u_2(t)$. The following analysis is need. Assume the control law $u_c(t)$ is activated at time t_0 on system (6.3.6). During time $t \in [t_0, t_0 + t_1]$, the behavior of the system is governed by the following equation:

$$\begin{aligned} x(t_0 + t_1) &= e^{A(t_1)}x(t_0) + \int_0^{t_1} e^{A(t_1-\tau)}d\tau Bu(t_0) \\ &+ \sum_{i=1}^p \int_0^{t_1} e^{A(t_1-\tau)}(\tau I)d\tau B_i\beta_i \end{aligned} \quad (6.3.10)$$

$$\begin{aligned} y_c(t_0 + t_1) &= C_c x(t_0 + t_1) \\ &= C_c e^{A(t_1)}x(t_0) + C_c \int_0^{t_1} e^{A(t_1-\tau)}d\tau Bu(t_0) \\ &+ \sum_{i=1}^p C_c \int_0^{t_1} e^{A(t_1-\tau)}(\tau I)d\tau B_i\beta_i. \end{aligned} \quad (6.3.11)$$

Define

$$f_0(t_1) = C_c e^{A(t_1)} \quad (6.3.12)$$

$$f_u(t_1) = C_c \int_0^{t_1} e^{A(t_1-\tau)} d\tau B \quad (6.3.13)$$

$$f_i(t_1) = C_c \int_0^{t_1} e^{A(t_1-\tau)} (\tau I) d\tau B_i. \quad (6.3.14)$$

We have

$$y_c(t_0 + t_1) = f_0(t_1)x(t_0) + f_u(t_1)u(t_0) + \sum_{i=1}^p f_i(t_1)\beta_i. \quad (6.3.15)$$

Similarly if we consider N points at $t + t_1, t + t_2, \dots, t + t_N$,

$$\begin{aligned} y_c(t_0 + t_1) &= f_0(t_1)x(t_0) + f_u(t_1)u(t_0) + \sum_{i=1}^p f_i(t_1)\beta_i \\ y_c(t_0 + t_2) &= f_0(t_2)x(t_0) + f_u(t_2)u(t_0) + \sum_{i=1}^p f_i(t_2)\beta_i \\ &\vdots \\ y_c(t_0 + t_N) &= f_0(t_N)x(t_0) + f_u(t_N)u(t_0) + \sum_{i=1}^p f_i(t_N)\beta_i, \end{aligned} \quad (6.3.16)$$

where t_N is the prediction time horizon.

Consider the system with upper boundary $P_{max} \in \mathbb{R}^{q^2}$, and lower boundary $P_{min} \in \mathbb{R}^{q^2}$. At each horizon time step, there exists the following inequality.

$$P_{min}(i) \leq y_{ci}(t) \leq P_{max}(i), i = 1, 2, \dots, N, \forall t \geq 0. \quad (6.3.17)$$

The equation (6.3.17) can be written into the following matrix form:

$$P_{min} \leq A_1\beta + A_2 \leq P_{max}, \quad (6.3.18)$$

where

$$A_1 = \begin{bmatrix} f_1(\tau_1) & f_2(\tau_1) & \cdots & f_p(\tau_1) \\ f_1(\tau_2) & f_2(\tau_2) & \cdots & f_p(\tau_2) \\ \vdots & \vdots & \ddots & \vdots \\ f_1(\tau_N) & f_2(\tau_N) & \cdots & f_p(\tau_N) \end{bmatrix} \quad (6.3.19)$$

$$A_2 = \begin{bmatrix} f_0(t_1)x(t_0) + f_u(t_1)u(t_0) \\ f_0(t_2)x(t_0) + f_u(t_2)u(t_0) \\ \vdots \\ f_0(t_N)x(t_0) + f_u(t_N)u(t_0) \end{bmatrix}. \quad (6.3.20)$$

Defining a feasible set S ,

$$S = \{\beta | \beta \in R^p, P_{min} \leq A_1\beta + A_2 \leq P_{max}, |\beta(i)| \leq \beta_{max}(i)\}, \quad (6.3.21)$$

where $|\beta(i)|$ represents the absolute value of the i^{th} element in β , and $\beta_{max}(i)$ indicates the rate saturation of the i^{th} control signal. The volume of the feasible set is given by

$$V_\beta(t) = \int \int \cdots \int_{\beta \in S} dV \quad (6.3.22)$$

at time instant t .

For the switching logic, we need to introduce a state variable λ to indicate the mode:

$$\lambda(t) = \begin{cases} 1, & V_\beta > 0 \\ 2, & V_\beta = 0, \end{cases} \quad (6.3.23)$$

where $\lambda(t) = 1$ indicates tracking mode and $\lambda(t) = 2$ indicates holding mode. The switching time t_0 records the most recent time at which $\lambda(t)$ switches from 1 to 2 as follows:

$$t_0 = \max\{t_0 | t_0 \in R, \lambda(t_0) = 2, \lambda(t_{0-}) = 1\}. \quad (6.3.24)$$

The switching logic of the control signal is given by

$$\begin{aligned} u_2(t) &= u_r(t), & \text{if } \lambda(t) &= 1, \\ u_2(t) &= u_c(t, t_0), & \text{if } \lambda(t) &= 2, \end{aligned} \quad (6.3.25)$$

where

$$u_r(t) = k_g r(t) \quad (6.3.26)$$

$$u_c(t, t_0) = u_2(t_0) + \beta(t - t_0). \quad (6.3.27)$$

k_g in (6.3.26) is a gain matrix to ensure that the diagonal elements of the transfer matrix $C(sI - A)^{-1}Bk_g$ have DC gain equal to one, while the off-diagonal elements have zero DC gain.

Remark 6.3.2. The switching logic is based on the existence of the feasible set S in (6.3.21). An online linear optimization algorithm is employed to indicate if the feasible set S is empty or non-empty.

The linear optimization is defined as follows:

$$\begin{aligned} \text{Maximize:} \quad & w^T \beta \\ \text{Subject to:} \quad & P_{min} \leq A_1 \beta + A_2 \leq P_{max}, \end{aligned} \quad (6.3.28)$$

where the vector w can be any non-zero value. In this chapter, we concern more about the volume of the feasible set S . The switch time t_0 happens when the linear optimization in (6.3.28) returns no solution, which means the volume of the feasible set S is zero from time

t_0 . As a result, the control law u_2 switches to u_c at time t_0 . The control parameter β can be chosen from the last non-empty set S .

Assumption 6.3.3. When the switching occurs at t_0 , there exists a finite time t_c with upper bound $t_c < t_N$ that the control signal $u_c(t, t_0)$, $t \in [t_0, t_0 + t_c]$ in (6.3.27) can make the feasible set S nonempty at t_c , where S is defined in (6.3.21) and t_N is the prediction horizon in (6.3.16).

Assumption 6.3.4. (Stability Condition) The choice of the low-pass filter $C(s)$ needs to ensure

$$\|(sI - A)^{-1}B(1 - C(s))\|_{\mathcal{L}_1} L < 1. \quad (6.3.29)$$

Remark 6.3.5. Assumption 6.3.3 ensures that the switching occurs in a finite time. So that the control signal can switch back to u_r in (6.3.26) within a finite time. Assumption 3.1 also introduces a restriction on the system for the approach in this chapter. Assumption 6.3.3 is required to derive the stability of the system by small gain theorem. Other than this stability condition, the sampling time T needs to satisfy (6.4.59) and (6.4.60), the existence of T will be proved in Lemma 6.4.6.

6.4 Analysis of \mathcal{L}_1 Adaptive Controller with Output Constraints

This section is organized as follows:

- Lemma 6.4.1 derives the bounds of the system $\hat{x}_2(t)$ defined in (6.3.6), the ideal system dynamic without disturbances, which is governed by the constraint violation avoidance control u_2 .
- Lemma 6.4.2 and Lemma 6.4.3 analyze the performance of the adaptive law under the condition $\|x_{t_1}\|_{\mathcal{L}_\infty} \leq \rho_x$ and $\|u_{t_1}\|_{\mathcal{L}_\infty} \leq \rho_u$. Later in theorem 6.4.8, this condition

is proved to be true.

- Lemma 6.4.5 analyzes the bound of the system state under the condition $\|x_{t_1}\|_{\mathcal{L}_\infty} \leq \rho_x$ and $\|u_{t_1}\|_{\mathcal{L}_\infty} \leq \rho_u$.
- Lemma 6.4.6 gives the stability condition of the controller.
- Theorem 6.4.8 proves the stability of the system $\|x\|_{\mathcal{L}_\infty} \leq \rho_x$. Therefore, making the condition $\|x_{t_1}\|_{\mathcal{L}_\infty} \leq \rho_x$ and $\|u_{t_1}\|_{\mathcal{L}_\infty} \leq \rho_u$ true in Lemma 6.4.2 and Lemma 6.4.3. Hence, Lemma 6.4.2 gives the bounds of $\|\tilde{x}\|_{\mathcal{L}_\infty}$.
- Theorem 6.4.9 analyzes the bound of the constraint violation.

Define

$$\rho_{x2} := \|(s\mathbb{I} - A)^{-1}B\|_{\mathcal{L}_1}(\|k_g\|_{\mathcal{L}_1}B_r + t_N B_\beta) \quad (6.4.1)$$

where B_β is the maximum absolute value of $\beta_{max}(i)$ $i = 1, 2, \dots, p$ and t_N is defined in (6.3.16).

Lemma 6.4.1. *For the system defined in (6.3.6) is subject to the control law of $u_2(t)$ defined in (6.3.25),*

$$\|\hat{x}_2\|_{\mathcal{L}_\infty} \leq \rho_{x2}. \quad (6.4.2)$$

Proof From Assumption 6.2.2, and the constraint on the control signal u_c in (6.3.21) we can derive that the control signal u_c is bounded as

$$\|u_c(t)\|_\infty \leq \|u_2(t_0)\|_\infty + t_N B_\beta, \quad (6.4.3)$$

where $B_\beta = \sum_{i=1}^N |\beta_{max}(i)|$ and $\beta_{max}(i)$ is the rate limitation of the control signal $u_c(t)$. Since $u_2(t_0) = k_g r(t_0)$, Substituting the bound of $r(t)$ in (6.2.4), we have

$$\|u_2(t_0)\|_\infty \leq \|k_g\|_{\mathcal{L}_1} B_r. \quad (6.4.4)$$

It follows from (6.4.3) that

$$\|u_c(t)\|_\infty \leq \|k_g\|_{\mathcal{L}_1} B_r + t_N B_\beta. \quad (6.4.5)$$

Using the definition of u_2 in (6.3.25), we have $u_2 = u_r$ or $u_2 = u_c$. The bound of u_r has already been considered in the first term of (6.4.5). We have the bound of u_2 :

$$\|u_2\|_{\mathcal{L}_\infty} \leq \|k_g\|_{\mathcal{L}_1} B_r + t_N B_\beta. \quad (6.4.6)$$

Since A is Hurwitz, the system in (6.3.6) has the following performance bound

$$\|\hat{x}_2\|_{\mathcal{L}_\infty} \leq \|(s\mathbb{I} - A)^{-1} B\|_{\mathcal{L}_1} (\|k_g\|_{\mathcal{L}_1} B_r + t_N B_\beta). \quad (6.4.7)$$

Considering the definition of ρ_{x2} in (6.4.1), thus $\|\hat{x}_2\|_{\mathcal{L}_\infty} \leq \rho_{x2}$ which completes the proof. \square

Let:

$$\eta(T) := \int_0^T \|e^{A\alpha} B\| d\alpha \quad (6.4.8)$$

$$\gamma_0(T, \rho_x) := \eta(T)(L\rho_x + L_0)\sqrt{n}, \quad (6.4.9)$$

$$b_{d\sigma}(\rho_x) := L(\|A\|_{\mathcal{L}_1} \rho_x + \|B\|_{\mathcal{L}_1} \rho_u + L\rho_x + L_0) + L_2(L\rho_x + L_0) + L_3, \quad (6.4.10)$$

$$\gamma_1(T, \rho_x, \rho_u) := \sqrt{\lambda_{\max}(A_m^\top A_m)} \gamma_0(T, \rho_x) + 2b_{d\sigma}(\rho_x) T \sqrt{n}, \quad (6.4.11)$$

$$\gamma_2(T, \rho_x, \rho_u) := \sqrt{n} \| (sI - A)^{-1} \|_{\mathcal{L}_1} \gamma_1(T, \rho_x, \rho_u), \quad (6.4.12)$$

$$\tilde{\sigma} := \hat{\sigma}(t) - B\sigma(t). \quad (6.4.13)$$

Lemma 6.4.2. *Considering the system described in (6.2.1) together with the state predictor (6.3.1), adaptive law (6.3.2) and control law (6.3.3), if the truncated \mathcal{L}_∞ norm $\|x_{t_1}\|_{\mathcal{L}_\infty} \leq \rho_x$, $\|u_{t_1}\|_{\mathcal{L}_\infty} \leq \rho_u$ for any time $t_1 \geq 0$, we have*

$$\|\tilde{\sigma}_{t_1}\| \leq \gamma_1(T, \rho_x, \rho_u), \quad \|\tilde{x}_{t_1}\| \leq \gamma_2(T, \rho_x, \rho_u). \quad (6.4.14)$$

Proof Subtracting (6.3.1) from (6.2.1), we get

$$\dot{\tilde{x}}(t) = A\tilde{x}(t) + \hat{\sigma}(t) - B\sigma(t). \quad (6.4.15)$$

The solution of (6.4.15) in $[(i-1)T, (i-1)T+t], t \in [0, T]$ is

$$\begin{aligned} \tilde{x}((i-1)T+t) &= e^{At} \tilde{x}((i-1)T) + \int_{(i-1)T}^{(i-1)T+t} e^{A((i-1)T+t-\tau)} \hat{\sigma}((i-1)T) d\tau \\ &\quad - \int_{(i-1)T}^{(i-1)T+t} e^{A((i-1)T+t-\tau)} B\sigma(\tau) d\tau \end{aligned} \quad (6.4.16)$$

$$\begin{aligned} &= e^{At} \tilde{x}((i-1)T) + \int_0^t e^{A(t-\tau)} \hat{\sigma}((i-1)T) d\tau - \\ &\quad \int_{(i-1)T}^{(i-1)T+t} e^{A((i-1)T+t-\tau)} B\sigma(\tau) d\tau. \end{aligned} \quad (6.4.17)$$

When $t = T$, it follows from (6.4.17) that

$$\begin{aligned} \tilde{x}(iT) &= e^{AT} \tilde{x}((i-1)T) + \int_{(i-1)T}^{iT} e^{A((i-1)T+t-\tau)} \hat{\sigma}((i-1)T) d\tau \\ &\quad - \int_{(i-1)T}^{iT} e^{A((i-1)T+t-\tau)} B\sigma(\tau) d\tau. \end{aligned} \quad (6.4.18)$$

According to the choice of adaptive law in (6.3.2), we have

$$e^{AT} \tilde{x}((i-1)T) + \int_{(i-1)T}^{iT} e^{A(iT-\tau)} \hat{\sigma}(iT) d\tau = 0. \quad (6.4.19)$$

It follows from (6.4.18) that

$$\tilde{x}(iT) = - \int_{(i-1)T}^{iT} e^{A(iT-\tau)} B \sigma(\tau) d\tau. \quad (6.4.20)$$

Taking the norm of (6.4.20)

$$\begin{aligned} \|\tilde{x}(iT)\| &= \left\| \int_{(i-1)T}^{iT} e^{A(iT-\tau)} B \sigma(\tau) d\tau \right\| \\ &\leq \int_{(i-1)T}^{iT} \left\| e^{A(iT-\tau)} B \right\| \|\sigma(\tau)\| d\tau \\ &\leq \int_{(i-1)T}^{iT} \left\| e^{A(iT-\tau)} B \right\| \|\sigma(\tau)\|_{\infty} \sqrt{n} d\tau \\ &\leq \int_{(i-1)T}^{iT} \left\| e^{A(iT-\tau)} B \right\| d\tau \|\sigma_{iT}\|_{\mathcal{L}_{\infty}} \sqrt{n} \\ &= \int_0^T \left\| e^{A\alpha} B \right\| d\alpha \|\sigma_{iT}\|_{\mathcal{L}_{\infty}} \sqrt{n} \\ &= \eta(T) \|\sigma_{iT}\|_{\mathcal{L}_{\infty}} \sqrt{n}, \end{aligned} \quad (6.4.21)$$

where $\|\sigma_{iT}\|_{\mathcal{L}_{\infty}}$ represents the truncated \mathcal{L}_{∞} norm of signal $\sigma(t)$ at time iT and $\tau \in [(i-1)T, iT]$.

In eqn (6.4.21), we have a conversion between the ∞ norm and 2 norm. The following inequalities are used:

$$\begin{aligned} \|x\|_{\infty} &\leq \|x\|_2, \\ \|x\|_2 &\leq \|x\|_{\infty} \sqrt{n}, \end{aligned} \quad (6.4.22)$$

where n is the dimension of the vector x . The Assumption 6.2.1 is defined in ∞ norm. Hence, we need to convert it to 2 norm by multiplying \sqrt{n} in the end of the equation

(6.4.21).

Using Assumption 6.2.1 in (6.4.21)

$$\|\tilde{x}(iT)\| \leq \eta(T)(L\|x_{iT}\|_{\mathcal{L}_\infty} + L_0)\sqrt{n}. \quad (6.4.23)$$

Using the condition of this Lemma $\|x_{t_1}\|_{\mathcal{L}_\infty} \leq \rho_x$ in (6.4.23), for all i while $iT < t_1$, we have

$$\|\tilde{x}(iT)\|_\infty \leq \eta(T)(L\rho_x + L_0)\sqrt{n}, \quad (6.4.24)$$

Using the definition of $\gamma_0(T, \rho_x)$

$$\|\tilde{x}(iT)\|_\infty < \gamma_0(T, \rho_x) \quad (6.4.25)$$

For all $iT < t_1$.

In what follows, we prove the upper-bound of $\tilde{\sigma}(t)$.

According to Assumption 6.2.2, $\dot{\sigma}(t)$ is subject to the following equations:

$$\begin{aligned} \|\dot{\sigma}_{t_1}\|_{\mathcal{L}_\infty} &\leq L_1\|\dot{x}_{t_1}\|_{\mathcal{L}_\infty} + L_2\|x_{t_1}\|_{\mathcal{L}_\infty} + L_3 \\ &\leq L\|(Ax + Bu + \sigma)_{t_1}\|_{\mathcal{L}_\infty} + L_2\|x_{t_1}\|_{\mathcal{L}_\infty} + L_3 \\ &\leq L(\|A\|_{\mathcal{L}_1}\rho_x + \|B\|_{\mathcal{L}_1}\rho_u + L\rho_x + L_0) + L_2(L\rho_x + L_0) + L_3. \end{aligned}$$

Using the definition of $b_{d\sigma}$ in (6.4.10) then

$$\|\dot{\sigma}_{t_1}\|_{\mathcal{L}_\infty} \leq b_{d\sigma}. \quad (6.4.26)$$

It follows from (6.4.20) that

$$\tilde{x}(iT) = - \int_{(i-1)T}^{iT} e^{A(iT-\tau)} B \sigma(\tau) d\tau. \quad (6.4.27)$$

It follows from (6.4.19) that

$$\begin{aligned} \tilde{x}(iT) &= (I - e^{AT}) \tilde{x}(iT) - \int_{iT}^{(i+1)T} e^{A((i+1)T-\tau)} \hat{\sigma}(iT) d\tau \\ &= - \int_{iT}^{(i+1)T} e^{A((i+1)T-\tau)} A \tilde{x}(iT) d\tau - \int_{iT}^{(i+1)T} e^{A((i+1)T-\tau)} \hat{\sigma}(iT) d\tau \\ &= - \int_{iT}^{(i+1)T} e^{A((i+1)T-\tau)} (\hat{\sigma}(iT) + A \tilde{x}(iT)) d\tau. \end{aligned} \quad (6.4.28)$$

Hence, (6.4.27) and (6.4.28) imply that

$$\int_{(i-1)T}^{iT} e^{A(iT-\tau)} B \sigma(\tau) d\tau = \int_{iT}^{(i+1)T} e^{A((i+1)T-\tau)} (\hat{\sigma}(iT) + A \tilde{x}(iT)) d\tau, \quad (6.4.29)$$

and hence there exists $t_p \in [(i-1)T, iT]$ such that

$$\hat{\sigma}(iT) + A \tilde{x}(iT) = B \sigma(t_p). \quad (6.4.30)$$

For any $t < t_1$, there exists $t_p \in [(i-1)T, iT]$ such that $|t - t_p| \leq 2T$ which satisfies (6.4.30). The time relationship of t_p , iT and t_1 as shown in Figure 6.4.1. Equation (6.4.30) implies

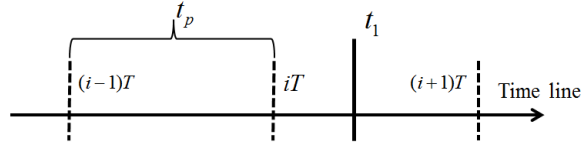


FIGURE 6.4.1: Time relationship between $(i-1)T$, iT , t_1 .

that

$$\begin{aligned}
\|\hat{\sigma}(t) - B\sigma(t)\| &\leq \|\hat{\sigma}(t) - B\sigma(t_p)\| + \|B\sigma(t) - B\sigma(t_p)\| \\
&\leq \|\hat{\sigma}(iT) - B\sigma(t_p)\| + \|B\sigma(t) - B\sigma(t_p)\| \\
&\leq A\|\tilde{x}(iT)\| + \int_{t_p}^t \|B\dot{\sigma}(\tau)\| d\tau.
\end{aligned} \tag{6.4.31}$$

The bound of $\dot{\sigma}(t)$ is derived in (6.4.26). Then we have

$$\|\hat{\sigma}(t) - B\sigma(t)\| \leq \sqrt{\lambda_{\max}(A^\top A)} \gamma_0(T, \rho_x) + 2b_{d\sigma} T \sqrt{n}. \tag{6.4.32}$$

It follows from the definition of $\tilde{\sigma}(t)$ and $\gamma_1(T, \rho_x, \rho_u)$ in (6.4.13), (6.4.11) that

$$\|\tilde{\sigma}_{t_1}\| \leq \gamma_1(T, \rho_x, \rho_u). \tag{6.4.33}$$

Using the dynamics in (6.4.15), we have

$$\tilde{x}(s) = (sI - A)^{-1}(\hat{\sigma}(s) - B\sigma(s)). \tag{6.4.34}$$

Hence, we have

$$\begin{aligned}
\|\tilde{x}_{t_1}\|_{\mathcal{L}_\infty} &\leq \|sI - A\|_{\mathcal{L}_1} \|(\hat{\sigma} - B\sigma)_t\|_{\mathcal{L}_\infty}, \\
\|\tilde{x}_{t_1}\|_{\mathcal{L}_\infty} &\leq \|(sI - A)^{-1}\|_{\mathcal{L}_1} \gamma_1(T, \rho_x, \rho_u).
\end{aligned} \tag{6.4.35}$$

Using the norm property $\|\tilde{x}_{t_1}\| \leq \sqrt{n} \|\tilde{x}_{t_1}\|_{\mathcal{L}_\infty}$, we have

$$\|\tilde{x}_{t_1}\| \leq \sqrt{n} \|(sI - A)^{-1}\|_{\mathcal{L}_1} \gamma_1(T, \rho_x, \rho_u) \tag{6.4.36}$$

where n is the dimension of $x(t)$.

It follows from the definition of $\gamma_2(T, \rho_x, \rho_u)$ in (6.4.12)

$$\|\tilde{x}_{t_1}\| \leq \gamma_2(T, \rho_x, \rho_u) \quad (6.4.37)$$

which completes the proof. \square

Lemma 6.4.3. *For any given bounded ρ_x, ρ_u*

$$\lim_{T \rightarrow 0} \gamma_0(T, \rho_x) \rightarrow 0 \quad (6.4.38)$$

$$\lim_{T \rightarrow 0} \gamma_1(T, \rho_x, \rho_u) \rightarrow 0 \quad (6.4.39)$$

$$\lim_{T \rightarrow 0} \gamma_2(T, \rho_x, \rho_u) \rightarrow 0. \quad (6.4.40)$$

Proof Recall the definition of $\gamma_0(T, \rho_x)$ in (6.4.9), everything inside the integration is a bounded value and function. So when $T \rightarrow 0$ we can get $\gamma_0(T, \rho_x) \rightarrow 0$.

Similarly we can prove the limits of $\gamma_1(T, \rho_x, \rho_u)$ and $\gamma_2(T, \rho_x, \rho_u)$ go to zero.

The proof is complete. \square

Remark 6.4.4. Lemma 6.4.2 and Lemma 6.4.3 show the performance of the state predictor. The estimated state variable $\hat{x}(t)$ has a bounded error $\gamma_1(T, \rho_x, \rho_u)$ with the real system state variable $x(t)$, and T can be chosen small enough to make this bound $\gamma_1(T, \rho_x, \rho_u)$ arbitrarily small. Similarly, the real disturbance $B\sigma(t)$ and estimated disturbance $\hat{\sigma}(t)$ also have a bounded error $\gamma_2(T, \rho_x, \rho_u)$ and can also be arbitrarily small.

Lemma 6.4.5. *For the system in (6.2.1) with the \mathcal{L}_1 adaptive controller in (6.3.1), (6.3.2) and (6.3.3), if the truncated \mathcal{L}_∞ norm $\|x_{t_1}\|_{\mathcal{L}_\infty} \leq \rho_x$, $\|u_{t_1}\|_{\mathcal{L}_\infty} \leq \rho_u$ for any time $t_1 \geq 0$,*

then

$$\begin{aligned} \|u_{t_1}\|_{\mathcal{L}_\infty} &\leq \|k_g\|_{\mathcal{L}_1} B_r + t_N B_\beta + \|C(s)\|_{\mathcal{L}_1} (L\rho_x + L_0) + \\ &\quad \left\| C(s)(B^T B)^{-1} B^T \right\|_{\mathcal{L}_1} \gamma_1(T, \rho_x, \rho_u) \end{aligned} \quad (6.4.41)$$

and

$$\begin{aligned} \|x_{t_1}\|_{\mathcal{L}_\infty} &\leq \frac{\|(sI - A)^{-1}\|_{\mathcal{L}_1}}{1 - \|(sI - A)^{-1} B(1 - C(s))\|_{\mathcal{L}_1} L} \left(\gamma_2(T, \rho_x, \rho_u) + \|k_g\|_{\mathcal{L}_1} B_r + t_N B_\beta + \right. \\ &\quad \left. \|(I - BC(s)(B^T B)^{-1} B^T)\|_{\mathcal{L}_\infty} \gamma_1(T, \rho_x, \rho_u) + L_0 \right) + \gamma_2(T, \rho_x, \rho_u). \end{aligned} \quad (6.4.42)$$

Proof Using the definition of $\tilde{\sigma}(t)$ in (6.4.13), $\hat{\sigma}(t)$ can be rewritten as

$$\hat{\sigma}(t) = B\sigma(t) + \tilde{\sigma}(t). \quad (6.4.43)$$

Plugging (6.4.43) into the definition of $u_1(t)$ in (6.3.8)

$$\begin{aligned} u_1(s) &= -C(s)(B^T B)^{-1} B^T (B\sigma(s) + \tilde{\sigma}(s)) \\ &= -C(s)\sigma(s) - C(s)(B^T B)^{-1} B^T \tilde{\sigma}(s). \end{aligned} \quad (6.4.44)$$

Taking the norm of eqn (6.4.44)

$$\|(u_1)_{t_1}\|_{\mathcal{L}_\infty} \leq \|C(s)\|_{\mathcal{L}_1} \|\sigma_{t_1}\|_{\mathcal{L}_\infty} + \left\| C(s)(B^T B)^{-1} B^T \right\|_{\mathcal{L}_1} \|\tilde{\sigma}_{t_1}\|_{\mathcal{L}_\infty}. \quad (6.4.45)$$

Using Assumption 6.2.1 and the results of Lemma 6.4.1, we have

$$\|(u_1)_{t_1}\|_{\mathcal{L}_\infty} \leq \|C(s)\|_{\mathcal{L}_1} (L\rho_x + L_0) + \left\| C(s)(B^T B)^{-1} B^T \right\|_{\mathcal{L}_1} \gamma_1(T, \rho_x, \rho_u). \quad (6.4.46)$$

Consider the result in (6.4.6) we have

$$\begin{aligned} \|u_{t_1}\|_{\mathcal{L}_\infty} &\leq \|k_g\|_{\mathcal{L}_1} B_r + t_N B_\beta + \|C(s)\|_{\mathcal{L}_1} (L\rho_x + L_0) + \\ &\quad \left\| C(s)(B^T B)^{-1} B^T \right\|_{\mathcal{L}_1} \gamma_1(T, \rho_x, \rho_u). \end{aligned} \quad (6.4.47)$$

Substituting the control law (6.3.3) and (6.3.8) into the state predictor (6.3.1), we have

$$\dot{\hat{x}}(t) = A\hat{x}(t) + B(u_2(t) - C(s)(B^T B)^{-1} B^T \hat{\sigma}(t)) + \hat{\sigma}(t). \quad (6.4.48)$$

Taking the laplace transform of (6.4.48)

$$\hat{x}(s) = (sI - A)^{-1} \left(B(u_2(s) - C(s)(B^T B)^{-1} B^T \hat{\sigma}(s)) + \hat{\sigma}(s) \right). \quad (6.4.49)$$

Plugging (6.4.43) into (6.4.49), we have

$$\hat{x}(s) = (sI - A)^{-1} \left(B(u_2(s) - C(s)(B^T B)^{-1} B^T (B\sigma(s) + \tilde{\sigma}(s))) + B\sigma(s) + \tilde{\sigma}(s) \right) \quad (6.4.50)$$

$$= (sI - A)^{-1} \left(B(1 - C(s))\sigma(s) + Bu_2(s) + (I - BC(s)(B^T B)^{-1} B^T)\tilde{\sigma}(s) \right) \quad (6.4.51)$$

where $\sigma(s)$, $r(s)$, $\tilde{\sigma}(s)$ are the laplace transforms of $\sigma(t)$, $r(t)$, $\tilde{\sigma}(t)$ respectively. Using the definition of $\sigma(t) = f(x(t))$ and rewriting $x(s)$ as $x(s) = \hat{x}(s) + \tilde{x}(s)$, we have

$$\sigma(s) = f(s, x(s)) \quad (6.4.52)$$

$$= f(s, (\hat{x}(s) + \tilde{x}(s))). \quad (6.4.53)$$

Substituting (6.4.53) into (6.4.51)

$$\begin{aligned} \hat{x}(s) &= (sI - A)^{-1} \left(B(1 - C(s))f(s, (\hat{x}(s) + \tilde{x}(s))) + Bu_2(s) \right. \\ &\quad \left. + (I - BC(s)(B^T B)^{-1} B^T)\tilde{\sigma}(s) \right), \end{aligned} \quad (6.4.54)$$

eqn (6.4.54) can be considered as two interconnected systems shown in Figure 6.4.2.

Utilizing the small gain theorem, the stability condition for the loop shown in Figure 6.4.2

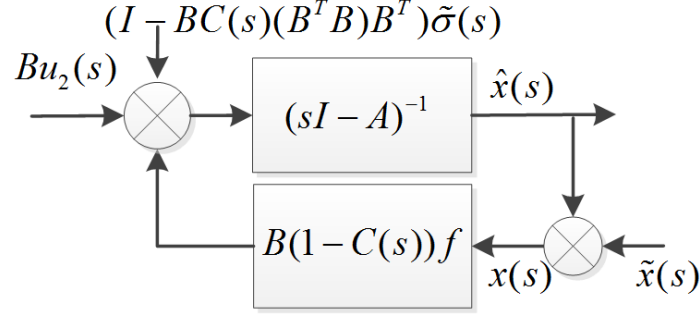


FIGURE 6.4.2: Signal flow chart of \mathcal{L}_1 adaptive controller

is as follows:

$$\|(sI - A)^{-1}B(1 - C(s))\|_{\mathcal{L}_1}L < 1. \quad (6.4.55)$$

It follows from (6.4.51) and the boundary of $u_2(t)$ in (6.4.6) that

$$\begin{aligned} \|\hat{x}_t\|_{\mathcal{L}_\infty} \leq & \frac{\|(sI - A)^{-1}\|_{\mathcal{L}_1}}{1 - \|(sI - A)^{-1}B(1 - C(s))\|_{\mathcal{L}_1}L} (\|\tilde{x}_t\|_{\mathcal{L}_\infty} + \|k_g\|_{\mathcal{L}_1}B_r + t_N B_\beta \\ & + \|(I - BC(s)(B^T B)^{-1}B^T)\|_{\mathcal{L}_1}\|\tilde{\sigma}_t\|_{\mathcal{L}_\infty} + L_0) \end{aligned} \quad (6.4.56)$$

for any $t > 0$.

For $t < t_1$, it follows Lemma 6.4.1 and (6.2.4) that (6.4.56) can be rewritten as the following

$$\begin{aligned} \|\hat{x}_t\|_{\mathcal{L}_\infty} \leq & \frac{\|(sI - A)^{-1}\|_{\mathcal{L}_1}}{1 - \|(sI - A)^{-1}B(1 - C(s))\|_{\mathcal{L}_1}L} (\gamma_2(T, \rho_x, \rho_u) + \|k_g\|_{\mathcal{L}_1}B_r + t_N B_\beta + \\ & \|(I - BC(s)(B^T B)^{-1}B^T)\|_{\mathcal{L}_1}\gamma_1(T, \rho_x, \rho_u) + L_0). \end{aligned} \quad (6.4.57)$$

It follows from the definition of \tilde{x} after (6.3.2) and (6.4.57) that

$$\begin{aligned}
\|x_t\|_{\mathcal{L}_\infty} &\leq \|\hat{x}_t\|_{\mathcal{L}_\infty} + \|\tilde{x}_t\|_{\mathcal{L}_\infty} \\
&\leq \frac{\|(sI - A)^{-1}\|_{\mathcal{L}_1}}{1 - \|(sI - A)^{-1}B(1 - C(s))\|_{\mathcal{L}_1}L} (\gamma_2(T, \rho_x, \rho_u) + \|k_g\|_{\mathcal{L}_1}B_r + t_N B_\beta + \\
&\quad \|(I - BC(s)(B^T B)^{-1}B^T)\|_{\mathcal{L}_1}\gamma_1(T, \rho_x, \rho_u) + L_0) + \gamma_2(T, \rho_x, \rho_u). \quad (6.4.58)
\end{aligned}$$

□

Lemma 6.4.6. *There exist $\rho_x > 0$, $\rho_u > 0$ and $T > 0$ such that*

$$\|k_g\|_{\mathcal{L}_1}B_r + t_N B_\beta + \|C(s)\|_{\mathcal{L}_1}(L\rho_x + L_0) + \left\|C(s)(B^T B)^{-1}B^T\right\|_{\mathcal{L}_1}\gamma_1(T, \rho_x, \rho_u) < \rho_x \quad (6.4.59)$$

and

$$\begin{aligned}
&\frac{\|(sI - A)^{-1}\|_{\mathcal{L}_1}}{1 - \|(sI - A)^{-1}B(1 - C(s))\|_{\mathcal{L}_1}L} (\gamma_2(T, \rho_x, \rho_u) + \|k_g\|_{\mathcal{L}_1}B_r + t_N B_\beta + \\
&\|(I - BC(s)(B^T B)^{-1}B^T)\|_{\mathcal{L}_1}\gamma_1(T, \rho_x, \rho_u) + L_0) + \gamma_2(T, \rho_x, \rho_u) < \rho_x. \quad (6.4.60)
\end{aligned}$$

Proof Let us choose ρ_x such that

$$\rho_x = \frac{\|(sI - A)^{-1}\|_{\mathcal{L}_1}}{1 - \|(sI - A)^{-1}B(1 - C(s))\|_{\mathcal{L}_1}L} (\|k_g\|_{\mathcal{L}_1}B_r + t_N B_\beta + L_0) + \Delta_1, \quad (6.4.61)$$

$$\rho_u = \|k_g\|_{\mathcal{L}_1}B_r + t_N B_\beta + \|C(s)\|_{\mathcal{L}_1}(L\rho_x + L_0) + \Delta_2 \quad (6.4.62)$$

where $\Delta_1 > 0$ and $\Delta_2 > 0$ are any positive constant. It follows from Lemma 4.3 that there exists some T to make

$$\frac{\|(sI - A)^{-1}\|_{\mathcal{L}_1}}{1 - \|(sI - A)^{-1}B(1 - C(s))\|_{\mathcal{L}_1}L} \|I - BC(s)(B^T B)^{-1}B^T\|_{\mathcal{L}_1} \gamma_1(T, \rho_x, \rho_u) + \gamma_2(T, \rho_x, \rho_u) < \Delta_1. \quad (6.4.63)$$

and

$$\|C(s)(B^T B)^{-1}B^T\|_{\mathcal{L}_1} \gamma_1(T, \rho_x, \rho_u) < \Delta_2 \quad (6.4.64)$$

hold true. Combine (6.4.61), (6.4.62), (6.4.63) and (6.4.64) which leads to (6.4.59) and (6.4.60) which completes the proof. \square

Remark 6.4.7. It follows from Lemma 6.4.6 that we can always choose T to satisfy (6.4.59) and (6.4.60).

Theorem 6.4.8. For the system in (6.2.1) together with the \mathcal{L}_1 adaptive controller in (6.3.1), (6.3.2) and (6.3.3) being stable, and choosing T as in Lemma 4.5, then

$$\|x\|_{\mathcal{L}_\infty} < \rho_x, \quad \|u\|_{\mathcal{L}_\infty} < \rho_u. \quad (6.4.65)$$

Proof The proof will be done by contradiction. From (6.2.1) and (6.3.1) where $x(0) = 0$, $\hat{x}(0) = x_0$. It follows from the definition of ρ_x and ρ_u in (6.4.61) and (6.4.62) that

$$x(0) < \rho_x, \quad u(0) < \rho_u. \quad (6.4.66)$$

Assume (6.4.65) is not true, since $x(t)$ and $u(t)$ are continuous. There exists some $t' \leq 0$ where

$$\|x(t')\|_\infty = \rho_x \quad \text{or} \quad \|u(t')\|_\infty = \rho_u, \quad (6.4.67)$$

while

$$\|x_{t'}\|_{\mathcal{L}_\infty} \leq \rho_x, \quad \|u_{t'}\|_{\mathcal{L}_\infty} \leq \rho_u. \quad (6.4.68)$$

Letting $t_1 = t'$ it follows from Lemma 6.4.5 and 6.4.6 that

$$\begin{aligned} \|x_{t'}\|_{\mathcal{L}_\infty} &\leq \frac{\|(sI - A)^{-1}\|_{\mathcal{L}_1}}{1 - \|(sI - A)^{-1}B(1 - C(s))\|_{\mathcal{L}_1}L} (\gamma_2(T, \rho_x, \rho_u) + \|k_g\|_{\mathcal{L}_1}B_r + t_N B_\beta + \\ &\quad \|(I - BC(s)(B^T B)^{-1}B^T)\|_{\mathcal{L}_1} \gamma_1(T, \rho_x, \rho_u) + L_0) + \gamma_2(T, \rho_x, \rho_u) \quad (6.4.69) \\ &< \rho_x \end{aligned}$$

and

$$\begin{aligned} \|u_{t'}\|_{\mathcal{L}_\infty} &\leq \|k_g\|_{\mathcal{L}_1}B_r + t_N B_\beta + \|C(s)\|_{\mathcal{L}_1} (L\rho_x + L_0) + \left\| C(s)(B^T B)^{-1}B^T \right\|_{\mathcal{L}_1} \gamma_1(T, \rho_x, \rho_u) \\ &< \rho_u \quad (6.4.70) \end{aligned}$$

This clearly contradicts the statements in (6.4.67). Therefore t' does not exist. As a result $\|x_t\|_{\mathcal{L}_\infty} < \rho_x$ and $\|u_t\|_{\mathcal{L}_\infty} < \rho_u$ holds for all $t > 0$ which proves Theorem 6.4.8. \square

Let $y_e > 0$ denote the uniform error bound for the constraint controller such that $P_{min}(i) - y_e \leq y_{ci}(t) \leq P_{max}(i) + y_e$, $y_e = \|Cc\|_{\mathcal{L}_1}\|x_e\|_{\mathcal{L}_\infty}$, where $x_e(t)$ is the error signal of the system state which will be derived in Theorem 4.2.

Theorem 6.4.9. For the system in (6.2.1) together with the \mathcal{L}_1 adaptive controller in (6.3.1), (6.3.2) and (6.3.3), and choosing T as in Lemma 4.5, then

$$\begin{aligned} \|x_e\|_{\mathcal{L}_\infty} &\leq \left\| (sI - A)^{-1}(I - BC(s)(B^T B)^{-1}B^T) \right\|_{\mathcal{L}_1} \gamma_1(T, \rho_x, \rho_u) + \gamma_2(T, \rho_x, \rho_u) + \\ &\quad \left\| (sI - A)^{-1}B(1 - C(s)) \right\|_{\mathcal{L}_1} (L\rho_x + L_0) + \\ &\quad \frac{t_{max}}{2} (\|A\|_{\mathcal{L}_1}\rho_x + \|B\|_{\mathcal{L}_1}\rho_u + L\rho_x + L_0). \quad (6.4.71) \end{aligned}$$

Proof The error $x_e(t)$ results from two contributing factors: The first being from the $\hat{x}_2(t)$ system in (6.3.6). The control law $u_2(t)$ is derived from the nominal system $\hat{x}_2(t)$ and can only ensure that the N discrete points in the prediction horizon t_1, t_2, \dots, t_p do not violate the output constraints. Though, between different check points there are still errors. The second contributing factor is from $x_1(t)$ in eqn (6.3.7), the difference of the real system state and $\hat{x}_2(t)$. In what follows we analyze these two error bounds respectively.

6.4.1 The Behavior of $\hat{x}_2(t)$:

Eqn (6.3.6) is an ideal system for the control law $u_2(t)$. The control law switches between two modes as described in (6.3.25): When $\lambda(t) = 1$, the feasible set defined in (6.3.21) is nonempty. That means the current constrained output is within the range $P_{min}(i) \leq y_{ci}(t) \leq P_{max}(i)$. When $\lambda(t) = 2$, $u_2(t) = u_c(t, t_0)$. According to the definition of the feasible set in (6.3.21), the control signal $u_c(t, t_0)$ will make (6.3.17) hold true at discrete times t_1, t_2, \dots, t_N . The current time t must fall between two discrete times in the set t_1, t_2, \dots, t_N . Since the state variable $x(t)$ is continuous, the maximum error between two points can be characterized as the following.

Using $x_{e2}(t)$ to denote this prediction error, $x_{e2}(t)$ is governed by the following equation

$$x_{e2}(t) \leq \frac{t_{max}}{2} dx_{max} \quad (6.4.72)$$

where dx_{max} is the maximum value of the $\|\dot{x}(t)\|_\infty$ over all time $t > 0$ and $t_{max} = \max(t_{i+1} - t_i), i \in 1, 2, \dots, p - 1$.

$$\|\dot{x}(t)\|_\infty = \|Ax(t) + Bu(t) + \sigma(t)\|_\infty. \quad (6.4.73)$$

Using the result of Theorem 6.4.8, we have

$$dx_{max} \leq \|A\|_{\mathcal{L}_1} \rho_x + \|B\|_{\mathcal{L}_1} \rho_u + L\rho_x + L_0. \quad (6.4.74)$$

It follows from (6.4.72) that

$$\|x_{e2}(t)\|_\infty \leq \frac{t_{max}}{2} (\|A\|_{\mathcal{L}_1} \rho_x + \|B\|_{\mathcal{L}_1} \rho_u + L\rho_x + L_0). \quad (6.4.75)$$

6.4.2 Bound of the Error Signal $x_1(t)$:

It follows from (6.4.51) and the definition of $x(t) = \hat{x} + \tilde{x}$ that

$$\begin{aligned} x(s) &= \hat{x}(s) + \tilde{x}(s) \\ &= (sI - A)^{-1} (B(1 - C(s))\sigma(s) + Bu_2(s) + (I - BC(s)(B^T B)^{-1} B^T) \tilde{\sigma}(s)) + \tilde{x}(s). \end{aligned}$$

Using the definition of $x_1(t)$ in (6.3.7) we have

$$\begin{aligned} x_1(s) &= (sI - A)^{-1} (I - BC(s)(B^T B)^{-1} B^T) \tilde{\sigma}(s) + \tilde{x}(s) \\ &\quad + (sI - A)^{-1} B(1 - C(s))\sigma(s). \end{aligned} \quad (6.4.76)$$

Substituting the boundary of $\tilde{\sigma}(t)$, $\tilde{x}(t)$ and $\sigma(t)$ from Lemma 6.4.1 and Assumption 2.1 into (6.4.76), we have

$$\begin{aligned} \|x_1\|_{\mathcal{L}_\infty} &\leq \left\| (sI - A)^{-1} (I - BC(s)(B^T B)^{-1} B^T) \right\|_{\mathcal{L}_1} \gamma_1(T, \rho_x, \rho_u) + \gamma_2(T, \rho_x, \rho_u) + \\ &\quad \left\| (sI - A)^{-1} B(1 - C(s)) \right\|_{\mathcal{L}_1} (L\rho_x + L_0). \end{aligned} \quad (6.4.77)$$

Combing the results in (6.4.75) and (6.4.77) we have

$$\begin{aligned} \|x_e\|_{\mathcal{L}_\infty} &\leq \|x_{e2}\|_{\mathcal{L}_\infty} + \|x_1\|_{\mathcal{L}_\infty} \\ &\leq \left\| (sI - A)^{-1} (I - BC(s)(B^T B)^{-1} B^T) \right\|_{\mathcal{L}_1} \gamma_1(T, \rho_x, \rho_u) + \gamma_2(T, \rho_x, \rho_u) + \\ &\quad \left\| (sI - A)^{-1} B(1 - C(s)) \right\|_{\mathcal{L}_1} (L\rho_x + L_0) + \\ &\quad \frac{t_{max}}{2} (\|A\|_{\mathcal{L}_1} \rho_x + \|B\|_{\mathcal{L}_1} \rho_u + L\rho_x + L_0) \end{aligned} \quad (6.4.78)$$

which completes the proof. \square

Remark 6.4.10. The error signal $y_e = \|C_c\|_{\mathcal{L}_1} \|x_e\|_{\mathcal{L}_\infty}$ is the boundary layer of the output constraint. According to Theorem 6.4.9 and Lemma 4.3, the boundary layer can be reduced by changing the prediction step size t_{max} , sampling time T and the bandwidth of the low-pass filter $C(s)$. In practice, we can pick tighter bounds offset by this boundary layer. Then the real $y_{ci}(t)$ will still be kept within $(P_{min}(i), P_{max}(i))$. Notice this boundary layer can be arbitrarily small.

6.5 Simulation Results

This section presents simulation results for several scenarios to demonstrate the effectiveness of the controller presented in this chapter. The system under consideration is second order and single-input single-output. The system dynamics are described by equation (6.2.1) in the problem formulation section. The parameters are as follows:

$$A = \begin{bmatrix} -2 & -10 \\ 1 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 2 \\ 0.5 \end{bmatrix} \quad C_r = \begin{bmatrix} 1 & 0 \end{bmatrix} \quad C_c = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0.5 & -0.25 \end{bmatrix},$$

where C_r denotes the tracking output matrix and C_c denotes the output matrix subject to constraints. In regards to the constrained output matrix, the first and third outputs are non-minimum phase while the second output is minimum-phase. In addition, the first output is tracking a reference signal. Also, the system is subject to time-varying non-linear uncertainties in the input channel (matched uncertainties). The uncertainty is designated as follows:

$$\sigma = \sin(t)(\sqrt{|x_1|} + \sqrt{|x_2|}),$$

where x_1 and x_2 are the first and second system states respectively. A rate limit is

applied to the control signal i.e. $-50 \leq \dot{u} \leq 50$. The simulation sampling time is 5×10^{-3} second. The parameters for the prediction horizon are a total horizon time of 2 seconds and a time step of 0.1 seconds ($N=20$). At each sampling time the system dynamics and the current control and state values are used to project the future trajectory of each output. The definition of feasible set S in equation 6.3.21 can easily be formulated into a linear program which is executed online at each sampling time.

Example 5.1 In this example the system is subject to the following constraints:

$$-10 \leq y_1 \leq 40 \quad -20 \leq y_2 \leq 5 \quad -5 \leq y_3 \leq 18,$$

where y_1 , y_2 , and y_3 are the first, second, and third system outputs respectively. In this scenario there is only a violation in the transient of output 3, though when the system reaches steady state no violations occur. We wanted to verify that the algorithm can successfully avoid a transient violation while still fully maintaining tracking. Figure 6.6.1 shows each outputs' response in just the tracking mode (no switching to the constraint violation avoidance mode (CVAM) occurs). Output 1 is tracking a step input $r(t) = 25$ which is applied at $t = 0.25$ seconds.

In Figure 6.6.1 output 1 successfully tracks the reference signal. Though, during the transient, at approximately $t = 2$ seconds output 3 violates the upper constraint. Figure 6.6.2 shows the results in the CVAM.

In Figure 6.6.2 as output 3 approached the constraint the feasible set reduced to zero initiating a switch from the tracking mode to a first order hold determined online by the linear programming. As a result, the constraint violation avoidance mode (CVAM) successfully prevents output 3 from violating the upper constraint during the transient.

Example 5.2 In the second example the simulation parameters are the same as example one though the constraints are made much more stringent:

$$-5 \leq y_1 \leq 24 \quad -14 \leq y_2 \leq 5 \quad -5 \leq y_3 \leq 15. \quad (6.5.1)$$

The purpose of this scenario is to confirm that when multiple outputs, both non-minimum phase and minimum phase, are approaching their constraints concurrently the CVAM can maintain multiple constraints without conflict. Also, we wanted to demonstrate that the CVAM can successfully eliminate steady state violations as well. Figure 6.6.3 shows that all three outputs violate constraints in both the transient and the steady state when only in the tracking mode.

Figure 6.6.4 demonstrates that the CVAM can prevent all three outputs from violating the defined constraints. As the outputs approach their respective constraint the feasible set reduces to zero and the algorithm switches to the appropriate first order hold determined by the linear program so that all constraints are maintained.

Example 5.3 In the final example we look at how the CVAM performs with a changing reference signal. We wanted to verify that the CVAM can maintain constraints especially during the transient of an aggressive change in reference command. $r(t)$ is defined as follows:

$$r(t) = \begin{cases} 0, & 0 \leq t < 0.25 \text{ seconds,} \\ 25, & 0.25 \leq t < 5 \text{ seconds,} \\ -25, & 5 \leq t \leq 15 \text{ seconds,} \end{cases}$$

For example 5.3 the output constraints become.

$$-29 \leq y_1 \leq 27 \quad -16 \leq y_2 \leq 16 \quad -18 \leq y_3 \leq 18.$$

Figure 6.6.5 displays outputs in the tracking mode. All three outputs violate their constraints in the transient. Special attention should be brought to output 1 at approximately $t = 5.5$ seconds, in this time frame output 1 demonstrates non-minimum phase behavior and the algorithm has a relatively small amount of time to prevent this violation.

Figure 6.6.6 shows that the algorithm was capable of eliminating all of the transient violations including the transient regarding the aggressive change in reference. The algorithm determined an appropriate first order hold with a lower time restriction while maintaining tracking with improved settling time.

6.6 Summary

In this chapter, we present a framework which incorporates \mathcal{L}_1 adaptive control coupled with an online optimization procedure. The \mathcal{L}_1 controller provides the necessary feedback to compensate for matched non-linear time-varying uncertainties while future trajectories of system outputs are predicted online. In the case that the finite prediction horizon identifies output constraint violations and there is no feasible set the controller switches from the tracking mode to a first order hold based on the linear program optimization. Theoretical analysis of the adaptive law and constraint control component was conducted to prove the stability of the system. Also, the prediction error in between check points of the prediction horizon was characterized and found to be finite and bounded. Simulation results demonstrate the effectiveness of the controller framework presented in this chapter.

Predictions in the finite time horizon are based on a set of control signals projected into the future. For the sake of computation complexity, we confine the control signals to be first order hold. It is noted that a larger subset of control signals such as higher order polynomials can be considered in the future. This will expand the feasible set, though at the expense of increased optimization complexity.

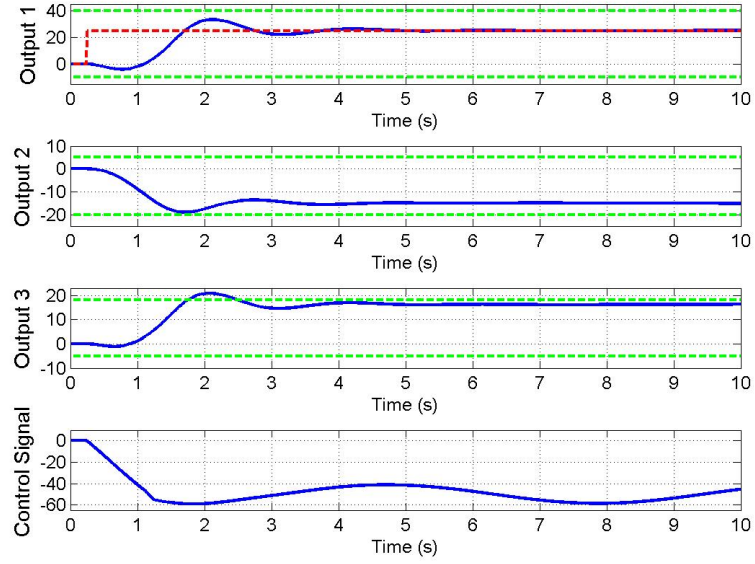


FIGURE 6.6.1: Unit step output response in tracking mode only, scenario 1. $y(t)$ = blue/dark solid, $r(t)$ = red/dark dashed, constraints = green/light dashed

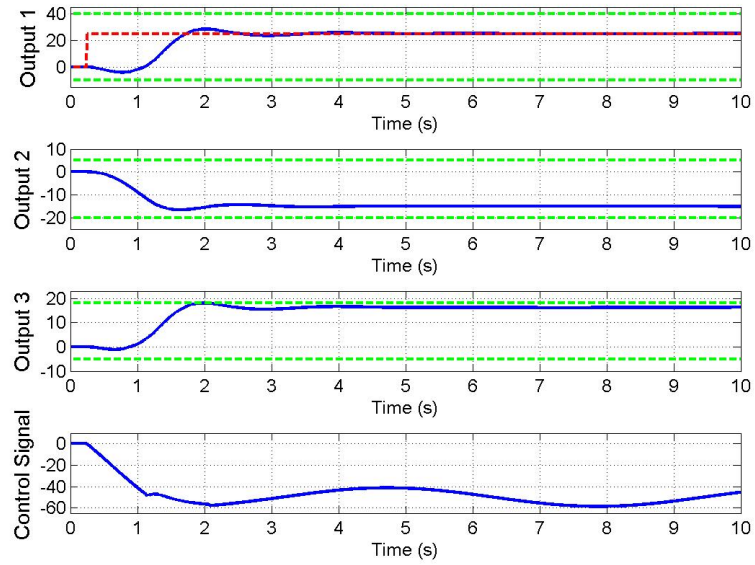


FIGURE 6.6.2: Unit step output response in constraint avoidance mode (CVAM), scenario 1. $y(t)$ = blue/dark solid, $r(t)$ = red/dark dashed, constraints = green/light dashed

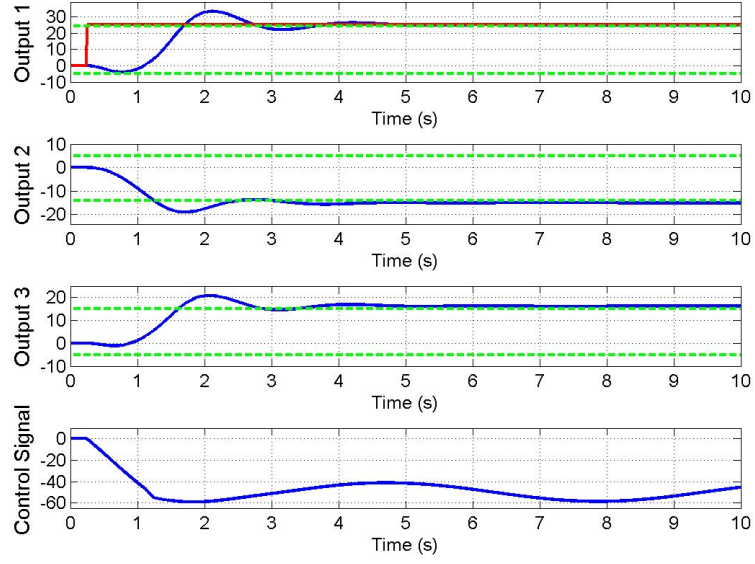


FIGURE 6.6.3: Unit step output response in tracking mode only, scenario 2. $y(t)$ = blue/dark solid, $r(t)$ = red/dark solid step input, constraints = green/light dashed

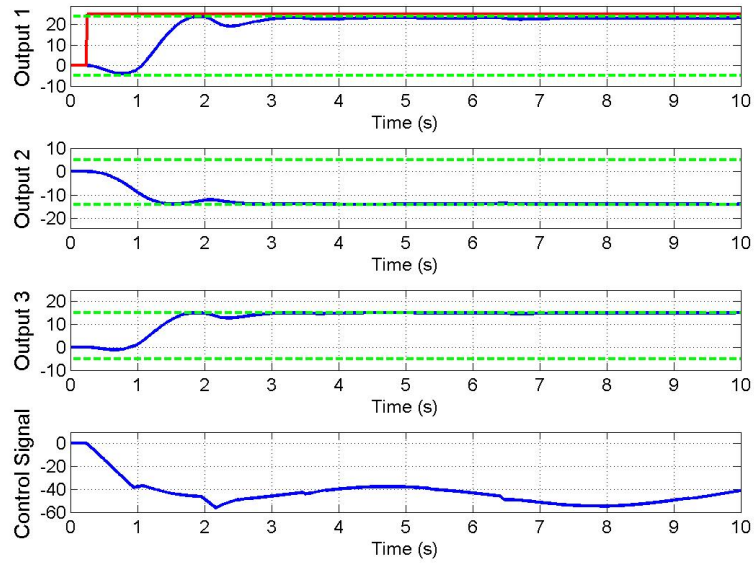


FIGURE 6.6.4: Unit step output response in constraint avoidance mode (CVAM), scenario 2. $y(t)$ = blue/dark solid, $r(t)$ = red/dark solid step input, constraints = green/light dashed

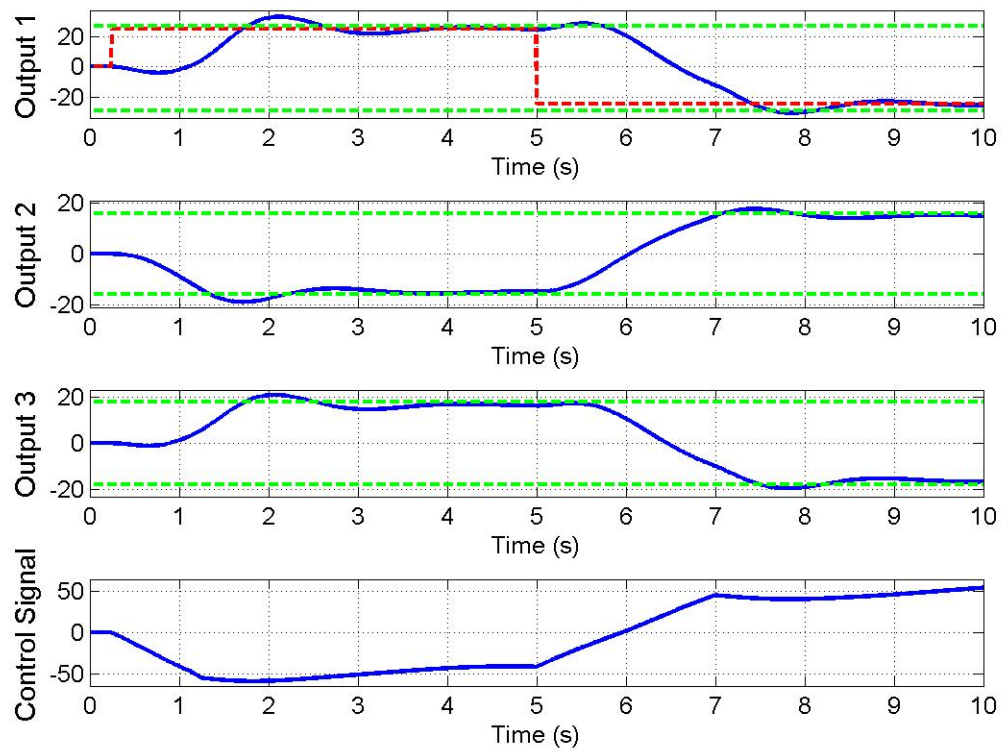
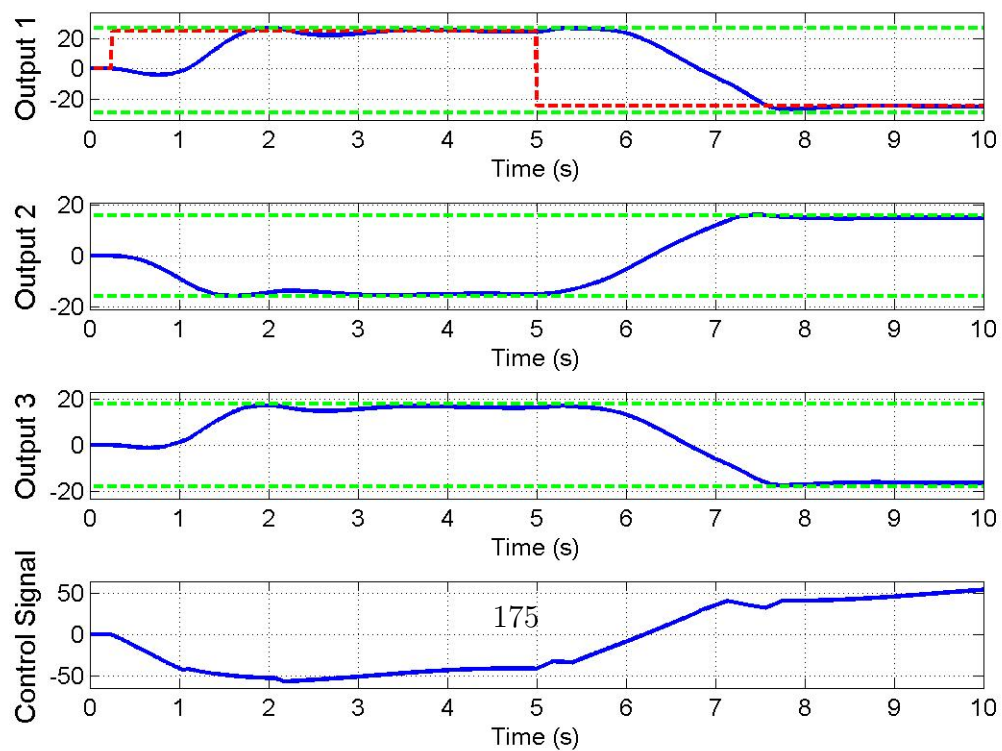


FIGURE 6.6.5: Changing unit step output response in tracking mode only, scenario 3.
 $y(t)$ = blue/dark solid, $r(t)$ = red/dark dashed, constraints = green/light dashed



Chapter 7

Conclusion and Future Work

7.1 Summary of Main Results

The main contribution of this dissertation is to extend the framework of \mathcal{L}_1 adaptive control theory and applied in various of applications. It can be summarized with 3 different parts: The first one is the extension of \mathcal{L}_1 adaptive to time-varying system and non-minimum phase system by using eigenvalue assignment method. This approach has been demonstrated by both theoretical model as well as high fidelity models such as flexible wing aircraft model from NASA and also the supersonic glider model developed from the supersonic lab in Austria. The 2nd part focus on filter bandwidth adaptation in the \mathcal{L}_1 adaptive control architecture. The stability condition of the low-pass filter in control is relaxed by introducing an additional Lyapunov-based adaptation mechanism which results in a more systematic design with minimized tuning efforts. adaptability for arbitrarily large nonlinear time-varying uncertainties without redesign parameters. The over all system is a non-LTI design even in the limiting case. The 3rd part introduce the concept of predictive horizon and online optimization into \mathcal{L}_1 adaptive control. This approach enable \mathcal{L}_1 adaptive control to solve the output limitation even for the non-minimum phase system.

7.2 Future Work

All of the extensions and applications of this dissertation are relatively new and promising research focus. However, as with any area of research, there are still some challenges that exist. In this section, we will introduce several potential extensions and applications to our research.

The control extensions discussed in chapter 2 - 4 has already applied in the challenging field like flexible wing aircraft and supersonic aircraft in a form of High Fidelity simulation. The real flight test needs to be further conducted in order to verify the performance of this approach.

The methodology introduced in chapter 5 needs to be further tested in a more detailed simulation and applications.

The approach in chapter 6 is still not beyond simulations. Since optimization method has great potentials to solve a lot of real application. For this method in this chapter, we plan to apply it to the case of engine control application and flight envelope protection. Since constraint and optimization are greatly encountered in those area.

Appendices

Bibliography

- [1] P. C. Gregory, “Air research and development command plans and programs,” in *Proc. Self Adaptive Flight Control Symp.* OH: Wright-Patterson Air Force Base, 1959, pp. 8–15.
- [2] E. Mishkin and L. Braun, *Adaptive control systems.* McGraw-Hill, 1961.
- [3] U. S. W. A. D. D. F. control laboratory and P. Gregory, *Proceedings of the Self Adaptive Flight Control Systems Symposium.* Wright air development center, Air research and development command, US Air Force, Wright-Patterson air force base, Ohio, 1959.
- [4] O. H. Schuck, “Honeywell History and Philosophy in the Adaptive Control Field (Including a description of flight tests of the first Adaptive Flight Control System).”
- [5] N. Hovakimyan and C. Cao, *L1 adaptive control theory: guaranteed robustness with fast adaptation.* Siam, 2010, vol. 21.
- [6] C. Rohrs, L. Valavani, M. Athans, and G. Stein, “Robustness of continuous-time adaptive control algorithms in the presence of unmodeled dynamics,” *IEEE Transactions on Automatic Control*, vol. 30, no. 9, pp. 881–889, 1985.
- [7] P. Ioannou and P. Kokotovic, “Robust redesign of adaptive control,” *IEEE Transactions on Automatic Control*, vol. 29, no. 3, pp. 202–211, 1984.

- [8] K. Narendra and A. Annaswamy, "A new adaptive law for robust adaptation without persistent excitation," *IEEE Transactions on Automatic control*, vol. 32, no. 2, pp. 134–145, 1987.
- [9] G. Arslan and T. Başar, "Disturbance attenuating controller design for strict-feedback systems with structurally unknown dynamics," *Automatica*, vol. 37, no. 8, pp. 1175–1188, 2001.
- [10] Z. Ding, "Adaptive control of triangular systems with nonlinear parameterization," *IEEE Transactions on Automatic Control*, vol. 46, no. 12, pp. 1963–1968, 2001.
- [11] P. Krishnamurthy and F. Khorrami, "A high-gain scaling technique for adaptive output feedback control of feedforward systems," *IEEE transactions on automatic control*, vol. 49, no. 12, pp. 2286–2292, 2004.
- [12] M. Krstic, I. Kanellakopoulos, and P. V. Kokotovic, *Nonlinear and adaptive control design*. Wiley, 1995.
- [13] Z. Pan and T. Basar, "Adaptive controller design for tracking and disturbance attenuation in parametric strict-feedback nonlinear systems," *IEEE Transactions on Automatic Control*, vol. 43, no. 8, pp. 1066–1083, 1998.
- [14] Z. Pan, K. Ezal, A. J. Krener, and P. V. Kokotovic, "Backstepping design with local optimality matching," *IEEE Transactions on Automatic Control*, vol. 46, no. 7, pp. 1014–1027, 2001.
- [15] B. Aloliwi and H. K. Khalil, "Adaptive output feedback regulation of a class of nonlinear systems: convergence and robustness," *IEEE transactions on automatic control*, vol. 42, no. 12, pp. 1714–1716, 1997.
- [16] M. Arcak and P. Kokotović, "Robust nonlinear control of systems with input unmodeled dynamics," *Systems & control letters*, vol. 41, no. 2, pp. 115–122, 2000.

- [17] F. Ikhouane and M. Krstic, "Robustness of the tuning functions adaptive backstepping design for linear systems," *IEEE transactions on Automatic Control*, vol. 43, no. 3, pp. 431–437, 1998.
- [18] F. Ikhouane, A. Rabeh, and F. Giri, "Transient performance analysis in robust nonlinear adaptive control," *Systems & control letters*, vol. 31, no. 1, pp. 21–31, 1997.
- [19] Z.-P. Jiang and D. J. Hill, "A robust adaptive backstepping scheme for nonlinear systems with unmodeled dynamics," *IEEE Transactions on Automatic Control*, vol. 44, no. 9, pp. 1705–1711, 1999.
- [20] V. O. Nikiforov and K. V. Voronov, "Nonlinear adaptive controller with integral action," *IEEE Transactions on Automatic Control*, vol. 46, no. 12, pp. 2035–2037, 2001.
- [21] M. Jankovic, "Adaptive nonlinear output feedback tracking with a partial high-gain observer and backstepping," *IEEE Transactions on Automatic Control*, vol. 42, no. 1, pp. 106–113, 1997.
- [22] I. Kanellakopoulos, P. V. Kokotovic, and A. S. Morse, "Adaptive output-feedback control of systems with output nonlinearities," *IEEE Transactions on Automatic Control*, vol. 37, no. 11, pp. 1666–1682, 1992.
- [23] P. Krishnamurthy, F. Khorrami, and R. S. Chandra, "Global high-gain-based observer and backstepping controller for generalized output-feedback canonical form," *IEEE Transactions on Automatic Control*, vol. 48, no. 12, pp. 2277–2283, 2003.
- [24] P. Krishnamurthy, F. Khorrami, and Z.-P. Jiang, "Global output feedback tracking for nonlinear systems in generalized output-feedback canonical form," *IEEE Transactions on Automatic Control*, vol. 47, no. 5, pp. 814–819, 2002.
- [25] R. Marino and P. Tomei, "Global adaptive output-feedback control of nonlinear systems. I. Linear parameterization," *IEEE Transactions on Automatic Control*, vol. 38, no. 1, pp. 17–32, 1993.

- [26] R. Hotzel and L. Karsenti, “Adaptive tracking strategy for a class of nonlinear systems,” *IEEE transactions on automatic control*, vol. 43, no. 9, pp. 1272–1279, 1998.
- [27] R. Marino and P. Tomei, “An adaptive output feedback control for a class of nonlinear systems with time-varying parameters,” *IEEE Transactions on Automatic Control*, vol. 44, no. 11, pp. 2190–2194, 1999.
- [28] R. Ordóñez and K. M. Passino, “Adaptive control for a class of nonlinear systems with a time-varying structure,” *IEEE Transactions on Automatic Control*, vol. 46, no. 1, pp. 152–155, 2001.
- [29] Y. Zhang, B. Fidan, and P. A. Ioannou, “Backstepping control of linear time-varying systems with known and unknown parameters,” *IEEE Transactions on Automatic Control*, vol. 48, no. 11, pp. 1908–1925, 2003.
- [30] N. Hovakimyan, B.-J. Yang, and A. J. Calise, “Adaptive output feedback control methodology applicable to non-minimum phase nonlinear systems,” *Automatica*, vol. 42, no. 4, pp. 513–522, 2006.
- [31] P. A. Ioannou and J. Sun, *Robust adaptive control*. Courier Corporation, 2012.
- [32] M. Krstić, P. V. Kokotović, and I. Kanellakopoulos, “Transient-performance improvement with a new class of adaptive controllers,” *Systems & Control Letters*, vol. 21, no. 6, pp. 451–461, 1993.
- [33] V. Dobrokhodov, I. Kaminer, I. Kitsios, E. Xargay, C. Cao, I. M. Gregory, N. Hovakimyan, and L. Valavani, “Experimental validation of L1 adaptive control: The rohrrs counterexample in flight,” *Journal of Guidance, Control, and Dynamics*, vol. 34, no. 5, pp. 1311–1328, 2011.
- [34] C. Cao, V. V. Patel, C. K. Reddy, N. Hovakimyan, E. Lavretsky, and K. Wise, “Are Phase and Time-Delay Margin Always Adversely Affected by High Gains?” in *AIAA Guidance, Navigation, and Control Conference, AIAA-2006-6347*, 2006.

- [35] C. Cao and N. Hovakimyan, “Design and analysis of a novel L1 adaptive controller, Part I: Control signal and asymptotic stability,” in *2006 American Control Conference*. IEEE, 2006, pp. 3397–3402.
- [36] C. Cao and N. Hovakimyan, “Design and analysis of a novel l1 adaptive controller, part ii: Guaranteed transient performance,” in *2006 American Control Conference*. IEEE, 2006, pp. 3403–3408.
- [37] C. Cao and N. Hovakimyan, “Guaranteed transient performance with L1 adaptive controller for systems with unknown time-varying parameters and bounded disturbances: Part I,” in *American Control Conference, 2007. ACC’07*. IEEE, 2007, pp. 3925–3930.
- [38] C. Cao and N. Hovakimyan, “Stability Margins of L_1 Adaptive Controller: Part II,” in *American Control Conference, 2007. ACC’07*. IEEE, 2007, pp. 3931–3936.
- [39] C. Cao and N. Hovakimyan, “Design and Analysis of a Novel L1 Adaptive Control Architecture With Guaranteed Transient Performance,” *Automatic Control, IEEE Transactions on*, vol. 53, no. 2, pp. 586–591, 2008.
- [40] C. Cao and N. Hovakimyan, “Stability margins of adaptive control architecture,” *IEEE Transactions on Automatic Control*, vol. 55, no. 2, pp. 480–487, 2010.
- [41] N. Hovakimyan, C. Cao, E. Kharisov, E. Xargay, and I. M. Gregory, “Adaptive control for safety-critical systems,” *IEEE Control Systems*, vol. 31, no. 5, pp. 54–104, 2011.
- [42] J. Luo, C. Cao, and N. Hovakimyan, “ \mathcal{L}_1 adaptive controller for a class of systems with unknown nonlinearities,” in *American Control Conference*, Baltimore, MD, Jul. 2010, pp. 1659–1664.
- [43] X. Zou, C. Cao, and N. Hovakimyan, “ \mathcal{L}_1 adaptive controller for systems with hysteresis uncertainties,” in *American Control Conference*, Baltimore, MD, Jul. 2010, pp. 6662–6667.

- [44] J. Luo and C. Cao, " \mathcal{L}_1 Adaptive Controller for a Class of Nonlinear Systems," *Journal of Dynamic Systems, Measurement and Control*, vol. 136, no. 3, pp. 031 023–1 – 031 023–8, May 2014.
- [45] X. Zou, J. Luo, and C. Cao, "Adaptive Control for Uncertain Hysteretic Systems," *Journal of Dynamic Systems, Measurement and Control*, vol. 136, no. 1, pp. 011 011–1 – 011 011–7, Oct. 2013.
- [46] J. Luo, C. Cao, and Q. Yang, "L1 adaptive controller for a class of non-affine multi-input multi-output nonlinear systems," *International Journal of Control*, vol. 86, no. 2, pp. 348–359, 2013.
- [47] R. Choe, E. Xargay, and N. Hovakimyan, "Adaptive Control for a Class of Uncertain Nonaffine-in-Control Nonlinear Systems," *IEEE Transactions on Automatic Control*, vol. 61, no. 3, pp. 840–846, 2016.
- [48] C. Cao and N. Hovakimyan, " \mathcal{L}_1 adaptive output feedback controller to systems of unknown dimension," in *American Control Conference*, New York, Jul. 2007, pp. 1191–1196.
- [49] C. Cao and N. Hovakimyan, " \mathcal{L}_1 adaptive output-feedback controller for non-strictly-positive-real reference systems: Missile longitudinal autopilot design," *Journal of Guidance, Control, and Dynamics*, vol. 32, no. 3, pp. 717–726, 2009.
- [50] J. Luo and C. Cao, " \mathcal{L}_1 adaptive output feedback controller for a class of nonlinear systems," in *50th IEEE Conference on Decision and Control and European Control Conference*, Orlando, FL, Dec. 2011, pp. 5425–5430.
- [51] C. Cao and N. Hovakimyan, "Design and Analysis of a Novel \mathcal{L}_1 Adaptive Control Architecture With Guaranteed Transient Performance," *IEEE Transactions on Automatic Control*, vol. 53, no. 2, pp. 586–591, 2008.
- [52] J. Luo, X. Zou, and C. Cao, "Eigenvalue assignment for linear time-varying systems with disturbances," *IET Control Theory & Applications*, vol. 6, no. 3, pp. 365–374, 2012.

- [53] J. Luo, J. Cooper, C. Cao, and K. Pham, “Cooperative adaptive control of a two-agent system,” in *American Control Conference*, Montréal, Canada, Jun. 2012, pp. 2413–2418.
- [54] J. Luo, J. Cooper, C. Cao, and K. Pham, “Cooperative adaptive control of a two-agent system,” *International Journal of Control*, vol. 86, no. 1, pp. 127–138, Jan. 2013.
- [55] J. Luo and C. Cao, “Consensus in Multi-Agent Systems with Nonlinear Uncertainties Under a Fixed Undirected Graph,” *International Journal of Control, Automation and Systems*, vol. 12, no. 2, pp. 231–240, April 2013.
- [56] J. Luo and C. Cao, “Flocking for multi-agent systems with unknown nonlinear time-varying uncertainties under a fixed undirected graph,” *International Journal of Control*, vol. 88, no. 5, pp. 1051–1062, 2015.
- [57] J. Luo and C. Cao, “L1 adaptive control with sliding-mode based adaptive law,” *Control Theory and Technology*, vol. 13, no. 3, pp. 221–229, 2015.
- [58] C. Cao and N. Hovakimyan, “adaptive controller for systems with unknown time-varying parameters and disturbances in the presence of non-zero trajectory initialization error,” *International Journal of Control*, vol. 81, no. 7, pp. 1147–1161, 2008.
- [59] D. Li, V. V. Patel, C. Cao, N. Hovakimyan, and K. Wise, “Optimization of the time-delay margin of L1 adaptive controller via the design of the underlying filter,” in *AIAA Guidance, Navigation and Control Conference*, 2007.
- [60] C. Cao and N. Hovakimyan, “L1 adaptive controller for systems in the presence of unmodelled actuator dynamics,” in *IEEE Conference on Decision and Control*, 2007, pp. 891–896.
- [61] C. Cao and N. Hovakimyan, “L1 adaptive controller for a class of systems with unknown nonlinearities: Part I,” in *American Control Conference, 2008*. IEEE, 2008, pp. 4093–4098.

- [62] I. Kitsios, V. Dobrokhodov, I. Kaminer, K. D. Jones, E. Xargay, N. Hovakimyan, C. Cao, M. I. Lizarraga, I. M. Gregory, N. T. Nguyen, *et al.*, “Experimental Validation of a Metrics Driven L1 Adaptive Control in the Presence of General Unmodeled Dynamics,” in *AIAA Guidance, Navigation, and Control Conference*, vol. 6188, 2009.
- [63] S. Yoo, N. Hovakimyan, and C.-y. Cao, “Decentralised L1 adaptive control for large-scale non-linear systems with interconnected unmodelled dynamics,” *IET control theory & applications*, vol. 4, no. 10, pp. 1972–1988, 2010.
- [64] C. Cao and N. Hovakimyan, “L1 adaptive output-feedback controller for non-strictly-positive-real reference systems: missile longitudinal autopilot design,” *Journal of guidance, control, and dynamics*, vol. 32, no. 3, pp. 717–726, 2009.
- [65] C. Cao and N. Hovakimyan, “L1 adaptive output feedback controller for non-strictly positive real reference systems with applications to aerospace examples,” in *AIAA Guidance, Navigation, and Control Conference and Exhibit*, 2008.
- [66] R. Hindman, C. Cao, and N. Hovakimyan, “Designing a High Performance, Stable 1 Adaptive Output Feedback Controller,” 2007.
- [67] C. Cao and N. Hovakimyan, “L1 adaptive output feedback controller for systems with time-varying unknown parameters and bounded disturbances,” in *Proc. of American Control Conference*, 2007, pp. 486–491.
- [68] C. Cao and N. Hovakimyan, “L 1 adaptive controller for multi-input multi-output systems in the presence of unmatched disturbances,” in *2008 American Control Conference*. IEEE, 2008, pp. 4105–4110.
- [69] E. Xargay, N. Hovakimyan, and C. Cao, “L1 adaptive controller for multi-input multi-output systems in the presence of nonlinear unmatched uncertainties,” in *2010 American Control Conference*, 2010, pp. 874–879.
- [70] C. Cao and N. Hovakimyan, “L1 adaptive output-feedback controller for non-strictly-positive-real reference systems: missile longitudinal autopilot design,” *Journal of guidance, control, and dynamics*, vol. 32, no. 3, pp. 717–726, 2009.

- [71] J. Luo, C. Cao, and N. Hovakimyan, "L1 adaptive controller for a class of systems with unknown nonlinearities," in *Proceedings of the 2010 American Control Conference*. IEEE, 2010, pp. 1659–1664.
- [72] X. Zou, J. Luo, and C. Cao, "Adaptive control for uncertain hysteretic systems," *Journal of Dynamic Systems, Measurement, and Control*, vol. 136, no. 1, p. 011011, 2014.
- [73] J. Luo, J. Cooper, C. Cao, and K. Pham, "Cooperative adaptive control of a two-agent system," *International Journal of Control*, vol. 86, no. 1, pp. 127–138, 2013.
- [74] J. Luo and C. Cao, "L1 adaptive output feedback controller for a class of nonlinear systems," in *2011 50th IEEE Conference on Decision and Control and European Control Conference*. IEEE, 2011, pp. 5425–5430.
- [75] J. Luo, C. Cao, and Q. Yang, "L1 adaptive controller for a class of non-affine multi-input multi-output nonlinear systems," *International Journal of Control*, vol. 86, no. 2, pp. 348–359, 2013.
- [76] J. Luo, X. Zou, and C. Cao, "Eigenvalue assignment for linear time-varying systems with disturbances," *Control Theory & Applications, IET*, vol. 6, no. 3, pp. 365–374, 2012.
- [77] J. Cooper, J. Che, and C. Cao, "The use of learning in fast adaptation algorithms," *International Journal of Adaptive Control and Signal Processing*, vol. 28, no. 3-5, pp. 325–340, 2014.
- [78] C. Cao and N. Hovakimyan, "Novel neural network adaptive control architecture with guaranteed transient performance," *IEEE Transactions on Neural Networks*, vol. 18, no. 4, pp. 1160–1171, 2007.
- [79] L. Jie and C. Chengyu, "L1 adaptive control with sliding-mode based adaptive law," *()*, no. 2015-03, pp. 221–229, 2015.

- [80] E. Kharisov, N. Hovakimyan, J. Wang, and C. Cao, “L 1 adaptive controller for time-varying reference systems in the presence of unmodeled nonlinear dynamics,” in *Proceedings of the 2010 American Control Conference*. IEEE, 2010, pp. 886–891.
- [81] J. Cooper and C. Cao, “L 1 adaptive controller with additional uncertainty bias estimation,” in *2013 25th Chinese Control and Decision Conference (CCDC)*. IEEE, 2013, pp. 607–611.
- [82] C. Wang, R. Sharifi, and C. Cao, “L1 Adaptive Control of Uncertain Nonlinear Systems in the Presence of Output Limits,” 2011.
- [83] A. Thompson, J. Hacker, and C. Cao, “Adaptive engine control in the presence of output limits,” *AIAA Paper*, vol. 3492, p. 2010, 2010.
- [84] R. Sharifi, J. Hacker, and C. Cao, “Adaptive Engine Control with Multiple Constraints.” AIAA-2011-1592, AIAA Infotech@ Aerospace Conference, St. Louis, Missouri, 29-31 March, 2011.
- [85] D. Li, N. Hovakimyan, and C. Cao, “Positive invariance set of L1 adaptive controller in the presence of input saturation,” in *AIAA Guidance, Navigation, and Control Conference, Toronto, Canada*, 2010.
- [86] D. Li, N. Hovakimyan, and C. Cao, “L1 adaptive controller in the presence of input saturation,” in *AIAA Guidance, Navigation, and Control Conf*, 2009.
- [87] S. J. Yoo, N. Hovakimyan, and C. Cao, “Decentralized L 1 adaptive control for large-scale systems with unknown time-varying interaction parameters,” in *Proceedings of the 2010 American Control Conference*. IEEE, 2010, pp. 5590–5595.
- [88] X. Zou, C. Cao, and N. Hovakimyan, “L1 adaptive controller for systems with hysteresis uncertainties,” in *American Control Conference*, 2010, pp. 6662–6667.
- [89] J. Luo and C. Cao, “Consensus in multi-agent systems with nonlinear uncertainties under a fixed undirected graph,” *International Journal of Control, Automation and Systems*, vol. 12, no. 2, pp. 231–240, 2014.

- [90] L. Ma, C. Cao, N. Hovakimyan, C. Woolsey, and W. E. Dixon, “Fast estimation for range identification in the presence of unknown motion parameters,” *IMA journal of applied mathematics*, p. hxq008, 2010.
- [91] A. Young, C. Cao, N. Hovakimyan, and E. Lavretsky, “An adaptive approach to nonaffine control design for aircraft applications,” in *AIAA Guidance, Navigation and Control Conference and Exhibit, Keystone, CO*, 2006.
- [92] M. C. Cotting, C. Cao, N. Hovakimyan, L. C. R. J. Kraus, and W. C. Durham, “Simulator Testing of Longitudinal Flying Qualities with L1 Adaptive Control,” in *AIAA Atmospheric Flight Mechanics Conference and Exhibit*, 2008, p. 6551.
- [93] J. Luo, L. Pan, and C. Cao, “ \mathcal{L}_1 Adaptive Controller for a Nonlinear Boiler-Turbine System,” in *53rd Annual ISA POWID Symposium*, Summerlin, NV, May 2010.
- [94] L. Pan, J. Luo, C. Cao, and J. Shen, “L1 adaptive control for improving load-following capability of nonlinear boiler-turbine units in the presence of unknown uncertainties,” *Simulation Modelling Practice and Theory*, vol. 57, pp. 26–44, 2015.
- [95] L. P. J. Luo and C. Cao, “L1 Adaptive Controller for a Nonlinear Boiler-Turbine System,” in *53th Annual ISA POWID Symposium, Summerlin, NV*. IEEE, 2010.
- [96] Q. Yang, J. Luo, C. Cao, J. Lu, and Y. Sun, “Automated manipulation of micro-nano objects with SPM by using L1 adaptive controller,” *International Journal of Advanced Robotic Systems*, vol. 9, no. 228, 2012.
- [97] Z. Li, N. Hovakimyan, C. Cao, and G.-O. Kaasa, “Integrated estimator and L1 adaptive controller for well drilling systems,” in *2009 American Control Conference*. IEEE, 2009, pp. 1958–1963.
- [98] R. W. Beard, N. B. Knoebel, C. Cao, N. Hovakimyan, and J. S. Matthews, “An L1 adaptive pitch controller for miniature air vehicles,” in *AIAA Guidance, Navigation, and Control Conference, Keystone, CO*, 2006.

- [99] C. Cao, N. Hovakimyan, and E. Lavretsky, "Application of L1 adaptive controller to wing rock," in *AIAA Guidance, Navigation, and Control Conference, Keystone, CO*, 2006.
- [100] B. J. Guerreiro, C. Silvestre, R. Cunha, C. Cao, and N. Hovakimyan, "L 1 adaptive control for autonomous rotorcraft," in *2009 American Control Conference*. IEEE, 2009, pp. 3250–3255.
- [101] J. Wang, C. Cao, N. Hovakimyan, R. Hindman, and D. B. Ridgely, "L1 adaptive controller for a missile longitudinal autopilot design," in *AIAA Guidance, Navigation and Control Conference*, 2008, pp. 18–21.
- [102] C. Wong, I. Gregory, and C. Cao, "L1 Adaptive Control with Output Constraints Applied to Flight Envelope Limiting," in *Proceedings of the AIAA Modeling and Simulation Technologies Conference, Minneapolis, MN, USA*, vol. 1316, 2012.
- [103] C. Wang, M. Santone, and C. Cao, "Pilot-induced oscillation suppression by using L 1 adaptive control," *Journal of Control Science and Engineering*, vol. 2012, p. 2, 2012.
- [104] R. Choe, E. Xargay, N. Hovakimyan, C. Cao, and I. M. Gregory, "L1 Adaptive Control under Anomaly: Flying Qualities and Adverse Pilot Interaction," in *Proceedings of AIAA guidance, navigation and control conference*, 2010.
- [105] J. Che, C. Cao, and I. M. Gregory, "Integrated Flight/Structural Mode Control for Very Flexible Aircraft Using L1 Adaptive Output Feedback Controller," 2012.
- [106] Y. Lei, C. Cao, E. Cliff, N. Hovakimyan, A. Kurdila, and K. Wise, "L 1 adaptive controller for air-breathing hypersonic vehicle with flexible body dynamics," in *2009 American Control Conference*. IEEE, 2009, pp. 3166–3171.
- [107] I. M. Gregory, C. Cao, V. V. Patel, and N. Hovakimyan, "Adaptive control laws for flexible semi-span wind tunnel model of high-aspect ratio flying wing," in *AIAA Guidance, Navigation and Control Conference and Exhibit*, 2007, pp. 20–23.

- [108] Y. Lei, C. Cao, E. Cliff, N. Hovakimyan, and A. Kurdila, "Design of an L1 adaptive controller for air-breathing hypersonic vehicle model in the presence of unmodeled dynamics," in *Proc. AIAA Guidance, Navigation and Control Conf*, 2007.
- [109] V. V. Patel, C. Cao, N. Hovakimyan, K. A. Wise, and E. Lavretsky, "L1 adaptive controller for tailless unstable aircraft," in *2007 American Control Conference*. IEEE, 2007, pp. 5272–5277.
- [110] V. V. Patel, C. Cao, N. Hovakimyan, K. A. Wise, and E. Lavretsky, "L 1 adaptive controller for tailless unstable aircraft in the presence of unknown actuator failures," *International Journal of Control*, vol. 82, no. 4, pp. 705–720, 2009.
- [111] I. M. Gregory, C. Cao, E. Xargay, N. Hovakimyan, and X. Zou, "L1 adaptive control design for NASA AirSTAR flight test vehicle," in *AIAA guidance, navigation, and control conference*, vol. 5738, 2009.
- [112] I. Gregory, E. Xargay, C. Cao, and N. Hovakimyan, "Flight test of L1 adaptive controller on the NASA AirSTAR flight test vehicle," in *AIAA Guidance, Navigation and Control Conference, Toronto, Ontario*, 2010.
- [113] V. Dobrokhodov, I. Kitsios, I. Kaminer, K. D. Jones, E. Xargay, N. Hovakimyan, C. Cao, M. Lizarraga, and I. M. Gregory, "Flight validation of metrics driven L1 adaptive control," in *AIAA Guidance, Navigation and Control Conference and Exhibit*, 2008, p. 6987.
- [114] V. Dobrokhodov, I. Kitsios, I. Kaminer, K. D. Jones, E. Xargay, N. Hovakimyan, C. Cao, M. I. Lizarraga, and I. M. Gregory, "Preliminary Results of Development, System Integration and Flight Validation of a Metrics Driven L1 Adaptive Control," *AIAA Infotech@ Aerospace, Seattle, WA, AIAA Paper*, vol. 2053, 2009.
- [115] V. Dobrokhodov, E. Xargay, N. Hovakimyan, I. Kaminer, C. Cao, and I. M. Gregory, "Multicriteria analysis of an L1 adaptive flight control system," *Proceedings of the Institution of Mechanical Engineers, Part I: Journal of systems and control engineering*, vol. 227, no. 4, pp. 413–427, 2013.

- [116] J. Wang, N. Hovakimyan, and C. Cao, “Racetrack Maneuver of Aerial Refueling by L1 Adaptive Controller,” in *AIAA Infotech@ Aerospace Conference and AIAA Unmanned... Unlimited Conference*, 2009, p. 1888.
- [117] J. Wang, V. V. Patel, C. Cao, N. Hovakimyan, and E. Lavretsky, “L1 adaptive neural network controller for autonomous aerial refueling with guaranteed transient performance,” in *AIAA Guidance, Navigation, and Control Conference, Keystone, CO*, 2006.
- [118] J. Wang, V. V. Patel, C. Cao, N. Hovakimyan, and E. Lavretsky, “Novel L1 adaptive control methodology for aerial refueling with guaranteed transient performance,” *Journal of guidance, control, and dynamics*, vol. 31, no. 1, pp. 182–193, 2008.
- [119] J. Wang, C. Cao, N. Hovakimyan, and E. Lavretsky, “Novel L1 adaptive control approach to autonomous aerial refueling with guaranteed transient performance,” in *Proceedings of the 2006 American control conference, Minneapolis, Minnesota*, 2006, pp. 14–16.
- [120] J. Wang, V. V. Patel, C. Cao, N. Hovakimyan, and E. Lavretsky, “Verifiable L1 adaptive controller for aerial refueling,” in *AIAA Guidance, Navigation and Control Conference and Exhibit, Hilton Head, South Carolina*, 2007.
- [121] J. Wang, N. Hovakimyan, and C. Cao, “L1 adaptive augmentation of gain-scheduled controller for racetrack maneuver in aerial refueling,” in *AIAA Guidance, Navigation, and Control Conference*, vol. 5739, 2009.
- [122] K. Wise, E. Lavretsky, N. Hovakimyan, C. Cao, and J. Wang, “Verifiable adaptive flight control: UcaV and aerial refueling,” in *AIAA Guidance, Navigation, and Control Conference, AIAA*, vol. 6658, 2008.
- [123] A. P. Aguiar, I. Kaminer, R. Ghabcheloo, A. M. Pascoal, E. Xargay, N. Hovakimyan, C. Cao, and V. Dobrokhodov, “Time-coordinated path following of multiple UAVs over time-varying networks using L1 adaptation,” in *Proceedings of the AIAA Guidance, Navigation and Control Conference and Exhibit*, 2008.

- [124] A. P. Aguiar, I. Kaminer, R. Ghabcheloo, A. M. Pascoal, E. Xargay, N. Hovakimyan, C. Cao, and V. Dobrokhodov, "Coordinated path following of multiple UAVs for time-critical missions in the presence of time-varying communication topologies," *IFAC Proceedings Volumes*, vol. 41, no. 2, pp. 16 015–16 020, 2008.
- [125] I. Kaminer, A. Pascoal, E. Xargay, C. Cao, N. Hovakimyan, and V. Dobrokhodov, "3D path following for small uavs using commercial autopilots augmented by L1 adaptive control," *J. Guid. Control Dyn.*, vol. 33, no. 2, pp. 550–564, 2010.
- [126] I. Kaminer, O. Yakimenko, V. Dobrokhodov, A. Pascoal, N. Hovakimyan, C. Cao, A. Young, and V. Patel, "Coordinated path following for time-critical missions of multiple UAVs via L1 adaptive output feedback controllers," in *AIAA Guidance, Navigation and Control Conference and Exhibit*, 2007, p. 6409.
- [127] I. Kaminer, N. Hovakimyan, V. Patel, C. Cao, A. Young, A. Pascoal, and V. Dobrokhodov, "Time-critical coordinated path following for multiple UAVs via L1 adaptive output feedback controllers," in *Control Conference (ECC), 2007 European*. IEEE, 2007, pp. 281–286.
- [128] C. Cao, N. Hovakimyan, I. Kaminer, V. V. Patel, and V. Dobrokhodov, "Stabilization of Cascaded Systems via L1 Adaptive Controller with Application to," 2007.
- [129] L. Ma, C. Cao, N. Hovakimyan, V. Dobrokhodov, and I. Kaminer, "Adaptive Vision-Based Guidance Law with Guaranteed Performance Bounds for Tracking a Ground Target with Time-Varying Velocity," in *AIAA Guidance, Navigation, and Control Conference and Exhibit, Honolulu, HI*, 2008.
- [130] L. Ma, C. Cao, N. Hovakimyan, V. Dobrokhodov, and I. Kaminer, "Adaptive vision-based guidance law with guaranteed performance bounds," *Journal of guidance, control, and dynamics*, vol. 33, no. 3, pp. 834–852, 2010.
- [131] L. Ma, C. Cao, N. Hovakimyan, C. Woolsey, V. Dobrokhodov, and I. Kaminer, "Development of a vision-based guidance law for tracking a moving target," in *AIAA Guidance, Navigation and Control Conference and Exhibit*, 2007, p. 6744.

- [132] B. Sease, Q. Yang, Y. Xu, J. Che, and C. Cao, "L 1 adaptive attitude control for a picoscale satellite test bed," *IEEE Transactions on Aerospace and Electronic Systems*, vol. 51, no. 2, pp. 1147–1154, 2015.
- [133] A. Elahidoost, J. Cooper, C. Cao, and K. Pham, "Satellite Orbit Stabilization Using L1 Adaptive Control," in *AIAA Guidance, Navigation, and Control Conference*, 2012, p. 4923.
- [134] Q. Yang, J. Luo, C. Cao, J. Lu, and Y. Sun, "Automated manipulation of micro-nano objects with SPM by using L1 adaptive controller," *International Journal of Advanced Robotic Systems*, vol. 9, 2012.
- [135] J. R. Cooper, C. Cao, and J. Tang, "Control of a Nonlinear Pressure-Regulating Engine Bleed Valve in Aircraft Air Management Systems," in *ASME 2013 Dynamic Systems and Control Conference*. American Society of Mechanical Engineers, 2013, pp. V001T15A013–V001T15A013.
- [136] S. Mallikarjunan, B. Nesbitt, E. Kharisov, E. Xargay, N. Hovakimyan, C. Cao, *et al.*, "L1 adaptive controller for attitude control of multirotors," in *AIAA Guidance, Navigation and Control Conference, Minneapolis, AIAA-2012-48312012*, 2012.
- [137] J. Che, J. Cernio, J. Prainito, M. Zuba, C. Cao, J.-H. Cui, and K. Kazerounian, "An advanced autonomous underwater vehicle design and control strategy," in *2012 Oceans*. IEEE, 2012, pp. 1–6.
- [138] H. K. Khalil, *Nonlinear systems*. Prentice hall Upper Saddle River, 2002, vol. 3.
- [139] B. Friedland, "Treatment of bias in recursive filtering," *Automatic Control, IEEE Transactions on*, vol. 14, no. 4, pp. 359–367, 1969.
- [140] M. Bodson and S. Douglas, "Adaptive algorithms for the rejection of sinusoidal disturbances with unknown frequency," *Automatica*, vol. 33, no. 12, pp. 2213–2221, 1997.
- [141] C. Johnson, "Theory of disturbance-accommodating controllers," *Control and Dynamic Systems*, vol. 12, pp. 387–489, 1976.

- [142] J. Han, “A class of extended state observers for uncertain systems,” *Control and Decision*, vol. 10, no. 1, pp. 85–88, 1995.
- [143] R. Miklosovic, A. Radke, and Z. Gao, “Discrete implementation and generalization of the extended state observer,” in *American Control Conference, 2006*. IEEE, 2006, pp. 6–pp.
- [144] C. Cao and N. Hovakimyan, “ \mathcal{L}_1 adaptive output feedback controller for non strictly positive real Multi-input Multi-output systems in the presence of unknown nonlinearities,” in *American Control Conference, 2009. ACC’09*. IEEE, 2009, pp. 5138–5143.
- [145] J. Cooper, J. Che, and C. Cao, “The use of learning in fast adaptation algorithms,” *International Journal of Adaptive Control and Signal Processing*, pp. 4823–4828, 2013.
- [146] J. Luo and C. Cao, “ \mathcal{L}_1 adaptive output feedback controller for a class of nonlinear systems,” in *Decision and Control and European Control Conference (CDC-ECC), 2011 50th IEEE Conference on*. IEEE, 2011, pp. 5425–5430.
- [147] C. Cao and N. Hovakimyan, “ \mathcal{L}_1 adaptive controller for systems with unknown time-varying parameters and disturbances in the presence of non-zero trajectory initialization error,” in *International Journal of Control*. IEEE, 2008, pp. 1147–1161.
- [148] C. Cao and N. Hovakimyan, “ \mathcal{L}_1 adaptive controller for a class of systems with unknown nonlinearities: Part I,” in *American Control Conference, 2008*. IEEE, 2008, pp. 4093–4098.
- [149] C. Chen, *Linear system theory and design*. Oxford University Press, Inc., 1998.
- [150] B. Francis and W. Wonham, “The internal model principle of control theory,” *Automatica*, vol. 12, no. 5, pp. 457–465, 1976.
- [151] C. Garcia and M. Morari, “Internal model control. A unifying review and some new results,” *Industrial & Engineering Chemistry Process Design and Development*, vol. 21, no. 2, pp. 308–323, 1982.

- [152] D. Chen and B. PADEN, “Stable inversion of nonlinear non-minimum phase systems,” *International Journal of Control*, vol. 64, no. 1, pp. 81–97, 1996.
- [153] L. Qiu and E. Davison, “Performance limitations of non-minimum phase systems in the servomechanism problem,” *Automatica*, vol. 29, no. 2, pp. 337–349, 1993.
- [154] Y. Q. C. C. Zou, Xiaotian, “Eigenvalue Assignment Control Method for Linear Time-Varying System with Disturbances,” *3rd Annual Dynamic Systems and Control Conference*, 2010.
- [155] J. Che and C. Cao, “ \mathcal{L}_1 adaptive control of system with unmatched disturbance by using eigenvalue assignment method,” in *Decision and Control (CDC), 2012 IEEE 51st Annual Conference on*. IEEE, 2012, pp. 4823–4828.
- [156] B. Perry, III, S. R. Cole, and G. D. Miller, “Summary of an active flexible wing program,” *Journal of Aircraft*, vol. 32, no. 1, pp. 10–15, 1995.
- [157] V. Mukhopadhyay, “Flutter suppression control law design and testing for the active flexible wing,” *Journal of Aircraft*, vol. 32, no. 1, pp. 45–51, 1995.
- [158] I. M. Gregory, “Dynamic inversion to control large flexible transport aircraft,” in *AIAA Guidance, Navigation and Control Conference*, no. 98-4323, 1998.
- [159] I. M. Gregory, “Modified dynamic inversion to control large flexible aircraft: what is going on,” in *AIAA Guidance, Navigation and Control Conference*, no. 99-3998, 1999.
- [160] E. Kharisov, N. Hovakimyan, and K. Åström, “Comparison of several adaptive controllers according to their robustness metrics,” in *AIAA Guidance, Navigation and Control Conference, Toronto, Ontario*, 2010.
- [161] C. Cao and N. Hovakimyan, “ \mathcal{L}_1 Adaptive Output Feedback Controller for Systems of Unknown Dimension,” *Automatic Control, IEEE Transactions on*, vol. 53, no. 3, pp. 815–821, 2008.

- [162] C. Cao and N. Hovakimyan, “L1 Adaptive Output Feedback Controller for Non-Strictly Positive Real Reference Systems with Applications to Aerospace Examples,” in *AIAA Guidance, Navigation, and Control Conference and Exhibit*, 2008.
- [163] N. Hovakimyan, B. Yang, and A. Calise, “An adaptive output feedback control methodology for non-minimum phase systems,” in *Decision and Control, 2002, Proceedings of the 41st IEEE Conference on*, vol. 1. IEEE, 2002, pp. 949–954.
- [164] J. Luo and C. Cao, “L 1 adaptive output feedback controller for a class of nonlinear systems,” in *Decision and Control and European Control Conference (CDC-ECC), 2011 50th IEEE Conference on*. IEEE, 2011, pp. 5425–5430.
- [165] C. Cao and N. Hovakimyan, “L1 adaptive output feedback controller for non strictly positive real Multi-input Multi-output systems in the presence of unknown nonlinearities,” *American Control Conference*, pp. 5138 – 5143, 2009.
- [166] W. A. Silva, E. Vartio, A. Shimko, E. Segundo, R. Kvaternik, K. Eure, and R. Scott, “Development of aeroservoelastic analytical models and gust load alleviation control laws of a sensorcraft wind-tunnel model using measured data,” *AIAA Paper*, vol. 1935, p. 2006, 2006.
- [167] G. Zames, “On the input-output stability of time-varying nonlinear feedback systems part one: Conditions derived using concepts of loop gain, conicity, and positivity,” *Automatic Control, IEEE Transactions on*, vol. 11, no. 2, pp. 228–238, 1966.
- [168] S. Sastry, *Nonlinear systems: analysis, stability, and control*. Springer New York, 1999, vol. 10.
- [169] P. Dorato, “Short-time stability in linear time-varying systems,” DTIC Document, Tech. Rep., 1961.
- [170] C. C. Nguyen, “Arbitrary eigenvalue assignments for linear time-varying multivariable control systems,” *International Journal of Control*, vol. 45, no. 3, pp. 1051–1057, 1987.

- [171] H. Lee and J. Choi, "Ackermann-like eigenvalue assignment formulae for linear time-varying systems," *IEE Proceedings-Control Theory and Applications*, vol. 152, no. 4, pp. 427–434, 2005.
- [172] J. J. Zhu, "PD-spectral theory for multivariable linear time-varying systems," in *Decision and Control, 1997., Proceedings of the 36th IEEE Conference on*, vol. 4. IEEE, 1997, pp. 3908–3913.
- [173] W. J. Rugh and J. S. Shamma, "Research on gain scheduling," *Automatica*, vol. 36, no. 10, pp. 1401–1425, 2000.
- [174] J. S. Shamma, "Performance limitations in sensitivity reduction for nonlinear plants," *Systems & control letters*, vol. 17, no. 1, pp. 43–47, 1991.
- [175] J. Che and C. Cao, "L1 adaptive control of system with unmatched disturbance by using eigenvalue assignment method," in *2012 IEEE 51st IEEE Conference on Decision and Control (CDC)*. IEEE, 2012, pp. 4823–4828.
- [176] H. Kwakernaak and R. Sivan, *Linear optimal control systems*. Wiley-Interscience New York, 1972, vol. 172.
- [177] L. Silverman, "Transformation of time-variable systems to canonical (phase-variable) form," *Automatic Control, IEEE Transactions on*, vol. 11, no. 2, pp. 300–303, 1966.
- [178] S. Banerjee, Z. Wang, B. Baur, F. Holzapfel, J. Che, and C. Cao, "L 1 Adaptive Control Augmentation for the Longitudinal Dynamics of a Hypersonic Glider," *Journal of Guidance, Control, and Dynamics*, vol. 39, no. 2, pp. 275–291, 2015.
- [179] P. A. Ioannou and J. Sun, *Robust Adaptive Control*. Upper Saddle River, NJ: Prentice-Hall, 1996.
- [180] A. Datta and M. Ho, "On modifying model reference adaptive control schemes for performance improvement," *IEEE Transactions on Automatic Control*, vol. 39, no. 9, pp. 1977–1980, 1994.

- [181] J. Sun, “A modified model reference adaptive control scheme for improved transient performance,” *IEEE Transactions on Automatic Control*, vol. 38, no. 8, pp. 1255–1259, 1993.
- [182] R. Costa, “Improving transient behavior of model-reference adaptive control,” in *American Control Conference*, 1999, pp. 576–580.
- [183] A. Datta and P. Ioannou, “Performance analysis and improvement in model reference adaptive control,” *IEEE Transactions on Automatic Control*, vol. 39, no. 12, pp. 2370–2387, 1994.
- [184] T. Zhang, S. Ge, and C. Hang, “Adaptive output feedback control for general nonlinear systems using multilayer neural networks,” in *American Control Conference, 1998. Proceedings of the 1998*, vol. 1. IEEE, 1998, pp. 520–524.
- [185] S. S. Ge and J. Zhang, “Neural-network control of nonaffine nonlinear system with zero dynamics by state and output feedback,” *Neural Networks, IEEE Transactions on*, vol. 14, no. 4, pp. 900–918, 2003.
- [186] N. Hovakimyan and C. Cao, \mathcal{L}_1 *Adaptive Control Theory: guaranteed robustness with fast adaptation*. Philadelphia, PA: Siam, 2010.
- [187] N. Hovakimyan, C. Cao, E. Kharisov, E. Xargay, and I. M. Gregory, “ \mathcal{L}_1 adaptive control for safety-critical systems,” *IEEE Control Systems Magazine*, vol. 31, no. 5, pp. 54–104, 2011.
- [188] C. Cao and N. Hovakimyan, “Stability Margins of \mathcal{L}_1 Adaptive Control Architecture,” *IEEE Transactions on Automatic Control*, vol. 55, no. 2, p. 480, 2010.
- [189] C. Cao and N. Hovakimyan, “Guaranteed transient performance with \mathcal{L}_1 adaptive controller for systems with unknown time-varying parameters and bounded disturbances: Part I,” in *American Control Conference*, New York, Jul. 2007, pp. 3925–3930.

- [190] C. Cao and N. Hovakimyan, " \mathcal{L}_1 adaptive controller for systems with unknown time-varying parameters and disturbances in the presence of non-zero trajectory initialization error," *International Journal of Control*, vol. 81, no. 7, pp. 1147–1161, 2008.
- [191] C. Cao and N. Hovakimyan, "Design and analysis of a novel \mathcal{L}_1 adaptive control architecture with guaranteed transient performance," *IEEE Transactions on Automatic Control*, vol. 53, no. 2, pp. 586–591, 2008.
- [192] J. Luo, C. Cao, and Q. Yang, "L1 Adaptive Controller for a Class of Non-affine Multi-Input Multi-Output Nonlinear Systems," *International Journal of Control*, vol. 86, no. 2, pp. 348–359, Feb. 2013.
- [193] C. Cao and N. Hovakimyan, " \mathcal{L}_1 adaptive output-feedback controller for non-strictly-positive-real reference systems: Missile longitudinal autopilot design," *Journal of guidance, control, and dynamics*, vol. 32, no. 3, pp. 717–726, 2009.
- [194] J. Luo and C. Cao, " \mathcal{L}_1 adaptive output feedback controller for a class of nonlinear systems," in *Decision and Control and European Control Conference (CDC-ECC), 2011 50th IEEE Conference on*. IEEE, 2011, pp. 5425–5430.
- [195] J. F. Horn, A. J. Calise, and J. Prasad, "Flight envelope cueing on a tilt-rotor aircraft using neural network limit prediction," *Journal of the American Helicopter Society*, vol. 46, no. 1, pp. 23–31, 2001.
- [196] C. Tomlin, J. Lygeros, and S. Sastry, "Aerodynamic envelope protection using hybrid control," in *American Control Conference, 1998. Proceedings of the 1998*, vol. 3. IEEE, 1998, pp. 1793–1796.
- [197] J. Horn, A. J. Calise, and J. Prasad, "Flight envelope limit detection and avoidance for rotorcraft," *Journal of the American Helicopter Society*, vol. 47, no. 4, pp. 253–262, 2002.
- [198] S. Qin and T. Badgwell, "A survey of industrial model predictive control technology," *Control engineering practice*, vol. 11, no. 7, pp. 733–764, 2003.

- [199] M. Morari and J. H Lee, “Model predictive control: past, present and future,” *Computers & Chemical Engineering*, vol. 23, no. 4, pp. 667–682, 1999.
- [200] E. Zafriou, “Robust model predictive control of processes with hard constraints,” *Computers & Chemical Engineering*, vol. 14, no. 4, pp. 359–371, 1990.
- [201] J. Rawlings and K. Muske, “The stability of constrained receding horizon control,” *Automatic Control, IEEE Transactions on*, vol. 38, no. 10, pp. 1512–1516, 1993.
- [202] D. Q. Mayne, J. B. Rawlings, C. V. Rao, and P. O. Scokaert, “Constrained model predictive control: Stability and optimality,” *Automatica*, vol. 36, no. 6, pp. 789–814, 2000.
- [203] E. Zafriou and H. Chiou, “Output constraint softening for SISO model predictive control,” in *American Control Conference, 1993*. IEEE, 1993, pp. 372–376.
- [204] N. de Oliveira and L. Biegler, “Constraint handing and stability properties of model-predictive control,” *AIChE journal*, vol. 40, no. 7, pp. 1138–1155, 2004.
- [205] A. Zheng and M. Morari, “Stability of model predictive control with mixed constraints,” *Automatic Control, IEEE Transactions on*, vol. 40, no. 10, pp. 1818–1823, 1995.
- [206] S. L. de Oliveira Kothare and M. Morari, “Contractive model predictive control for constrained nonlinear systems,” *Automatic Control, IEEE Transactions on*, vol. 45, no. 6, pp. 1053–1071, 2000.
- [207] M. Hovd and R. Braatz, “Handling state and output constraints in MPC using time-dependent weights,” in *American Control Conference, 2001. Proceedings of the 2001*, vol. 3. IEEE, 2001, pp. 2418–2423.
- [208] Z. Q. Zheng and M. Morari, “Robust stability of constrained model predictive control,” in *American Control Conference, 1993*. IEEE, 1993, pp. 379–383.

- [209] M. V. Kothare, V. Balakrishnan, and M. Morari, “Robust constrained model predictive control using linear matrix inequalities,” *Automatica*, vol. 32, no. 10, pp. 1361–1379, 1996.
- [210] F. A. Cuzzola, J. C. Geromel, and M. Morari, “An improved approach for constrained robust model predictive control,” *Automatica*, vol. 38, no. 7, pp. 1183–1189, 2002.
- [211] L. Chisci, J. Rossiter, and G. Zappa, “Systems with persistent disturbances: predictive control with restricted constraints,” *Automatica*, vol. 37, no. 7, pp. 1019–1028, 2001.
- [212] D. Mayne, S. Raković, R. Findeisen, and F. Allgöwer, “Robust output feedback model predictive control of constrained linear systems,” *Automatica*, vol. 42, no. 7, pp. 1217–1222, 2006.
- [213] D. Mayne, S. Raković, R. Findeisen, and F. Allgöwer, “Robust output feedback model predictive control of constrained linear systems: Time varying case,” *Automatica*, vol. 45, no. 9, pp. 2082–2087, 2009.
- [214] Z. Wan and M. V. Kothare, “An efficient off-line formulation of robust model predictive control using linear matrix inequalities,” *Automatica*, vol. 39, no. 5, pp. 837–846, 2003.
- [215] Y. Wang and S. Boyd, “Fast model predictive control using online optimization,” *Control Systems Technology, IEEE Transactions on*, vol. 18, no. 2, pp. 267–278, 2010.