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Stabilization by Noise of Systems of Complex-valued ODEs

Fan Ny Shum, Ph.D.

University of Connecticut, 2016

ABSTRACT

D. Herzog and J. Mattingly have shown that a $\mathbb{C}$-valued polynomial ODE with finite-time blow-up solutions may be stabilized by the addition of $\mathbb{C}$-valued Brownian noise. In this paper, we extend their results to $\mathbb{C}^2$-valued systems of coupled ODEs with finite-time blow-up solutions. We show analytically and numerically that stabilization can be achieved in our setting by adding a suitable Brownian noise, and that the resulting systems of SDEs are ergodic. For one of the systems, the proof uses the Girsanov theorem to induce a time change from that $\mathbb{C}^2$-system to a quasi-$\mathbb{C}$-system similar to the one studied by Herzog and Mattingly.

Stabilization by Noise of Systems of Complex-valued ODEs

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Doctor of Philosophy Dissertation

Stabilization by Noise of Systems of Complex-valued ODEs

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Contents

Chapter 1

Introduction

1.1 Introduction

The main purpose of this dissertation is to study the stability of dynamical systems with the addition of *noise* in the multivariate setting. Specifically, we consider the $\mathbb{C}^2$-valued system of ODEs

$$\begin{cases} \dot{z}_t = -\nu z_t + \alpha z_t w_t \\ \dot{w}_t = -\nu w_t + \beta z_t w_t \end{cases} \tag{1.1.1}$$

with initial condition $(z_0, w_0) \in \mathbb{C}^2$. Here $\nu \in \mathbb{R}^+$ and $\alpha, \beta \in \mathbb{R}$. This system has a pair of fixed points: a sink at the origin and a saddle point at $(\nu/\beta, \nu/\alpha)$ (see Figure 3.1.1). Trajectories which lie on the unstable manifold associated with the saddle point, but not in the basin of attraction near the origin, will blow up in finite time. So the question is: Which types of complex-valued Brownian motions can one add to

stabilize these explosive trajectories? In particular, for what $\kappa_1, \kappa_2 \in \mathbb{R}$ and Brownian motions W_t^1, W_t^2 will the system of stochastic differential equations (SDEs)

$$\begin{cases} dz_t = (-\nu z_t + \alpha z_t w_t)\, dt + \kappa_1\, dW_t^1 \\ dw_t = (-\nu w_t + \beta z_t w_t)\, dt + \kappa_2\, dW_t^2 \end{cases} \tag{1.1.2}$$

be stable; more specifically, the processes z_t, w_t in (1.1.2) exist for all finite times and initial conditions, and the dynamics converges to a unique steady state with a corresponding invariant measure? We will refer to the SDE (1.1.2) as the *toy model of the stochastic Burgers' equation*. It is a simplification of the stochastic Burgers' equation of Example 3.8 in [HM15c].

D. Herzog and J. Mattingly studied the stability of the SDE

$$dz_t = (a_{n+1} z_t^{n+1} + a_n z_t^n + \dots + a_0)\, dt + \sigma\, dB_t \tag{1.1.3}$$

with initial condition $z_0 \in \mathbb{C}$, where $B_t = B_t^{(1)} + iB_t^{(2)}$, $B_t^{(1)}$ and $B_t^{(2)}$ are independent real-valued standard Brownian motions, and $\sigma \in \mathbb{R}^+$. When $\sigma = 0$, the system (1.1.3) has solutions that blow up in finite time. Herzog and Mattingly showed that, when $\sigma \neq 0$, the system is stable using Lyapunov theory in [Her11, HM15a, HM15b].

Through the use of a coordinate transformation, the system (1.1.1) can be partially decoupled; as a result, it is comparable to (1.1.3) for n=1. We will show that the system (1.1.2) is stable by using the stability of the system (1.1.3) for $\sigma \neq 0$. In particular, we will prove

Theorem 1.1.1 (Theorem 1.3 in [CFK+15]). *Consider the system of SDEs*

$$\begin{cases} dz_t = (-\nu z_t + \alpha z_t w_t)\ dt + \sigma\, dB_t \\ dw_t = (-\nu w_t + \beta z_t w_t)\ dt + \frac{\beta}{\alpha}\sigma\, dB_t \end{cases} \tag{1.1.4}$$

with initial condition $X_0 = (z_0, w_0) \in \mathbb{C}^2$, where $\nu \in \mathbb{R}^+$, $\alpha, \beta \in \mathbb{R} \setminus \{0\}$, $\sigma \in \mathbb{R} \setminus \{0\}$, and $B_t = B_t^{(1)} + iB_t^{(2)}$ is a $\mathbb{C}$-valued standard Brownian motion. Then the process $X_t = (z_t, w_t)$ is nonexplosive, and moreover, possesses a unique ergodic (invariant) measure.

Refer to Chapter 2 for the precise definition of nonexplosive, unique invariant measure, and ergodic.

In fact, we can say more about the invariant measure. To do so, we first introduce the shorthands

$$x_1 = \mathrm{Re}(z), \quad x_2 = \mathrm{Im}(z), \quad x_3 = \mathrm{Re}(w), \quad x_4 = \mathrm{Im}(w). \tag{1.1.5}$$

Then let

$$y_1 = \frac{1}{2}\left(x_1 + \frac{\alpha}{\beta}x_3\right),\ y_2 = \frac{1}{2}\left(x_1 - \frac{\alpha}{\beta}x_3\right),\ y_3 = \frac{1}{2}\left(x_2 + \frac{\alpha}{\beta}x_4\right),\ y_4 = \frac{1}{2}\left(x_2 - \frac{\alpha}{\beta}x_4\right). \tag{1.1.6}$$

Finally, let $\tilde{z} = y_1 + iy_3$ and $\tilde{w} = y_2 + iy_4$.

Proposition 1.1.2 (Invariant measure in [CFK+15]). *The system (1.1.4) has the unique invariant measure $\pi(\tilde{z})\delta_0(\tilde{w})$, where π is the unique invariant measure for the*

$\mathbb{C}$-valued system

$$d\tilde{z}_t = (-\nu\tilde{z}_t + \beta\tilde{z}_t^2)\,dt + \sigma\,dB_t, \tag{1.1.7}$$

and δ_0 is the delta measure at 0.

Remark 1.1.3. The delta measure δ_0, sometimes referred to as the Dirac measure, on a set S is defined for any measurable set $A \subseteq S$ by

$$\delta_0(A) = \mathbb{1}_A(0) = \begin{cases} 0, & 0 \notin A \\ 1, & 0 \in A \end{cases}$$

where $\mathbb{1}_A$ is the indicator function of A.

Note that (1.1.7) is of the form (1.1.3) with $n = 1$, $a_2 = \beta$, $a_1 = -\nu$, and $a_0 = 0$. As mentioned previously, Herzog and Mattingly have established the stability of (1.1.3) in [HM15a, HM15b].

The proofs of Theorem 1.1.1 and Proposition 1.1.2 involve a change of coordinates and the Girsanov theorem, which reduces the $\mathbb{C}^2$-system (1.1.4) to a quasi-$\mathbb{C}$-system similar to (1.1.7). We also present numerical evidence that supports Theorem 1.1.1 (in the case when an isotropic Brownian noise is added; meaning, noise is applied in all directions), as well as the case where an anisotropic Brownian noise is added (noise is added in specific directions, not all).

1.2 Motivation

An *explosive* system is a system of differential equations with trajectories that blow up in finite time. (Refer to Definition 2.2.2 for the explicit definition.) Some systems of this type have been shown to be stable by adding a random noise; that is, by adding a small amount of randomness transversal to an explosive trajectory, which pushes the trajectory onto a dynamically stable path. While there are examples where the addition of noise does not guarantee stability (see Scheutzow's construction in [Sch93]), usually one can stabilize an explosive system by adding a suitable Brownian noise.

An idea of stabilization was influenced by the study of turbulence. In [Bec05, BCH07], Bec *et al.* modeled the flow of certain fluids. This model can be written as an SDE with a polynomial drift term through the use of a certain substitution. Specifically, this model resembles the SDE

$$dz_t = (z_t^2 + az_t + b)\, dt + \sigma\, dB_t. \tag{1.2.1}$$

To see this substitution, refer to [GHW11]. This inspired Herzog and Mattingly, in [Her11, HM15a, HM15b], to study the stability of the complex-valued SDE

$$dz_t = (a_{n+1} z_t^{n+1} + a_n z_t^n + ... + a_0)\, dt + \sigma\, dB_t \tag{1.2.2}$$

with initial condition $z_0 \in \mathbb{C}$, where $B_t = B_t^{(1)} + iB_t^{(2)}$, $B_t^{(1)}$ and $B_t^{(2)}$ are independent real-valued standard Brownian motions, and $\sigma \in \mathbb{R}^+$. They showed that the SDE (1.2.2) has a solution for all finite times and initial conditions, and its solution possesses a unique invariant measure. In particular, they showed the system (1.2.2) is

ergodic; roughly speaking, it has the same behavior averaged over time as averaged over the space of all the system's states. For the precise definition, see Definition 2.3.6.

For example, the ODE

$$\dot{z}_t = z_t^2 \tag{1.2.3}$$

has solutions that blow up in finite time with the initial condition $z_0 > 0$. The addition of a complex-valued Brownian motion perturbs these explosive solutions onto one of the stable solutions. Thus, the SDE

$$dz_t = z_t^2\, dt + \sigma\, dB_t \tag{1.2.4}$$

is nonexplosive and its dynamics resembles those of the ODE 1.2.3. See Figure 2.5.2a and Section 2.5 for more details. Note that the SDE 1.2.2 is non-Lipschitz; hence, we cannot verify its stability through the standard existence and uniqueness theorem for SDEs (see Theorem 2.1.1).

One of the main tools Herzog and Mattingly used to prove the stability of system (1.2.2) is finding a *Lyapunov function* $\varphi \in C^2(\mathbb{C} : [0, \infty))$ such that

1. $\varphi(z) \to \infty$ as $|z| \to \infty$; and

2. $\mathcal{L}\varphi(z) \to -\infty$ as fast as possible as $|z| \to \infty$,

where $\mathcal{L}$ is the infinitesimal generator of the SDE (1.2.2). If such a function exists for the process z_t in (1.2.2), then the system is nonexplosive, and there exists an invariant measure. Intuitively, this Lyapunov function guarantees that our process decays rapidly. However, identifying the "correct" Lyapunov functions is difficult in

general; see [Her11, HM15a, HM15b] for their explicit construction. We will expand on this in Section 2.4.

From the physics perspective, equation (1.2.2) is interesting because it corresponds to complex-valued Langevin equations arising from minimization of certain complex-valued energy functionals, also known as *path integrals* (but with non-positive-definite integrands). Making sense of these path integrals will lead to better understanding of lattice gauge quantum chromodynamics (QCD) models. Recently, G. Aarts and collaborators have studied special cases of (1.2.2) from the numerical point-of-view, *cf.* [AGS13, ABSS14] and references therein. They have numerically calculated the invariant measure of the SDE, and obtained an approximate spectrum of the infinitesimal generator of the SDE. However, there is still work to do to bridge the gap between the stochastic analysis (ergodicity, exponential convergence to equilibrium) and the numerical calculations (spectral simulations, physics implications) for these SDEs.

The layout of the dissertation is as follows. In Chapter 2, we introduce some of the methods needed to attain the ergodic property for time-homogeneous stochastic differential equations in $\mathbb{R}^d$. Note that Chapter 2 is a compilation of results from various sources, which will be referenced appropriately. Sections 2.1-2.3 provide standard techniques and Section 2.4 gives an outline of the construction of the Lyapunov function. In particular, we will see the construction of the Lyapunov function for a specific example. In Section 2.5, we have the application of those methods to the SDE (1.2.2). In Chapter 3, we will prove the main theorems stated in Section 1.1. We describe the linear transformation which reduces our toy model of the Burgers' equation to a deterministic quasi-$\mathbb{C}$-valued ODE. We also perform a dynamical analysis to identify the explosive regions of the deterministic system. This serves as preparation

for Section 3.3, where we add an isotropic Brownian noise and show rigorously the reduction of our $\mathbb{C}^2$-valued SDE (1.1.4) to a quasi-$\mathbb{C}$-valued SDE similar to (1.1.7). From this we can deduce the ergodic properties of our system (1.1.4) *à la* Herzog, and thus prove Theorem 1.1.1 and Proposition 1.1.2. In Chapter 4, we give numerical evidence for the stabilization of our $\mathbb{C}^2$ system by adding an isotropic or an anisotropic Brownian noise. Some concluding remarks and future directions are given in Chapter 5.

Chapter 2

Stability of Stochastic Differential Equations

This chapter informs the reader of the techniques and terminologies used to attain stability for the systems in question and some more general cases. We will start with some notation.

Let $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ be a probability space equipped with the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ and a probability measure P on $\mathcal{F}$. Then let $(X_t)_{t\geq 0}$ be a $\mathbb{R}^d$-valued stochastic process adapted to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$. Let $W_t = (W_t^1, \ldots, W_t^m)$ be an m-dimensional standard real-valued Brownian motion.

Remark 2.0.1. The dimensions of $\mathbb{R}^d$ and W_t do not need to be the same. However, for the examples discussed in this paper, we will assume $m = d$.

2.1 Stochastic Differential Equation

A *stochastic differential equation* (SDE) is of the form

$$dX_t = b(t, X_t)\, dt + \sigma(t, X_t)\, dW_t \tag{2.1.1}$$

with initial condition $X_0 = x \in \mathbb{R}^d$, where $b(\cdot,\cdot) : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma(\cdot,\cdot) : [0,T] \times \mathbb{R}^d \to \mathbb{R}^{d\times d}$ for $T > 0$. We will call b the *drift coefficient* and σ (or, sometimes, $\frac{1}{2}\sigma\sigma^T$) the *diffusion coefficient*. In particular, solutions to this SDE are referred to as *Itô diffusions*.

For systems of SDEs of the form (2.1.1), the existence and uniqueness of solutions are determined by the following theorem. We will skip its proof (see [Øks03]).

Theorem 2.1.1 (Existence and Uniqueness Theorem)**.** *Let b and σ, defined above, be measurable functions such that for some constant C and for any $x, y \in \mathbb{R}^d$*

$$|b(t,x) - b(t,y)| + |\sigma(t,x) - \sigma(t,y)| \le C|x - y| \tag{2.1.2}$$

for $t \in [0,T]$, where $|\sigma| = \sum |\sigma_{ij}|^2$. Then the SDE (2.1.1) has a unique solution X_t with continuous paths such that $X_t(\omega)$ is adapted to $\mathcal{F}_t$ of W_t.

Remark 2.1.2. We refer to equation (2.1.2) as the *Lipschitz* condition. Other versions of this theorem can be found in [Øks03] as Theorem 5.2.1. and in [Her11] as Theorem 2.2.

If we can show the coefficients of an SDE satisfy the Lipschitz condition, then the SDE has a unique solution. However, the coefficients of the SDEs that we are studying are at most locally Lipschitz; we can only guarantee unique solutions locally.

This does not imply that we do not have global stability. For example, consider the real-valued one-dimensional SDE

$$dx_t = -x_t^3\,dt + dW_t. \tag{2.1.3}$$

For any initial condition $x_0 = x \neq 0$, the dynamics sinks into the origin. However, $b(x) = -x^3$ is not globally Lipschitz. Since we want to show that the SDEs in question have a solution for all finite time, we will need to explore alternative approaches to show stability. An alternative method is the use of *Lyapunov functions* proposed in [Kha12]. It has been shown that if such a function exists for an SDE, then it guarantees the solutions exist for all finite time and moreover, there exists an invariant measure. In other words, the dynamics converges to a limiting distribution and by understanding what this distribution looks like, we will have an idea of the "long-time" behavior of our solutions. This invariant measure tells us how stable the solutions are. Before we define what a Lyapunov function is, we need to understand what it means to be explosive and what is an invariant measure.

Solutions to SDEs are known to be *strongly Markovian*; meaning, the future only depends on the present time, including stopping times. We will look at what that means precisely. Denote by P_x the law of X_t starting at $x \in \mathbb{R}^d$ and E_x the corresponding expectation.

Definition 2.1.3. For any $A \in \mathcal{B}(\mathbb{R}^d)$, $P(t, x, A) := P_x(X_t \in A)$ is a *Markov transition function*, or *transition kernel*. In particular, it satisfies the *Chapman-Kolmogorov equation*: for any $t, s > 0$ and $x \in \mathbb{R}^d$,

$$P(t+s, x, A) = \int_{\mathbb{R}^d} P(s, y, A) P(t, x, dy). \tag{2.1.4}$$

Lemma 2.1.4 (Markov Property [Øks03]). *Let $\varphi : \mathbb{R}^d \to \mathbb{R}$ be a bounded, Borel measurable function. We have for $s, t \geq 0$,*

$$E_x[\varphi(X_{s+t})|\mathcal{F}_s] = E_y[\varphi(X_t)]|_{y=X_s}.$$

Lemma 2.1.5 (Strong Markov Property [Øks03]). *Let $\varphi : \mathbb{R}^d \to \mathbb{R}$ be a bounded, Borel measurable function and τ be an almost surely bounded stopping time with respect to $\{\mathcal{F}_t\}_{t\geq 0}$. Then for $t \geq 0$,*

$$E_x[\varphi(X_{\tau+t})|\mathcal{F}_\tau] = E_y[\varphi(X_t)]|_{y=X_\tau},$$

where $\mathcal{F}_\tau$ is the sigma algebra generated by $\{W_{s\wedge\tau}\}_{s\geq 0}$.

In other words, the future behavior of the process $X_{\tau+t}$, given what has happened up to time τ, only depends on where the process X_τ is at time τ (i.e. the future only depends on the present, not the past). Since we will work with Markov processes, we need to fix some definitions to give us a better idea about these processes.

We will define an operator P_t for all $t \geq 0$ by

$$P_t\varphi(x) = \int P(t, x, dy)\varphi(y), \tag{2.1.5}$$

where $\varphi : \mathbb{R}^d \to \mathbb{R}$ is a bounded measurable function, and

$$\mu P_t(A) = \int \mu(dy)P(t, y, A), \tag{2.1.6}$$

where μ is a finite Borel measure on $\mathbb{R}^d$ and $A \in \mathcal{B}(\mathbb{R}^d)$.

Remark 2.1.6. By (2.1.4), $\{P_t\}_{t\geq 0}$ forms a *semigroup* on $\mathcal{B}(\mathbb{R}^d)$; that is,

$$P_{t+s} = P_t P_s. \tag{2.1.7}$$

Using (2.1.6), we can define a *dual semigroup* acting on σ-finite measures on $\mathbb{R}^d$:

$$S_t\mu(A) = \int \mu(dx)P(t,x,A) \tag{2.1.8}$$

For more details, see Section 3.1 in [RB06].

This semigroup has the following properties:

Proposition 2.1.7. *Let φ be a bounded Borel measurable function. Then*

1. $P_t\varphi(x) \geq 0$ *if* $\varphi(x) \geq 0$ *and*

2. $P_tC = C$ *for constants* C.

Definition 2.1.8. We say P_t is *weak Feller* if P_t maps bounded continuous functions to bounded continuous functions.

We will use this definition in Section 2.3 to state the existence of an invariant measure.

2.2 Conditions for Nonexplosion

Recall that one of our goals is to show a process X_t does not have solutions that blow up in finite time. This concept is referred to as *nonexplosive*. To do so, we need to show its *explosion time* is infinite for any initial condition $X_0 = x$. We will define this rigorously below.

Definition 2.2.1. For each fixed $n \geq 0$, let

$$\xi_n = \inf\{t \geq 0 : |X_t| \geq n\} \tag{2.2.1}$$

be the exit time of X_t from the ball of radius n centered at the origin. Then the *explosion time* ξ of the process X_t is defined as

$$\xi := \sup_n \xi_n. \tag{2.2.2}$$

Definition 2.2.2. A process X_t is said to be *nonexplosive* if

$$P_x(\xi < \infty) = 0 \tag{2.2.3}$$

for all $x \in \mathbb{R}^d$.

We will utilize a theorem from [Kha12] to obtain nonexplosivity. First, we need to introduce the *generator* of the process X_t and *Dynkin's Formula.* For any $A \in \mathcal{B}([0,\infty))$ and $U \in \mathcal{B}(\mathbb{R}^d)$, denote the set of functions that are once continuously differentiable on A and k times continuously differentiable on U by $C_1^k(A \times U)$, denote the set of functions that are k times continuously differentiable on U and compactly supported in U by $C_0^k(U)$, and denote the set of functions that are k times continuously differentiable on U by $C^k(U)$. For any F in $\mathcal{F}$, denote the *conditional probability* $P_{t,x}(F) = P[F|X(t) = x]$ and the associated *conditional expectation* $E_{t,x}Y = E[Y|X(t) = x]$ for any $x \in \mathbb{R}^d, t \geq 0$, and any random variable $Y = Y(\omega)$, $\omega \in \Omega$.

Definition 2.2.3. Let X_t be an Itô diffusion defined in (2.1.1). The (infinitesimal)

generator $\mathcal{L}$ of X_t is defined by

$$\mathcal{L}\varphi(t,x) = \lim_{s \searrow 0} \frac{E_{t,x}[\varphi(t+s, X_{t+s})] - \varphi(t,x)}{s}, \tag{2.2.4}$$

for $t \geq 0$ and $x \in \mathbb{R}^d$. In particular,

$$\mathcal{L}\varphi(t,x) = \frac{\partial \varphi}{\partial t}(t,x) + \sum_{i=1}^{d} b^{(i)}(t,x) \frac{\partial \varphi}{\partial x^{(i)}}(t,x) + \frac{1}{2} \sum_{i,j=1}^{d} (\sigma\sigma^T)^{(ij)}(t,x) \frac{\partial^2 \varphi}{\partial x^{(i)} \partial x^{(j)}}(t,x), \tag{2.2.5}$$

for $t \geq 0$, $x \in \mathbb{R}^d$, and $\varphi \in C_1^2([0,\infty) \times \mathbb{R}^d)$.

Remark 2.2.4. The infinitesimal generator is typically defined on a time-homogeneous system by

$$\mathcal{L}\varphi(x) = \sum_{i=1}^{d} b^{(i)}(x) \frac{\partial \varphi}{\partial x^{(i)}}(x) + \frac{1}{2} \sum_{i,j=1}^{d} (\sigma\sigma^T)^{(ij)}(x) \frac{\partial^2 \varphi}{\partial x^{(i)} \partial x^{(j)}}(x), \tag{2.2.6}$$

for $x \in \mathbb{R}^d$ and $\varphi \in C_0^2(\mathbb{R}^d)$. See Theorem 7.3.3. in [Øks03].

The generator of X_t is essentially a partial differential operator. The following lemma will provide a connection between the probabilistic theory of SDEs and the classical theory of PDEs.

Lemma 2.2.5 (Dynkin's Formula [Her11])**.** *Let* $\varphi \in C_1^2([0,\infty) \times \mathbb{R}^d)$*. Then*

$$E_x \varphi(\xi_n \wedge t, X_{\xi_n \wedge t}) - \varphi(0, X_0) = E_x \left[\int_0^{\xi_n \wedge t} \mathcal{L}\varphi(s, X_s)\, ds \right] \tag{2.2.7}$$

for all $t \geq 0$ *and for all* $x \in \mathbb{R}^d$*.*

Proof. See [Her11]. □

Remark 2.2.6. A time-homogeneous version of Dynkin's formula can be found in [Øks03] as Theorem 7.4.1.

With Dynkin's formula, we can use an alternative method to show the existence and uniqueness of (2.1.1) when it fails the Lipschitz condition. To do so, we need to find a suitable function φ utilized in (2.2.7); specifically, we need a function $\varphi \to \infty$ as $|X_t| \to \infty$. If we can control the growth rate of φ, then we can control the rate of X_t. Hence, we want an upper bound for the right-hand side of (2.2.7).

Theorem 2.2.7 (Theorem 2.8 in [Her11])**.** *Let $\varphi \in C^2(\mathbb{R}^d)$ be a nonnegative function and suppose*

$$\varphi(x) \to \infty$$

as $|x| \to \infty$, and there exist positive constants C, D such that

$$\mathcal{L}\varphi(x) \leq C\varphi(x) + D \tag{2.2.8}$$

for all $x \in \mathbb{R}^d$. Then the process X_t is nonexplosive.

Proof. Let $\Phi(t,x) = e^{-Ct}(\varphi(x) + D/C)$. There exists an $N \in \mathbb{N}$ such that $\varphi(y) \geq 1$ for all $|y| \geq N$. Then for all $n \geq N$,

$$\begin{aligned} E_x(\Phi(\xi_n \wedge t, X_{\xi_n \wedge t})) - \Phi(0, X_0) &= E_x\left[\int_0^{\xi_n \wedge t} \mathcal{L}\Phi(s, X_s)\, ds\right] \\ &= E_x\left[\int_0^{\xi_n \wedge t} \mathcal{L}(e^{-Cs}(\varphi(X_s) + D/C))\, ds\right] \\ &= E_x\left[\int_0^{\xi_n \wedge t} -C\Phi(s, X_s) + e^{-Cs}\mathcal{L}\varphi(X_s)\, ds\right] \\ &\leq E_x\left[\int_0^{\xi_n \wedge t} -C\Phi(s, X_s) + C\Phi(s, X_s)\, ds\right] \\ &= 0. \end{aligned}$$

Hence, we have

$$\begin{aligned}\Phi(0, X_0) &\geq E_x(\Phi(\xi_n \wedge t, X_{\xi_n \wedge t})) \\ &\geq E_x\left[\mathbb{1}_{\{\xi_n \leq t\}} e^{-C(\xi_n \wedge t)} (\varphi(X_{\xi_n \wedge t}) + D/C)\right] \\ &\geq e^{-Ct} \inf_{|y| \geq n} \varphi(y) P_x(\xi_n \leq t).\end{aligned}$$

Finally, we have

$$P_x(\xi_n \leq t) \leq \frac{e^{Ct}\Phi(0, X_0)}{\inf_{|y|\geq n} \varphi(y)}. \tag{2.2.9}$$

Taking $n \to \infty$, we have $P_x(\xi \leq t) = 0$ for all $t \geq 0$. Then $P_x(\xi < \infty) = 0$ for all $x \in \mathbb{R}^d$, which means that the process X_t is nonexplosive. $\square$

Remark 2.2.8. A similar version of Theorem 2.2.7 can be found in [Kha12] as Theorem 3.5.

As one can see, if we can find a suitable function φ that satisfies (2.2.9), then we can guarantee nonexplosivity of the process X_t with the addition of noise. However, we also want to show that we obtain the same deterministic system after the perturbation (see Section 2.5). We need a more restrictive condition on this function φ to ensure that there is also an invariant measure that the dynamics converges to.

2.3 Invariant Measures

We will now assume that the Markov process X_t is nonexplosive. In order to show that the process X_t possesses a unique invariant measure, we will use the definitions in Section 2.1 to define an invariant measure.

Definition 2.3.1. Let μ be a Borel measure. We say μ is an *invariant measure* for the semigroup $\{P_t\}_{t\geq 0}$ if, for all $t \geq 0$,

$$\mu P_t = \mu. \tag{2.3.1}$$

If $\mu(\mathbb{R}^d) < \infty$, then it can be normalized to a probability measure π that also satisfies (2.3.1) respectively. We say π is an *invariant probability measure* for the semigroup $\{P_t\}_{t\geq 0}$. (See (2.1.6) for the definition of μP_t.)

Theorem 2.3.2 gives a necessary and sufficient condition for the existence of an invariant measure. Once we know its existence, we can start to understand what it looks like to describe the "long-time" behavior of the process X_t.

Theorem 2.3.2 (Theorem 2.32 in [Her11])**.** *Suppose that P_t is weak Feller. Then there exists an invariant probability measure if and only if for some $x \in \mathbb{R}^d$,*

$$\lim_{r\to\infty} \liminf_{t\to\infty} \frac{1}{t}\int_0^t P(s, x, B_r(0)^C)\, ds = 0, \tag{2.3.2}$$

where $B_r(0)^C$ denotes the complement of the ball of radius r centered at the origin.

Recall in Section 2.2 we needed a suitable function φ to guarantee that the process X_t is nonexplosive. Now we want to utilize Theorem 2.3.2 to show that, with a suitable function φ, we can also guarantee that the process X_t possesses an invariant measure.

Theorem 2.3.3 (Theorem 2.34 in [Her11])**.** *Let $\varphi \in C^2(\mathbb{R}^d)$ be a nonnegative function such that*

$$\mathcal{L}\varphi(x) \to -\infty \tag{2.3.3}$$

as $|x| \to \infty$. Then there exists an invariant probability measure for $\{P_t\}_{t\geq 0}$.

Proof. There exists $R > 0$ such that for all $r \geq R$,

$$\begin{aligned} \mathcal{L}\varphi(X_s) &\leq \sup_{|x|>r} \mathcal{L}\varphi(x) \mathbb{1}_{\{|X_s|>r\}} + \sup_{x \in \mathbb{R}^n} \mathcal{L}\varphi(x) \\ &\leq -c_r \mathbb{1}_{\{|X_s|>r\}} + d, \end{aligned}$$

for some constants $c_r, d > 0$ such that $c_r \to \infty$ as $r \to \infty$. By Lemma 2.2.5,

$$\begin{aligned} c_r \int_0^{\xi_n \wedge t} P(s, x, B_r(0)^C)\, ds &= c_r E_x \int_0^{\xi_n \wedge t} \mathbb{1}_{\{|X_s|>r\}}\, ds \\ &\leq E_x \varphi(X_{\xi_n \wedge t}) + c_r E_x \int_0^{\xi_n \wedge t} \mathbb{1}_{\{|X_s|>r\}}\, ds \\ &\leq \varphi(X_0) + d(\xi_n \wedge t). \end{aligned}$$

Since X_t is nonexplosive, $\xi_n \wedge t \to t$ as $n \to \infty$ almost surely. Hence we have

$$\lim_{r\to\infty} \liminf_{t\to\infty} \frac{1}{t} \int_0^t P(s, x, B_r(0)^C)\, ds \leq \lim_{r\to\infty} \liminf_{t\to\infty} \frac{d}{c_r} + \frac{\varphi(X_0)}{c_r t} = 0.$$

By Theorem 2.3.2, there exists an invariant probability measure. $\square$

As one can see from Theorem 2.2.7 and Theorem 2.3.3, we only need to find a suitable function φ to show that the process X_t is nonexplosive and it has an invariant measure. We will state this result explicitly later. First, we will talk about ergodicity.

In Section 2.1, we defined a Markov transition function with respect to the probability distribution P_x, which corresponds to a stochastic process X_t such that

$$P(X_0 = x) = 1.$$

Similarly, given an *initial distribution* π, let π be a probability measure on $\mathbb{R}^d$ that

describes the initial state of the system at time $t = 0$. Denote by P_π and E_π the corresponding probability distribution and expectation.

Definition 2.3.4. Let X_t be a Markov process with initial distribution π. We say the process X_t is *stationary* if

$$S_t\pi = \pi. \tag{2.3.4}$$

In particular, we say π is *stationary.* (Refer to (2.1.8) for the definition of $S_t\pi$.)

We will now state a special case of the Birkhoff Ergodic Theorem for our process X_t. See Theorem 3.8 and Remark 3.9 in [RB06] for more details. Let $L^1(\pi)$ denote the set of functions that are π-integrable.

Theorem 2.3.5. *Let X_t be a stationary process with the initial distribution π. Then for any $f \in L^1(\pi)$, the limit*

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t f(X_s(\omega))\, ds = f^*(x) \tag{2.3.5}$$

exists P_π almost all $\omega \in \Omega$ and π almost all $X_0 = x \in \mathbb{R}^d$.

Definition 2.3.6. The process X_t is *ergodic* with respect to the measure π if for all $f \in L^1(\pi)$,

$$P_\pi\left(\lim_{t\to\infty} \frac{1}{t} \int_0^t f(X_s)\, ds = \int_{\mathbb{R}^d} f(x)\, \pi(dx)\right) = 1. \tag{2.3.6}$$

In particular, we say π is *ergodic.*

Definition 2.3.6 says that the time average equals the space average almost surely. For instance, suppose our measure space models particles of a gas and $f(x)$ denotes the velocity of the particle at position x. The ergodicity of X_t says that the average

velocity of all particles at some given time is equal to the average velocity of one particle over time. We will show the invariant measure of the process z_t defined by (1.1.4) is, in fact, ergodic.

Remark 2.3.7. Stationary distributions correspond to ergodic stationary processes. See Theorem 3.8 in [RB06] for more details.

Applying Lebesgue's dominated convergence theorem, (2.3.6) implies that

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t E_x f(X_s)\, ds = \int_{\mathbb{R}^n} f(x)\, \pi(dx), \tag{2.3.7}$$

for all $x \in \mathbb{R}^d$ and all bounded functions f. In particular, the invariant measure π can be obtained as the limit

$$\pi(\cdot) = \lim_{t\to\infty} \frac{1}{t} \int_0^t P_x(X_s \in \cdot)\, ds, \tag{2.3.8}$$

which is independent of $x \in \mathbb{R}^d$. We will use this to define our invariant measure for the process $\tilde{z}_t$ that satisfies the SDE (3.3.2) in Section 3.3.

There is also another way to define an invariant measure using classical PDE theory. We can define an invariant measure with respect to the generator, more precisely, the *adjoint* of the infinitesimal generator.

Definition 2.3.8. The *adjoint* of $\mathcal{L}$, denoted by $\mathcal{L}^*$, is defined by

$$\int_{\mathbb{R}^d} f(x)(\mathcal{L}g)(x)\, dx = \int_{\mathbb{R}^d} (\mathcal{L}^* f)(x) g(x)\, dx, \tag{2.3.9}$$

for $x \in \mathbb{R}^d$ and for all square-integrable functions $f \in C^2(\mathbb{R}^d)$ and $g \in C_0^2(\mathbb{R}^d)$.

Remark 2.3.9. For time-homogeneous systems and assuming that σ is independent of X_t, using the definition of $\mathcal{L}$ in (2.2.6), we can compute $\mathcal{L}^*$ directly by applying integration by parts to (2.3.9). Then we would get

$$\mathcal{L}^*\varphi(x) = -\sum_{i=1}^{d} \frac{\partial}{\partial x^{(i)}}(b^{(i)}(x)\varphi(x)) + \frac{1}{2}\sum_{i,j=1}^{d}(\sigma\sigma^T)^{(ij)}\frac{\partial^2\varphi}{\partial x^{(i)}\partial x^{(j)}}(x). \tag{2.3.10}$$

Based on Definition 2.2.3 and equation (2.3.1), we have the following equivalent definition for an invariant measure:

Definition 2.3.10. An invariant measure π associated to the SDE (2.1.1) is a solution to

$$\mathcal{L}^*\mu = 0, \tag{2.3.11}$$

where $\mu = d\pi$.

Remark 2.3.11. Definition 2.3.10 is based on the assumption that an invariant probability measure exists.

In Section 4.2, we will use this definition to numerically compute the density for the associated invariant measure of the process $\tilde{z}_t$ defined by the SDE (3.3.2).

2.4 Lyapunov Functions

In Sections 2.2 and 2.3, we stated conditions needed to show the process X_t is nonexplosive and it has an invariant measure. Based on Theorem 2.2.7 and Theorem 2.3.3, we only need to show there exists a suitable function φ. However, there is no general method to find such a function. In fact, it is quite difficult to show the function exists. We will explain this further below. First we state explicitly what such a function is.

Definition 2.4.1. We say $\varphi \in C^2(\mathbb{R}^d)$ is a *Lyapunov function* if $\varphi(x) \geq 1$ and

$$\lim_{|x|\to\infty} \varphi(x) = \infty. \tag{2.4.1}$$

Sometimes we require $\varphi \in C^\infty(U)$ for some $U \subset \mathbb{R}^d$. We will see later that φ only needs to be twice continuously differentiable. In particular, this is the definition given in Section 5 of [RB06]. In addition, Definition 2.4.1 says that φ has compact level sets. This property can be used to study how quickly the dynamics converges to the invariant measure. See Theorem 3.3 of [HM15a] for results concerning the process z_t satisfying the SDE (2.5.1). More general results can be found in [RB06]. We will not pursue the rate of convergence question in this paper.

Lemma 2.4.2. *If the process X_t possesses a Lyapunov function φ that satisfies*

$$\mathcal{L}\varphi(x) \leq -C\varphi(x) + D \tag{2.4.2}$$

for some positive constants C, D and for all $x \in \mathbb{R}^d$, then:

1. *X_t is nonexplosive.*

2. *X_t has an invariant probability measure.*

Proof. See Theorem 2.2.7 and Theorem 2.3.3. □

Remark 2.4.3. A different version of Lemma 2.4.2 can be found in [RB06] as Theorem 8.7. In addition, the condition on the Lyapunov function φ is stated as a definition in [Her11] as Definition 3.3 (it is also stated as Assumption 2.62) and in [HM15a] as Definition 4.1.

Essentially, we need to construct a function φ such that

1. $\varphi(x) \to \infty$ as $|x| \to \infty$, and

2. $\mathcal{L}\varphi(x) \to -\infty$ as fast as possible as $|x| \to \infty$.

Intuitively, this Lyapunov function guarantees that our process decays rapidly. Since the construction of φ depends on the generator $\mathcal{L}$ defined by (2.2.5), we only need $\varphi \in C^2(\mathbb{R}^d)$. For example, consider the SDE

$$dx_t = -x_t^3\, dt + \sigma\, dW_t, \tag{2.4.3}$$

where $x_t \in \mathbb{R}$ and $\sigma > 0$. We mentioned in Section 2.1 that this SDE is stable; however, its coefficients are only locally Lipschitz. Now we will verify its stability by the existence of a Lyapunov function. By Definition 2.2.3,

$$\mathcal{L} = -x^3 \frac{d}{dx} + \frac{\sigma^2}{2}\frac{d^2}{dx^2}. \tag{2.4.4}$$

Let $\varphi(x) = |x|^\beta$ for some large $\beta > 0$. We will see how large we need β to be:

$$\begin{aligned} \mathcal{L}\varphi(x) &= -\beta x^3 \mathrm{sgn}(x)|x|^{\beta-1} + \frac{\sigma^2}{2}\beta(\beta-1)|x|^{\beta-2} \\ &\sim -\beta|x|^{\beta+2} \end{aligned}$$

as $x \to \infty$. Observe that when $\sigma = 0$, the process x_t is nonexplosive to begin with. In addition to $x_t \in \mathbb{R}$, the Lyapunov function was easy to construct in this example. However, identifying the "correct" Lyapunov functions is difficult in general.

In [HM15a, HM15b], Herzog and Mattingly outlined a method to construct such functions for (2.5.2). Their method made use of three simplifying tactics. The first was to rewrite the system in polar coordinate $z = re^{i\theta}$. Since we care about the

behavior of our system at infinity, it makes sense to study it radially. The second was to use a time change which effectively slowed down the dynamics at large time scales. The third was an invertible scaling transformation which made studying the operator $\mathcal{L}$ tractable. Using these three tactics, they observed that the complex plane could be partitioned into regions, each of which a piece of the to-be Lyapunov function could be defined and understood. We start with a ball of finite radius and begin the partition outside this ball. The first region is where the trajectories point inward and we set the Lyapunov function as $\varphi_0 = r^\beta$ for some large enough $\beta > 0$. Then we start cutting wedges outward from this initial region until we get to the region where our explosive trajectories lie. For each region, we approximate the generator at infinity and we use that to construct the Lyapunov function. Then we glue everything together.

For example, consider the SDE

$$dz_t = z_t^2\,dt + \sigma\,dB_t, \tag{2.4.5}$$

where $z_t \in \mathbb{C}$, $\sigma > 0$, and B_t is a complex-valued standard Brownian motion. Let $z_t = r_t e^{i\theta_t}$. Then its generator is

$$\mathcal{L} = r^2 \cos\theta\,\partial_r + r\sin\theta\,\partial_\theta + \frac{\sigma^2}{2r}\partial_r + \frac{\sigma^2}{2}\partial_r^2 + \frac{\sigma^2}{2r^2}\partial_\theta^2. \tag{2.4.6}$$

Using the time change $\mathcal{L} = rL$, we will focus on L, which is

$$L = r\cos\theta\,\partial_r + \sin\theta\,\partial_\theta + \frac{\sigma^2}{2r^2}\partial_r + \frac{\sigma^2}{2r}\partial_r^2 + \frac{\sigma^2}{2r^3}\partial_\theta^2. \tag{2.4.7}$$

We can easily revert back to $\mathcal{L}$. Now we will “cut” our system into regions and

determine the dominant terms of L in each region near the point at infinity. To do so, we will utilize a scaling transformation that can help us understand the asymptotic behavior of φ when we apply L to φ. We define the transformation by

$$S_\alpha^\lambda : (r, \theta) \to (\lambda r, \lambda^{-\alpha}\theta), \tag{2.4.8}$$

for any $\lambda \geq 1, \alpha \geq 0$. Heuristically, we will determine the behavior of

$$L \circ S_\alpha^\lambda = r\cos(\theta\lambda^{-\alpha})\,\partial_r + \lambda^\alpha \sin(\theta\lambda^{-\alpha})\,\partial_\theta + \lambda^{-3}\frac{\sigma^2}{2r^2}\partial_r + \lambda^{-3}\frac{\sigma^2}{2r}\partial_r^2 + \lambda^{2\alpha-3}\frac{\sigma^2}{2r^3}\partial_\theta^2, \tag{2.4.9}$$

as $\lambda \to \infty$. We can analyze $L \circ S_\alpha^\lambda$ for three cases, when:

1. $\alpha = 0$ and $\theta \neq 0$,
2. $0 < \alpha < \frac{3}{2}$ and $|\theta| \neq 0$ is sufficiently small, and
3. $\alpha = \frac{3}{2}$ and $|\theta| \to 0$.

For case 1,

$$L \circ S_\alpha^\lambda \approx r\cos\theta\,\partial_r + \sin\theta\,\partial_\theta =: T_1, \tag{2.4.10}$$

as $\lambda \to \infty$. For case 2,

$$L \circ S_\alpha^\lambda \approx r\partial_r + \theta\partial_\theta =: T_2, \tag{2.4.11}$$

as $\lambda \to \infty$. For case 3,

$$L \circ S_\alpha^\lambda \approx r\partial_r + \theta\partial_\theta + \frac{\sigma^2}{2r^3}\partial_\theta^2 =: A, \tag{2.4.12}$$

as $\lambda \to \infty$. We can see the construction in the image (Figure 2.4.1) courtesy of Herzog.

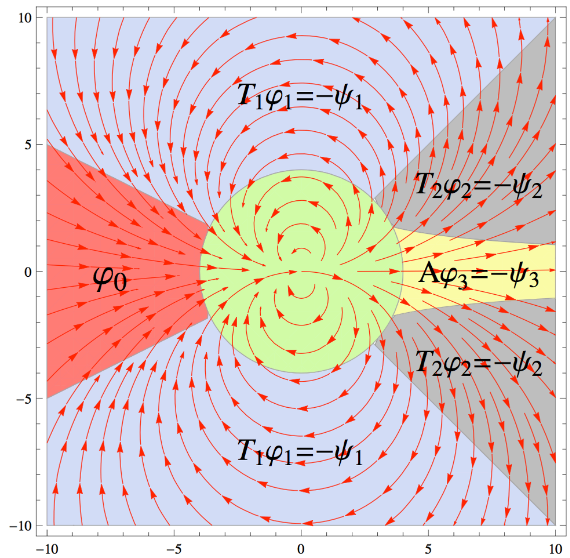


FIGURE 2.4.1: Lyapunov construction by region of $dz_t = z_t^2\, dt + \sigma\, dB_t$

Remark 2.4.4. After all these simplifying tactics, we still need to construct the Lyapunov function, which is specific for each region. After reducing the generator for every region, we will guess the function associated to each particular region. Then, we need to ensure that all functions can be glued smoothly together. This is not an easy task.

Now that we have an idea of how to construct a Lyapunov function, we can now state a theorem that summarizes the stability of the SDE (2.1.1) through the use of Lyapunov functions. First, we need to define *uniform ellipticity*.

Definition 2.4.5. An operator $\mathcal{L}$ is *uniformly elliptic* if there exists a constant $\lambda > 0$ such that

$$\sum_{i,j=1}^{d} a_{ij}(x)\zeta_i\zeta_j \geq \lambda \sum_{i=1}^{d} \zeta_i^2,$$

for $(\zeta_1, \ldots, \zeta_d) \in \mathbb{R}^d$ and $a = (a_{ij})$ are second-order terms.

In other words, if the eigenvalues of $\sigma\sigma^T$ are bounded away from the origin, then

the generator $\mathcal{L}$ is uniformly elliptic.

Theorem 2.4.6 (Theorem 4.6 in [HM15a])**.** *Suppose that X_t has a uniformly elliptic diffusion matrix σ and a Lyapunov function φ. Then X_t has a unique invariant probability measure π. Moreover, π is ergodic, satisfies*

$$\int_{\mathbb{R}^d} \varphi(x)\pi(dx) < \infty, \tag{2.4.13}$$

and has a smooth and everywhere positive density with respect to the Lebesgue measure on $\mathbb{R}^d$.

Remark 2.4.7. The uniform ellipticity assumption guarantees uniqueness; however, this assumption may not be necessary. There are examples where we have uniqueness when the generator is *hypoelliptic*; that is, for every π defined on an open subset of $\mathbb{R}^n$ such that $\mathcal{L}\pi$ is C^∞, π must also be C^∞. Recall that if the diffusion satisfies *Hörmander's condition*, then X_t admits a smooth density with respect to the Lebesgue measure [Bel06, H6̈7]. In particular, it is referred to as the *parabolic Hörmander condition*, where the diffusion generates $\mathbb{R}^d$ under the operation of Lie brackets. See [Her11, GHW11].

2.5 Stabilization by Noise

We will briefly talk about the results of Herzog and Mattingly in [HM15a, HM15b]. They analyzed the SDE

$$dz_t = (a_{n+1}z_t^{n+1} + a_n z_t^n + ... + a_0)\, dt + \sigma\, dB_t \tag{2.5.1}$$

with initial condition $z_0 \in \mathbb{C}$, where $n \geq 1$ is an integer, $a_i \in \mathbb{C}$, $a_{n+1} \neq 0$, $\sigma \geq 0$ is constant, and $B_t = B_t^{(1)} + iB_t^{(2)}$ is a complex Brownian motion defined on the probability space $(\Omega, \mathcal{F}, P)$. When $\sigma = 0$, the system (2.5.1) is explosive. With the use of Lyapunov theory, we will see that the system (2.5.1) is nonexplosive when $\sigma \neq 0$. For the purpose of this paper, we will only focus on $a_i \in \mathbb{R}$.

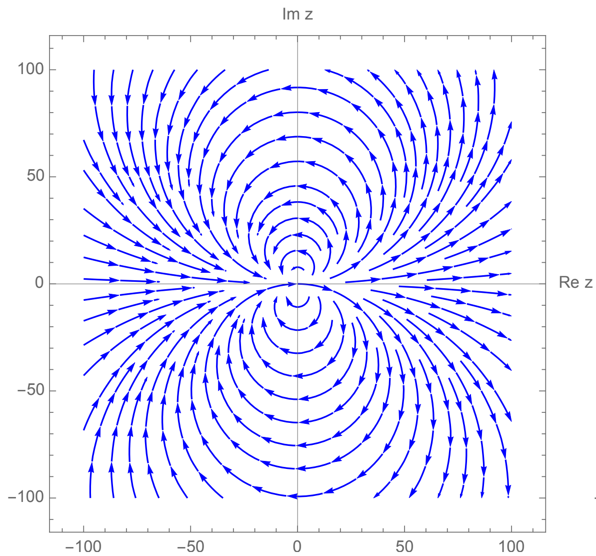
To observe the overall behavior of these systems in the "long-time" limit, we can analyze the leading term of the drift coefficient. For instance, in Figure 2.5.1, we have the phase portraits of $dz_t = z_t^2\,dt$, $dz_t = z_t^6\,dt$, and $dz_t = (z_t^6 + z_t^2)\,dt$, where we can compare their solutions locally and globally. In Figure 2.5.1c, we see the effect of the lower order term z^2 of $dz_t = (z_t^6 + z_t^2)\,dt$ near the origin; the system resembles $dz_t = z_t^2\,dt$ locally. When we look at the trajectories globally, we see that $dz_t = (z_t^6 + z_t^2)\,dt$ resembles $dz_t = z_t^6\,dt$ in Figure 2.5.1d.

Heuristically, we will analyze the SDE

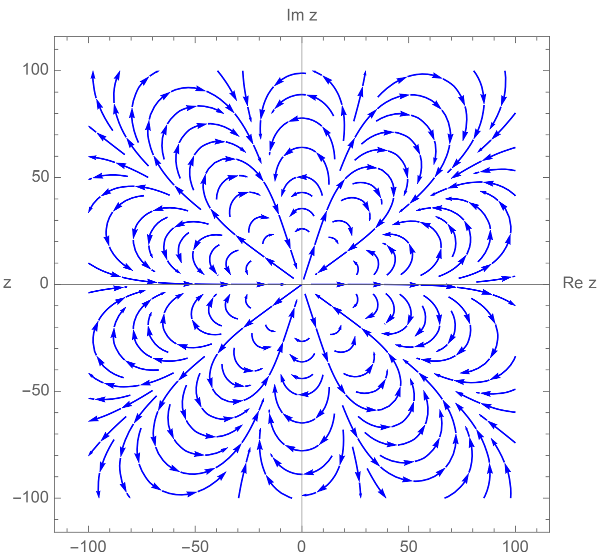
$$dz_t = z_t^{n+1}\,dt + \sigma\,dB_t. \tag{2.5.2}$$

When $\sigma = 0$, it is easy to verify that the SDE (2.5.2) has explosive trajectories which lie along n rays, specifically on the rays $\arg(z) = 2\pi k/n$ for $k \in \{0, 1, \ldots, n-1\}$. For example, on the phase portraits in Figure 2.5.1a and Figure 2.5.1b, we can see there are one explosive ray and five explosive rays, respectively. More precisely, the systems have explosive solutions when the initial conditions are $z_0 > 0$ for $n = 1$ and $z_0 = r_0 e^{2\pi k i/5}$, where $r_0 > 0$ and $k \in \{0, 1, 2, 3, 4\}$, for $n = 5$.

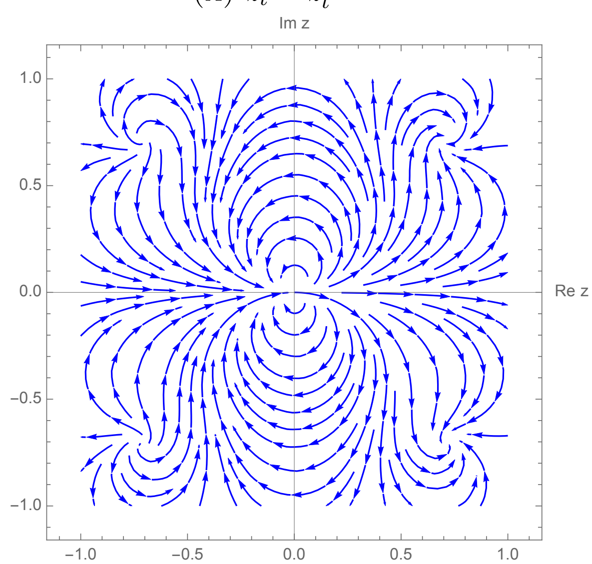
The goal is to verify the SDE (2.5.2) is indeed stable when $\sigma > 0$. In other words, we no longer have the explosive solutions we previously observed when $\sigma = 0$. Intuitively, the Brownian motion "kicks" the explosive solution off its original



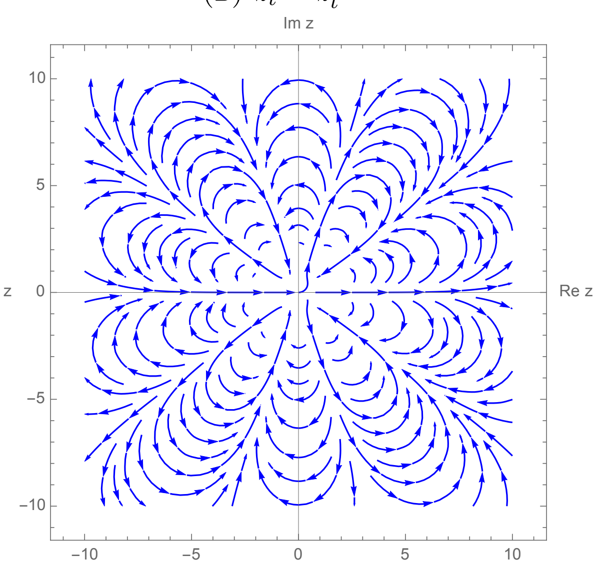
(A) $\dot{z}_t = z_t^2$



(B) $\dot{z}_t = z_t^6$



(C) $\dot{z}_t = z_t^6 + z_t^2$ (locally)



(D) $\dot{z}_t = z_t^6 + z_t^2$ (globally)

FIGURE 2.5.1: Phase Portraits

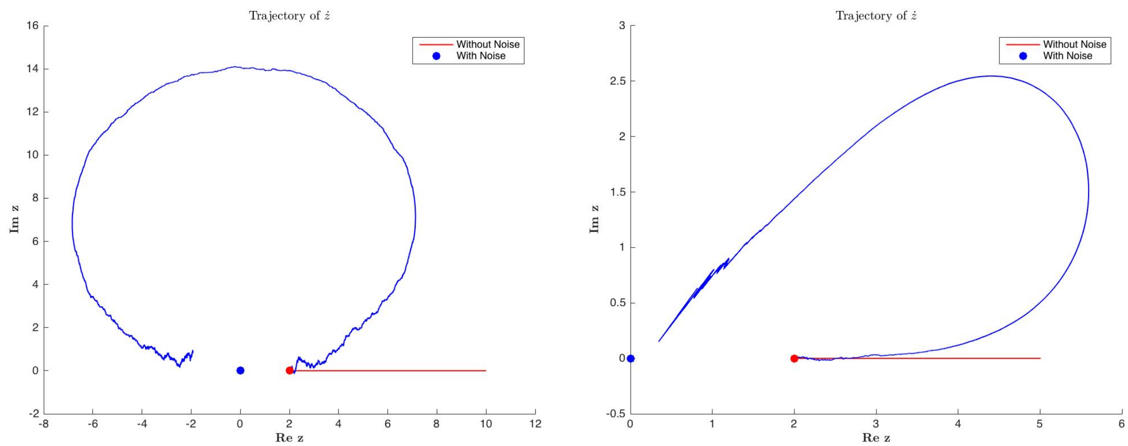
trajectory and onto one of the other stable solution curves. In Figure 2.5.2, we have a simulation for both the systems $dz_t = z_t^2\,dt + \sigma\,dB_t$ and $dz_t = z_t^6\,dt + \sigma\,dB_t$ (done using Euler's method). When $\sigma = 0$, for the initial condition $z_0 = 2$, the solutions blow up in finite time as we would expect based on their phase portraits in Figure 2.5.1 (shown in red). When $\sigma > 0$, we have the blue trajectories shown in Figure 2.5.2 with the same initial condition $z_0 = 2$. Notice that our solution curves are approaching the origin rather than infinity with the addition of noise. In particular, the effect of the noise is evident based on the "jagged" curves, especially near the origin. We can see the additive noise prevents the solutions from blowing up in finite time. The most interesting aspect of the stabilization is the solutions resemble one of the original stable curves from the phase portraits in Figure 2.5.1. In addition, if we were to simulate the solutions with any initial condition in one of the nonexplosive regions, we would get the same results. To summarize, the additive noise prevents the system from being explosive and it does not change the overall deterministic behavior of the "pre-noise" system. Hence, we expect the SDE (2.5.2), similarly for (2.5.1), to be nonexplosive and its solutions possess a unique (ergodic) invariant measure.

We have shown how to construct a Lyapunov function in Section 2.4. This outline can be generalized for the SDE (2.5.1); hence we can show the existence of such a function. Since the diffusion of the SDE (2.5.1) is uniformly elliptic, it is ergodic. More precisely,

Theorem 2.5.1. *Consider the SDEs*

$$dz_t = (a_{n+1} z_t^{n+1} + a_n z_t^n + ... + a_0)\,dt + \sigma\,dB_t$$

with initial condition $z_0 \in \mathbb{C}$, *where* $n \geq 1$ *is an integer,* $a_i \in \mathbb{R}$, $a_{n+1} \neq 0$, $\sigma > 0$ *is*



(A) $dz_t = z_t^2\,dt + \sigma\,dB_t$ (B) $dz_t = z_t^6\,dt + \sigma\,dB_t$

FIGURE 2.5.2: Simulation of solutions for the initial condition $z_0 = 2$. When $\sigma = 0$, we have the red trajectory and when $\sigma > 0$, we have the blue trajectory.

constant, and $B_t = B_t^{(1)} + iB_t^{(2)}$ *is a complex Brownian motion. Then the process* z_t *is nonexplosive, and moreover, has a unique (ergodic) invariant measure* π*.*

Proof. First, we need to construct the Lyapunov function as outlined in Section 2.4. We will skip this construction; see [HM15b] for the details. Once we have the Lyapunov functions, it follows from Theorem 2.4.2 and Theorem 2.4.6. □

Remark 2.5.2. Observe that in Theorem 2.5.1, we only require $a_i \in \mathbb{R}$ to prove the main results in this paper. However, this theorem still holds for $a_i \in \mathbb{C}$. For the explicit construction of the Lyapunov function in the more general case of the SDE (2.5.1), see [HM15b].

Chapter 3

Proof of Main Theorem

We will now prove Theorem 1.1.1 and Proposition 1.1.2. To do so, we will first introduce a coordinate transformation that will reduce system (1.1.1) and help us identify its explosive regions. Then we will see, with the suitable additive noise, our SDE is stable by extension of Theorem 2.5.1. This will be done using the Girsanov transformation.

3.1 Reduction via a Change of Coordinates

We begin by rewriting the system (1.1.1) in terms of the coordinate $\mathbf{x} = (x_1, x_2, x_3, x_4)$, where

$$x_1 = \mathrm{Re}(z), \quad x_2 = \mathrm{Im}(z), \quad x_3 = \mathrm{Re}(w), \quad x_4 = \mathrm{Im}(w). \tag{3.1.1}$$

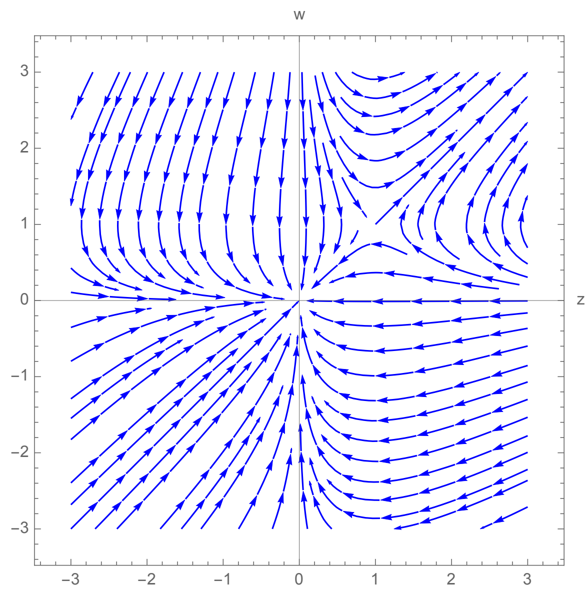


FIGURE 3.1.1: Phase portrait of system (3.1.2) restricted to $z, w \in \mathbb{R}$.

This results in the system of equations

$$\begin{cases} \dot{x}_1 = -\nu x_1 + \alpha(x_1 x_3 - x_2 x_4) \\ \dot{x}_2 = -\nu x_2 + \alpha(x_2 x_3 + x_1 x_4) \\ \dot{x}_3 = -\nu x_3 + \beta(x_1 x_3 - x_2 x_4) \\ \dot{x}_4 = -\nu x_4 + \beta(x_2 x_3 + x_1 x_4), \end{cases} \tag{3.1.2}$$

with initial condition $(x_1(0), x_2(0), x_3(0), x_4(0))$. In what follows, we shall assume that $\alpha > 0$ and $\beta > 0$ without loss of generality. If α or β vanishes, then the system degenerates. For all other cases, one can replace some of the x_i by $-x_i$ to effectively make α and β positive.

By setting the time derivatives to 0, we can find the two equilibrium points of the system: $\mathbf{0} = (0,0,0,0)$ and $\mathbf{p} = (\nu/\beta, 0, \nu/\alpha, 0)$. We then linearize (3.1.2) about each

of the equilibrium points:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} -\nu & 0 & 0 & 0 \\ 0 & -\nu & 0 & 0 \\ 0 & 0 & -\nu & 0 \\ 0 & 0 & 0 & -\nu \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + O(\mathbf{x}^2), \tag{3.1.3}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & \nu\alpha/\beta & 0 \\ 0 & 0 & 0 & \nu\alpha/\beta \\ \nu\beta/\alpha & 0 & 0 & 0 \\ 0 & \nu\beta/\alpha & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 - \nu/\beta \\ x_2 \\ x_3 - \nu/\alpha \\ x_4 \end{bmatrix} + O((\mathbf{x}-\mathbf{p})^2). \tag{3.1.4}$$

It is clear from the Jacobian in (3.1.3) that $\mathbf{0}$ is an attracting equilibrium point. On the other hand, the Jacobian in (3.1.4) has eigensolutions

$$\lambda_1 = -\nu, \quad \mathbf{e}_1 = \begin{bmatrix} 0 \\ -\alpha/\beta \\ 0 \\ 1 \end{bmatrix}; \quad \lambda_2 = -\nu, \quad \mathbf{e}_2 = \begin{bmatrix} -\alpha/\beta \\ 0 \\ 1 \\ 0 \end{bmatrix}; \tag{3.1.5}$$

$$\lambda_3 = +\nu, \quad \mathbf{e}_3 = \begin{bmatrix} 0 \\ +\alpha/\beta \\ 0 \\ 1 \end{bmatrix}; \quad \lambda_4 = +\nu, \quad \mathbf{e}_4 = \begin{bmatrix} +\alpha/\beta \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

This implies that a 2-dimensional unstable manifold and a 2-dimensional stable man-

ifold are associated to the saddle point $\mathbf{p}$. See Figure 3.1.1 for the phase portrait.

The above linearization analysis shows that only ν and α/β have an impact on system (3.1.2). Moreover, the symmetry of the eigensolutions suggests that the effective dynamics may be simpler than the 4-dimensional nature of the system. To this end, let us make a change of coordinates so that the new coordinate directions agree with the eigendirections in (3.1.5):

$$y_1 = \frac{1}{2}\left(x_1 + \frac{\alpha}{\beta}x_3\right),\ y_2 = \frac{1}{2}\left(x_1 - \frac{\alpha}{\beta}x_3\right),\ y_3 = \frac{1}{2}\left(x_2 + \frac{\alpha}{\beta}x_4\right),\ y_4 = \frac{1}{2}\left(x_2 - \frac{\alpha}{\beta}x_4\right). \tag{3.1.6}$$

By replacing the x_i by the y_i, we rewrite (1.1.1) as

$$\begin{cases} \dot{y_1} = -\nu y_1 + \beta\left[(y_1^2 - y_2^2) - (y_3^2 - y_4^2)\right] \\ \dot{y_2} = -\nu y_2 \\ \dot{y_3} = -\nu y_3 + 2\beta(y_1 y_3 - y_2 y_4) \\ \dot{y_4} = -\nu y_4 \end{cases} \tag{3.1.7}$$

with initial condition $(y_1(0), y_2(0), y_3(0), y_4(0))$. Observe that y_2 and y_4 evolve autonomously under (3.1.7), with solutions

$$y_2(t) = y_2(0)e^{-\nu t}, \quad y_4(t) = y_4(0)e^{-\nu t}. \tag{3.1.8}$$

Plugging these back into (3.1.7) yields the 2-dimensional system

$$\begin{cases} \dot{y_1}(t) = -\nu y_1(t) + \beta\left(y_1^2(t) - y_3^2(t)\right) - \beta\left(y_2^2(0) - y_4^2(0)\right)e^{-2\nu t} \\ \dot{y_3}(t) = -\nu y_3(t) + 2\beta y_1(t) y_3(t) - 2\beta y_2(0) y_4(0) e^{-2\nu t} \end{cases}. \tag{3.1.9}$$

Notice that by setting $\tilde{z} = y_1 + iy_3$, and without the exponentially decaying terms of order $e^{-2\nu t}$, (3.1.9) resembles the system

$$\dot{\tilde{z}} = -\nu\tilde{z} + \beta\tilde{z}^2, \tag{3.1.10}$$

the stochastic stabilization of which was studied by Herzog and Mattingly in [HM15a]. So on a heuristic level, our stabilization problem in $\mathbb{C}^2$,

$$\begin{cases} dz_t = (-\nu z_t + \alpha z_t w_t)\ dt + \text{Brownian noise} \\ dw_t = (-\nu w_t + \beta z_t w_t)\ dt + \text{Brownian noise} \end{cases}, \tag{3.1.11}$$

can be reduced to the stabilization problem in $\mathbb{C}$,

$$d\tilde{z}_t = \left(-\nu\tilde{z}_t + \beta\tilde{z}_t^2\right)\ dt + \text{Brownian noise.} \tag{3.1.12}$$

In Section 3.3, we will make this heuristic rigorous for a class of Brownian noises.

3.2 Conditions for Explosion

Analyzing (3.1.9) for sets of initial conditions corresponding to explosive solutions is now more tractable, since (1.1.1) has been reduced to a system evolving over $\mathbb{R}^2$ rather than $\mathbb{R}^4$. The following proposition gives us estimates on the boundaries of the explosive regions.

For the remainder of this section, we will denote $(y_1(t), y_2(t), y_3(t), y_4(t))$ by $\mathbf{y}(t)$ and its initial condition $(y_1(0), y_2(0), y_3(0), y_4(0))$ by $\mathbf{y}_0$. Let $I_{\max}$ be the largest interval $[0, T)$ on which $\mathbf{y}(t)$ is defined. By the existence and uniqueness theorem for

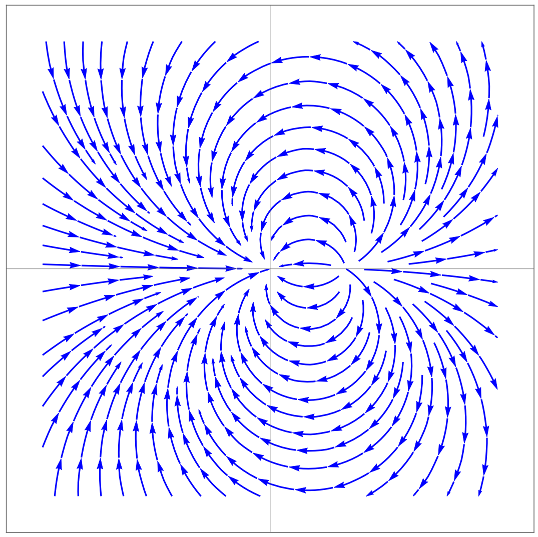


FIGURE 3.1.2: Phase Portrait of $\dot{z}_t = -\nu z_t + \beta z_t^2$.

first-order ODEs, $I_{\max} \in (0, \infty]$. Also, since the RHS of (3.1.9) is real analytic, the solutions to (3.1.9) are also real analytic. It suffices to show that they are continuously differentiable.

We now state a sufficient condition for explosivity.

Proposition 3.2.1 (Proposition 2.1 in [CFK$^+$15])**.** *The solution* $\mathbf{y}(t)$ *of (3.1.9) with initial condition* $\mathbf{y}_0$ *is explosive if all of the following conditions hold:*

(I) $y_3(0) = 0$*.*

(II) Either $y_2(0) = 0$ *or* $y_4(0) = 0$*.*

(III) $y_1(0) > C$ *for some large enough constant* C *which depends on* $\beta, \nu, y_2(0), y_4(0)$*.*

Remark 3.2.2. Conditions (I) and (II) imply that $y_3(t) = 0$ for all $t \in I_{\max}$. Since our stabilization problem is reduced to (3.1.12), based on Figure 3.1.2, explosion should occur when $y_3(t) = 0$, given a sufficiently large constant C in condition (III).

Proof of Proposition 3.2.1. Let $y_3(0) = 0$. Suppose $y_2(0) = 0$, so that (3.1.9) reduces to

$$\dot{y_1}(t) = -\nu y_1(t) + \beta y_1^2(t) + \beta y_4^2(0)e^{-2\nu t}, \tag{3.2.1}$$

with initial condition $y_1(0)$. The solutions of (3.2.1) are bounded below by the solutions of

$$\begin{cases} \dot{x}(t) = -\nu x(t) + \beta x^2(t) \\ x(0) = y_1(0) \end{cases}, \tag{3.2.2}$$

since $\dot{y}(t) > \dot{x}(t)$ for all times t. It is easy to verify that solutions of (3.2.2) are explosive whenever $y_1(0) > \frac{\nu}{\beta}$. Therefore, solutions of (3.2.1) are explosive under the same initial condition $y_1(0) > \frac{\nu}{\beta}$.

Next suppose $y_4(0) = 0$. Then (3.1.9) reduces to

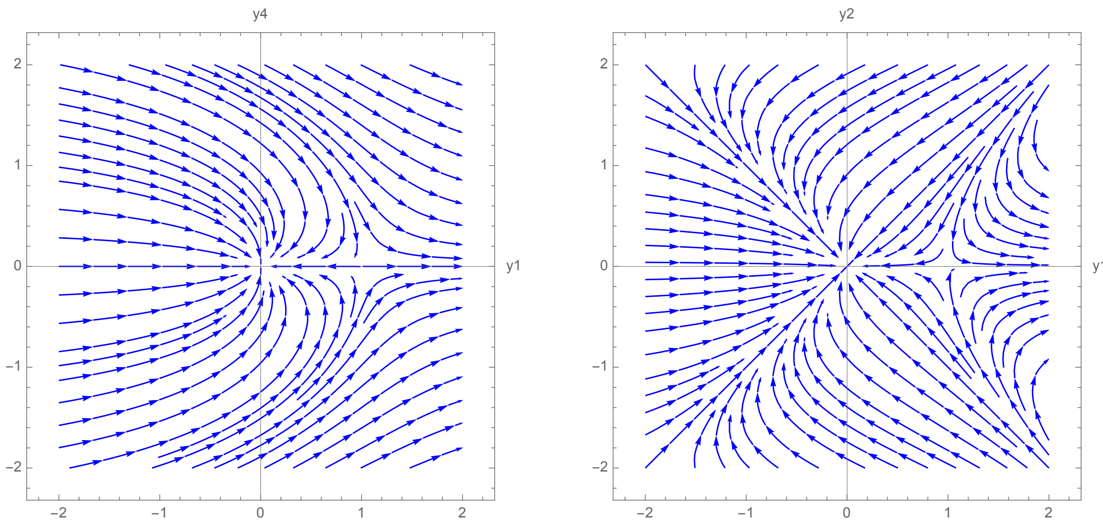
$$\dot{y_1}(t) = -\nu y_1(t) + \beta y_1^2(t) - \beta y_2^2(0)e^{-2\nu t}, \tag{3.2.3}$$

with initial condition $y_1(0)$. Similar to the arguments given above, we see that the solutions of (3.2.3) are bounded below by the solutions of

$$\begin{cases} \dot{x}(t) = -\nu x(t) + \beta x^2(t) - \beta y_2^2(0) \\ x(0) = y_1(0) \end{cases}, \tag{3.2.4}$$

which explode whenever $y_1(0) > \frac{1}{2}\left(\frac{\nu}{\beta} + \sqrt{(\frac{\nu}{\beta})^2 + (2y_2(0))^2}\right)$. $\square$

Remark 3.2.3. For condition (III), it is difficult to analytically pin down the constant C for the initial condition $y_2(0) = y_3(0) = 0$ (see Figure 3.2.1a). On the other hand, for the initial condition $y_3(0) = y_4(0) = 0$, the estimates on C are at least qualitatively



(A) $y_2(0) = y_3(0) = 0$ (B) $y_3(0) = y_4(0) = 0$

FIGURE 3.2.1: Phase Portraits of $\dot{\mathbf{y}}(t)$ for $\beta = \nu = 1$

correct (see Figure 3.2.1b).

Now we will show all the explosive regions are contained in the region where $y_3(0) = 0$.

Proposition 3.2.4 (Proposition 2.4 in [CFK$^+$15])**.** *If* $y_3(0) \neq 0$*, then* $\mathbf{y}(t)$ *is nonexplosive.*

Proof. We prove this by contradiction. Suppose $\mathbf{y}(t)$ is explosive. Then at least one of $y_1(t)$ and $y_3(t)$ blows up in finite time. Let t^* be the (finite) explosion time of $\mathbf{y}(t)$.

We make two observations from (3.1.9). If $y_1(t)$ blows up at time t^*, then $y_3(t)$ must also blow up at time t^*, unless $y_3(t^*) = 0$. Based on (3.1.10), we know $\mathbf{y}(t)$ resembles the solutions shown in Figure 2.5.1a. Hence, it is not difficult to check that if $y_3(0) \neq 0$, then $y_3(t)$ can not be zero for any finite $t > 0$. Therefore, the only logical conclusion is $y_3(0) = 0$.

On the other hand, if $y_3(t)$ blows up at time t^*, then $y_1(t)$ must blow up at time

t^* (could explode to $\pm\infty$).

So it remains to consider the case where both $y_1(t)$ and $y_3(t)$ blow up at time t^*. Without loss of generality, we may assume that $y_1(t) \nearrow +\infty$ and $y_3(t) \nearrow +\infty$ as $t \nearrow t^*$. Let $u(t) = y_1(t) + iy_3(t)$. We choose $\epsilon > 0$ such that not only

$$\begin{cases} \nu y_1(t^* - \epsilon) \geq \beta(y_2^2(0) - y_4^2(0)) \\ \nu y_3(t^* - \epsilon) \geq 2\beta y_2(0) y_4(0) \end{cases}, \tag{3.2.5}$$

but also that there exists a $\delta > 0$ such that

$$|u(t^* - \epsilon)| \leq (\beta\epsilon)^{-1} - \delta. \tag{3.2.6}$$

Consider the time interval $I = [t^* - \epsilon, t^*)$. By combining (3.1.9) and (3.2.5), we deduce that

$$\begin{cases} \mathrm{Re}(\dot{u}(t)) \leq \beta\,\mathrm{Re}(u(t)^2) \\ \mathrm{Im}(\dot{u}(t)) \leq \beta\,\mathrm{Im}(u(t)^2) \end{cases} \quad \text{on } I.$$

Writing $u(t)$ in polar coordinates, $u(t) = |u(t)|e^{i\theta_t}$ where $\theta_t = \arg(u(t))$, we then get

$$\begin{aligned} \frac{d}{dt}|u(t)|^2 &= \frac{d}{dt}(\mathrm{Re}(u(t)))^2 + \frac{d}{dt}(\mathrm{Im}(u(t)))^2 \\ &= 2[\mathrm{Re}(u(t))\mathrm{Re}(\dot{u}(t)) + \mathrm{Im}(u(t))\mathrm{Im}(\dot{u}(t))] \\ &\leq 2\beta[\mathrm{Re}(u(t))\mathrm{Re}(u(t)^2) + \mathrm{Im}(u(t))\mathrm{Im}(u(t)^2)] \\ &= 2\beta|u(t)|^3(\cos\theta_t \cos 2\theta_t + \sin\theta_t \sin 2\theta_t) \\ &= 2\beta|u(t)|^3 \cos\theta_t \\ &\leq 2\beta|u(t)|^3 \quad \text{on } I. \end{aligned}$$

By Gronwall's inequality, $|u(t)|^2$ is bounded above by the solution of

$$\frac{d}{dt}|u(t)|^2 = 2\beta|u(t)|^3$$

on I. This implies that

$$|u(t)| \le \frac{1}{|u(t^* - \epsilon)|^{-1} - \beta\epsilon + \beta(t^* - t)} \tag{3.2.7}$$

for all $t \in I$. Since $|u(t^* - \epsilon)|$ is bounded away from $(\beta\epsilon)^{-1}$ by assumption (3.2.6), the RHS of (3.2.7) is bounded by a finite constant for all $t \in I$. It follows that u is not an explosive solution. □

3.3 Ergodicity of the $\mathbb{C}^2$-valued SDEs

In this section, we make rigorous the heuristics stated towards the end of Section 3.1, and prove Theorem 1.1.1 and Proposition 1.1.2. Recall that our objective is to add a complex-valued Brownian noise to stabilize the deterministic system (1.1.1). For the proofs, we will assume that the Brownian noise is of the form $(\sigma\, B_t, \frac{\beta}{\alpha}\sigma\, B_t)$ in the (z, w)-coordinates, where $\sigma > 0$ is a constant and B_t is a complex-valued standard Brownian motion. In particular, B_t is the same Brownian motion in both coordinates. The corresponding SDE is (1.1.4). A direct calculation shows that (1.1.4) can be rewritten as

$$\begin{cases} dy_1(t) = -\nu y_1(t) + \beta\left(y_1^2(t) - y_3^2(t)\right) - \beta\left(y_2^2(0) - y_4^2(0)\right)e^{-2\nu t} + \sigma\, dB_t^{(1)} \\ dy_3(t) = -\nu y_3(t) + 2\beta y_1(t)y_3(t) - 2\beta y_2(0)y_4(0)e^{-2\nu t} + \sigma\, dB_t^{(2)} \end{cases} \tag{3.3.1}$$

in the "reduced" coordinates $\mathbf{y}(t) = (y_1(t), y_2(t), y_3(t), y_4(t))$ defined in (3.1.6). Here $B_t^{(1)}$ and $B_t^{(2)}$ are independent *real*-valued standard Brownian motions, and $B_t = B_t^{(1)} + iB_t^{(2)}$. Furthermore, if we define $\tilde{z}_t = y_1(t) + iy_3(t)$ and $\tilde{w}_0 = y_2(0) + iy_4(0)$, then $\tilde{z}_t$ satisfies the SDE

$$d\tilde{z}_t = (-\nu \tilde{z}_t + \beta \tilde{z}_t^2 - \beta \tilde{w}_0^2 e^{-2\nu t})\, dt + \sigma\, dB_t. \tag{3.3.2}$$

Observe that if the term of order $e^{-2\nu t}$ vanishes, then (3.3.2) is a special case of the $\mathbb{C}$-valued SDE with polynomial drift (2.5.1), studied in [HM15a].

In any case, the key step is to justify the connection between (2.5.1) and (3.3.2) so that the ergodic properties of the former can be transferred to the latter. This will be achieved using the Girsanov transform. See Appendix A for the standard Girsanov Theorem.

Lemma 3.3.1 (Girsanov transform [CFK$^+$15])**.** *Let B_t be a $\mathbb{C}$-valued standard Brownian motion on $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$, and z_t and $\tilde{z}_t$ be Itô processes of respective forms*

$$dz_t = \left(-\nu z_t + \beta z_t^2\right)\, dt + \sigma\, dB_t, \tag{3.3.3}$$

$$d\tilde{z}_t = (-\nu \tilde{z}_t + \beta \tilde{z}_t^2 - \beta \tilde{w}_0^2 e^{-2\nu t})\, dt + \sigma\, dB_t, \tag{3.3.4}$$

both of which have the same initial condition $z_0 = \tilde{z}_0 \in \mathbb{C}$. For each $t \in (0, \infty)$, let

$$\theta(t) = -\frac{\beta \tilde{w}_0^2}{\sigma} e^{-2\nu t}, \tag{3.3.5}$$

$$M_t = \exp\left(-\int_0^t \mathrm{Re}[\theta(s)]\, dB_s^{(1)} - \int_0^t \mathrm{Im}[\theta(s)]\, dB_s^{(2)} - \frac{1}{2}\int_0^t |\theta(s)|^2\, ds \right), \tag{3.3.6}$$

$$dQ_t = M_t\, dP \quad \text{on } \mathcal{F}_t, \tag{3.3.7}$$

$$\widehat{B}_t = \int_0^t \theta(s)\, ds + B_t. \tag{3.3.8}$$

Then:

1. $\{M_t : t \geq 0\}$ *is a uniformly integrable martingale.*
2. *There exists a probability measure Q on $\mathcal{F}_\infty$ such that $Q|_{\mathcal{F}_t} = Q_t$. Moreover P and Q are equivalent measures.*
3. $\widehat{B}_t$ *is a $\mathbb{C}$-valued standard Brownian motion under Q.*
4. *The Q-law of $\tilde{z}_t$ is the same as the P-law of z_t for all $t \in [0, \infty]$.*

Proof. From standard SDE theory, we know that the Girsanov transform from $\tilde{z}_t$ to z_t holds on the time interval $[0, T]$ for some finite $T > 0$ if Novikov's condition,

$$E^P\left[\exp\left(\frac{1}{2}\int_0^T |\theta(s)|^2\, ds\right)\right] < \infty, \tag{3.3.9}$$

is satisfied. By (3.3.5),

$$E^P\left[\exp\left(\frac{1}{2}\int_0^T |\theta(s)|^2\, ds\right)\right] = E^P\left[\exp\left(\frac{1}{2}\beta^2 |\tilde{w}_0|^4 \int_0^T e^{-4\nu s}\, ds\right)\right].$$

Since

$$|\tilde{w}_0|^4 = |y_2(0) + iy_4(0)|^4 = [(y_2(0))^2 + (y_4(0))^2]^2 \le 2[(y_2(0))^4 + (y_4(0))^4],$$

we get

$$\begin{aligned}\frac{1}{2}\beta^2|\tilde{w}_0|^4 \int_0^T e^{-4\nu s}\, ds &\le \beta^2[(y_2(0))^4 + (y_4(0))^4] \int_0^T e^{-4\nu s}\, ds \\ &= \beta^2 \left[(y_2(0))^4 + (y_4(0))^4\right] \frac{1 - e^{-4\nu T}}{4\nu} < \infty.\end{aligned}$$

This verifies Novikov's condition (3.3.9).

In order to extend the Girsanov transform to $T = \infty$, we need to verify that the martingale $\{M_t : t \ge 0\}$ is uniformly integrable, *cf.* Item 1 of the Lemma. By the preceding calculation, we see that there exists a finite constant C (taken to be $\frac{\beta^2}{4\nu}[(y_2(0))^4 + (y_4(0))^4]$) such that for all $t > 0$, the first moment of M_t satisfies

$$E^P[M_t] = E^P\left[\exp\left(\frac{1}{2}\int_0^t |\theta(s)|^2\, ds\right)\right] \le C. \tag{3.3.10}$$

Meanwhile, the second moment of M_t satisfies

$$\begin{aligned}E^P[M_t^2] &= E^P\left[\exp\left(-\int_0^t 2\mathrm{Re}[\theta(s)]\, dB_s^{(1)} - \int_0^t 2\mathrm{Im}[\theta(s)]\, dB_s^{(2)} - \int_0^t |\theta(s)|^2\, ds\right)\right] \\ &= \exp\left(\int_0^t 2|\theta(s)|^2\, ds - \int_0^t |\theta(s)|^2\, ds\right) = \exp\left(\int_0^t |\theta(s)|^2\, ds\right) \le C^2.\end{aligned}$$

Applying the Cauchy-Schwarz inequality, we obtain that for any measurable subset

A of $\mathbb{C}$,

$$E^P[|M_t|\mathbb{1}_A] \leq \left(E^P[|M_t|^2]\right)^{1/2} \left(E^P[\mathbb{1}_A]\right)^{1/2} \leq C[P(A)]^{1/2}. \tag{3.3.11}$$

The estimates (3.3.10) and (3.3.11) together imply that $\{M_t\}$ is uniformly integrable. Items 2 through 4 of the Lemma now follow from Proposition VIII.1.1, Proposition VIII.1.1', and Theorem VIII.1.4 of [RY99] (see also Proposition VIII.1.15 of [RY99] for the statement of Novikov's condition on the time interval $[0, \infty]$). See Appendix A. $\square$

Remark 3.3.2. The proof of Lemma 3.3.1 involves more than the standard Girsanov Theorem. The standard Girsanov Theorem applies only for finite time $t \geq 0$. Since the SDE (3.3.2) is time-inhomogeneous, we needed $t \in [0, \infty]$ to make a connection with the SDE (3.3.3). Hence we needed to extend the Girsanov transform to $T = \infty$, as shown in the proof.

Proposition 3.3.3 (Proposition 3.3 in [CFK+15])**.** *$\tilde{z}_t$ is nonexplosive.*

Proof. Let ξ (resp. $\tilde{\xi}$) be the explosion time of z_t (resp. $\tilde{z}_t$) as defined in Definition 2.2.1. By Items 2 and 4 of Lemma 3.3.1, we have the equivalence

$$P_{z_0}(\tilde{\xi} < \infty) = 0 \iff Q_{z_0}(\tilde{\xi} < \infty) = 0 \iff P_{z_0}(\xi < \infty) = 0.$$

Since $P_{z_0}(\xi < \infty) = 0$ for all $z_0 \in \mathbb{C}$ by Theorem 2.5.1, we deduce that $P_{z_0}(\tilde{\xi} < \infty) = 0$ for all $z_0 \in \mathbb{C}$. This proves the nonexplosivity of (3.3.2). $\square$

The ensuing computation allows us to identify the limiting distribution of the process $\tilde{z}_t$.

Lemma 3.3.4 (Lemma 3.4 in [CFK+15])**.** *Suppose (2.3.8) holds. Then for each $z_0 \in \mathbb{C}$,*

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t P_{z_0}(\tilde{z}_s \in \cdot)\, ds = \pi(\cdot), \tag{3.3.12}$$

where π is as in (2.3.8).

Proof. To begin, we fix $r \in (0,t)$ and use the Girsanov transform to write

$$\begin{aligned}
\frac{1}{t}\int_0^t P_{z_0}(\tilde{z}_s \in \cdot)\, ds &= \frac{1}{t}\int_0^t E_{z_0}^Q\left[\mathbb{1}_{\{\tilde{z}_s\in\cdot\}} M_s^{-1}\right] ds \\
&= \frac{1}{t}\int_0^r E_{z_0}^Q\left[\mathbb{1}_{\{\tilde{z}_s\in\cdot\}} M_s^{-1}\right] ds + \frac{1}{t}\int_r^t E_{z_0}^Q\left[\mathbb{1}_{\{\tilde{z}_s\in\cdot\}} M_s^{-1}\right] ds \\
&= \frac{1}{t}\int_0^r E_{z_0}^Q\left[\mathbb{1}_{\{\tilde{z}_s\in\cdot\}} M_s^{-1}\right] ds + \frac{1}{t}\int_r^t E_{z_0}^Q\left[\mathbb{1}_{\{\tilde{z}_s\in\cdot\}} M_r^{-1}\right] ds \\
&\quad + \frac{1}{t}\int_r^t E_{z_0}^Q\left[\mathbb{1}_{\{\tilde{z}_s\in\cdot\}} M_r^{-1}(R(r,s)-1)\right] ds,
\end{aligned} \tag{3.3.13}$$

where

$$\begin{aligned}
M_s^{-1} &:= \exp\left(\int_0^s \mathrm{Re}[\theta(\xi)]\, dB_\xi^{(1)} + \int_0^s \mathrm{Im}[\theta(\xi)]\, dB_\xi^{(2)} - \frac{1}{2}\int_0^s |\theta(\xi)|^2\, d\xi\right), \\
R(r,s) &:= \exp\left(\int_r^s \mathrm{Re}[\theta(\xi)]\, dB_\xi^{(1)} + \int_r^s \mathrm{Im}[\theta(\xi)]\, dB_\xi^{(2)} - \frac{1}{2}\int_r^s |\theta(\xi)|^2\, d\xi\right).
\end{aligned}$$

We denote the three integrals in the RHS of (3.3.13) by I_1, I_2, and I_3, respectively. To complete the proof, we will show that in the limit $t \to \infty$ followed by $r \to \infty$, $I_1 \to 0$, $I_2 \to \pi(\cdot)$, and $I_3 \to 0$.

First of all, using the fact that M_s^{-1} is a mean-1 martingale, we get

$$\overline{\lim_{t\to\infty}}\, |I_1| \le \overline{\lim_{t\to\infty}}\, \frac{1}{t}\int_0^r E_{z_0}^Q[M_s^{-1}]\, ds = \overline{\lim_{t\to\infty}}\, \frac{1}{t}\int_0^r 1\, ds = \overline{\lim_{t\to\infty}}\, \frac{r}{t} = 0.$$

Next, using the Markov property of $\tilde{z}_t$, a change of variables, and Tonelli's theorem, we can write

$$
\begin{aligned}
I_2 &= \frac{1}{t}\int_r^t E^Q_{z_0}\left[E^Q_{\tilde{z}_r}\left[\mathbb{1}_{\{\tilde{z}_{s-r}\in\cdot\}}\right]M_r^{-1}\right]ds \\
&= \frac{t-r}{t}\cdot\frac{1}{t-r}\int_0^{t-r} E^Q_{z_0}\left[Q_{\tilde{z}_r}(\tilde{z}_s\in\cdot)M_r^{-1}\right]ds \\
&= \frac{t-r}{t}\cdot E^Q_{z_0}\left[\left(\frac{1}{t-r}\int_0^{t-r} Q_{\tilde{z}_r}(\tilde{z}_s\in\cdot)\,ds\right)M_r^{-1}\right].
\end{aligned}
$$

Recall that the Q-law of $\tilde{z}_t$ is equal to the P-law of z_t, and the definition of π in (2.3.8). By Reverse Fatou's lemma,

$$
\varlimsup_{t\to\infty} I_2 \le \left(\varlimsup_{t\to\infty}\frac{t-r}{t}\right)\cdot E^Q_{z_0}\left[\varlimsup_{t\to\infty}\left(\frac{1}{t-r}\int_0^{t-r} Q_{\tilde{z}_r}(\tilde{z}_s\in\cdot)\,ds\right)M_r^{-1}\right] = E^Q_{z_0}\left[\pi(\cdot)M_r^{-1}\right].
$$

Similarly, by Fatou's lemma,

$$
\varliminf_{t\to\infty} I_2 \ge E^Q_{z_0}\left[\pi(\cdot)M_r^{-1}\right].
$$

Since M_r^{-1} is a uniformly integrable martingale, it follows that

$$
\lim_{r\to\infty}\lim_{t\to\infty} I_2 = \pi(\cdot)E^Q_{z_0}[M_\infty^{-1}] = \pi(\cdot).
$$

Finally, for I_3 we apply the Cauchy-Schwarz inequality twice, first with respect to

the Q-expectation and then with respect to the s-integral, to find

$$\begin{aligned}|I_3| &\le \frac{t-r}{t}\cdot\frac{1}{t-r}\int_r^t \left(E_{z_0}^Q[M_r^{-2}]\right)^{1/2}\left(E_{z_0}^Q|R(r,s)-1|^2\right)^{1/2}\,ds\\ &\le \frac{t-r}{t}\cdot(E_{z_0}^Q[M_r^{-2}])^{1/2}\cdot\left(\frac{1}{t-r}\int_r^t E_{z_0}^Q|R(r,s)-1|^2\,ds\right)^{1/2}.\end{aligned}$$

Note that

$$\begin{aligned}&E_{z_0}^Q[M_r^{-2}]\\ &= E_{z_0}^Q\left[\exp\left(\int_0^r \mathrm{Re}[2\theta(\xi)]\,dB_\xi^{(1)}+\int_0^r \mathrm{Im}[2\theta(\xi)]\,dB_\xi^{(2)}-\frac{1}{2}\int_0^r |2\theta(\xi)|^2\,d\xi+\int_0^r|\theta(\xi)|^2\,d\xi\right)\right]\\ &= \exp\left(\int_0^r |\theta(\xi)|^2\,d\xi\right),\end{aligned}$$

and $E_{z_0}^Q[M_\infty^{-2}] < \infty$ because $\theta \in L^2([0,\infty])$. An analogous calculation gives that $E_{z_0}^Q[R(r,s)] = 1$ and

$$E_{z_0}^Q|R(r,s)-1|^2 = E_{z_0}^Q[R(r,s)]^2 - 1 = \exp\left(\int_r^s |\theta(\xi)|^2\,d\xi\right) - 1.$$

Thus

$$\begin{aligned}\overline{\lim_{t\to\infty}}\,\frac{1}{t-r}\int_r^t E_{z_0}^Q|R(r,s)-1|^2\,ds &= \overline{\lim_{t\to\infty}}\left[\frac{1}{t-r}\int_r^t \exp\left(\int_r^s |\theta(\xi)|^2\,d\xi\right)\right]-1\\ &\le \overline{\lim_{t\to\infty}}\exp\left(\int_r^t|\theta(\xi)|^2\,d\xi\right)-1\\ &= \exp\left(\int_r^\infty |\theta(\xi)|^2\,d\xi\right)-1.\end{aligned}$$

Putting everything together we get

$$\overline{\lim_{r\to\infty}}\,\overline{\lim_{t\to\infty}}\,|I_3| \le \overline{\lim_{r\to\infty}}\left[\exp\left(\int_0^r |\theta(\xi)|^2\,d\xi\right)\right]^{1/2}\cdot\overline{\lim_{r\to\infty}}\left[\exp\left(\int_r^\infty |\theta(\xi)|^2\,d\xi\right)-1\right]^{1/2} = 0.$$

This proves (3.3.12). □

Proofs of Theorem 1.1.1 and Proposition 1.1.2. Since the nonexplosivity of $\tilde{z}_t$ is proved in Proposition 3.3.3, we concentrate on the ergodic theorem. We already showed in Lemma 3.3.4 that, under P, the dynamics of $\tilde{z}_t$ converges to the measure π. We now strengthen this convergence to the P-a.s. sense: for every $f \in L^1(\pi)$,

$$P\left(\lim_{t\to\infty}\frac{1}{t}\int_0^t f(\tilde{z}_s)\,ds = \int_{\mathbb{C}} f(\tilde{z})\,d\pi(\tilde{z})\right) = 1. \tag{3.3.14}$$

Our approach here is to exploit the equivalence of the probability measures P and Q on $\mathcal{F}_\infty$ [Item 3 of Lemma 3.3.1], as well as the equivalence of the Q-law of $\tilde{z}_t$ and the P-law of z_t [Item 4 of Lemma 3.3.1]. Using these two observations, we have that for every Borel measurable subset A of $\mathbb{R}$,

$$\begin{aligned} P\left(\lim_{t\to\infty}\frac{1}{t}\int_0^t f(\tilde{z}_s)\,ds \in A\right) = 0 &\Longleftrightarrow Q\left(\lim_{t\to\infty}\frac{1}{t}\int_0^t f(\tilde{z}_s)\,ds \in A\right) = 0 \\ &\Longleftrightarrow P\left(\lim_{t\to\infty}\frac{1}{t}\int_0^t f(z_s)\,ds \in A\right) = 0. \end{aligned}$$

By (2.3.8), we deduce that

$$P\left(\lim_{t\to\infty}\frac{1}{t}\int_0^t f(\tilde{z}_s)\,ds \in A\right) = P\left(\lim_{t\to\infty}\frac{1}{t}\int_0^t f(z_s)\,ds \in A\right) = 0$$

unless $\int_{\mathbb{C}} f(x)\,\pi(dz) \in A$. This implies (3.3.14).

Referring back to the notation $\tilde{z}$ and $\tilde{w}$ introduced immediately prior to Proposition 1.1.2, let Π be the probability measure on $\mathbb{C}^2$ defined by $\Pi(\tilde{z},\tilde{w}) = \pi(\tilde{z})\delta_0(\tilde{w})$, where δ_0 is the delta measure. We are going to show that for all $g \in L^1(\Pi)$,

$$P\left(\lim_{t\to\infty}\frac{1}{t}\int_0^t g(\tilde{z}_s,\tilde{w}_s)\,ds = \int_{\mathbb{C}^2} g(\tilde{z},\tilde{w})\,d\Pi(\tilde{z},\tilde{w})\right) = 1. \tag{3.3.15}$$

Since the space $C_c(\mathbb{C}^2)$ of continuous functions with compact support is dense in $L^1(\Pi)$, it suffices to prove (3.3.15) for all $g \in C_c(\mathbb{C}^2)$. Using that $\tilde{w}_s = \tilde{w}_0 e^{-\nu s} \to 0$ as $s \to \infty$, as well as the continuity of g, we see that for every $\epsilon > 0$, there exists a $\kappa > 0$ such that if $\|(\tilde{z}_s,\tilde{w}_s) - (\tilde{z}_s,0)\| = |\tilde{w}_s| < \kappa$, then $|g(\tilde{z}_s,\tilde{w}_s) - g(\tilde{z}_s,0)| < \epsilon$. Fix an $r > 0$ such that $|\tilde{w}_0|e^{-\nu r} \le \kappa$. Then

$$\begin{aligned}\frac{1}{t}\left|\int_0^t \left(g(\tilde{z}_s,\tilde{w}_s) - g(\tilde{z}_s,0)\right)\,ds\right| &\le \frac{1}{t}\left[\int_0^r |g(\tilde{z}_s,\tilde{w}_s) - g(\tilde{z}_s,0)|\,ds + \int_r^t |g(\tilde{z}_s,\tilde{w}_s) - g(\tilde{z}_s,0)|\,ds\right] \\ &< \frac{r}{t}\|g\|_\infty + \frac{t-r}{t}\epsilon.\end{aligned}$$

Taking the limsup as $t \to \infty$ on both sides yields

$$\limsup_{t\to\infty}\frac{1}{t}\left|\int_0^t \left(g(\tilde{z}_s,\tilde{w}_s) - g(\tilde{z}_s,0)\right)\,ds\right| < \epsilon.$$

Since $\epsilon > 0$ is arbitrary, and the limit $\lim_{t\to\infty}\frac{1}{t}\int_0^t g(\tilde{z}_s,0)\,ds$ exists P-a.s., we deduce that

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t g(\tilde{z}_s,\tilde{w}_s)\,ds = \lim_{t\to\infty}\frac{1}{t}\int_0^t g(\tilde{z}_s,0)\,ds \qquad P\text{-a.s.} \tag{3.3.16}$$

Meanwhile, by (3.3.14) and the definition of Π,

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t g(\tilde{z}_s, 0)\, ds = \int_{\mathbb{C}} g(\tilde{z}, 0)\, d\pi(\tilde{z}) = \int_{\mathbb{C}^2} g(\tilde{z}, \tilde{w})\, d\Pi(\tilde{z}, \tilde{w}) \qquad P\text{-a.s.} \qquad (3.3.17)$$

Putting (3.3.16) and (3.3.17) together yields (3.3.15).

We have thus proved that the system of time-homogeneous SDEs (1.1.4) converges to a unique ergodic measure Π. $\square$

Chapter 4

Numerical Results

In this section we provide a numerical perspective for solving our stabilization by noise problem, and expand upon the analysis conducted in previous sections.

4.1 Suitable Brownian Noise for Nonexplosion

We have shown that a necessary condition for the deterministic system (1.1.1) to have solutions that blow up in finite time is when $y_3(t) = 0$ for all $t \geq 0$ (Remark 3.2.2). Hence, to stabilize this system, we add a Brownian noise which ensures that $y_3(t) \neq 0$ for all $t \geq 0$. Our simulations, described below, suggest that it is enough to add a real-valued Brownian noise in the $\mathrm{Im}(z)$ direction; that is, the corresponding

SDE reads

$$\begin{cases} dx_1 = (-\nu x_1 + \alpha(x_1x_3 - x_2x_4))dt \\ dx_2 = (-\nu x_2 + \alpha(x_1x_4 + x_2x_3))dt + dB_t \\ dx_3 = (-\nu x_3 + \beta(x_1x_3 - x_2x_4))dt \\ dx_4 = (-\nu x_4 + \beta(x_1x_4 + x_2x_3))dt \end{cases} \tag{4.1.1}$$

in the $\mathbf{x}$ coordinates, or

$$\begin{cases} dy_1 = (-\nu y_1 + \beta[(y_1^2 - y_2^2) - (y_3^2 - y_4^2)])dt \\ dy_2 = (-\nu y_2)dt \\ dy_3 = (-\nu y_3 + 2\beta(y_1y_3 - y_2y_4))dt + \frac{1}{2}dB_t \\ dy_4 = (-\nu y_4)dt + \frac{1}{2}dB_t \end{cases} \tag{4.1.2}$$

in the $\mathbf{y}$ coordinates.

Observe that if we complexify the coordinates in (4.1.2) by taking $\tilde{w}_t = y_2(t) + iy_4(t)$, then $\tilde{w}_t$ satisfies the SDE $d\tilde{w}_t = -\nu\tilde{w}\,dt + \frac{i}{2}\,dB_t$, which is a 2-dimensional Ornstein-Uhlenbeck process. (Compare this against the choice of additive Brownian noise in (1.1.4), where $\tilde{w}_t$ satisfies the deterministic equation $d\tilde{w}_t = -\nu\tilde{w}\,dt$.) Since the Ornstein-Uhlenbeck process is ergodic and has an explicit invariant measure, we believe that the system (4.1.1) should be nonexplosive, and may be ergodic. As of this writing, we are not in a position to prove these statements, due to some technicality involved in carrying out a time change similar to the one done in Section 3.3.

That said, we have numerical evidence for nonexplosivity of system (4.1.2). We first simulated the trajectories of the ODE (3.1.7) (without noise). Using MATLAB, we created a function that takes in a set of initial conditions $y_1(0), y_2(0), y_3(0), y_4(0), \alpha, \beta$, and ν, and produces a discrete-time approximate solution of (3.1.7) via Euler's method. We then created a program that fixes two of the initial coordinates (for

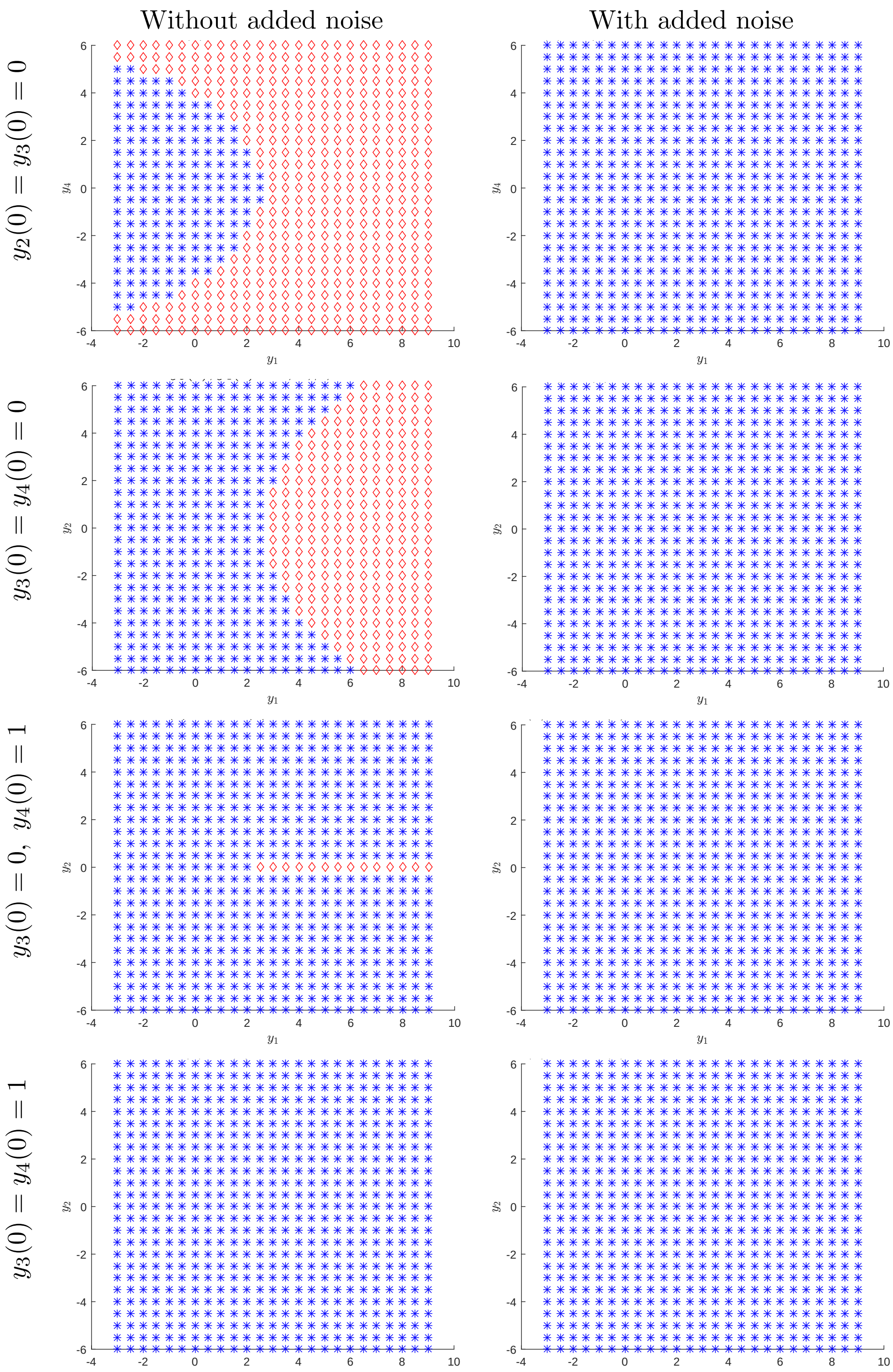


FIGURE 4.1.1: Simulation results of various initial conditions which give rise to nonexplosive solutions (indicated by $*$) and explosive solutions (indicated by $\diamondsuit$) of our $\mathbb{C}^2$-valued coupled system (1.1.1), with $\beta = \nu = 1$. The results on the left panel are for the system without added Brownian noise, while the results on the right panel are with added Brownian noise of the form (4.1.2).

instance, $y_3(0) = y_4(0) = 0$) and α, β, and ν, while varying the other two coordinates. The program then ran a series of simulations, testing every set of inputs in a grid of the varying initial coordinates. Thus, this program examined a two dimensional grid in the 4-dimensional space of the ODE. For each set of initial conditions, our program recorded whether or not that trajectory blows up in a set period of time. It then recorded the result in a 2-dimensional plot (whose axes are the two varying initial conditions). Our MATLAB code is available for download at [CFK$^+$].

On the left panel in Figure 4.1.1 we present our results of simulating trajectories where $\beta = \nu = 1$, and two of the four coordinates are initially fixed, while the other two are being varied. The red diamonds indicate initial conditions which lead to finite-time blow-up trajectories, while blue stars indicate those that give rise to nonexplosive trajectories. Observe that without added noise, our simulations for the fixed initial conditions $y_3(0) = y_4(0) = 0$ and $y_2(0) = y_3(0) = 0$ correspond, respectively, with the phase portraits in Figures 3.2.1a and 3.2.1b. Also, for the fixed initial condition $y_3(0) = 0$ and $y_4(0) = 1$ (without added noise), the explosive trajectory lies on the line $y_2 = 0$. Contrast this with the fixed initial condition $y_3(0) = y_4(0) = 1$, which do not give rise to explosive trajectories. Our numerical simulations verify the analysis in Chapter 3.

We used the same procedure to verify our analysis of the SDE (4.1.2). We ran this program again, this time with our trajectory function programmed with Brownian noise added to y_3 and y_4. The Brownian noise is modeled by a normally distributed random variable, scaled by the square root of the time step, to each step of the iterated Euler's method. Then we ran the simulations and generate the explosive and nonexplosive initial conditions as before; see the right panel in Figure 4.1.1. It appears evident that the SDE (4.1.2) is stable globally. We ran this computation

many times on the same set of initial conditions to ensure that the probability of a stable trajectory is near 1.

While not shown in Figure 4.1.1, we have tested a large variety of initial conditions to ensure that SDE (4.1.2) is stable everywhere in the 4-dimensional space, for all values of $\nu > 0$ and $\alpha, \beta \in \mathbb{R}$.

4.2 Estimate of Invariant Measure

In Section 3.3, we proved that the system (3.3.2) has a unique invariant measure. However, characterizing this invariant measure analytically is challenging, so we take a numerical approach here.

Consider the SDEs

$$\begin{cases} dy_1 = (-\nu y_1 + \beta[(y_1^2 - y_2^2) - (y_3^2 - y_4^2)])\,dt + \sigma_1\,dB_t^1 \\ dy_2 = (-\nu y_2)\,dt + \sigma_2\,dB_t^1 \\ dy_3 = (-\nu y_3 + 2\beta(y_1 y_3 - y_2 y_4))\,dt + \sigma_3\,dB_t^2 \\ dy_4 = (-\nu y_4)\,dt + \sigma_4\,dB_t^2, \end{cases}$$

where

$$\sigma = \begin{pmatrix} \sigma_1 & 0 \\ \sigma_2 & 0 \\ 0 & \sigma_3 \\ 0 & \sigma_4 \end{pmatrix}.$$

We will consider only the case $\sigma_2 = \sigma_4 = 0$ that corresponds to adding an isotropic Brownian noise, which was the case analyzed in Section 3.3.

In Section 3.3, we proved that the system (3.3.2) has the same invariant measure

as the system (3.3.3). To compute for the invariant measure of the system (3.3.2), we will compute the invariant measure for the system

$$\begin{cases} dy_1 = (-\nu y_1 + \beta(y_1^2 - y_3^2))\, dt + \sigma_1\, dB_t^1 \\ dy_3 = (-\nu y_3 + 2\beta y_1 y_3)\, dt + \sigma_3\, dB_t^2. \end{cases} \tag{4.2.1}$$

To find the invariant measure for (4.2.1), we solve the following non-elliptic PDE

$$\begin{cases} \mathcal{L}^* f = 0 \\ f(y_1, y_3) \to 0 \quad \text{as } ||(y_1, y_3)|| \to +\infty, \end{cases} \tag{4.2.2}$$

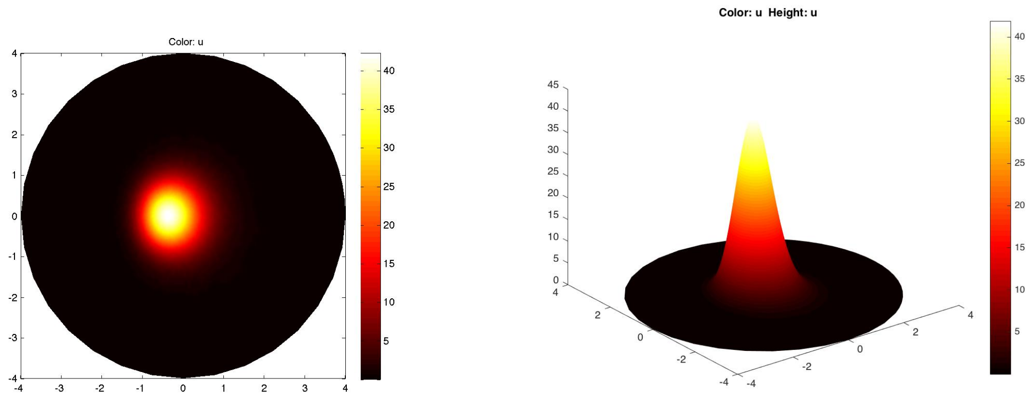
where $f = \frac{d\pi}{d\lambda}$, λ is the 2−dimensional Lebesgue measure, and the $\mathcal{L}^*$, the adjoint of $\mathcal{L}$, is given in this case by

$$\mathcal{L}^* = -\partial_{y_1}((-\nu y_1 + \beta y_1^2 - \beta y_3^2)(\cdot)) - \partial_{y_3}((-\nu y_3 + 2\beta y_1 y_3)(\cdot)) + \frac{1}{2}(\sigma_1^2 \partial_{y_1 y_1} + \sigma_3^2 \partial_{y_3 y_3}).$$

This gives the steady-state solution to the forward Kolmogorov (or Fokker-Planck) equation associated with the SDE (4.2.1). We employ the MATLAB PDE Toolbox to solve this PDE using the finite-element method. We approximate the solutions by solving

$$\begin{cases} \mathcal{L}^* f = 0 & \text{in } B_4(0) \\ f(y_1, y_3) = .1 & \text{on } \partial B_4(0) \end{cases}, \tag{4.2.3}$$

where $B_4(0)$ is the ball of radius 4 centered at the origin. The size of the boundary conditions and radius of the ball are mostly irrelevant. Altering them would roughly be equivalent to rescaling the units of the resulting measure. See the details of the computation in Appendix B.



(A) Intensity plot (B) Height plot

FIGURE 4.2.1: Density plot for the invariant measure for $\beta = \nu = 1$

After a number of mesh refinements, we obtain an approximate density plot for the invariant measure for (4.2.1), see Figure 4.2.1. Observe that the invariant measure has a peak which is symmetric about $y_3 = 0$, and slightly skewed toward the left half plane ($y_1 < 0$). Also, the measure appears to have a heavy-tailed distribution (higher moments may be infinite), which is consistent with the result of Herzog and Mattingly in their analysis of (2.5.1) in [HM15a].

Chapter 5

Future Directions

Extending existing research on the stabilization of $\mathbb{C}$-valued polynomial ODEs [Her11, HM15a, HM15b], we have ascertained that the addition of a Brownian noise to our prototype multivariable system of ODEs stabilizes explosive (and thus all) trajectories with probability one. This may be seen as a first step toward understanding higher-dimensional stochastic Burgers' equations [HM15c], as well as higher-dimensional analogs of complex Langevin equations studied by Aarts *et al.* [AGS13, ABSS14].

While we have analytically and numerically verified conditions for stabilization of our coupled ODEs, there remain many open questions. How would our results differ if we change ODE (1.1.1) in any of the following manners: (1) make the drift parameter ν negative; (2) if the drift parameters for the two complex coordinates ν_1 and ν_2 are distinct (in which case it may be difficult to find a similar coordinate transformation)? Additionally, we would like to go beyond our system and consider the stabilization problem in more general nonlinear systems. For instance, would our methods still apply to systems in higher dimensions, say in $\mathbb{C}^n$, $n \geq 3$? What about

coupled ODEs wherein the coupling terms are higher than quadratic order?

Due to the reduction of (1.1.1), one may want to consider a system where such a reduction can not be done. For instance, consider the $\mathbb{C}^2$-valued system of ODEs

$$\begin{cases} \dot{z}_t = z_t^2 + \alpha z_t w_t \\ \dot{w}_t = w_t^2 + \beta z_t w_t. \end{cases} \tag{5.0.1}$$

The change of coordinates does not reduce the $\mathbb{C}^2$-system (5.0.1) to a quasi-$\mathbb{C}$-system. An alternative approach is to apply the same Lyapunov method outlined in [HM15a, HM15b]. If we observe the behavior of the system (5.0.1) as $t \to \infty$, we should see two main scenarios: when one process explodes faster than the other and when we cannot distinguish which process explodes faster than the other. The first case implies that one process is dominant over the other, say the process z_t is the dominant one. Then it can be analyzed similarly to the noise-induced stabilization of $dz_t = z_t^2\, dt$. However, the second case requires a new approach.

From the numerical analysis done in Section 4.1, we can confidently say SDE (4.1.1) is nonexplosive. Due to the coordinate transformation, it also contains the Ornstein-Uhlenbeck process. However, we can not be certain that this SDE is ergodic. Note that the focus of the simulations is to ascertain whether or not the system is explosive. We did not do further numerics, such as those done in Figure 2.5.2, to determine the behavior of the solutions. It would be interesting to explore in the future whether this system and similar models are ergodic.

Appendix A

The Girsanov Theorem

Theorem A.0.1 (The Girsanov theorem I in [Øks03]). *Let $\omega \in \Omega$ and $Y_t \in \mathbb{R}^d$ be an Itô process that satisfies*

$$dY_t = a(t, \omega)\, dt + dW_t \tag{A.0.1}$$

with initial condition $Y_0 = 0$ for $t \leq T$, where $T \leq \infty$ is a given constant, $a(t, \omega) \in \mathbb{R}^d$, and W_t is a d-dimensional Brownian motion. Put

$$M_t = \exp\left(-\int_0^t a(s, \omega)\, dW_s - \frac{1}{2} \int_0^t a^2(s, \omega)\, ds \right) \tag{A.0.2}$$

for $t \leq T$. Assume that $a(s, \omega)$ satisfies Novikov's condition

$$E\left[\exp\left(\frac{1}{2} \int_0^T a^2(s, \omega)\, ds \right) \right] < \infty, \tag{A.0.3}$$

where $E = E^P$ is the expectation with respect to P. Define the measure Q on $(\Omega, \mathcal{F}_T)$ by

$$dQ(\omega) = M_T(\omega)dP(\omega). \tag{A.0.4}$$

Then Y_t is a d-dimensional Brownian motion with respect to the probability law Q for $t \leq T$.

Proof. See Theorem 8.6.3 in [Øks03]. $\square$

Remark A.0.2. The Novikov condition (A.0.3) is sufficient to guarantee that $\{M_t\}_{t\leq T}$ is a martingale with respect to $\mathcal{F}_t$ and P. In particular, we need $\{M_t\}_{t\leq T}$ to be a martingale for the result to hold.

Notice the SDE (A.0.1) assumes the diffusion coefficient to be one. In our case, we have an arbitrary diffusion. Hence, we need a slight variation to Theorem A.0.1.

Let $\mathcal{H} \subset \mathcal{F}$ be a σ-algebra. Define $\mathcal{W}^n_{\mathcal{H}}(S,T)$ be the class of processes $f(t,\omega) \in \mathbb{R}^n$ such that

1. $(t,\omega) \to f(t,\omega)$ is $\mathcal{B}([0,\infty)) \times \mathcal{H}$-measurable,
2. $f(t,\omega)$ is $\mathcal{H}_t$-adapted, and
3. $E\left[\int_S^T f^2(t,\omega)\,dt\right] < \infty$.

Let $\mathcal{W}^n_{\mathcal{H}} = \cap_{T>0}\mathcal{W}_{\mathcal{H}}(0,T)$.

Theorem A.0.3 (The Girsanov theorem II in [Øks03])**.** *Let $\omega \in \Omega$ and $Y_t \in \mathbb{R}^d$ be an Itô process of the form*

$$dY_t = \beta(t,\omega)\,dt + \theta(t,\omega)\,dW_t \tag{A.0.5}$$

for $t \le T$*, where* $W_t \in \mathbb{R}^m, \beta(t,\omega) \in \mathbb{R}^d$*, and* $\theta(t,\omega) \in \mathbb{R}^{d\times m}$*. Suppose there exist processes* $u(t,\omega) \in \mathcal{W}_{\mathcal{H}}^m$ *and* $\alpha(t,\omega) \in \mathcal{W}_{\mathcal{H}}^d$ *such that*

$$\theta(t,\omega)u(t,\omega) = \beta(t,\omega) - \alpha(t,\omega) \tag{A.0.6}$$

and assume that $u(t,\omega)$ *satisfies Novikov's condition*

$$E\left[\exp\left(\frac{1}{2}\int_0^T u^2(s,\omega)\,ds\right)\right] < \infty. \tag{A.0.7}$$

Put

$$M_t = \exp\left(-\int_0^t u(s,\omega)\,dW_s - \frac{1}{2}\int_0^t u^2(s,\omega)\,ds\right), \tag{A.0.8}$$

and

$$dQ(\omega) = M_T(\omega)\,dP(\omega) \tag{A.0.9}$$

on $\mathcal{F}_T$*. Then*

$$\widehat{W}_t := \int_0^t u(s,\omega)\,ds + W_t, \tag{A.0.10}$$

for $t \le T$*, is a Brownian motion with respect to* Q *and in terms of* $\widehat{W}_t$ *the process* Y_t *has the stochastic integral representation*

$$dY_t = \alpha(t,\omega)\,dt + \theta(t,\omega)\,d\widehat{W}_t. \tag{A.0.11}$$

Proof. See Theorem 8.6.4 in [Øks03]. $\square$

Remark A.0.4. Theorem A.0.3 is the version used to prove Lemma 3.3.1.

Appendix B

Numerical Computation of the Invariant Measure

B.1 Kolmogorov Forward Equation

Let X_t be an Itô diffusion in $\mathbb{R}^d$ satisfying the SDE (2.1.1). If X_t has a probability density $p(t,x)$, then it is said to satisfy the *Kolmogorov forward equation*, also known as the *Fokker-Plank equation*,

$$\frac{\partial p(t,x)}{\partial t} = -\frac{\partial}{\partial x}[b(t,x)p(t,x)] + \frac{1}{2}\frac{\partial^2}{\partial x^2}[\sigma^2(t,x)p(t,x)]. \tag{B.1.1}$$

Observe that the Kolmogorov forward equation is equivalent to

$$\frac{\partial p(t,x)}{\partial t} = \mathcal{L}^* p(t,x) \tag{B.1.2}$$

for the adjoint of $\mathcal{L}$, $\mathcal{L}^*$, defined in (2.3.10). Note we assume σ is independent of

X_t because we have σ be a constant for our coupled SDEs. A similar version of the Kolmogorov forward equation can be found as Exercise 8.3 in [Øks03]. Given this equivalent form, we want to solve the non-elliptic PDE (4.2.2) to numerically solve for the invariant measure.

B.2 MATLAB PDE Toolbox

To compute the invariant measure for the system (4.2.1), we utilize the PDE Toolbox in MATLAB. To do so, we have to adjust the non-elliptic PDE equation (4.2.2) to satisfy the *elliptic* PDE

$$-\nabla \cdot (c\nabla u) + au = f, \tag{B.2.1}$$

where the coefficients a and c are functions of $y_1, y_3 \in \mathbb{R}$, f can be a function of u and its derivatives as well as y_1, y_3, and u is the function we want to solve for. We will need to identify the coefficients a and c, and f.

Recall in Section 2.3, we have

$$\mathcal{L}^* = -\partial_{y_1}((-\nu y_1 + \beta y_1^2 - \beta y_3^2)(\cdot)) - \partial_{y_3}((-\nu y_3 + 2\beta y_1 y_3)(\cdot)) + \frac{1}{2}(\sigma_1^2 \partial_{y_1 y_1} + \sigma_3^2 \partial_{y_3 y_3}).$$

Then

$$\begin{aligned} 0 = \mathcal{L}^* u \quad &= \quad -(-\nu y_1 + \beta y_1^2 - \beta y_3^2)\partial_{y_1} u - (-\nu y_3 + 2\beta y_1 y_3)\partial_{y_3} u \\ &\quad -2(-\nu + 2\beta y_1)u + \frac{1}{2}(\sigma_1^2 \partial_{y_1}^2 u + \sigma_3^2 \partial_{y_3}^2 u); \end{aligned}$$

when reordered, we have

$$\begin{aligned}\frac{1}{2}(\sigma_1^2\partial_{y_1}^2 u + \sigma_3^2\partial_{y_3}^2 u) &- 2(-\nu + 2\beta y_1)u \\ &= (-\nu y_1 + \beta y_1^2 - \beta y_3^2)\partial_{y_1} u + (-\nu y_3 + 2\beta y_1 y_3)\partial_{y_3} u.\end{aligned}$$

To apply this to the toolbox, we let $\nu, \beta, \sigma = 1$ and label $y_1 = x$ and $y_3 = y$. Thus, we have

$$\begin{aligned} c &= -.5 \\ a &= 2(1 - 2x) \\ f &= (-x + x^2 - y^2)u_x + (-y + 2xy)u_y. \end{aligned}$$

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