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# Three Essays in Asset Bubbles, Banking and Macroeconomics

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# **Three Essays in Asset Bubbles, Banking and Macroeconomics**

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University of Connecticut, 2015

This dissertation examines theoretically the macroeconomic effects of asset bubbles and bank competition. The first two essays study the aggregate impacts of bubbles and crashes by extending the standard rational bubbles model with endogenous labor supply. By explicitly considering the labor choice, the studies generate results that asset bubbles can promote economic expansion as opposed to the contractionary effect predicted by previous studies. In addition, when bubbles crash, the transmission to the real economy is much faster than the economy without labor choice. The third essay discusses the role of bank competition on capital accumulation. Within a dynamic general equilibrium framework with oligopolistic financial intermediaries and asymmetric information between lenders and borrowers, the study provides conditions under which a more competitive banking structure is beneficial to capital accumulation.

**Three Essays in Asset Bubbles, Banking and Macroeconomics**

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B.A., Guangdong University of Foreign Studies, China, 2008

M.A., University of Connecticut, USA, 2011

A Dissertation

Submitted in Partial Fulfillment of the

Requirements for the Degree of

Doctor of Philosophy

at the

University of Connecticut

2015

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Lisi Shi

2015

# APPROVAL PAGE

Doctor of Philosophy Dissertation

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# Chapter 1

## Overview

My dissertation is comprised of three studies that examine theoretically the relation between asset bubbles and the macroeconomics, and that between bank competition and the aggregate economy. All three essays are built on a dynamic general equilibrium framework. The first two chapters focus on the aggregate impact of bubbles and crashes while the third chapter investigates the role of competition in the financial sector.

The cause and consequences of asset bubbles have long been studied in the literature. One influential strand is the rational bubbles theory, which considers asset bubbles as a remedy of dynamic inefficiency. Yet, among these studies, little effort has been spent on exploring how the labor decision responds to asset price boom and how the labor market helps propagate the impact of asset market fluctuation to the real economy, despite of ample evidence indicating a close relation between the labor market and the asset bubbles. In addition, the standard rational bubble theory predicts economic contraction when bubbles emerge but the recent macroeconomic statistics show the opposite.

Responding to above concerns, chapter 2 employs an overlapping generations model with endogenous labor supply to study the effects of asset bubbles. By allowing flexibility

in labor supply, the model shows that bubbles can promote expansion in steady-state capital, investment, employment and output as long as labor supply responds strongly and positively to increase in interest rate. The new result helps to reconcile the inconsistency between theory and the empirical finding.

Chapter 3 continues to investigate the implication of endogenous labor supply on the role of asset bubbles but emphasizes the event of bubble crash. It develops a stochastic environment to assume that the bubbles may randomly crash in any period after they are formed. Comparing with the stochastic setup with no labor choice, our model generates more realistic results in following aspects: the major economy in terms of the interest rate, aggregate employment and total output experience an immediate drop when bubble burst, while these variables won't be affected in the model without labor choice. This is because in the economy without labor choice, the only effective determinant of production is capital. In the overlapping generations setting, current period's capital is predetermined by saving from last period. Thus if the bubbles crash, current capital stock won't be affected, nor will the interest rate or total output. On the other hand, when considering labor decision, the output and interest rate are depended on both capital and labor. If bubbles burst, even though capital level is predetermined and not subject to the change, labor supply will be affected immediately by the crash, so are interest rate and total output. Moreover, the second chapter finds that the stochastic bubbles can be expansionary to the economy as well, for similar reason in the case of deterministic bubbles.

The main question Chapter 4 attempts to address is the implication of bank competition on saving and the deposit interest, capital accumulation, and the borrowing lending activities. Recent studies have started to examine the macroeconomic effects of the industrial organization of the banking system, but these studies typically focus on two

extreme cases: perfectly competitive banking system and monopoly banking system. A less extreme market structure has not been thoroughly examined yet. On the other hand, oligopoly banking system is prevalent in many economies, which stresses the importance of investigating such particular market structure. The present study is intended to fill this gap.

To achieve that, chapter 4 presents a dynamic general equilibrium model with banks engaged in Cournot competition in the loan market as well as the deposit market. Lending activities take place under asymmetric information and are subject to costly monitoring. By affecting the deposit rate as well as the loan rate, the banks control the volume of saving and borrowing, which jointly determine the capital level. In addition, due to the asymmetric information problem, the monitoring intensity is positively associated with leverage ratio. Therefore, when the financial market becomes less concentrated, a higher volume of credits will be issued to entrepreneurs, leading to more capital investment. Meanwhile, banks demand more active monitoring, which aggravates the inefficiency. Further analysis has been done to examine how the severity of asymmetry, the share of entrepreneurs or the intertemporal elasticity of substitution affect the capital accumulation. The results show a negative, positive and mixing effect respectively.

## Chapter 2

### Asset Bubbles in an Overlapping Generations Model with Endogenous Labor Supply

#### 2.1 Introduction

The existence and consequences of asset bubbles have long been a subject of interest to economists. In a seminal paper, Tirole (1985) showed that asset bubbles can exist in an overlapping generations economy with rational consumers and exogenous labor supply. A central implication of Tirole's model is that asset bubbles will always crowd out investment in productive capital and reduce capital accumulation. Since labor supply is inelastic, this will also lead to a reduction in aggregate output. This negative relationship between asset bubbles and aggregate economic activities, however, is in contrast with empirical evidence. As pointed out by Martin and Ventura (2012), episodes of asset bubbles in the U.S. and Japan are typically associated with periods of robust economic expansions. In the present study, we show that this conflict between theory and evidence can be resolved when labor supply is endogenous.<sup>1</sup> More specifically, we show that asset bubbles can

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<sup>1</sup>Olivier (2000), Farhi and Tirole (2012) and Martin and Ventura (2012) have explored other channels through which asset bubbles can crowd in productive investment and foster economic growth in overlapping

induce an expansion in steady-state capital, investment, employment and output if labor supply responds strongly and positively to changes in interest rate. This type of response is possible when the intertemporal elasticity of substitution (IES) in consumption is small and the Frisch elasticity of labor supply is large. The intuition of this result will be explained later. We also provide a specific numerical example to illustrate our findings.

## 2.2 The Model

The model economy under study is essentially the one considered in Tirole (1985, Section 2), except that labor supply is now endogenously determined. Specifically, consider an overlapping generations model in which each consumer lives two periods: young and old. In each period  $t \geq 0$ , a new generation of identical consumers is born. The size of generation  $t$  is given by  $N_t = (1 + n)^t$ , with  $n > 0$ . All consumers have one unit of time endowment which can be allocated between work and leisure. Retirement is mandatory in the second period of life, so the labor supply of old consumers is zero.

Consider a consumer who is born at time  $t \geq 0$ . Let  $c_{y,t}$  and  $c_{o,t+1}$  denote his consumption when young and old, respectively, and let  $l_t$  denote his labor supply when young. The consumer's preferences are represented by

$$\mathcal{U}(c_{y,t}, l_t, c_{o,t+1}) = \frac{c_{y,t}^{1-\sigma}}{1-\sigma} - A \frac{l_t^{1+\psi}}{1+\psi} + \beta \frac{c_{o,t+1}^{1-\sigma}}{1-\sigma}, \quad (2.1)$$

where  $\sigma > 0$  is the inverse of the IES in consumption,  $\psi \geq 0$  is the inverse of the Frisch elasticity of labor supply,  $\beta \in (0, 1)$  is the subjective discount factor and  $A$  is a positive constant. Let  $w_t$  be the market wage rate at time  $t$ . Then the consumer's labor income when young is  $w_t l_t$ . The consumer can save in two types of assets: physical capital and

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generations models. Miao and Wang (2012) have developed an infinite-horizon model in which asset bubbles can promote total factor productivity. None of these studies have examined the connections between endogenous labor supply and asset bubbles.

an intrinsically worthless asset.<sup>2</sup> The total supply of the intrinsically worthless asset is constant over time and is denoted by  $M \geq 0$ .<sup>3</sup> Denote savings in the form of physical capital by  $s_t$ , and savings in the form of intrinsically worthless asset by  $m_t$ . The gross return from physical capital between time  $t$  and  $t + 1$  is given by  $R_{t+1}$ . The price of the intrinsically worthless asset at time  $t$  is  $p_t$ . No-arbitrage means that these two types of assets must yield the same return in every period, so that  $R_{t+1} = p_{t+1}/p_t$  for all  $t \geq 0$ . Taking  $\{w_t, p_t, p_{t+1}, R_{t+1}\}$  as given, the consumer's problem is to choose an allocation  $\{c_{y,t}, l_t, c_{o,t+1}, s_t, m_t\}$  so as to maximize his lifetime utility in (4.1), subject to the budget constraints:

$$c_{y,t} + s_t + p_t m_t = w_t l_t, \quad \text{and} \quad c_{o,t+1} = R_{t+1} s_t + p_{t+1} m_t.$$

The first-order conditions for this problem are given by

$$w_t c_{y,t}^{-\sigma} = A l_t^\psi, \quad \text{and} \quad c_{y,t}^{-\sigma} = \beta R_{t+1} c_{o,t+1}^{-\sigma}. \quad (2.2)$$

Using these equations, we can obtain

$$\begin{aligned} c_{y,t} &= \frac{c_{o,t+1}}{(\beta R_{t+1})^{\frac{1}{\sigma}}} = \frac{w_t l_t}{1 + \beta^{\frac{1}{\sigma}} R_{t+1}^{\frac{1}{\sigma}-1}}, \\ l_t &= A^{-\frac{1}{\sigma+\psi}} \left( 1 + \beta^{\frac{1}{\sigma}} R_{t+1}^{\frac{1}{\sigma}-1} \right)^{\frac{\sigma}{\sigma+\psi}} w_t^{\frac{1-\sigma}{\sigma+\psi}}, \\ s_t + p_t m_t &= \Sigma(R_{t+1}) w_t l_t, \quad \text{where} \quad \Sigma(R_{t+1}) \equiv \frac{\beta^{\frac{1}{\sigma}} R_{t+1}^{\frac{1}{\sigma}-1}}{1 + \beta^{\frac{1}{\sigma}} R_{t+1}^{\frac{1}{\sigma}-1}}. \end{aligned} \quad (2.3)$$

An increase in  $R_{t+1}$  has two opposing effects on saving. These effects are captured by the function  $\Sigma : \mathbb{R}_+ \rightarrow [0, 1]$  defined in (4.3). First, an increase in  $R_{t+1}$  means that for

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<sup>2</sup>The second type of asset is called “intrinsically worthless” because it has no consumption value and cannot be used for production. The only motivation for holding this type of asset is to resell it at a higher price in the next period.

<sup>3</sup>At time 0, all assets are owned by a group of “initial-old” consumers. The decision problem of these consumers is trivial and does not play any role in the following analysis.

the same level of total savings, the consumer will receive more interest income when old. This creates an income effect which encourages consumption when young and discourages saving. Second, an increase in interest rate also lowers the relative price of future consumption. This creates an intertemporal substitution effect which discourages consumption when young and promotes saving. The relative strength of these two effects depends on the value of  $\sigma$ . In particular, the intertemporal substitution effect dominates when  $\sigma < 1$ . In this case, an increase in  $R_{t+1}$  will always increase the savings rate so that  $\Sigma(\cdot)$  is a strictly increasing function. When  $\sigma > 1$ , the income effect dominates so that  $\Sigma(\cdot)$  is strictly decreasing. The two effects exactly cancel out when  $\sigma = 1$ . In this case,  $\Sigma(\cdot)$  is a constant.

On the supply side of the economy, there is a large number of identical firms. In each period, each firm hires labor and physical capital from the competitive factor markets, and produces output according to

$$Y_t = K_t^\alpha L_t^{1-\alpha}, \quad \text{with } \alpha \in (0, 1),$$

where  $Y_t$  denotes output produced at time  $t$ ,  $K_t$  and  $L_t$  denote capital input and labor input, respectively. Since the production function exhibits constant returns to scale, we can focus on the choices made by a single price-taking firm. We assume that physical capital is fully depreciated after one period, so that  $R_t$  coincides with the rental price of physical capital at time  $t \geq 0$ . The representative firm's problem is given by

$$\max_{K_t, L_t} \{ K_t^\alpha L_t^{1-\alpha} - R_t K_t - w_t L_t \},$$

and the first-order conditions are  $R_t = \alpha K_t^{\alpha-1} L_t^{1-\alpha}$ , and  $w_t = (1 - \alpha) K_t^\alpha L_t^{-\alpha}$ .

Given  $M \geq 0$ , a competitive equilibrium of this economy consists of sequences of allocations  $\{c_{y,t}, l_t, c_{o,t+1}, s_t, m_t\}_{t=0}^\infty$ , aggregate inputs  $\{K_t, L_t\}_{t=0}^\infty$ , and prices  $\{w_t, p_t, R_t\}_{t=0}^\infty$



such that (i) given  $\{w_t, p_t, p_{t+1}, R_{t+1}\}$ , the allocation  $\{c_{y,t}, l_t, c_{o,t+1}, s_t, m_t\}$  is optimal for the consumers in generation  $t \geq 0$ , (ii) given  $\{w_t, R_t\}$ , the aggregate inputs  $\{K_t, L_t\}$  solve the representative firm's problem at time  $t \geq 0$ , and (iii) all markets clear in every period, so that  $L_t = N_t l_t$ ,  $N_t m_t = M$  and

$$K_{t+1} = N_t s_t = \left[ \frac{\beta^{\frac{1}{\sigma}} R_{t+1}^{\frac{1}{\sigma}-1}}{1 + \beta^{\frac{1}{\sigma}} R_{t+1}^{\frac{1}{\sigma}-1}} \right] w_t l_t - p_t m_t, \quad \text{for all } t \geq 0. \quad (2.4)$$

Let  $k_t \equiv K_t/N_t$  be the quantity of physical capital per worker at time  $t$ , and let  $a_t \equiv p_t m_t$  be the quantity of unproductive savings per young consumer. Then the equilibrium wage rate can be expressed as  $w_t = (1 - \alpha) k_t^\alpha l_t^{-\alpha}$ , and (4.4) can be rewritten as

$$(1 + n) k_{t+1} = (1 - \alpha) \left[ \frac{\beta^{\frac{1}{\sigma}} R_{t+1}^{\frac{1}{\sigma}-1}}{1 + \beta^{\frac{1}{\sigma}} R_{t+1}^{\frac{1}{\sigma}-1}} \right] \left( \frac{k_t}{l_t} \right)^\alpha l_t - a_t. \quad (2.5)$$

The dynamics of  $a_t$  is determined by

$$a_{t+1} = p_{t+1} m_{t+1} = \frac{p_{t+1} m_{t+1}}{p_t m_t} a_t = \frac{R_{t+1}}{1 + n} a_t.$$

## 2.3 Stationary Equilibrium

### 2.3.1 Economy without Intrinsically Worthless Assets

Before analyzing the effects of asset bubbles, we first characterize the stationary equilibrium of an economy with zero supply of intrinsically worthless asset, i.e.,  $M = 0$  and  $a_t = 0$  for all  $t \geq 0$ . A stationary equilibrium is a competitive equilibrium in which  $k_t = k^*$ ,  $l_t = l^*$  and  $R_t = R^*$  for all  $t \geq 0$ . Substituting these conditions into (4.5) gives

$$\begin{aligned} \frac{\beta^{\frac{1}{\sigma}} (R^*)^{\frac{1}{\sigma}-1}}{1 + \beta^{\frac{1}{\sigma}} (R^*)^{\frac{1}{\sigma}-1}} \left( \frac{k^*}{l^*} \right)^{\alpha-1} &= \frac{1 + n}{1 - \alpha} \\ \Rightarrow \Lambda(R^*) &\equiv \frac{\beta^{\frac{1}{\sigma}} (R^*)^{\frac{1}{\sigma}}}{1 + \beta^{\frac{1}{\sigma}} (R^*)^{\frac{1}{\sigma}-1}} = \frac{(1 + n) \alpha}{1 - \alpha}. \end{aligned} \quad (2.6)$$

Equation (4.6) follows from the fact that  $R^* = \alpha (k^*/l^*)^{\alpha-1}$ . For any  $\sigma > 0$ , the function  $\Lambda : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is strictly increasing with  $\Lambda(0) = 0$  and  $\lim_{R \rightarrow \infty} \Lambda(R) = \infty$ . Hence, there exists a unique  $R^* > 0$  that solves (4.6). The steady-state value of all other variables can be uniquely determined by

$$w^* = (1 - \alpha) \left( \frac{\alpha}{R^*} \right)^{\frac{\alpha}{1-\alpha}}, \quad (2.7)$$

$$l^* = A^{-\frac{1}{\sigma+\psi}} \left[ 1 + \beta^{\frac{1}{\sigma}} (R^*)^{\frac{1}{\sigma}-1} \right]^{\frac{\sigma}{\sigma+\psi}} (w^*)^{\frac{1-\sigma}{\sigma+\psi}}, \quad (2.8)$$

$$k^* = l^* \left( \frac{\alpha}{R^*} \right)^{\frac{1}{1-\alpha}}, \quad \text{and} \quad c_y^* = \frac{c_o^*}{(\beta R^*)^{\frac{1}{\sigma}}} = \frac{w^* l^*}{1 + \beta^{\frac{1}{\sigma}} (R^*)^{\frac{1}{\sigma}-1}}. \quad (2.9)$$

This establishes the following result.

**Proposition 2.3.1.** A unique bubbleless steady state exists for any  $\sigma > 0$ . The steady-state values

$\{R^*, w^*, k^*, l^*, c_y^*, c_o^*\}$  are determined by (4.6)-(4.10).

### 2.3.2 Economy with Intrinsically Worthless Assets

Suppose now the economy has a strictly positive supply of intrinsically worthless assets, i.e.,  $M > 0$ . In the following analysis, we focus on stationary equilibria in which the price of these assets exceeds their fundamental value, i.e.,  $p_t = p^* > 0$ . Formally, a “bubbly” steady state is a set of values  $\{\tilde{a}^*, \tilde{R}^*, \tilde{w}^*, \tilde{k}^*, \tilde{l}^*, \tilde{c}_y^*, \tilde{c}_o^*\}$  that satisfies the following conditions:

$$\tilde{a}^* > 0, \quad \tilde{R}^* = 1 + n,$$

$$\tilde{a}^* + (1 + n) \tilde{k}^* = (1 - \alpha) \left[ \frac{\beta^{\frac{1}{\sigma}} (\tilde{R}^*)^{\frac{1}{\sigma}-1}}{1 + \beta^{\frac{1}{\sigma}} (\tilde{R}^*)^{\frac{1}{\sigma}-1}} \right] \left( \frac{\tilde{k}^*}{\tilde{l}^*} \right)^{\alpha} \tilde{l}^*, \quad (2.10)$$

and (4.7)-(4.10).<sup>4</sup> Substituting  $\tilde{a}^* > 0$  and  $\tilde{R}^* = 1 + n$  into (4.11) gives

$$\frac{1 + n}{1 - \alpha} < \left[ \frac{\beta^{\frac{1}{\sigma}} (1 + n)^{\frac{1}{\sigma}-1}}{1 + \beta^{\frac{1}{\sigma}} (1 + n)^{\frac{1}{\sigma}-1}} \right] \left( \frac{\tilde{k}^*}{\tilde{l}^*} \right)^{\alpha-1} \Rightarrow \frac{(1 + n) \alpha}{1 - \alpha} < \Lambda(1 + n). \quad (2.11)$$

---

<sup>4</sup>Note that equations (4.7)-(4.10) must be satisfied in any steady state, regardless of the existence of asset bubbles.

Since  $\Lambda(\cdot)$  is strictly increasing, (4.6) and (4.8) together imply that  $R^* < 1 + n$ . This shows that  $R^* < 1 + n$  is a necessary condition for the existence of bubbly steady state. Suppose this condition is satisfied. Then substituting  $\tilde{R}^* = 1 + n$  into (4.7)-(4.10) yields a unique set of values for  $\{\tilde{w}^*, \tilde{k}^*, \tilde{l}^*, \tilde{c}_y^*, \tilde{c}_o^*\}$ . Using (4.11), we can obtain a unique value of  $a^*$ , which is strictly positive as  $R^* < 1 + n$  and  $\Lambda(\cdot)$  is strictly increasing. Hence, a unique bubbly steady state exists. This proves the following result.

**Proposition 2.3.2.** A unique bubbly steady state exists if and only if  $R^* < 1 + n$ .

Similar to Tirole (1985), our model predicts that equilibrium interest rate will increase in the presence of asset bubbles. When labor supply is exogenous, the steady-state value of per-worker capital is determined by  $k^* = (\alpha/R^*)^{\frac{1}{1-\alpha}}$ . Thus, a higher interest rate in the bubbly steady state means that there is fewer per-worker capital than in the bubbleless steady state, i.e.,  $\tilde{k}^* < k^*$ . When labor supply is endogenous, the value of  $k^*$  is jointly determined by  $l^*$  and  $R^*$  as shown in (4.10). If the existence of asset bubbles can induce young consumers to work more (i.e.,  $\tilde{l}^* > l^*$ ), and if this effect is strong enough to overcome the increase in interest rate, then more capital will be accumulated in the bubbly steady state than in the bubbleless one, i.e.,  $\tilde{k}^* > k^*$ . The rest of this paper is intended to formalize this idea. Two remarks are in order before we proceed. First, the above description makes clear that  $\tilde{k}^* > k^*$  can happen *only if* labor supply is adjustable. This highlights the importance of introducing endogenous labor into Tirole's model. Second, suppose  $\tilde{l}^* > l^*$  and  $\tilde{k}^* > k^*$  are true. Then per-worker output in the bubbly steady state must also be higher than in the bubbleless steady state.

Suppose  $R^* < 1 + n$ . Then using (4.7)-(4.10), which are valid in both bubbleless and bubbly steady states, we can obtain

$$k^* = (1 - \alpha)^{\frac{1-\sigma}{\sigma+\psi}} A^{-\frac{1}{\sigma+\psi}} \left[ 1 + \beta^{\frac{1}{\sigma}} (R^*)^{\frac{1}{\sigma}-1} \right]^{\frac{\sigma}{\sigma+\psi}} \left( \frac{\alpha}{R^*} \right)^{\phi},$$

$$\tilde{k}^* = (1 - \alpha)^{\frac{1-\sigma}{\sigma+\psi}} A^{-\frac{1}{\sigma+\psi}} \left[ 1 + \beta^{\frac{1}{\sigma}} (1 + n)^{\frac{1}{\sigma}-1} \right]^{\frac{\sigma}{\sigma+\psi}} \left( \frac{\alpha}{1 + n} \right)^{\phi},$$

where  $\phi \equiv \frac{1}{1-\alpha} \left[ 1 + \frac{\alpha(1-\sigma)}{\sigma+\psi} \right] > 0$  for any  $\sigma > 0$ . Hence,  $\tilde{k}^* > k^*$  if and only if

$$\begin{aligned} \left[ 1 + \beta^{\frac{1}{\sigma}} (1 + n)^{\frac{1}{\sigma}-1} \right]^{\frac{\sigma}{\sigma+\psi}} \left( \frac{\alpha}{1 + n} \right)^{\phi} &> \left[ 1 + \beta^{\frac{1}{\sigma}} (R^*)^{\frac{1}{\sigma}-1} \right]^{\frac{\sigma}{\sigma+\psi}} \left( \frac{\alpha}{R^*} \right)^{\phi} \\ \Leftrightarrow \left( \frac{R^*}{1 + n} \right)^{\phi} &> \left[ \frac{1 + \beta^{\frac{1}{\sigma}} (R^*)^{\frac{1}{\sigma}-1}}{1 + \beta^{\frac{1}{\sigma}} (1 + n)^{\frac{1}{\sigma}-1}} \right]^{\frac{\sigma}{\sigma+\psi}}. \end{aligned} \quad (2.12)$$

Note that this condition *cannot* be satisfied if  $\sigma \geq 1$ . Since  $R^* < 1 + n$ , we have  $(R^*)^{\frac{1}{\sigma}-1} \geq (1 + n)^{\frac{1}{\sigma}-1}$ , whenever  $\sigma \geq 1$ . Condition (4.2) then implies

$$\left( \frac{R^*}{1 + n} \right)^{\phi} > \left[ \frac{1 + \beta^{\frac{1}{\sigma}} (R^*)^{\frac{1}{\sigma}-1}}{1 + \beta^{\frac{1}{\sigma}} (1 + n)^{\frac{1}{\sigma}-1}} \right]^{\frac{\sigma}{\sigma+\psi}} \geq 1,$$

which contradicts  $R^* < 1 + n$ . Thus, a necessary condition for  $\tilde{k}^* > k^*$  is  $\sigma < 1$ . The intuition underlying this result is straightforward: In the presence of asset bubbles, equilibrium interest rate rises from  $R^*$  to  $\tilde{R}^* = 1 + n$ . Such an increase will create an income effect and an intertemporal substitution effect on the young's consumption. Since consumption and labor supply is inversely related, the income effect will discourage young consumers from working, whereas the intertemporal substitution effect will induce them to work more.<sup>5</sup> Since  $\tilde{k}^* > k^*$  can happen only if  $\tilde{l}^* > l^*$ , it is necessary to have the

intertemporal substitution effect dominates the income effect, i.e.,  $\sigma < 1$ .<sup>6</sup>

<sup>5</sup>The inverse relationship between  $c_{y,t}$  and  $l_t$  can be seen by combining the first-order condition  $w_t c_{y,t}^{-\sigma} = A l_t^{\psi}$  with the expression for the equilibrium wage rate  $w_t = (1 - \alpha) (k_t / l_t)^{\alpha}$ .

<sup>6</sup>In infinite-horizon models, it is typical to assume that  $\sigma$  is greater than or equal to one. However, in overlapping generations model, it is typical to assume that the intertemporal substitution effect is greater than the income effect. Galor and Ryder (1989) shows that this assumption plays an important role in establishing the existence, uniqueness and stability of both stationary and non-stationary equilibria in

We now derive a sufficient condition for  $\tilde{k}^* > k^*$ . Suppose  $R^* < 1 + n$  and  $\sigma < 1$  are satisfied. Using (4.6), we can get

$$1 + \beta^{\frac{1}{\sigma}} (R^*)^{\frac{1}{\sigma}-1} = \frac{(1 - \alpha) \beta^{\frac{1}{\sigma}} (R^*)^{\frac{1}{\sigma}}}{\alpha (1 + n)}.$$

Substituting this into (4.2) and rearranging terms gives

$$\underbrace{\left( \frac{R^*}{1 + n} \right)^{\phi(\sigma + \psi) - 1}}_{\Upsilon} > (1 + n)^{1 - \sigma} \left\{ \frac{(1 - \alpha) \beta^{\frac{1}{\sigma}}}{\alpha \left[ 1 + \beta^{\frac{1}{\sigma}} (1 + n)^{\frac{1}{\sigma}-1} \right]} \right\}^{\sigma}, \quad (2.13)$$

where  $\phi(\sigma + \psi) - 1 = \frac{\psi + \alpha}{1 - \alpha} - (1 - \sigma)$ . Note that the parameter  $\psi$  does not affect the value of  $R^*$  nor the expression on the right-hand side of (4.13). Since  $R^* < 1 + n$ , lowering the value of  $\psi$  will raise the value of  $\Upsilon$ . Thus, holding other parameters constant,  $\tilde{k}^* > k^*$  is more likely to occur when the value of  $\psi$  is low (i.e., close to zero). A low value of  $\psi$  means that the Frisch elasticity of labor supply is large. This, together with  $\sigma < 1$ , ensures that young consumers will significantly increase their labor supply when interest rate rises. A low value of  $\psi$  is not uncommon in macroeconomic studies. In the extreme case when  $\psi = 0$ , the preferences in (4.1) become quasi-linear in labor. Hansen (1985) shows that this type of utility function can arise in a model with indivisible labor. Quasi-linear utility function is now commonly used in business cycle models and monetary-search models.

The main results of this paper are summarized in Proposition 3.<sup>7</sup>

**Proposition 2.3.3.** (i) Suppose  $R^* < 1 + n$ . Then a necessary condition for  $\tilde{k}^* > k^*$  is  $\sigma < 1$ . (ii) Suppose  $R^* < 1 + n$  and  $\sigma < 1$  are satisfied. Then  $\tilde{k}^* > k^*$  if (4.13) is satisfied.

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an overlapping generations model with exogenous labor supply. Nourry (2001) uses similar conditions to examine the local stability of stationary equilibria in a model with endogenous labor supply. In a well-known study on stochastic bubbles, Weil (1987) focuses on the case when the interest elasticity of savings is positive. Under a constant-relative-risk-aversion utility function, this assumption is equivalent to  $\sigma < 1$ . There is also some empirical support for  $\sigma < 1$ . See, for instance, the results in Table III and Table IV of Gourinchas and Parker (2002).

<sup>7</sup>Following Tirole (1985) and Weil (1987), we state our main results in terms of  $R^*$ , which is an endogenous variable. In the next subsection, we provide a set of parameter values under which the conditions in Proposition 3 are satisfied.

### 2.3.3 Numerical Example

We now provide a specific numerical example to illustrate the results in Proposition 3. Suppose one model period takes 30 years. Set the annual subjective discount rate to 0.9950 and the annual employment growth rate to 1.6%.<sup>8</sup> Then we have  $\beta = (0.9950)^{30} = 0.8604$  and  $n = (1.0160)^{30} - 1 = 0.6099$ . We also set  $\alpha = 0.30$  and  $\psi = 0$ . The value of  $A$  is calibrated so that  $l^*$  is about one-third. Under this calibration procedure,  $\tilde{k}^*$  is greater than  $k^*$  for any  $\sigma \in [0, 0.16]$ . In Table 1, we report the results obtained under  $\sigma = 0.15$  and  $A = 0.5862$ . Under these parameter values, the bubbly steady state has a higher level of employment, per-worker capital and per-worker output than the bubbleless steady state.<sup>9</sup>

Table 1: Numerical Results

	Bubbleless Steady State	Bubbly Steady State
$R$	1.2416	1.6099
$k$	0.0438	0.0461
$a$	0	0.0721
$l$	0.3333	0.5084
$w$	0.3808	0.3407
$y$	0.1813	0.2474

Note: The notation  $y$  denotes per-worker capital, i.e.,  $y = k^\alpha l^{1-\alpha}$ .

<sup>8</sup>The latter coincides with the average annual growth rate of employed workers (over age 16) in the United States over the period 1953-2008.

<sup>9</sup>Similar results can be obtained for other values of  $\{\alpha, \beta, n\}$  and some non-zero values of  $\psi$ . In general, one can extend the range of  $\sigma$  under which  $\tilde{k}^* > k^*$  by either raising the value of  $\beta$  or lowering the value of  $\alpha$ . On the other hand, changing the value of  $A$  has no effect on the relative magnitude between  $\tilde{k}^*$  and  $k^*$ .

## 2.4 Conclusions

In this paper, we show that a simple modification of the Tirole (1985) model can lead to a drastically different conclusion. Specifically, we show that when labor supply is elastic, deterministic rational bubbles can induce an expansion in aggregate economic activities under certain conditions. In the present study, specific forms of utility function and production function have been used . This allows us to deliver our main results in a clear and concise manner. One direction for future research is to extend our results to general utility functions and production technologies. Another possibility is to extend the model to allow for financial market frictions and agency costs as in Azariadis and Chakraborty (1998).

## Chapter 3

### The Macroeconomic Consequences of Asset Bubbles and Crashes

#### 3.1 Introduction

In this paper, we present a stylized model of asset bubbles and crashes, and analyze the effects of these phenomena on the macroeconomy. The model is an extended version of the stochastic bubble model in Weil (1987) that takes into account the effects of asset bubbles on labor supply decisions. Using this model, we demonstrate how labor market responses to asset price fluctuations can help propagate the effects of bubbles and crashes to the aggregate economy.

Since the seminal work of Tirole (1985), it has been known that asset price bubbles — defined as substantial positive deviations of an asset’s market price from its fundamental value — can emerge and grow indefinitely in an overlapping generations (OLG) economy. Weil (1987) generalizes the main results in this study to an environment in which asset bubbles may randomly crash in any period. These studies provide an important conceptual framework for understanding the effects of bubbles and crashes, based on rational



expectations and general equilibrium analysis. There are, however, two features of these models that are at odd with empirical evidence. First, both Tirole (1985) and Weil (1987) assume that labor supply is exogenously given. Thus, the implicit assumption is that labor market variables, such as total employment and aggregate labor hours, are unrelated to and unaffected by fluctuations in asset prices. This assumption is at odd with the observation that total employment and aggregate labor hours tend to move closely with asset prices in the actual data. In particular, the bursting of asset bubbles is often followed by a noticeable decline in these labor market variables (see Section 2 for details). Second, both studies suggest that the formation of asset bubbles will crowd out investment in physical capital and impede economic growth, while the bursting of these bubbles will have the opposite effects. These predictions are also difficult to reconcile with empirical evidence. For instance, private nonresidential fixed investment in the U.S. has increased significantly during the formation of the internet bubble in the 1990s and the formation of the housing bubble in the 2000s, and has dropped markedly when these bubbles burst. Chirinko and Schaller (2001, 2011) and Gan (2007) provide formal empirical evidence showing that asset bubbles have positive effects on private investment in the U.S. and Japan. Martin and Ventura (2012) also observe that asset bubbles in these countries are often associated with robust economic growth.

In a previous study (Shi and Suen, 2014), we show that these conflicts between theory and evidence can potentially be resolved by relaxing the assumption of exogenous labor supply. More specifically, we show that when labor supply is endogenously determined in Tirole's (1985) model, asset bubbles can potentially lead to an expansion in steady-state capital, investment, employment and output. This happens when the inverse of the intertemporal elasticity of substitution (IES) for consumption is small and the Frisch

elasticity of labor supply is large, so that individual labor supply will respond strongly and positively to changes in interest rate. This result highlights the importance of labor supply decisions in analyzing the effects of asset bubbles. This study, however, does not take into account one salient feature of asset bubbles, namely that they will crash at some point but the timing of this cannot be predicted with certainty. Allowing for bubble crashes is important for the issue at hand because, as history attests, these incidents can often lead to great disturbances in the aggregate economy. Motivated by this, the present study extends the analysis in Shi and Suen (2014) to the case of stochastic bubbles and explores the circumstances under which our model can account for the empirical evidence mentioned above.

Similar to our prior work, we consider a two-period OLG model in which consumers can choose how much time to work, and how much to save and consume in their first period of life. There are two types of assets in this economy: physical capital and an intrinsically worthless asset. The latter is similar in nature to fiat money and unbacked government debt. Asset bubble is said to occur when this type of asset is traded across generations at a positive price. The main point of departure from our previous study is the assumption that asset bubbles may randomly crash as in the model of Weil (1987).<sup>1</sup> A crash in this context refers to the situation in which the price of the intrinsically worthless asset falls abruptly and unexpectedly to its fundamental value which is zero. The prospect of this happening means that investment in asset bubbles is subject to considerable risks. A key question is whether this type of risk will spawn uncertainty at the aggregate level. We show that the answer to this question depends crucially on the endogeneity of labor supply. To see this, suppose an asset bubble exists in the current period and it will either

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<sup>1</sup>This type of stochastic bubble is also considered in Caballero and Krishnamurthy (2006), Farhi and Tirole (2012, Section 4.2) and Ventura (2012, Section 3.3).

survive or crash in the next period. Whether this type of uncertainty will affect the aggregate economy depends on the effects of asset bubbles on the inputs of production. Since the next-period stock of capital is determined by the savings in the current period, it is unaffected by the future state of the bubble. If labor supply is exogenous as in Weil's (1987) model, then both capital and labor inputs (as well as their marginal products and aggregate output) are independent of the state of the bubble. Thus, the bursting of asset bubble will have *no immediate impact* on aggregate quantities and factor prices, and the risky investment in asset bubbles will not generate aggregate uncertainty.<sup>2</sup> This implication of Weil's model is no longer valid once we allow for an endogenous labor supply. In this case, individual labor hours will in general depend on the state of the asset bubble. As a result, the uncertain prospect of the bubble will create uncertainty in future labor inputs and future prices, which will in turn affect consumers' choices in the current period. This provides a simple and intuitive mechanism through which bubbles and crashes can affect the wider economy. The present study provides the first attempt to analyze this mechanism in a rational bubble model. The main results of this paper are largely in line with those obtained from our previous work. Specifically, we show that the existence of stochastic bubbles can potentially crowd in productive investment, but this happens only if the bubbles can induce a significant expansion in labor supply. Again this scenario is likely to occur when the inverse of the IES for consumption is small and the Frisch elasticity of labor supply is large.

Several recent studies have explored other channels through which asset bubbles can crowd in productive investment and foster economic growth using OLG models. For instance, Martin and Ventura (2012) and Ventura (2012) present models in which asset

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<sup>2</sup>In the present study, the factor markets are assumed to be competitive so that factor prices (i.e., the rental price of capital and wage rate) are determined by the marginal products of capital and labor.

bubbles can improve investment efficiency by shifting resources from less productive firms or countries to more productive ones. Caballero and Krishnamurthy (2006) and Farhi and Tirole (2012) develop models in which asset bubbles can facilitate investment by providing liquidity to financially constrained firms. These existing studies, however, choose to adopt some strongly simplifying assumptions on consumer preferences which thwart both the intertemporal substitution in consumption and the intratemporal substitution between consumption and labor.<sup>3</sup> The present study complements the existing literature by showing that these forces are important for understanding the macroeconomic impact of bubbles and crashes.

The rest of this paper is organized as follows. Section 2 provides evidence showing that total employment, aggregate labor hours and private investment tend to move closely with asset prices during episodes of asset bubbles. Section 3 describes the structure of the model. Section 4 defines the equilibrium concepts and investigates the main properties of the model. Section 5 concludes.

### **3.2 Recent Cases of Asset Bubbles in the U.S.**

In this section, we use the two most recent episodes of asset bubbles in the United States as examples to show that total employment, aggregate labor hours and private investment tend to move closely with asset prices during the course of these episodes. The first case that we consider is the “internet bubble” or “dot-com bubble” which formed during the second half of the 1990s. The second one is the housing price bubble which formed during the first half of the 2000s. Figure 1 shows the monthly data of the Dow

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<sup>3</sup>In addition to an exogenous labor supply, these studies also assume that consumers (or investors) are risk neutral and only care about their consumption at the old age. Thus, the consumers will save all their income when young which is completely determined by the wage rate.

Jones Industrial Average and the Standard & Poor's 500 index between January 1995 and December 2003. Unless otherwise stated, all the data reported in this section were obtained from the Federal Reserve Economic Data (FRED) website. Both the Dow Jones index and the S&P 500 have tripled between January 1995 and January 2000, and have dropped significantly afterward. Ofek and Richardson (2002) and LeRoy (2004) provide detailed account on why the surge in stock prices between 1995 and 2000 cannot be explained by the growth in fundamentals (e.g., corporate earnings and dividends), and thus suggest the existence of an asset bubble. Figure 2 shows the monthly data of the Case-Shiller 20-City Home Price Index between June 2003 and June 2010. From June 2003 to June 2006, this index has increased by 46 percent. According to Shiller (2007) and other subsequent studies, this surge in home prices represents a substantial deviation from the fundamentals (e.g., rent and construction costs) and is thus generally regarded as a bubble.

The next three diagrams show the relationship between stock prices, employment and private nonresidential fixed investment during the internet bubble episode. Figure 3 shows the monthly data of total employment between January 1995 and December 2003, and compares it to the Dow Jones index. Total employment refers to the total number of employees in all private industries in the Current Employment Statistics (CES) data. Figure 4 shows the monthly data of the aggregate weekly hours index in the CES data over the same time period.<sup>4</sup> These two diagrams show that total employment and aggregate labor hours have moved closely with stock prices during the internet bubble episode. Between January 1995 and January 2000, both total employment and aggregate

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<sup>4</sup>The scale of these diagrams has been adjusted so as to highlight the timing of the rise and fall of these variables. This is necessary because otherwise the threefold increase in the Dow Jones index will dwarf the changes of employment in these diagrams.

labor hours have increased by 13 percent, which is equivalent to an average annual growth rate of 2.6 percent. This is significantly higher than the average annual growth rate of total employment between 1948 and 2013, which was 1.3 percent. The average annual growth rate of the aggregate hours index between 1964 and 2013 was 1.5 percent.<sup>5</sup>

Figures 3 and 4 also show a noticeable decline in aggregate labor input after the bursting of the internet bubble. Figure 5 shows the quarterly data of private nonresidential fixed investment (deflated by the GDP deflator) between 1995Q1 and 2003Q4. These data were obtained from the National Income and Product Accounts. Between 1995Q1 and 2000Q1, real nonresidential investment has increased by 41 percent which is equivalent to an average annual growth rate of 7.1 percent. As a point of reference, the average annual growth rate of the same variable between 1948 and 2012 was 3.5 percent.

Next, we turn to the relationship between home prices, employment and private nonresidential fixed investment during the housing price bubble episode. Figures 6 and 7 show the monthly data of total employment and aggregate labor hours between June 2003 and June 2010, and compare them to the Case-Shiller index. Between June 2003 and June 2006, total employment has increased by 5.3 percent while aggregate labor hours have increased by 7 percent. These are equivalent to an average annual growth rate of 1.7 percent and 2.4 percent, respectively, which are again higher than their long-term averages. Figure 8 shows the Case-Shiller index and private nonresidential fixed investment during the period 2003Q3 to 2010Q3. The starting value of these time series have been normalized to one so that the two are directly comparable. Between 2003Q3 and 2006Q3, real nonresidential investment has increased by 18 percent, which is equivalent to an average

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<sup>5</sup>Data on this index are only available from January 1964 onward.

annual growth rate of 5.6 percent. This is again significantly higher than the average annual growth rate between 1948 and 2012.

To summarize, total employment and aggregate labor hours (and also private investment) have moved closely with asset prices during the two most recent cases of asset bubbles in the United States. This provides a direct justification for endogenizing labor supply in the rational bubble model.

### 3.3 The Model

#### 3.3.1 The Environment

Consider an economy inhabited by an infinite sequence of overlapping generations. In each period  $t \in \{0, 1, 2, \dots\}$ , a new generation of identical consumers is born. The size of generation  $t$  is given by  $N_t = (1 + n)^t$ , with  $n > 0$ . Each consumer lives two periods, which we will refer to as the young age and the old age. In each period, each consumer has one unit of time which can be allocated between work and leisure. Retirement is mandatory in the old age, so the labor supply of old consumers is zero. Young consumers, on the other hand, can choose how much time to work, and how much to save and consume. There is a single commodity in this economy which can be used for consumption and capital accumulation. All prices are expressed in units of this commodity.

Consider a consumer who is born at time  $t \geq 0$ . Let  $c_{y,t}$  and  $c_{o,t+1}$  denote his consumption when young and old, respectively; and let  $l_t$  denote his labor supply when young. The consumer's expected lifetime utility is given by

$$E_t \left[ \frac{c_{y,t}^{1-\sigma}}{1-\sigma} - A \frac{l_t^{1+\psi}}{1+\psi} + \beta \frac{c_{o,t+1}^{1-\sigma}}{1-\sigma} \right], \quad (3.1)$$

where  $\sigma > 0$  is the coefficient of relative risk aversion and the inverse of the IES for consumption,  $\psi \geq 0$  is the inverse of the Frisch elasticity of labor supply,  $\beta \in (0, 1)$  is the subjective discount factor, and  $A$  is a positive constant.<sup>6</sup> The consumer can invest in two types of assets: the first one is physical capital and the second one is an intrinsically worthless asset. The latter is called “intrinsically worthless” because it has no consumption value and it cannot be used for production. The only motivation for holding this asset is to resell it at a higher price in the next period. The total supply of the intrinsically worthless asset is fixed and is denoted by  $M > 0$ .<sup>7</sup>

Let  $\tilde{p}_t \geq 0$  be the price of the intrinsically worthless asset in period  $t$ , which is a random variable. Since the fundamental value of this asset is zero, a strictly positive  $\tilde{p}_t$  signifies an overvaluation in period  $t$ , which we will refer to as an asset bubble. Following Weil (1987), we assume that  $\tilde{p}_t$  can be separated into a purely random component  $\varepsilon_t$  and a purely deterministic component  $p_t$ , so that  $\tilde{p}_t \equiv \varepsilon_t p_t$  for all  $t$ . The random component, or asset price shock, is assumed to follow a Markov chain with two possible states  $\{0, 1\}$ , transition probabilities

$$\Pr \{ \varepsilon_{t+1} = 1 | \varepsilon_t = 1 \} = q \in (0, 1),$$

$$\Pr \{ \varepsilon_{t+1} = 0 | \varepsilon_t = 0 \} = 1,$$

and initial value  $\varepsilon_0 = 1$ . The asset price shock is the only source of uncertainty in this economy. The time path of the deterministic component,  $\{p_t\}_{t=0}^{\infty}$ , is endogenously determined in equilibrium. At the beginning of each period  $t$ , the value of  $\varepsilon_t$  is revealed and publicly observed. Suppose  $\varepsilon_t = 1$  and  $p_t > 0$  so that an asset bubble exists in period  $t$ . Then, with probability  $q$ , the price of the intrinsically worthless asset will remain on the

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<sup>6</sup>If  $A = 0$ , then all consumers will supply one unit of labor inelastically when young. In this case, our model is essentially identical to the production economy in Weil (1987).

<sup>7</sup>At time 0, all assets are owned by a group of “initial-old” consumers. The decision problem of these consumers is trivial and does not play any role in the following analysis.



deterministic time path in period  $t + 1$  (i.e.,  $\tilde{p}_{t+1} = p_{t+1}$ ), and with probability  $(1 - q)$ , it will drop to zero in period  $t + 1$ . One can think of the latter case as the result of a sudden, unanticipated change in market sentiment which triggers a crash in the financial market. The parameter  $q$  can be interpreted as the persistence of asset bubbles.<sup>8</sup> Since the probability of moving from  $\varepsilon_t = 1$  to  $\varepsilon_{t+1} = 0$  is strictly positive in every period  $t$ , every asset bubble is destined to crash in the long run (technically, this means  $\tilde{p}_t$  will converge in probability to zero as  $t$  tends to infinity). The timing of the crash, however, is uncertain. Figure 9 shows the probability tree diagram for the asset price shock. The dark line in the diagram traces the time path of  $\varepsilon_t$  before the crash. We will refer to this as the *pre-crash economy* and the other parts of the diagram as the *post-crash economy*. Once the bubble bursts, the asset price  $\tilde{p}_t$  will remain zero from that point on. Hence, there is no incentive for the consumers to hold the intrinsically worthless asset in the post-crash economy.

### 3.3.2 Consumer's Problem

In this section, we will analyze the consumer's problem before and after the crash. To distinguish between these two scenarios, we use a hat (^) to indicate variables in the post-crash economy. First, consider the case when  $\varepsilon_t = 0$ . A young consumer at time  $t$  now faces a deterministic problem, which is given by

$$\max_{\hat{c}_{y,t}, \hat{s}_t, \hat{l}_t, \hat{c}_{o,t+1}} \left[ \frac{\hat{c}_{y,t}^{1-\sigma}}{1-\sigma} - A \frac{\hat{l}_t^{1+\psi}}{1+\psi} + \beta \frac{\hat{c}_{o,t+1}^{1-\sigma}}{1-\sigma} \right]$$

subject to the budget constraints:

$$\hat{c}_{y,t} + \hat{s}_t = \hat{w}_t \hat{l}_t, \quad \text{and} \quad \hat{c}_{o,t+1} = \hat{R}_{t+1} \hat{s}_t,$$

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<sup>8</sup>The deterministic model considered in Shi and Suen (2014) can be considered as a special case of this model with  $q = 1$ . In this case, an asset bubble will last forever.

where  $\hat{s}_t$  denotes savings in physical capital,  $\hat{w}_t$  is the market wage rate, and  $\hat{R}_{t+1}$  is the gross return from physical capital between time  $t$  and  $t + 1$ . The solution of this problem is characterized by

$$\hat{c}_{y,t} = \left( \beta \hat{R}_{t+1} \right)^{-\frac{1}{\sigma}} \hat{c}_{o,t+1} = \frac{\hat{w}_t \hat{l}_t}{1 + \beta^{\frac{1}{\sigma}} \left( \hat{R}_{t+1} \right)^{\frac{1}{\sigma}-1}}, \quad (3.2)$$

$$\hat{l}_t = A^{-\frac{1}{\sigma+\psi}} \left[ 1 + \beta^{\frac{1}{\sigma}} \left( \hat{R}_{t+1} \right)^{\frac{1}{\sigma}-1} \right]^{\frac{\sigma}{\sigma+\psi}} \hat{w}_t^{\frac{1-\sigma}{\sigma+\psi}}, \quad (3.3)$$

$$\hat{s}_t = \Sigma \left( \hat{R}_{t+1} \right) \hat{w}_t \hat{l}_t, \quad \text{where } \Sigma \left( \hat{R}_{t+1} \right) \equiv \frac{\beta^{\frac{1}{\sigma}} \left( \hat{R}_{t+1} \right)^{\frac{1}{\sigma}-1}}{1 + \beta^{\frac{1}{\sigma}} \left( \hat{R}_{t+1} \right)^{\frac{1}{\sigma}-1}}. \quad (3.4)$$

The function  $\Sigma : \mathbb{R}_+ \rightarrow [0, 1]$  defined in (4.5) summarizes the effects of interest rate on savings. First, a higher interest rate means that with the same amount of savings in the young age, there will be more interest income when old. This creates an income effect which encourages consumption when young and discourages saving. Second, an increase in interest rate also lowers the price of future consumption relative to current consumption. This creates an intertemporal substitution effect which discourages consumption when young and promotes saving. The relative strength of these two effects is determined by the value of  $\sigma$ . In particular, the intertemporal substitution effect dominates when  $\sigma < 1$ . In this case,  $\Sigma(\cdot)$  is a strictly increasing function. When  $\sigma > 1$ , the income effect dominates so that  $\Sigma(\cdot)$  is strictly decreasing. The two effects exactly cancel out when  $\sigma = 1$ . In this case,  $\Sigma(\cdot)$  is a positive constant which means the consumer will save (and consume) a constant fraction of his labor income when young.

Next, consider the case when  $\varepsilon_t = 1$ . Let  $m_t$  be the consumer's demand for the intrinsically worthless asset at time  $t$ . The consumer now faces the following budget constraint in the young age

$$c_{y,t} + s_t + p_t m_t = w_t l_t. \quad (3.5)$$

The gross return from physical capital between time  $t$  and  $t + 1$  is now a random variable, which means its value depends on the realization of  $\varepsilon_{t+1}$  (except under some special cases which we will discuss below). Let  $R_{t+1}$  be the value when  $\varepsilon_{t+1} = 1$ , and  $\widehat{R}_{t+1}$  be the value when  $\varepsilon_{t+1} = 0$ . The consumer's old-age consumption is now given by

$$c_{o,t+1} = \begin{cases} R_{t+1}s_t + p_{t+1}m_t & \text{with probability } q, \\ \widehat{R}_{t+1}s_t & \text{with probability } 1 - q. \end{cases} \quad (3.6)$$

Taking  $\{w_t, p_t, p_{t+1}, R_{t+1}, \widehat{R}_{t+1}\}$  as given, the consumer's problem is to choose an allocation  $\{c_{y,t}, s_t, l_t, m_t, c_{o,t+1}\}$  so as to maximize his expected lifetime utility in (4.1), subject to the budget constraints in (4.6) and (4.7), and the non-negativity constraint:  $m_t \geq 0$ .<sup>9</sup> The first-order conditions regarding  $s_t$  and  $l_t$  are given by

$$c_{y,t}^{-\sigma} = \beta \left[ q R_{t+1} (R_{t+1}s_t + p_{t+1}m_t)^{-\sigma} + (1 - q) \widehat{R}_{t+1} (\widehat{R}_{t+1}s_t)^{-\sigma} \right], \quad (3.7)$$

$$w_t c_{y,t}^{-\sigma} = A l_t^\psi. \quad (3.8)$$

Equation (4.9) is the standard Euler equation for consumption in the presence of aggregate uncertainty. Equation (4.10) is the optimality condition for labor supply. Conditional on  $\varepsilon_t = 1$ , the optimal choice of  $m_t$  is determined by

$$p_t c_{y,t}^{-\sigma} \geq \beta E_t [\widetilde{p}_{t+1} (c_{o,t+1})^{-\sigma}] = \beta q p_{t+1} (R_{t+1}s_t + p_{t+1}m_t)^{-\sigma}, \quad (3.9)$$

with equality holds in the first part if  $m_t > 0$ . This equation states that if the marginal cost of holding the intrinsically worthless asset (which is  $p_t c_{y,t}^{-\sigma}$ ) is greater than the marginal benefit of doing so (which is  $\beta E_t [\widetilde{p}_{t+1} (c_{o,t+1})^{-\sigma}]$ ), then the consumer will choose to have

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<sup>9</sup>Given a constant-relative-risk-aversion (CRRA) utility function, it is never optimal for the consumer to choose  $c_{y,t} = 0$  or  $c_{o,t+1} = 0$ , regardless of the existence of asset bubble. Hence, the non-negativity constraint for these variables is never binding. It is also never optimal to have  $s_t \leq 0$  and  $l_t = 0$ . Suppose the contrary that  $s_t \leq 0$ , then the consumer will end up having  $c_{o,t+1} \leq 0$  when  $\varepsilon_{t+1} = 0$ , which cannot be optimal. This, together with  $m_t \geq 0$ , means that consumers will never borrow. Finally, since labor income is the only source of income during the consumer's lifetime, it is never optimal to choose  $l_t = 0$ .

$m_t = 0$ . Equation (3.9) can be rewritten as

$$\Rightarrow p_t \geq E_t \left[ \beta \left( \frac{c_{o,t+1}}{c_{y,t}} \right)^{-\sigma} \tilde{p}_{t+1} \right],$$

which is the standard consumption-based asset pricing equation.

We now explore the conditions under which the optimal choice of  $m_t$  is strictly positive. Consider a young consumer who initially chooses  $m_t = 0$ . Suppose now he is considering increasing it to  $\xi/p_t > 0$ , where  $\xi > 0$  is infinitesimal. In order to balance his budget, the consumer will simultaneously reduce  $s_t$  by  $\xi$ . Define  $\pi_{t+1} \equiv p_{t+1}/p_t$  which is the gross return from the intrinsically worthless asset conditional on  $\varepsilon_{t+1} = 1$ . Increasing  $m_t$  from zero to  $\xi/p_t$  will generate an expected return of  $q\pi_{t+1}\xi$ , which will in turn increase expected future utility by  $q\pi_{t+1} (R_{t+1}s_t)^{-\sigma} \xi$ . At the same time, the reduction in  $s_t$  will lower expected future utility by

$$\left[ qR_{t+1} (R_{t+1}s_t)^{-\sigma} + (1-q) \hat{R}_{t+1} (\hat{R}_{t+1}s_t)^{-\sigma} \right] \xi. \quad (3.10)$$

Such an increase in  $m_t$  is desirable if and only if the marginal benefit of doing so outweighs the marginal cost, i.e.,

$$q\pi_{t+1} (R_{t+1}s_t)^{-\sigma} \xi > \left[ qR_{t+1} (R_{t+1}s_t)^{-\sigma} + (1-q) \hat{R}_{t+1} (\hat{R}_{t+1}s_t)^{-\sigma} \right] \xi.$$

This can be simplified to

$$q\pi_{t+1} > \left[ q + (1-q) \left( \frac{\hat{R}_{t+1}}{R_{t+1}} \right)^{1-\sigma} \right] R_{t+1}. \quad (3.11)$$

This means the consumer is willing to hold the intrinsically worthless asset if and only if the expected return  $q\pi_{t+1}$  exceeds a certain threshold. The threshold level is determined by three factors: (i) the persistence of asset bubble  $q$ ; (ii) the state-dependent returns from physical capital  $R_{t+1}$  and  $\hat{R}_{t+1}$ ; and (iii) the preference parameter  $\sigma$ . If the gross

return from physical capital is *not* state-dependent, i.e.,  $R_{t+1} = \widehat{R}_{t+1}$ , then the condition in (4.12) can be simplified to  $q\pi_{t+1} > R_{t+1}$ . If the utility function for consumption is logarithmic, i.e.,  $\sigma = 1$ , then the expression in (3.10) can be simplified to  $s_t^{-1}\xi$ . In this case, both the marginal benefit and the marginal cost of increasing  $m_t$  are independent of  $\widehat{R}_{t+1}$ , and the condition in (4.12) can again be simplified to become  $q\pi_{t+1} > R_{t+1}$ .

Suppose the condition in (4.12) is valid. Then the optimal investment in the intrinsically worthless asset, denoted by  $a_t \equiv p_t m_t$ , is given by

$$a_t \equiv p_t m_t = \frac{p_t}{p_{t+1}} \left( \Omega_{t+1} \widehat{R}_{t+1} - R_{t+1} \right) s_t, \quad (3.12)$$

where

$$\Omega_{t+1} \equiv \left[ \frac{q(\pi_{t+1} - R_{t+1})}{(1-q)\widehat{R}_{t+1}} \right]^{\frac{1}{\sigma}}.$$

It is straightforward to show that  $\Omega_{t+1} \widehat{R}_{t+1} > R_{t+1}$  is equivalent to (4.12). Further details of the consumer's problem in the pre-crash economy can be found in Appendix A.

### 3.3.3 Production

On the supply side of the economy, there are a large number of identical firms. In each period, each firm hires labor and physical capital from the competitive factor markets, and produces output according to a Cobb-Douglas production function

$$Y_t = K_t^\alpha L_t^{1-\alpha}, \quad \text{with } \alpha \in (0, 1),$$

where  $Y_t$  denotes output produced at time  $t$ ,  $K_t$  and  $L_t$  denote capital input and labor input, respectively. Since the production function exhibits constant returns to scale, we can focus on the problem faced by a single price-taking firm. We assume that physical capital is fully depreciated after one period, so that  $R_t$  coincides with the rental price of

physical capital at time  $t \geq 0$ . The representative firm's problem is given by

$$\max_{K_t, L_t} \{K_t^\alpha L_t^{1-\alpha} - R_t K_t - w_t L_t\},$$

and the first-order conditions are

$$R_t = \alpha K_t^{\alpha-1} L_t^{1-\alpha} \quad \text{and} \quad w_t = (1 - \alpha) K_t^\alpha L_t^{-\alpha}. \quad (3.13)$$

Note that neither the production function nor the representative firm's problem is directly affected by the asset price shock, so the above equations are valid both before and after the asset bubble crashes.<sup>10</sup>

### 3.4 Equilibria

In this section, we will define and characterize an equilibrium in which the intrinsically worthless asset is valued at some point in time, i.e.,  $\tilde{p}_t > 0$  for some  $t$ . We will refer to this as a *bubbly equilibrium*. Such an equilibrium will have to take into account the stochastic timing of the crash, and specify the conditions under which the economy is in equilibrium both before and after the crash. One crucial element of a bubbly equilibrium is the interactions between the pre-crash and the post-crash economies. First, given the chronological order of events, the equilibrium outcomes in the pre-crash economy will determine the initial state (more specifically, the initial value of physical capital) of the post-crash economy. Second, when consumers are making their decisions before the crash, say at some time  $t$ , the anticipated value of  $\hat{R}_{t+1}$  will have to be consistent with an equilibrium in the post-crash economy at time  $t + 1$ . In other words, the equilibrium quantities and prices in the post-crash economy will also affect the equilibrium outcomes prior the crash.<sup>11</sup>

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<sup>10</sup>In the post-crash economy, all the variables in the above equations will be decorated with a hat.

<sup>11</sup>For reasons that we will discuss below, the second type of interaction is not present in Weil's (1987) model.

### 3.4.1 Bubbleless Equilibrium

Suppose the crash happens at time  $T > 0$ , i.e.,  $\varepsilon_{T-1} = 1$  and  $\varepsilon_T = 0$ . Then the economy is free of asset bubbles from time  $T$  onward. Given an initial value  $\hat{K}_T > 0$ , a post-crash bubbleless equilibrium consists of sequences of allocation  $\{\hat{c}_{y,t}, \hat{s}_t, \hat{l}_t, \hat{c}_{o,t}\}_{t=T}^{\infty}$ , aggregate inputs  $\{\hat{K}_t, \hat{L}_t\}_{t=T}^{\infty}$ , and prices  $\{\hat{w}_t, \hat{R}_t\}_{t=T}^{\infty}$  such that for all  $t \geq T$ , (i) the allocation  $\{\hat{c}_{y,t}, \hat{s}_t, \hat{l}_t, \hat{c}_{o,t+1}\}$  solves the consumer's problem at time  $t$  given  $\hat{w}_t$  and  $\hat{R}_{t+1}$ ; (ii) the consumption of old consumers at time  $T$  is determined by

$$N_{T-1}\hat{c}_{o,T} = \hat{R}_T\hat{K}_T;$$

(iii) the aggregate inputs  $\{\hat{K}_t, \hat{L}_t\}$  solve the representative firm's problem at time  $t$  given  $\hat{w}_t$  and  $\hat{R}_t$ ; and (iv) all markets clear at time  $t$ , i.e.,  $\hat{L}_t = N_t\hat{l}_t$  and  $\hat{K}_{t+1} = N_t\hat{s}_t$ .

Define  $\hat{k}_t \equiv \hat{K}_t/N_t$ . Then the equilibrium dynamics of  $\hat{k}_t$  and  $\hat{R}_t$  are determined by<sup>12</sup>

$$\hat{k}_{t+1} = \frac{1-\alpha}{\alpha(1+n)} \left[ \frac{\beta^{\frac{1}{\sigma}} \left( \hat{R}_{t+1} \right)^{\frac{1}{\sigma}-1}}{1 + \beta^{\frac{1}{\sigma}} \left( \hat{R}_{t+1} \right)^{\frac{1}{\sigma}-1}} \right] \hat{R}_t \hat{k}_t, \quad (3.14)$$

$$\hat{R}_t^\eta \hat{k}_t = \alpha^\eta \left[ \frac{(1-\alpha)^{1-\sigma}}{A} \right]^{\frac{1}{\sigma+\psi}} \left[ 1 + \beta^{\frac{1}{\sigma}} \left( \hat{R}_{t+1} \right)^{\frac{1}{\sigma}-1} \right]^{\frac{\sigma}{\sigma+\psi}}, \quad (3.15)$$

where  $\eta \equiv \frac{1}{1-\alpha} + \frac{\alpha}{1-\alpha} \frac{1-\sigma}{\sigma+\psi} > 0$ . The initial value  $\hat{k}_T = \hat{K}_T/N_T$  is given. Once the equilibrium time path of  $\hat{k}_t$  and  $\hat{R}_t$  are known, all other variables in the bubbleless equilibrium can be uniquely determined.

For any  $\sigma > 0$ , the dynamical system in (4.13)-(4.14) has a unique steady state, which we will call a *bubbleless steady state*. This result is formally stated in Proposition 4.2.1. All proofs can be found in Appendix B.

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<sup>12</sup>The derivation of these equations can be found in Appendix A.

**Proposition 3.4.1.** A unique bubbleless steady state exists for any  $\sigma > 0$ . The steady-state values  $(\widehat{R}^*, \widehat{k}^*)$  are determined by

$$\frac{\beta^{\frac{1}{\sigma}} (\widehat{R}^*)^{\frac{1}{\sigma}}}{1 + \beta^{\frac{1}{\sigma}} (\widehat{R}^*)^{\frac{1}{\sigma}-1}} = \frac{(1+n)\alpha}{1-\alpha}, \quad (3.16)$$

$$\widehat{k}^* = (1-\alpha)^{\frac{1-\sigma}{\sigma+\psi}} A^{-\frac{1}{\sigma+\psi}} \left[ 1 + \beta^{\frac{1}{\sigma}} (\widehat{R}^*)^{\frac{1}{\sigma}-1} \right]^{\frac{\sigma}{\sigma+\psi}} \left( \frac{\alpha}{\widehat{R}^*} \right)^{\eta}. \quad (3.17)$$

Next, we consider the stability property of the bubbleless steady state. This type of property is crucial in determining the uniqueness of non-stationary bubbleless equilibrium. When the utility function for consumption is logarithmic, i.e.,  $\sigma = 1$ , the dynamical system in (4.13)-(4.14) is independent of  $\widehat{R}_{t+1}$ . In this case, (4.13) can be simplified to become  $\widehat{k}_{t+1} = B\widehat{k}_t^{\alpha}$ , where  $B$  is a positive constant, and the unique bubbleless steady state is globally stable. When  $\sigma < 1$ , the bubbleless steady state can be shown to be globally saddle-path stable. In both cases, any non-stationary bubbleless equilibrium that originates from a given initial value  $\widehat{k}_T > 0$  must be unique and converges to the bubbleless steady state. In addition, if the post-crash economy begins with an initial value  $\widehat{k}_T$  that is greater than the steady-state value  $\widehat{k}^*$ , then  $\widehat{k}_t$  will decline monotonically during the transition and  $\widehat{R}_t$  will rise monotonically towards  $\widehat{R}^*$ . In other words,  $\widehat{R}_t$  and  $\widehat{k}_t$  will always move in opposite directions on the saddle path. These results are summarized in Proposition 4.2.2.

**Proposition 3.4.2.** Suppose  $\sigma \leq 1$ . Then any non-stationary bubbleless equilibrium that originates from a given initial value  $\widehat{k}_T > 0$  must be unique and converges monotonically to the bubbleless steady state. In particular, the value of  $\widehat{R}_T$  is uniquely determined by  $\widehat{R}_T = \Phi(\widehat{k}_T)$ , where  $\Phi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is a strictly decreasing function. In the transitional



dynamics,  $\hat{R}_t$  and  $\hat{k}_t$  will move in opposite directions so that  $(\hat{k}_t - \hat{k}^*)(\hat{R}_t - \hat{R}^*) \leq 0$  for all  $t \geq T$ .

When  $\sigma > 1$ , the bubbleless steady state can be either a sink or a saddle (see Appendix A for more details). If it is a sink, then there exist multiple sets of equilibrium time paths that originate from the same initial value  $\hat{k}_T > 0$  and converge to the bubbleless steady state. In other words, local indeterminacy may occur when  $\sigma > 1$ . In this study, we confine our attention to bubbleless equilibria that are determinate. In particular, we focus on the case when  $\sigma \leq 1$ , which means the intertemporal substitution effect of a higher interest rate is no weaker than the income effect. This assumption is not uncommon in OLG models. For instance, Galor and Ryder (1989) show that this assumption plays an important role in establishing the existence, uniqueness and global stability of stationary equilibrium in a model with exogenous labor supply. Fuster (1999) uses this assumption to establish the existence and uniqueness of non-stationary equilibrium in a model with uncertain lifetime and accidental bequest. More recently, Andersen and Bhattacharya (2013) adopt the same assumption to analyze the welfare implications of unfunded pensions in a model with endogenous labor supply. In the rational bubble literature, Weil (1987, Section 2) focuses on equilibria in which the interest elasticity of savings is non-negative. Under a constant-relative-risk-aversion utility function, this assumption holds if and only if  $\sigma \leq 1$ . Other studies allow the per-period utility function to be different across age, and assume that the coefficient of relative risk aversion is no greater than one in the old age. For instance, Azariadis and Smith (1993) adopt this assumption to study the general equilibrium implications of credit rationing in a model with adverse selection. Morand and Reffett (2007) and Hillebrand (2014) use this assumption to establish the uniqueness of Markov equilibrium in a model with productivity shocks.

### 3.4.2 Bubbly Equilibrium

We now provide the formal definition of a bubbly equilibrium. Given the initial values  $K_0 > 0$  and  $\varepsilon_0 = 1$ , a bubbly equilibrium consists of two sets of sequences  $\{c_{y,t}, c_{o,t}, l_t, s_t, m_t, R_t, w_t, p_t, K_t, L_t\}_{t=0}^\infty$  and  $\{\hat{c}_{y,t}, \hat{c}_{o,t}, \hat{l}_t, \hat{s}_t, \hat{R}_t, \hat{w}_t, \hat{K}_t, \hat{L}_t\}_{t=0}^\infty$  that satisfy the following conditions in every period  $t \geq 0$ .

1. If  $\varepsilon_t = 0$ , then  $\{\hat{c}_{y,\tau}, \hat{c}_{o,\tau}, \hat{l}_\tau, \hat{s}_\tau, \hat{R}_\tau, \hat{w}_\tau, \hat{K}_\tau, \hat{L}_\tau\}_{\tau=t}^\infty$  constitutes a non-stationary bubbleless equilibrium with initial condition  $\hat{K}_t$ .
2. If  $\varepsilon_t = 1$ , then
  - (i) given  $\{w_t, p_t, p_{t+1}, R_{t+1}, \hat{R}_{t+1}\}$ , the allocation  $\{c_{y,t}, c_{o,t+1}, l_t, s_t, m_t\}$  solves the consumer's problem at time  $t$ , i.e., (4.6)-(3.9) are satisfied;
  - (ii) given  $R_t$  and  $w_t$ , the aggregate inputs  $K_t$  and  $L_t$  solve the firm's problem at time  $t$ , i.e., (4.2) is satisfied;
  - (iii) all markets clear at time  $t$ , i.e.,  $L_t = N_t l_t$ ,  $K_{t+1} = N_t s_t$  and  $N_t m_t = M$ ;
  - (iv) if  $\varepsilon_{t+1} = 0$ , then  $\hat{K}_{t+1} = K_{t+1}$ .

The last condition states that if the asset bubble crashes at time  $t + 1$ , then  $K_{t+1}$  will provide the initial condition for the ensuing bubbleless equilibrium.

Regardless of the existence of asset bubbles, the labor market clears when the total supply of labor by young consumers equals the total demand by firms (i.e.,  $\hat{L}_t = N_t \hat{l}_t$  when  $\varepsilon_t = 0$ , and  $L_t = N_t l_t$  when  $\varepsilon_t = 1$ ); and the market for physical capital clears when the productive savings made by young consumers equal the stock of aggregate capital in the next period (i.e.,  $\hat{K}_{t+1} = N_t \hat{s}_t$  when  $\varepsilon_t = 0$ , and  $K_{t+1} = N_t s_t$  when  $\varepsilon_t = 1$ ). Note that, regardless of the state of the asset bubble, the stock of capital at time  $t + 1$  is

predetermined at time  $t$ , and is thus independent of  $\varepsilon_{t+1}$ . This brings us back to one of the major differences between the present study and Weil (1987) that we have mentioned in the introduction. In the production economy of Weil (1987), every young consumer provides one unit of labor inelastically regardless of the existence of asset bubble. Thus, the equilibrium quantity of labor input at time  $t + 1$  is always determined by  $N_{t+1}$ , i.e.,  $L_{t+1} = \widehat{L}_{t+1} = N_{t+1}$ . Suppose the asset bubble crashes at time  $t + 1$ . Since neither  $K_{t+1}$  nor  $L_{t+1}$  depends on  $\varepsilon_{t+1}$ , *the crash will have no effect on aggregate output and factor prices at time  $t + 1$* . Thus, in Weil's (1987) model, the gross return from physical capital is not contingent on the realization of the asset price shock, i.e.,  $R_{t+1} = \widehat{R}_{t+1}$  for all  $t$ . When labor supply is endogenous, the equilibrium quantity of  $L_{t+1}$  will also depend on individual's choice of  $l_{t+1}$ . If this choice is contingent on the realization of  $\varepsilon_{t+1}$ , then this will open up a channel through which the asset price shock can affect the aggregate economy. Our next result shows that this channel is operative only if  $\sigma \neq 1$ .

**Proposition 3.4.3.** Suppose the utility function for consumption is logarithmic, i.e.,  $\sigma = 1$ . Then the optimal labor supply is constant over time and is identical before and after the crash. Specifically,

$$l_t = \widehat{l}_t = \left( \frac{1 + \beta}{A} \right)^{\frac{1}{1+\psi}}, \quad \text{for all } t \geq 0.$$

This result can be explained as follows: Regardless of the existence of asset bubble, the optimal choice of  $l_t$  is determined by (4.10). The expression  $w_t c_{y,t}^{-\sigma}$  on the left captures both the income and substitution effects of a higher wage rate on labor supply. Holding  $c_{y,t}$  constant, an increase in  $w_t$  raises the opportunity cost of leisure. This creates a substitution effect which discourages leisure and promotes labor supply. On the other hand, an increase in  $w_t$  also generates an income effect which promotes consumption and

discourages labor supply. These two effects exactly offset each other when  $\sigma = 1$ . This happens because in this case, the consumers will save (and consume) a constant fraction of their labor income in the young age. Consequently, the expression  $w_t c_{y,t}^{-1}$  in (4.10) is independent of  $w_t$ , which means individual labor supply is not affected by changes in wage rate. Thus, when  $\sigma = 1$ , our model is essentially identical to the production economy in Weil (1987).

When  $\sigma < 1$ , the optimal choice of  $l_t$  will not be a constant in general, and it will depend on the realization of the asset price shock. The rest of this paper is devoted to analyzing the effects of bubbles and crashes under this value of  $\sigma$ . To simplify the analysis, suppose the economy is in a *conditional bubbly steady state* before the crash happens. Formally, a conditional bubbly steady state is a set of stationary values  $\mathcal{S} \equiv \{c_y^*, c_o^*, l^*, s^*, a^*, R^*, \hat{R}_0^*, w^*, \pi^*, k^*\}$  such that conditional on  $\varepsilon_t = 1$ , we have  $p_{t+1}/p_t = \pi^*$ ,  $K_t = N_t k^*$ ,  $L_t = N_t l^*$ ,  $p_t m_t = a^* > 0$ , and  $(c_{y,t}, c_{o,t}, s_t, l_t, R_t, w_t) = (c_y^*, c_o^*, s^*, l^*, R^*, w^*)$  in a bubbly equilibrium.<sup>13</sup> The main ideas behind this definition are as follows:

Before the crash happens, the consumers face a stationary environment in which (i) the probability of having a crash in the next period is constant over time; (ii) the market wage rate ( $w^*$ ) and the expected return from the bubbly asset ( $q\pi^*$ ) are identical in every period; and (iii) the state-contingent returns for physical capital are also identical in every period (specifically the return is  $R^*$  if the asset bubble persists in the next period and  $\hat{R}_0^*$  otherwise). Thus, the consumers will make the same choices in every period before the crash happens. In particular, they will invest an amount  $a^* > 0$  in the asset bubble in the conditional steady state. Once the asset bubble crashes, the economy will follow the transition paths described in Proposition 4.2.2 and converge to the bubbleless steady

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<sup>13</sup>The concept of “conditional steady state” is not new in macroeconomics. For instance, Cole and Rogerson (1999) and Galor and Weil (2000) have defined a similar notion in different contexts.

state  $(\hat{R}^*, \hat{k}^*)$ . Note that, regardless of the timing of the crash, the dynamical system in (4.13)-(4.14) will always begin with the same initial values:  $k^*$  and  $\hat{R}_0^* \equiv \Phi(k^*)$ .<sup>14</sup>

We now summarize some of the main properties of a conditional bubbly steady state. Conditional on  $\varepsilon_t = 1$ , the market for the intrinsically worthless asset clears when  $N_t m_t = M$ . Using this and the stationary conditions  $p_{t+1}/p_t = \pi^*$  and  $p_t m_t = p_{t+1} m_{t+1} = a^*$ , we can obtain

$$\frac{p_{t+1}}{p_t} = \pi^* = \frac{m_t}{m_{t+1}} = \frac{N_{t+1}}{N_t} = 1 + n.$$

Thus, before the crash happens, the price of the intrinsically worthless asset is growing deterministically at rate  $n$ . Given  $\hat{R}_0^* > 0$ , the steady-state values  $\{R^*, w^*, l^*, k^*, a^*\}$  are uniquely determined by<sup>15</sup>

$$1 + \left[ 1 + (\beta q)^{-\frac{1}{\sigma}} (1 + n)^{1 - \frac{1}{\sigma}} \right] \left( \frac{q}{1 - q} \right)^{\frac{1}{\sigma}} \left( \frac{\hat{R}_0^*}{1 + n} \right)^{1 - \frac{1}{\sigma}} \left( 1 - \frac{R^*}{1 + n} \right)^{\frac{1}{\sigma}} = \frac{1}{\alpha} \frac{R^*}{1 + n}, \quad (3.18)$$

$$w^* = (1 - \alpha) \left( \frac{\alpha}{R^*} \right)^{\frac{\alpha}{1 - \alpha}}, \quad (3.19)$$

$$A(l^*)^{\psi + \sigma} = \beta q [(1 + n) w^*]^{1 - \sigma} \left[ \frac{(1 - \alpha) R^*}{\alpha \Omega^* \hat{R}_0^*} \right]^{\sigma}, \quad (3.20)$$

$$k^* = l^* \left( \frac{\alpha}{R^*} \right)^{\frac{1}{1 - \alpha}}, \quad (3.21)$$

$$a^* = \left( \Omega^* \hat{R}_0^* - R^* \right) k^*. \quad (3.22)$$

Once these values are known, the value of  $\{c_y^*, c_o^*, s^*\}$  can be uniquely determined from the consumer's budget constraints. Equations (3.18)-(4.19) essentially define a one-to-one mapping between  $\hat{R}_0^*$  and  $k^*$ , which we will denote by  $k^* = \Gamma(\hat{R}_0^*)$ . We now have a pair of equations,  $\hat{R}_0^* = \Phi(k^*)$  and  $k^* = \Gamma(\hat{R}_0^*)$ , which can be used to solve for  $k^*$  and  $\hat{R}_0^*$ . The first equation determines the initial value of  $\hat{R}_t$  in the post-crash bubbleless equilibrium.

<sup>14</sup>The variable  $\hat{R}_0^*$  is not to be confused with the bubbleless steady-state value  $\hat{R}^*$  defined in Proposition 4.2.1. In the post-crash economy,  $\hat{R}_0^*$  is the initial value of  $\hat{R}_t$  while  $\hat{R}^*$  is the long-run value.

<sup>15</sup>The derivation of these equations can be found in Appendix A.

The actual form of  $\Phi(\cdot)$  depends on the transitional dynamics in the bubbleless economy. The second equation states that, given  $\widehat{R}_0^*$ ,  $k^* = \Gamma(\widehat{R}_0^*)$  is the value of per-worker capital in the conditional bubbly steady state. The mapping  $\Gamma(\cdot)$  is determined by (3.18)-(4.19). These two equations can be combined to form a one-dimensional fixed point equation  $\widehat{R}_0^* = \Phi \circ \Gamma(\widehat{R}_0^*)$ , which provides the basis for computing the bubbly equilibrium.

Our next proposition states that when  $\sigma < 1$ , the gross return from physical capital in the conditional bubbly steady state ( $R^*$ ) is higher than the one in the bubbleless steady state ( $\widehat{R}^*$ ). This result is due to the combination of two factors. First, since aggregate uncertainty exists before the crash happens, consumers will require a higher return from savings in the conditional bubbly steady state. Second, even without any uncertainty, the existence of asset bubble tends to lower the capital-labor ratio and drives up the steady-state interest rate [see Shi and Suen (2014) Proposition 2].<sup>16</sup>

**Proposition 3.4.4.** Suppose  $\sigma < 1$ . Then the existence of asset bubble is associated with a higher level of steady-state interest rate, i.e.,  $R^* > \widehat{R}^*$ .

Our last set of results concerns the expansionary effects of asset bubbles. Specifically, we seek conditions under which the conditional bubbly steady state has more physical capital per worker and a higher labor supply than the bubbleless steady state, i.e.,  $k^* > \widehat{k}^*$  and  $l^* > \widehat{l}^*$ . Note that  $k^* > \widehat{k}^*$  implies that there is more physical capital per worker before the crash than after, i.e.,  $k^* \geq \widehat{k}_t$  for all  $t$ . To see this, suppose the post-crash economy begins at time  $T$  so that  $\widehat{k}_T = k^*$ . As shown in Proposition 4.2.2, if  $\widehat{k}_T = k^* > \widehat{k}^*$ , then  $\widehat{k}_t$  is strictly decreasing along the transition path so that  $\widehat{k}_T = k^* > \widehat{k}_t$  for all  $t > T$ .

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<sup>16</sup>This result is also consistent with the findings in other rational bubble models. For instance, the models of Tirole (1985), Weil (1987), Olivier (2000), and Farhi and Tirole (2012) all predict that the long-run interest rate is higher in the presence of asset bubble.

Using (4.19), which is valid both before and after the crash, we can obtain

$$k^* = l^* \left( \frac{\alpha}{R^*} \right)^{\frac{1}{1-\alpha}} > \widehat{l}^* \left( \frac{\alpha}{\widehat{R}^*} \right)^{\frac{1}{1-\alpha}} = \widehat{k}^* \quad \Leftrightarrow \quad \frac{l^*}{\widehat{l}^*} > \left( \frac{R^*}{\widehat{R}^*} \right)^{\frac{1}{1-\alpha}} > 1. \quad (3.23)$$

This shows that asset bubbles can potentially crowd in productive investment in the current framework, but this happens only if these bubbles can induce a sufficiently large expansion in labor supply.

Regardless of the existence of asset bubbles, individual labor supply is determined by equation (4.10), which can be rewritten as

$$Al_t^{\psi+\sigma} = w_t^{1-\sigma} \left( \frac{c_{y,t}}{w_t l_t} \right)^{-\sigma}. \quad (3.24)$$

The above equation shows how individual labor supply is determined by the current wage rate and the propensity to consume when young. Holding other things constant, labor supply increases when wage rate increases (as  $\sigma < 1$ ). Since  $R^* > \widehat{R}^*$  implies  $w^* < \widehat{w}^*$ , this effect in itself will *lower* labor supply in the presence of asset bubble. On the other hand, labor supply increases when the consumers allocate a smaller fraction of their labor income to young-age consumption. This captures the intratemporal substitution between consumption and labor. Thus,  $l^* > \widehat{l}^*$  is possible *only if* the consumers have a lower propensity to consume in the conditional bubbly steady state, i.e.,

$$\frac{\widehat{c}_y^*}{\widehat{w}^* \widehat{l}^*} > \frac{c_y^*}{w^* l^*}.$$

In the bubbleless steady state, this propensity is determined by

$$\frac{\widehat{c}_y^*}{\widehat{w}^* \widehat{l}^*} = \left[ 1 + \beta^{\frac{1}{\sigma}} \left( \widehat{R}^* \right)^{\frac{1}{\sigma}-1} \right]^{-1}, \quad (3.25)$$

which is strictly decreasing in the long-run interest rate when  $\sigma < 1$ . A similar expression can be obtained for its counterpart in the conditional bubbly steady state, which is

$$\frac{c_y^*}{w^* l^*} = \left[ 1 + \beta^{\frac{1}{\sigma}} (\rho^*)^{\frac{1}{\sigma}-1} \right]^{-1}, \quad (3.26)$$

where

$$(\rho^*)^{\frac{1}{\sigma}-1} \equiv \frac{[q(1+n)]^{\frac{1}{\sigma}}}{\Omega^* \widehat{R}_0^*} \left[ 1 + \frac{1}{1+n} \left( \Omega^* \widehat{R}_0^* - R^* \right) \right].$$

The variable  $\rho^*$  can be interpreted as the certainty equivalent return from investment in the conditional bubbly steady state. Specifically, this means a consumer in the conditional bubbly steady state will have the same amount of consumption  $(c_y^*, c_o^*)$  and labor supply  $(l^*)$  as a consumer in a deterministic bubbleless steady state where the gross return from savings is  $\rho^*$ . Under the assumption of  $\sigma < 1$ , an increase in interest rate will induce the consumers to save more and consume less when young. Thus, the consumers will have a lower propensity to consume in the conditional bubbly steady state if and only if  $\rho^* > \widehat{R}^*$ . After some manipulations, we can derive the following equivalent condition:

$$\frac{\widehat{c}_y^*}{\widehat{w}^* \widehat{l}^*} > \frac{c_y^*}{w^* l^*} \quad \Leftrightarrow \quad \left[ \frac{q(1+n)}{\widehat{R}^*} \right]^{\frac{1}{\sigma}} > \frac{\Omega^* \widehat{R}_0^*}{R^*} > 1. \quad (3.27)$$

Finally, using (4.17) and (4.21)-(4.23), we can derive a necessary and sufficient condition for  $l^* > \widehat{l}^*$  and one for  $k^* > \widehat{k}^*$ . The results are stated in Proposition 3.4.5.

**Proposition 3.4.5.** Suppose  $\sigma < 1$ . Then  $l^* > \widehat{l}^*$  if and only if

$$\left[ \frac{q(1+n)}{\widehat{R}^*} \right]^{\frac{1}{\sigma}} \left( \frac{R^*}{\widehat{R}^*} \right)^{-\frac{\alpha(1-\sigma)}{(1-\alpha)\sigma}} > \frac{\Omega^* \widehat{R}_0^*}{R^*},$$

and the asset bubble can crowd in productive investment, i.e.,  $k^* > \widehat{k}^*$ , if and only if

$$\left[ \frac{q(1+n)}{\widehat{R}^*} \right]^{\frac{1}{\sigma}} \left( \frac{R^*}{\widehat{R}^*} \right)^{-\left[1 + \frac{\psi+\sigma}{(1-\alpha)\sigma}\right]} > \frac{\Omega^* \widehat{R}_0^*}{R^*}.$$

### 3.4.3 Numerical Examples

We now present a set of numerical examples to illustrate how the key variables in our model respond to an asset bubble crash. Through these examples, we also want to highlight the importance of  $\sigma$  in determining the macroeconomic effects of asset bubbles. We stress



at the outset that these examples are only intended to demonstrate the working of the model and the results in the previous sections. For this reason, some of the parameter values are specifically chosen so that asset bubbles can crowd in productive investment in some cases.

Suppose one model period takes 30 years. Set the annual subjective discount factor to 0.9950 and the annual employment growth rate to 1.6 percent.<sup>17</sup> These values imply  $\beta = (0.9950)^{30} = 0.8604$  and  $n = (1.0160)^{30} - 1 = 0.6099$ . In addition, we set  $q = 0.90$ ,  $\alpha = 0.30$  so that the share of capital income in total output is 30 percent, and  $\psi = 0$  so that the utility function in (4.1) is quasi-linear in labor hours. As shown in Hansen (1985), this type of utility function is consistent with the assumption of indivisible labor. Our choice of  $q$  and  $n$  implies that the expected return from the intrinsically worthless asset is  $q(1+n) = 1.4490$ . To highlight the importance of  $\sigma$ , we consider four different values of this parameter between 0.10 and 0.30. For each value of  $\sigma$ , the parameter  $A$  is chosen so that  $\hat{l}^*$  is 0.50.<sup>18</sup> For each set of parameter values, we solve for the equilibrium time paths under the following scenario: Suppose the economy starts from a conditional bubbly steady state at time  $t = 0$ , and suppose the bubble bursts unexpectedly at time  $t = 3$ .<sup>19</sup> We then solve for the conditional bubbly steady state and the bubbleless steady state, and compute the transition path in the post-crash economy using backward shooting method.

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<sup>17</sup>The latter is consistent with the average annual growth rate of U.S. employment over the period 1953-2008.

<sup>18</sup>Under the assumption of indivisible labor, the variable  $l_t$  is more suitably interpreted as the labor force participation rate at time  $t$ . Thus, we choose a target value of  $\hat{l}^*$  based on the average labor force participation rate in the United States during the postwar period, which is about 0.50.

<sup>19</sup>In other words, we consider a particular sequence of asset price shocks in which  $\varepsilon_t = 1$  for  $t \in \{0, 1, 2\}$  and  $\varepsilon_t = 0$  for  $t \geq 3$ . As explained earlier, the non-stationary bubbleless equilibrium will always begin with the same initial values  $k^*$  and  $\hat{R}_0^*$  regardless of the timing of the crash. Thus, the exact time period when the crash happens is immaterial.

**Table 1**

Conditional Bubbly Steady State vs Bubbleless Steady State

	$\sigma = 0.10$		$\sigma = 0.15$		$\sigma = 0.20$		$\sigma = 0.30$	
Steady State	Bubbleless	Bubbly	Bubbleless	Bubbly	Bubbleless	Bubbly	Bubbleless	Bubbly
$R$	1.2176	1.4671	1.2416	1.4548	1.2637	1.4485	1.3036	1.4434
$\rho$	—	1.4402	—	1.4382	—	1.4381	—	1.4395
$c_y$	0.0832	0.0374	0.0846	0.0538	0.0858	0.0640	0.0878	0.0758
$l$	0.5000	0.7306	0.5000	0.5862	0.5000	0.5416	0.5000	0.5132
$k$	0.0676	0.0757	0.0657	0.0614	0.0641	0.0571	0.0613	0.0544
$y$	0.2743	0.3701	0.2720	0.2980	0.2700	0.2758	0.2664	0.2617
$a$	0	0.0998	0	0.0559	0	0.0371	0	0.0198

Note: The notation  $y$  denotes per-worker output, i.e.,  $y = k^\alpha l^{1-\alpha}$ .

Table 1 shows the key variables in the conditional bubbly steady state and the bubbleless steady state under different values of  $\sigma$ . In the first row, we report the value of  $\hat{R}^*$  and  $R^*$  in each case. In the second row, we report the certainty equivalent return from savings in the conditional bubbly steady state. In all four cases, we have  $\rho^* > \hat{R}^*$  and  $l^* > \hat{l}^*$ . In particular, the gap between  $l^*$  and  $\hat{l}^*$  widens as the value of  $\sigma$  decreases. This captures the effects of a stronger intertemporal substitution effect. When  $\sigma = 0.1$ , the difference between  $l^*$  and  $\hat{l}^*$  is sufficiently large so that asset bubble can crowd in productive investment (i.e.,  $k^* > \hat{k}^*$ ).

Figures 10-12 show the time path of interest rate ( $R$ ), labor supply ( $l$ ) and per-worker capital ( $k$ ) before and after the crash happens at  $t = 3$ . In all four cases, the crash induces

an immediate reduction in interest rate and labor supply. During the transition,  $\widehat{R}_t$  and  $\widehat{k}_t$  move in opposite directions as predicted by Proposition 4.2.2. In the more interesting case where asset bubble crowds in physical capital (i.e.,  $\sigma = 0.1$ ), labor supply and productive investment fall markedly at the time of the crash and continue to decline afterward. These patterns are qualitatively similar to those observed in the United States after the bursting of the internet bubble and the housing price bubble.

### 3.5 Concluding Remarks

The present study joins a growing body of literature that examines the effects of asset price bubbles and crashes on the aggregate economy. We contribute to this literature by demonstrating the importance of intratemporal and intertemporal substitution effects to the issue at hand. In particular, we show that the existence of asset bubbles can crowd in productive investment and induce an expansion in aggregate employment when these effects are sufficiently strong. We remark that the present study is mainly theoretical in nature and more effort is needed in order to generate realistic quantitative results. In particular, expanding the consumer's planning horizon (and thus reducing the length of each model period) is crucial for matching the model to the data. Introducing other model features, such as financial market imperfections and heterogeneity in firm productivity as in Martin and Ventura (2012) and Farhi and Tirole (2012), may also help expand the range of parameter values under which asset bubbles can crowd in productive investment. We leave these intriguing possibilities for future research.

## Appendix A: Mathematical Derivations

### Post-Crash Equilibrium

In this section, we provide a detailed characterization of a post-crash equilibrium. Since the consumer's problem in the post-crash economy is standard, the derivations of (4.3)-(4.5) are omitted. The dynamical system in (4.13)-(4.14) can be derived as follows. In equilibrium, the market wage rate and the gross return from physical capital are determined by  $\hat{w}_t = (1 - \alpha) \hat{K}_t^\alpha \hat{L}_t^{1-\alpha}$  and  $\hat{R}_t = \alpha \hat{K}_t^{\alpha-1} \hat{L}_t^{1-\alpha}$ , respectively. Using these, we can obtain

$$\hat{w}_t \hat{l}_t = \frac{1 - \alpha}{\alpha} \hat{R}_t \hat{k}_t, \quad (3.28)$$

$$\hat{w}_t = (1 - \alpha) \left( \frac{\alpha}{\hat{R}_t} \right)^{\frac{\alpha}{1-\alpha}}, \quad (3.29)$$

$$\hat{l}_t = \left( \frac{\hat{R}_t}{\alpha} \right)^{\frac{1}{1-\alpha}} \hat{k}_t. \quad (3.30)$$

where  $\hat{k}_t \equiv \hat{K}_t/N_t$  and  $\hat{l}_t \equiv \hat{L}_t/N_t$ . Then we can rewrite the capital market clearing condition as

$$(1 + n) \hat{k}_{t+1} = \left[ \frac{\beta^{\frac{1}{\sigma}} \left( \hat{R}_{t+1} \right)^{\frac{1}{\sigma}-1}}{1 + \beta^{\frac{1}{\sigma}} \left( \hat{R}_{t+1} \right)^{\frac{1}{\sigma}-1}} \right] \hat{w}_t \hat{l}_t \equiv \Sigma \left( \hat{R}_{t+1} \right) \hat{w}_t \hat{l}_t.$$

Substituting (A.2) into the above expression gives (4.13). Next, substituting (A.2) and (A.3) into (4.4) gives

$$\begin{aligned} \left( \frac{\hat{R}_t}{\alpha} \right)^{\frac{1}{1-\alpha}} \hat{k}_t &= A^{-\frac{1}{\sigma+\psi}} \left[ 1 + \beta^{\frac{1}{\sigma}} \left( \hat{R}_{t+1} \right)^{\frac{1}{\sigma}-1} \right]^{\frac{\sigma}{\sigma+\psi}} \left[ (1 - \alpha) \left( \frac{\alpha}{\hat{R}_t} \right)^{\frac{\alpha}{1-\alpha}} \right]^{\frac{1-\sigma}{\sigma+\psi}} \\ \Rightarrow \left( \frac{\hat{R}_t}{\alpha} \right)^{\eta} \hat{k}_t &= \left[ \frac{(1 - \alpha)^{1-\sigma}}{A} \right]^{\frac{1}{\sigma+\psi}} \left[ 1 + \beta^{\frac{1}{\sigma}} \left( \hat{R}_{t+1} \right)^{\frac{1}{\sigma}-1} \right]^{\frac{\sigma}{\sigma+\psi}}, \end{aligned} \quad (3.31)$$

where

$$\eta \equiv \frac{1}{1 - \alpha} + \frac{\alpha}{1 - \alpha} \frac{1 - \sigma}{\sigma + \psi} = \frac{\psi + \alpha + \sigma(1 - \alpha)}{(1 - \alpha)(\sigma + \psi)} > 0,$$

$$\eta - 1 = \frac{\alpha}{1 - \alpha} \frac{1 + \psi}{\sigma + \psi} > 0,$$

for any  $\sigma > 0$ . Equation (4.14) can be obtained by rearranging terms in (A.4).

### *Local Analysis*

We now explore the local stability property of the unique bubbleless steady state under different values of  $\sigma$ . To achieve this, we consider a linearized version of the dynamical system in (4.13)-(4.14). First, taking logarithms of both sides of these equations gives

$$\begin{aligned} \ln \hat{k}_{t+1} - \ln \Sigma(\hat{R}_{t+1}) &= \ln \left[ \frac{1 - \alpha}{\alpha(1 + n)} \right] + \ln \hat{R}_t + \ln \hat{k}_t, \\ \ln \left\{ \alpha^\eta \left[ \frac{(1 - \alpha)^{1 - \sigma}}{A} \right]^{\frac{1}{\sigma + \psi}} \right\} + \frac{\sigma}{\sigma + \psi} \ln \left( 1 + \beta^{\frac{1}{\sigma}} \hat{R}_{t+1}^{\frac{1}{\sigma} - 1} \right) &= \eta \ln \hat{R}_t + \ln \hat{k}_t. \end{aligned}$$

Next, taking the first-order Taylor expansion of these equations around  $(\hat{k}^*, \hat{R}^*)$  gives

$$\begin{aligned} \hat{\mathbf{k}}_{t+1} - \frac{\hat{R}^* \Sigma'(\hat{R}^*)}{\Sigma(\hat{R}^*)} \hat{\mathbf{R}}_{t+1} &= \hat{\mathbf{k}}_t + \hat{\mathbf{R}}_t, \\ \frac{1 - \sigma}{\sigma + \psi} \left[ \frac{\beta^{\frac{1}{\sigma}} (\hat{R}^*)^{\frac{1}{\sigma} - 1}}{1 + \beta^{\frac{1}{\sigma}} (\hat{R}^*)^{\frac{1}{\sigma} - 1}} \right] \hat{\mathbf{R}}_{t+1} &= \hat{\mathbf{k}}_t + \eta \hat{\mathbf{R}}_t, \end{aligned}$$

where  $\hat{\mathbf{k}}_t \equiv (\hat{k}_t - \hat{k}^*) / \hat{k}^*$  and  $\hat{\mathbf{R}}_t \equiv (\hat{R}_t - \hat{R}^*) / \hat{R}^*$  represent the percentage deviations of  $\hat{k}_t$  and  $\hat{R}_t$  from their steady-state values. Finally, rewrite the linearized system in matrix

form

$$\underbrace{\begin{bmatrix} 1 & b_{12} \\ 0 & b_{22} \end{bmatrix}}_B \begin{bmatrix} \hat{\mathbf{k}}_{t+1} \\ \hat{\mathbf{R}}_{t+1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & \eta \end{bmatrix} \begin{bmatrix} \hat{\mathbf{k}}_t \\ \hat{\mathbf{R}}_t \end{bmatrix}, \quad (3.32)$$

where

$$b_{12} = -\frac{\hat{R}^* \Sigma'(\hat{R}^*)}{\Sigma(\hat{R}^*)} = \left( 1 - \frac{1}{\sigma} \right) \left[ 1 + \beta^{\frac{1}{\sigma}} (\hat{R}^*)^{\frac{1}{\sigma} - 1} \right]^{-1},$$

$$b_{22} = \frac{1 - \sigma}{\sigma + \psi} \left[ \frac{\beta^{\frac{1}{\sigma}} (\hat{R}^*)^{\frac{1}{\sigma} - 1}}{1 + \beta^{\frac{1}{\sigma}} (\hat{R}^*)^{\frac{1}{\sigma} - 1}} \right].$$

The inverse of the matrix  $B$  is given by

$$B^{-1} = \frac{1}{b_{22}} \begin{bmatrix} b_{22} & -b_{12} \\ 0 & 1 \end{bmatrix}.$$

Using this, we can rewrite (A.5) as

$$\begin{bmatrix} \hat{\mathbf{k}}_{t+1} \\ \hat{\mathbf{R}}_{t+1} \end{bmatrix} = \underbrace{\frac{1}{b_{22}} \begin{bmatrix} b_{22} - b_{12} & b_{22} - \eta b_{12} \\ 1 & \eta \end{bmatrix}}_J \begin{bmatrix} \hat{\mathbf{k}}_t \\ \hat{\mathbf{R}}_t \end{bmatrix}, \quad (3.33)$$

where  $J$  is the Jacobian matrix of the linearized system. Let  $\rho_1$  and  $\rho_2$  be the characteristic roots of the linearized system. These can be obtained by solving

$$\Xi(\rho) \equiv \rho^2 - \left(1 - \frac{b_{12}}{b_{22}} + \frac{\eta}{b_{22}}\right) \rho + \frac{\eta - 1}{b_{22}} = 0.$$

If  $\sigma < 1$ , then we have  $b_{12} < 0$  and  $b_{22} > 0$  which imply

$$\Xi(\rho) > 0, \quad \text{for all } \rho < 0,$$

$$\Xi(0) = \frac{\eta - 1}{b_{22}} > 0, \quad \text{as } \eta > 1,$$

$$\Xi(1) \equiv 1 - \left(1 - \frac{b_{12}}{b_{22}} + \frac{\eta}{b_{22}}\right) + \frac{\eta - 1}{b_{22}} = \frac{b_{12} - 1}{b_{22}} < 0.$$

The last two inequalities ensure that one of the characteristic roots can be found within the interval of  $(0, 1)$ . This rules out the possibility of complex roots. Since  $\Xi(\rho) > 0$  for all  $\rho \leq 0$ , both  $\rho_1$  and  $\rho_2$  must be strictly positive. Finally, if both  $\rho_1$  and  $\rho_2$  are within the interval of  $(0, 1]$ , then we should have  $\Xi(1) \geq 0$  instead. Thus, the second root must be greater than one. This proves that the system in (A.6) is saddle-path stable within the neighborhood of the bubbleless steady state when  $\sigma < 1$ . Proposition 4.2.2 strengthens this result by showing that this steady state is globally saddle-path stable when  $\sigma < 1$ .

If  $\sigma > 1$ , then we have  $b_{12} \in (0, 1)$  and  $b_{22} < 0$  which imply  $\Xi(0) < 0 < \Xi(1)$ . Hence, one of the characteristic roots must lie within the interval of  $(0, 1)$ . Since the product of roots  $\Xi(0)$  is strictly negative, the second characteristic root must be strictly negative. If  $\Xi(-1) > 0$ , then the second root must lie within the interval of  $(-1, 0)$ . In this case, the linearized system has two stable roots which means the bubbleless steady state is a sink. If  $\Xi(-1) < 0$ , then the absolute magnitude of the second root is greater than one. In this case, the bubbleless steady state is again saddle-path stable. The value of  $\Xi(-1)$  is determined by

$$\Xi(-1) = 2 - \underbrace{\frac{b_{12}}{b_{22}}}_{(+)} + \underbrace{\frac{2\eta - 1}{b_{22}}}_{(-)}.$$

Unfortunately, the sign of this expression cannot be readily determined. Hence, the local stability property of the post-crash equilibrium is ambiguous when  $\sigma > 1$ .

### Bubbly Equilibrium

In this section, we will provide a detailed characterization of the consumer's problem in the pre-crash economy, and present the derivation of (3.18)-(4.20). Substituting (4.6) and (4.7) into the consumer's expected lifetime utility gives

$$\mathcal{L} = \frac{(w_t l_t - s_t - p_t m_t)^{1-\sigma}}{1-\sigma} - A \frac{l_t^{1+\psi}}{1+\psi} + \beta \left[ \frac{q (R_{t+1} s_t + p_{t+1} m_t)^{1-\sigma} + (1-q) (\hat{R}_{t+1} s_t)^{1-\sigma}}{1-\sigma} \right].$$

The first-order conditions with respect to  $s_t$ ,  $m_t$  and  $l_t$  are, respectively, given by

$$(w_t l_t - s_t - p_t m_t)^{-\sigma} = \beta \left[ q R_{t+1} (R_{t+1} s_t + p_{t+1} m_t)^{-\sigma} + (1-q) \hat{R}_{t+1} (\hat{R}_{t+1} s_t)^{-\sigma} \right], \quad (3.34)$$

$$(w_t l_t - s_t - p_t m_t)^{-\sigma} = \beta q \left( \frac{p_{t+1}}{p_t} \right) (R_{t+1} s_t + p_{t+1} m_t)^{-\sigma}, \quad (3.35)$$

$$A l_t^\psi = w_t (w_t l_t - s_t - p_t m_t)^{-\sigma}. \quad (3.36)$$

Here we only focus on interior solutions of  $m_t$ . Define  $\pi_{t+1} \equiv p_{t+1}/p_t$ . Combining (A.1) and (A.8) gives

$$\begin{aligned} q\pi_{t+1} (R_{t+1}s_t + p_{t+1}m_t)^{-\sigma} &= qR_{t+1} (R_{t+1}s_t + p_{t+1}m_t)^{-\sigma} + (1-q) \hat{R}_{t+1} \left( \hat{R}_{t+1}s_t \right)^{-\sigma}, \\ \Rightarrow q(\pi_{t+1} - R_{t+1}) (R_{t+1}s_t + p_{t+1}m_t)^{-\sigma} &= (1-q) \hat{R}_{t+1} \left( \hat{R}_{t+1}s_t \right)^{-\sigma}, \\ \Rightarrow R_{t+1}s_t + p_{t+1}m_t &= \underbrace{\left[ \frac{q(\pi_{t+1} - R_{t+1})}{(1-q) \hat{R}_{t+1}} \right]^{\frac{1}{\sigma}}}_{\Omega_{t+1}} \left( \hat{R}_{t+1}s_t \right), \end{aligned} \quad (3.37)$$

$$\begin{aligned} \Rightarrow m_t &= \frac{1}{p_{t+1}} \left( \Omega_{t+1} \hat{R}_{t+1} - R_{t+1} \right) s_t, \\ \Rightarrow s_t + p_t m_t &= \left[ 1 + \frac{R_{t+1}}{\pi_{t+1}} (\Lambda_{t+1} - 1) \right] s_t, \end{aligned} \quad (3.38)$$

where  $\Lambda_{t+1} \equiv \Omega_{t+1} \hat{R}_{t+1} / R_{t+1}$ . Using (A.8) and (A.10), we can get

$$\begin{aligned} R_{t+1}s_t + p_{t+1}m_t &= (\beta q \pi_{t+1})^{\frac{1}{\sigma}} (w_t l_t - s_t - p_t m_t) = \Omega_{t+1} \hat{R}_{t+1} s_t, \\ \Rightarrow s_t &= \left\{ \frac{(\beta q \pi_{t+1})^{\frac{1}{\sigma}}}{\Omega_{t+1} \hat{R}_{t+1} + (\beta q \pi_{t+1})^{\frac{1}{\sigma}} \left[ 1 + \frac{R_{t+1}}{\pi_{t+1}} (\Lambda_{t+1} - 1) \right]} \right\} w_t l_t. \end{aligned} \quad (3.39)$$

Using this and (A.6), we can obtain

$$c_{y,t} = w_t l_t - (s_t + p_t m_t) = \left\{ \frac{\Omega_{t+1} \hat{R}_{t+1}}{\Omega_{t+1} \hat{R}_{t+1} + (\beta q \pi_{t+1})^{\frac{1}{\sigma}} \left[ 1 + \frac{R_{t+1}}{\pi_{t+1}} (\Lambda_{t+1} - 1) \right]} \right\} w_t l_t. \quad (3.40)$$

Substituting this into (A.9) and rearranging terms give

$$A l_t^{\psi+\sigma} = (w_t)^{1-\sigma} \left\{ \frac{\Omega_{t+1} \hat{R}_{t+1} + (\beta q \pi_{t+1})^{\frac{1}{\sigma}} \left[ 1 + \frac{R_{t+1}}{\pi_{t+1}} (\Lambda_{t+1} - 1) \right]}{\Omega_{t+1} \hat{R}_{t+1}} \right\}^{\sigma}. \quad (3.41)$$

These equations characterize the optimal choice of  $c_{y,t}$ ,  $l_t$ ,  $s_t$  and  $m_t$  before the crash.

We now provide the derivation of (3.18)-(4.20). In equilibrium, the market for physical capital clears when

$$(1+n) k_{t+1} = s_t = \left\{ \frac{(\beta q \pi_{t+1})^{\frac{1}{\sigma}}}{\Omega_{t+1} \hat{R}_{t+1} + (\beta q \pi_{t+1})^{\frac{1}{\sigma}} \left[ 1 + \frac{R_{t+1}}{\pi_{t+1}} (\Lambda_{t+1} - 1) \right]} \right\} w_t l_t$$



$$\Rightarrow (1+n) k_{t+1} = \left\{ \frac{(\beta q \pi_{t+1})^{\frac{1}{\sigma}}}{\Omega_{t+1} \widehat{R}_{t+1} + (\beta q \pi_{t+1})^{\frac{1}{\sigma}} \left[ 1 + \frac{R_{t+1}}{\pi_{t+1}} (\Lambda_{t+1} - 1) \right]} \right\} \left( \frac{1-\alpha}{\alpha} \right) R_t k_t. \quad (3.42)$$

The second line uses the fact that  $\alpha w_t l_t = (1-\alpha) R_t k_t$ . Combining (A.7) and (A.9) gives

$$A l_t^{\psi+\sigma} = (w_t)^{1-\sigma} \left\{ \frac{(\beta q \pi_{t+1})^{\frac{1}{\sigma}}}{\Omega_{t+1} \widehat{R}_{t+1}} \left[ \frac{1-\alpha}{\alpha (1+n)} \right] \frac{R_t k_t}{k_{t+1}} \right\}^{\sigma}. \quad (3.43)$$

Upon setting  $k_{t+1} = k_t = k^*$ ,  $R_t = R_{t+1} = R^*$ ,  $\widehat{R}_{t+1} = \widehat{R}_0^*$  and  $\pi_{t+1} = 1+n$ , equation (A.9) becomes

$$1+n = \left\{ \frac{[\beta q (1+n)]^{\frac{1}{\sigma}}}{\Omega^* \widehat{R}_0^* + [\beta q (1+n)]^{\frac{1}{\sigma}} \left[ 1 + \frac{R^*}{1+n} (\Lambda^* - 1) \right]} \right\} \left( \frac{1-\alpha}{\alpha} \right) R^*, \quad (3.44)$$

where  $\Lambda^* = \Omega^* \widehat{R}_0^* / R^*$ . Rearranging terms in this equation gives

$$1 + \left[ 1 + (\beta q)^{-\frac{1}{\sigma}} (1+n)^{1-\frac{1}{\sigma}} \right] \left( \frac{\Omega^* \widehat{R}_0^*}{1+n} \right) = \frac{1}{\alpha} \frac{R^*}{1+n}$$

$$\Rightarrow 1 + \left[ 1 + (\beta q)^{-\frac{1}{\sigma}} (1+n)^{1-\frac{1}{\sigma}} \right] \left( \frac{q}{1-q} \right)^{\frac{1}{\sigma}} \left( \frac{\widehat{R}_0^*}{1+n} \right)^{1-\frac{1}{\sigma}} \left( 1 - \frac{R^*}{1+n} \right)^{\frac{1}{\sigma}} = \frac{1}{\alpha} \frac{R^*}{1+n},$$

which is equation (3.18) in the text. Similarly, after substituting the stationarity conditions into (A.11), we can obtain

$$A(l^*)^{\psi+\sigma} = (w^*)^{1-\sigma} \left\{ \frac{[\beta q (1+n)]^{\frac{1}{\sigma}}}{\Omega^* \widehat{R}_0^*} \left( \frac{1-\alpha}{\alpha} \right) \frac{R^*}{1+n} \right\}^{\sigma}.$$

Equation (4.18) follows immediately from this equation. Equations (4.17) and (4.19) can be obtained from (4.2). Finally, equation (4.20) can be obtained from (4.8).

Define  $\theta^* \equiv R^*/(1+n)$ . Then we can rewrite (3.18) as

$$\Psi(\theta^*) \equiv 1 + \left[ 1 + (\beta q)^{-\frac{1}{\sigma}} (1+n)^{1-\frac{1}{\sigma}} \right] \left( \frac{q}{1-q} \right)^{\frac{1}{\sigma}} \left( \frac{\widehat{R}_0^*}{1+n} \right)^{1-\frac{1}{\sigma}} (1-\theta^*)^{\frac{1}{\sigma}} = \frac{\theta^*}{\alpha}. \quad (3.45)$$

For any  $\widehat{R}_0^* > 0$  and  $\sigma > 0$ ,  $\Psi : [0, 1] \rightarrow \mathbb{R}_+$  is a strictly decreasing function that satisfies  $\Psi(0) > 0$  and  $\Psi(1) = 1 < 1/\alpha$ . Meanwhile, the right-hand side of the above equation is a

straight line that passes through the origin and  $1/\alpha$  (when  $\theta^* = 1$ ). Thus, for any  $\widehat{R}_0^* > 0$  and  $\sigma > 0$ , there exists a unique  $\theta^* \in (0, 1)$  that solves (A.12). Once  $\theta^*$  is determined, the value of  $\{k^*, w^*, l^*, a^*\}$  can be uniquely determined using (4.17)-(4.20).

### *Propensity to Consume When Young*

Using (A.13), we can get

$$\begin{aligned} \frac{c_y^*}{w^* l^*} &= \frac{\Omega^* \widehat{R}_0^*}{\Omega^* \widehat{R}_0^* + \beta^{\frac{1}{\sigma}} [q(1+n)]^{\frac{1}{\sigma}} \left[1 + \frac{R^*}{1+n} (\Lambda^* - 1)\right]} \\ &= \left\{ 1 + \beta^{\frac{1}{\sigma}} \frac{[q(1+n)]^{\frac{1}{\sigma}}}{\Omega^* \widehat{R}_0^*} \left[1 + \frac{R^*}{1+n} (\Lambda^* - 1)\right] \right\}^{-1} \equiv \left[1 + \beta^{\frac{1}{\sigma}} (\rho^*)^{\frac{1}{\sigma}-1}\right]^{-1}, \end{aligned}$$

where  $\rho^*$  is the certainty equivalent return defined in the text. An alternative expression for the propensity to consume can be obtained as follows. First, rewrite the above expression as

$$\frac{c_y^*}{w^* l^*} = \frac{\Omega^* \widehat{R}_0^*}{[\beta q(1+n)]^{\frac{1}{\sigma}}} \left\{ \frac{[\beta q(1+n)]^{\frac{1}{\sigma}}}{\Omega^* \widehat{R}_0^* + [\beta q(1+n)]^{\frac{1}{\sigma}} \left[1 + \frac{R^*}{1+n} (\Lambda^* - 1)\right]} \right\}. \quad (3.46)$$

Using (A.13), we can obtain

$$\frac{[\beta q(1+n)]^{\frac{1}{\sigma}}}{\Omega^* \widehat{R}_0^* + [\beta q(1+n)]^{\frac{1}{\sigma}} \left[1 + \frac{R^*}{1+n} (\Lambda^* - 1)\right]} = \frac{\alpha(1+n)}{1-\alpha} \frac{1}{R^*}.$$

Substituting this into (A.10) gives

$$\frac{c_y^*}{w^* l^*} = \frac{\Omega^* \widehat{R}_0^*}{[\beta q(1+n)]^{\frac{1}{\sigma}}} \left[ \frac{\alpha(1+n)}{1-\alpha} \frac{1}{R^*} \right].$$

On the other hand, in the bubbleless steady state, we have

$$\frac{\widehat{c}_y^*}{\widehat{w}^* \widehat{l}^*} = \left[1 + \beta^{\frac{1}{\sigma}} \left(\widehat{R}^*\right)^{\frac{1}{\sigma}-1}\right]^{-1} = \frac{\alpha(1+n)}{1-\alpha} \left(\beta \widehat{R}^*\right)^{-\frac{1}{\sigma}}.$$

The second equality follows from (4.15). Hence, we have

$$\frac{\widehat{c}_y^*}{\widehat{w}^* \widehat{l}^*} > \frac{c_y^*}{w^* l^*} \quad \Leftrightarrow \quad \left(\widehat{R}^*\right)^{-\frac{1}{\sigma}} > \frac{\Omega^* \widehat{R}_0^*}{[q(1+n)]^{\frac{1}{\sigma}} R^*} \quad \Leftrightarrow \quad \left[\frac{q(1+n)}{\widehat{R}^*}\right]^{\frac{1}{\sigma}} > \frac{\Omega^* \widehat{R}_0^*}{R^*}.$$

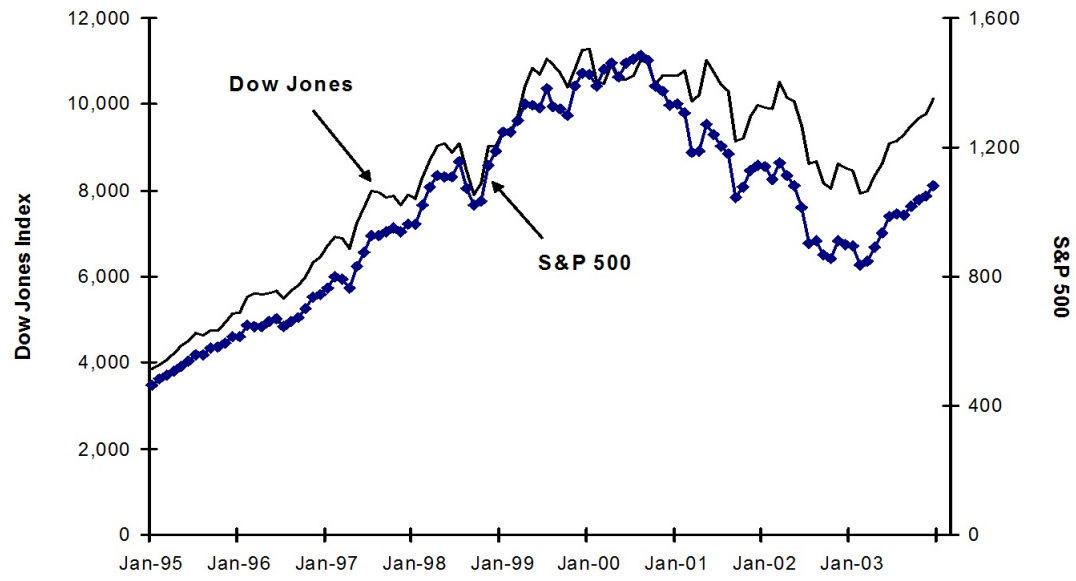


Figure 3.1: Dow Jones Industrial Average and S&P 500, 1995-2003.

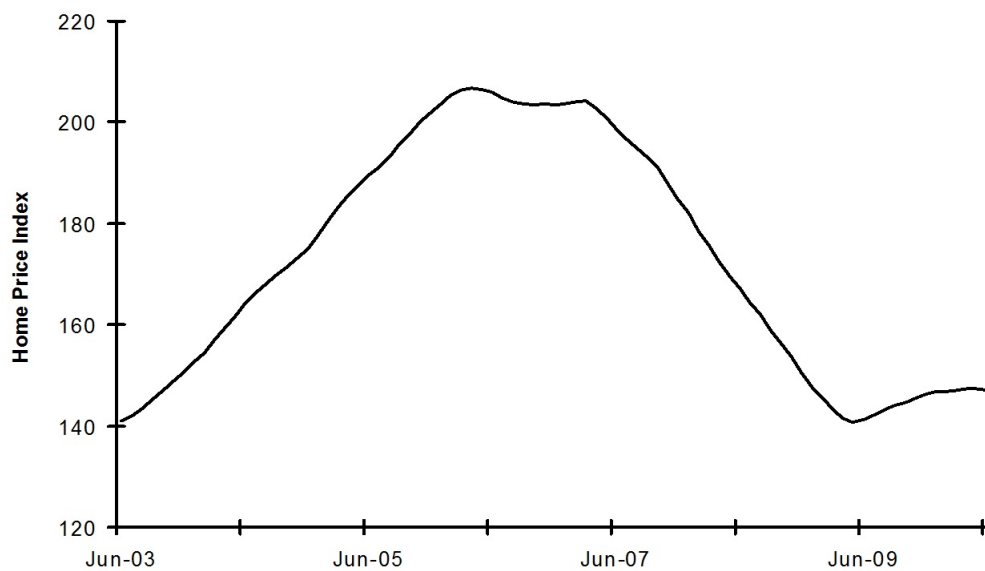


Figure 3.2: Case-Shiller 20-City Home Price Index, June 2003 to June 2010.

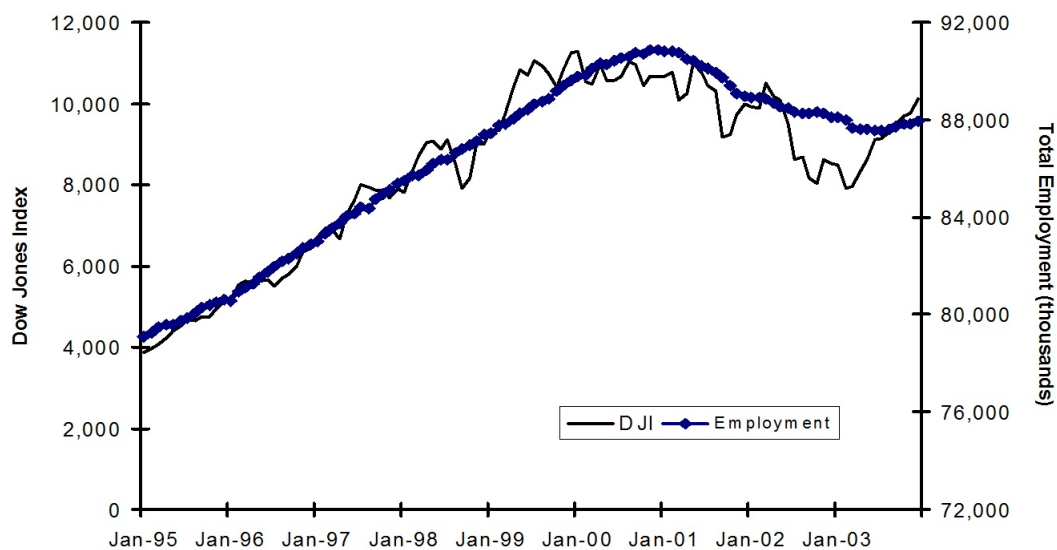


Figure 3.3: Total Employment and Dow Jones Index, 1995-2003.

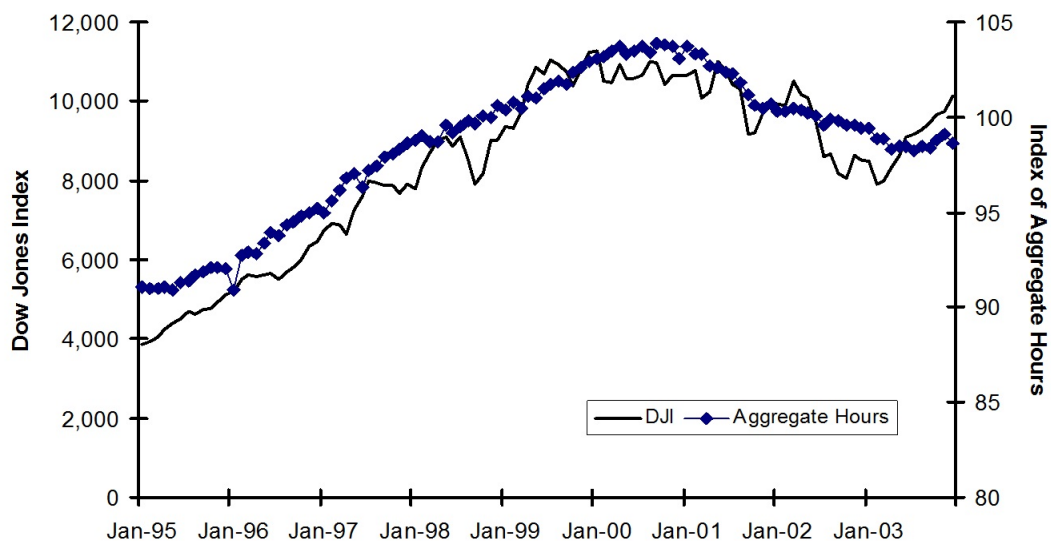


Figure 3.4: Aggregate Hours and Dow Jones Index, 1995-2003.

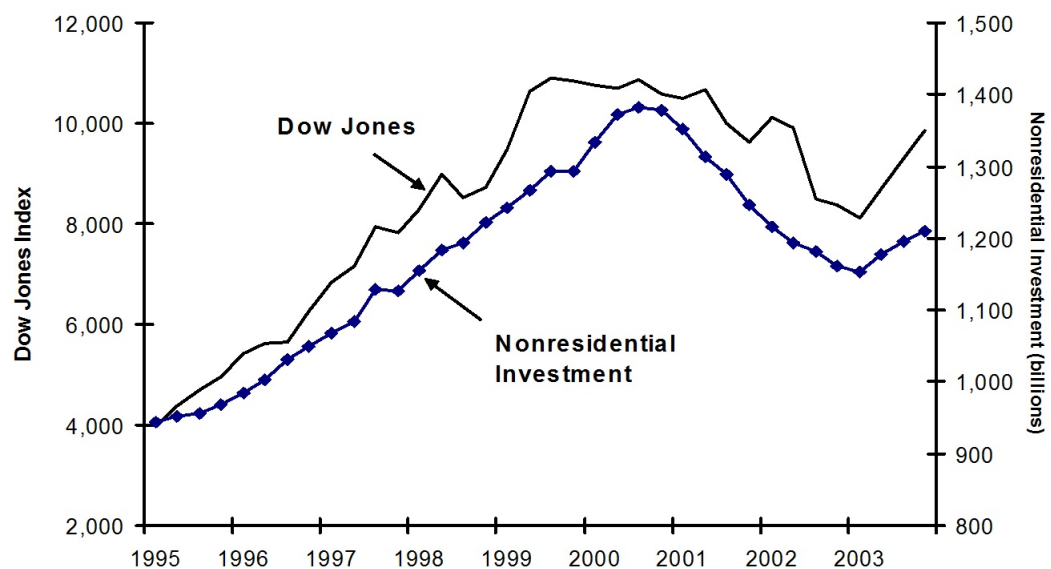


Figure 3.5: Private Nonresidential Fixed Investment and Dow Jones Index, 1995Q1 to 2003Q4.

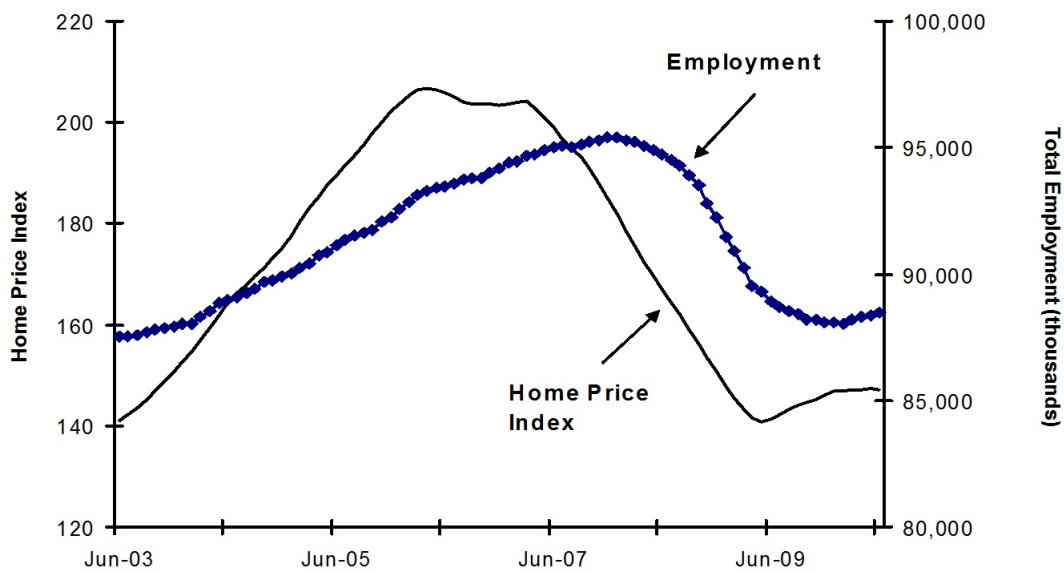


Figure 3.6: Total Employment and Home Price Index, June 2003 to June 2010.

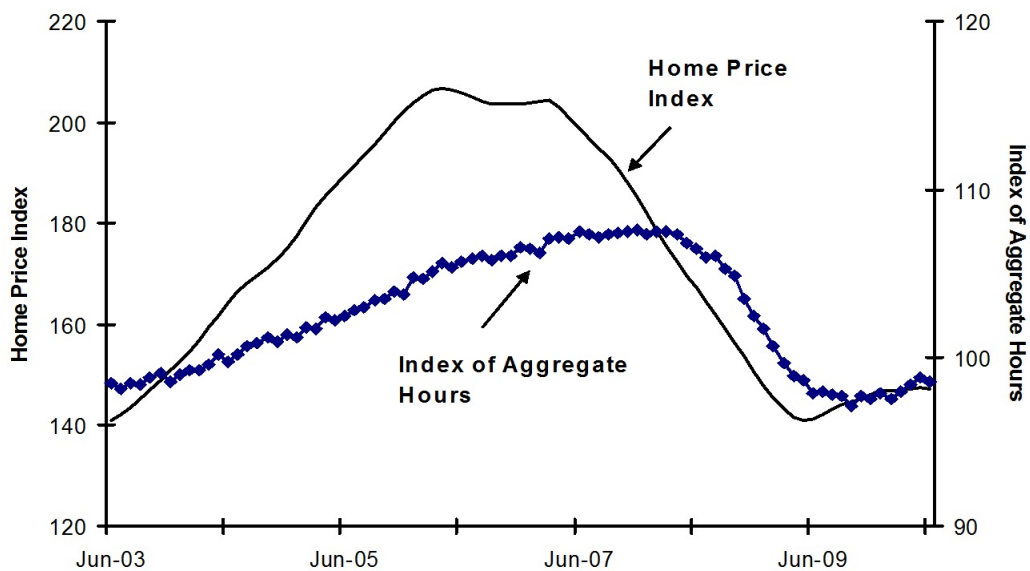


Figure 3.7: Aggregate Hours and Home Price Index, June 2003 to June 2010.

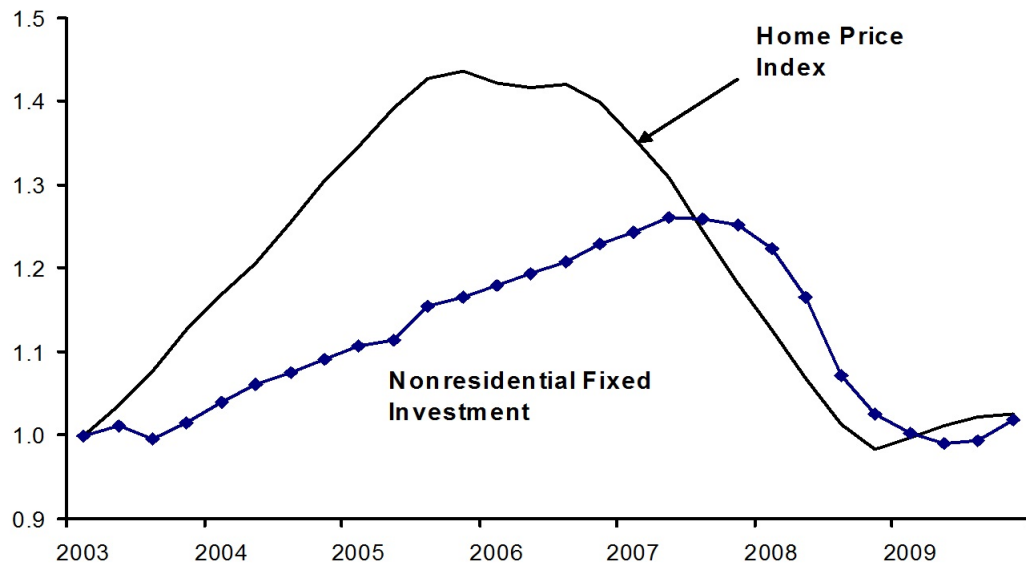


Figure 3.8: Private Nonresidential Fixed Investment and Home Price Index, 2003Q3 to 2010Q3.

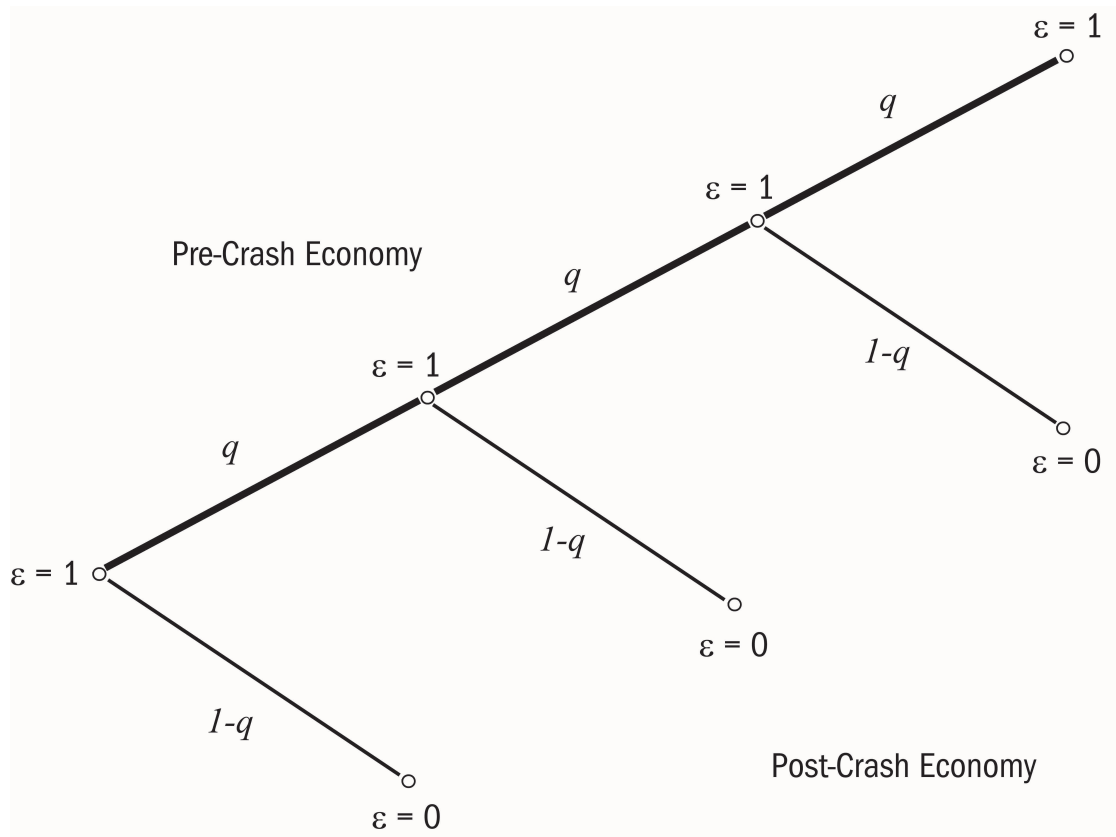


Figure 3.9: Probability Tree Diagram of the Asset Price Shock.

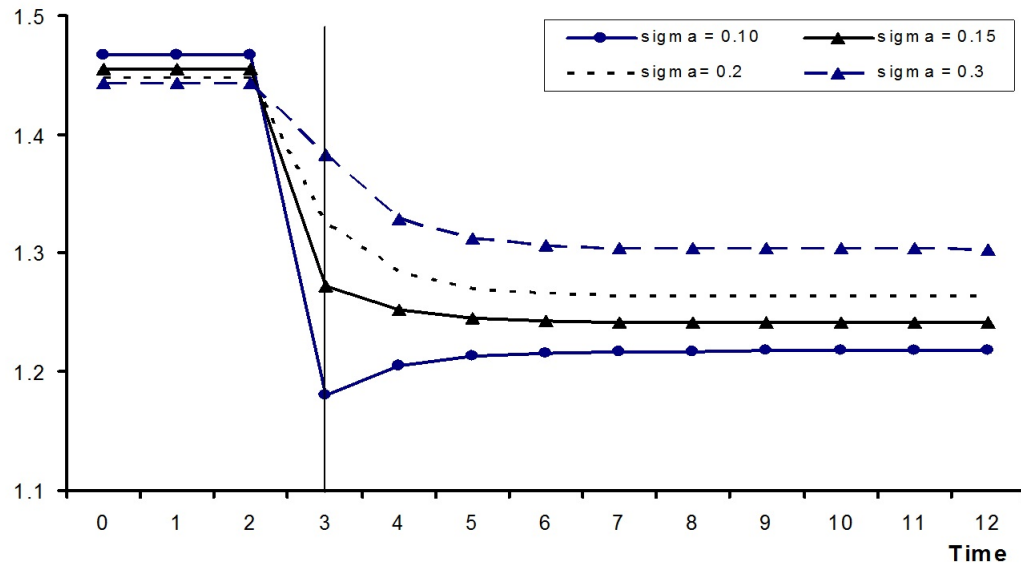


Figure 3.10: Time Paths of Interest Rate under Different Values of  $\sigma$ .

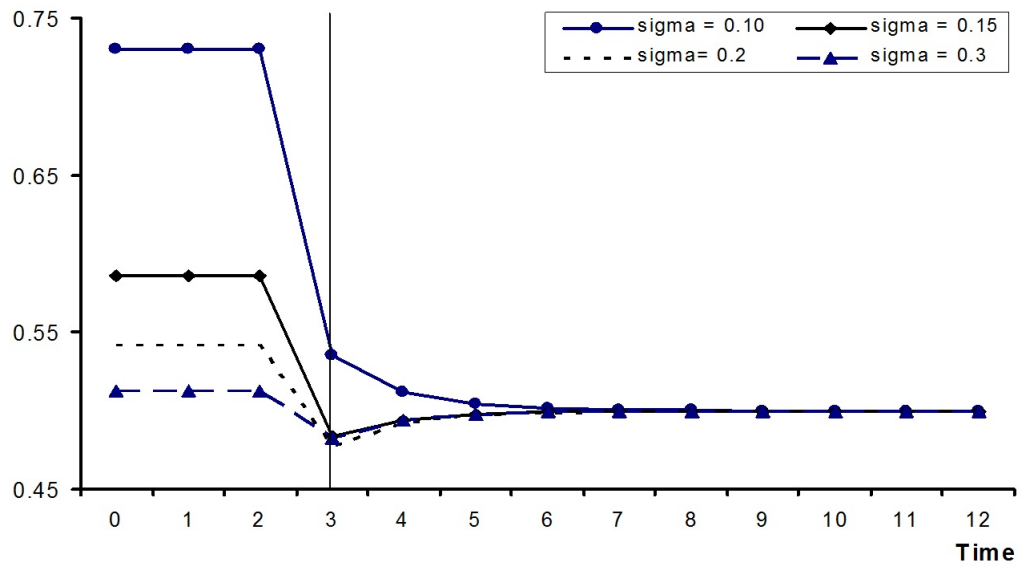


Figure 3.11: Time Paths of Labor Supply under Different Values of  $\sigma$ .



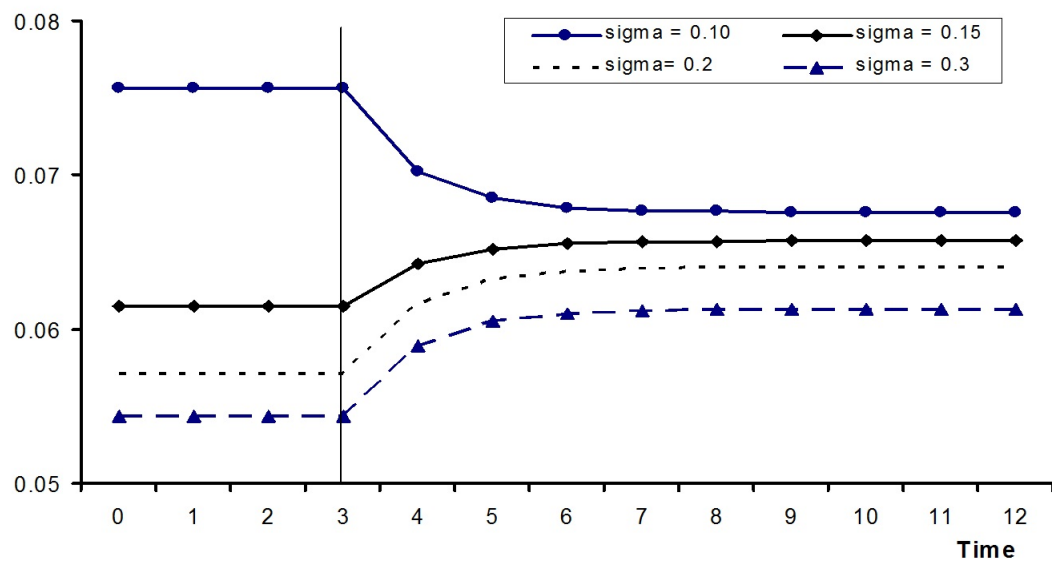


Figure 3.12: Time Paths of Capital under Different Values of  $\sigma$ .

## Chapter 4

### Bank Competition and Capital Accumulation in a Costly State Verification Model

#### 4.1 Introduction

This paper examines theoretically the effects of bank competition on capital accumulation. To achieve this, we develop a dynamic general equilibrium in which financial intermediaries or banks engage in Cournot competition in the loan market and the deposit market, and lending activities take place under asymmetric information and costly monitoring. Within this framework, we provide conditions under which a more competitive banking structure is beneficial to capital accumulation. It has long been recognized that development of financial intermediation can promote economic growth. But many of these arguments are based on a perfectly competitive financial market. To resolve this limit, recent studies have started to examine the macroeconomic effects of the industrial organization of the banking system. But these studies typically focus on two extreme cases: perfectly competitive banking system versus monopoly banking system. Still, not enough

attention has been paid to the banking structure in between, i.e., an oligopoly system. The present study is intended to fill this gap.

There is ample evidence showing that the banking sector in the United States has undergone significant changes and became increasingly concentrated over the past decades. According to Janicki and Prescott (2006), the number of independent banks in the United States has dropped from 13,000 to about 6,500 over the period 1960-2005. This dramatic reduction is largely due to the deregulations that took effect in the 1980s and the 1990s. At the same time, the bank size distribution (measured in terms of bank assets) has become much more concentrated. In 1960, the share of assets held by the ten largest banks was 21%. This increased to around 60% by 2005. Similarly, in the past three decades, the number of banks in operation has reduced substantially in many European countries. In Germany, this figure dropped from 3,717 to 1686, contracting by 55% between the year 1993 to 2012. How will bank's competition in a concentrated market impact the economy in terms of saving and the deposit interest, capital accumulation, and the borrowing lending activities?

This paper attempts to answer these questions in a dynamic general equilibrium framework. Specifically, we employ a variant of Diamond's two-period overlapping generations (OG) model as the analytical vehicle. Investment in capital accumulation can be funded internally or externally via financial intermediation. The borrowing and lending activities are subject to asymmetric information and costly state verification (CSV) problems as in Townsend (1979) and Williamson (1986, 1987). Thus, in our framework, banks not only intermediate the supply and demand of credit, they also serve as monitors of investment activities. We also extend the standard CSV model to allow for the use of collateral and

endogenize the leverage ratio. In the financial markets, banks engage in Cournot competitions. Specifically, they compete both in the deposit market to gather savings and in the loan market to lend to entrepreneurs. Our main findings are largely consistent with the common wisdom. All else being equal, a more competitive financial environment induces higher capital accumulation mainly because the competition encourages more savings as well as borrowing by driving up the deposit rate and depressing the charges on loans. An adverse consequence, however, is that as the entrepreneurs raise their leverage ratio, the agency problem becomes more severe. Banks therefore demand more frequent monitoring and more resources are wasted in the verification process.

In the numerical experiments, we compare the economies with different characteristics. One interesting result is the mixing effect of agents' intertemporal elasticity of substitution (IES). It is shown that IES might have opposite impact on the capital accumulation depending on the deposit rate the economy originally exhibits. A higher IES will increase (decrease) the capital accumulation if the deposit rate is high (low) at first. The main reason lies in the interaction between IES and the deposit rate. In a world with high deposit return, increase in IES tends to discourage saving, while the opposite is true when the deposit return is high. The change in IES also implies different ability for banks to extract profit. Savers with lower IES are less sensitive to deposit rate, which gives banks more power in the deposit market. As a result, we see a larger gap between the loan and deposit interest.

The present study provides the first attempt to connect two strands of literature. In macroeconomics, there is now a large number of studies that explore the interrelationship between financial intermediation and the real economy in the presence of CSV problem. Examples of these studies include Boyd and Smith (1998a, 1998b), Huybens and Smith

(1999), Guzman (2000), Khan (2001), and Paal *et al.* (2005) among many others. These studies, however, either focus on a perfectly competitive banking system or a monopolistic banking system. Few attention has been paid to the oligopolistic banking structure. In the finance literature, many studies have examined the effect of market power in the banking sector using a partial equilibrium framework. In an early study, Petersen and Rajan (1995) examine this issue from the perspective of relationship-based banking and lending. More recently, Allen and Gale (2004), Boyd and De Nicoló (2005) and Boyd *et al.* (2009) have investigated the relation between bank competition and the stability of the banking sector. Hauswald and Marquez (2006) stress bank's role of information collection when studying the consequence of competition on the efficiency credit market. All these studies, however, are abstracted from the aggregate economic effect.

Several attempts have also been conducted to analyze the concentration in the banking industry and the aggregate economic outcome, but with different focuses. For example, Deidda and Fattouh (2005) once assumed that in the process of intermediating credits, banks demands capital and they compete with real sector for the use of it. Thus on one hand, less concentrated banking sector promote specialization and enhances efficiency, on the other hand, it induces duplication of use of capital. A more recent work by Cetorelli and Peretto (2012) pointed out the "free riding" problem when competitive banks offer relationship services to firms in order to reduce default rate of the loans. The relationship services are beneficial to investments but may be depressed by bank competition. Unlike their works, this paper highlights the asymmetric information between lenders and borrowers and study how bank competition influences the borrowing and monitoring activities under the asymmetry, and consequently, the capital production.

The remainder of the paper is arranged as follows: the baseline model is outlined in Section 2, where we consider a banking sector that has market power only in the deposit market. Section 3 extends the analysis to the economy where banks have market power in both the deposit and the loan market. Section 4 presents the numerical results while Section 5 concludes.

## 4.2 The Baseline Model

Time is discrete and is denoted by  $t \in \{0, 1, 2, \dots\}$ . The economy under study is inhabited by an infinite sequence of overlapping generations. Each generation has a continuum of individuals who live two periods. The size of generation  $t$  is given by  $N_t = (1 + n)^t$ , where  $n > 0$  is a constant growth rate. All individuals are endowed with one unit of time in the young age, which they supply inelastically to work. All individuals are retired when old. Within each cohort there are two types of individuals, which we label as depositors and entrepreneurs. The share of entrepreneurs in each cohort is constant over time and is denoted by  $\alpha \in (0, 1)$ .

Depositors and entrepreneurs differ in two regards: First, depositors have standard concave preferences for consumption in both periods of life, whereas entrepreneurs are risk neutral and only care about old-age consumption. Second, depositors can only save by depositing funds in the banks, whereas entrepreneurs can choose either to save in the banks or invest in risky investment projects. The exact nature of these projects will be described later.

### 4.2.1 Final Good Production

There is a single final good in this economy which can be used for consumption and investment. In the final-good sector, there is a large number of identical firms. In each period, each firm hires workers, rents physical capital and produces final goods according to

$$Y_t = K_t^\phi L_t^{1-\phi}, \quad \text{with } \phi \in (0, 1),$$

where  $Y_t$  denotes output at time  $t$ ,  $K_t$  and  $L_t$  denote capital input and labor input, respectively. Markets for final goods and factors of production are assumed to be perfectly competitive.

Since the production function exhibits constant returns to scale, we can focus on the choices made by a single, price-taking firm. The representative firm's problem is given by

$$\max_{K_t, L_t} \left\{ K_t^\phi L_t^{1-\phi} - w_t L_t - \rho_t K_t \right\},$$

where  $w_t$  is the wage rate at time  $t$  and  $\rho_t$  is the rental price of physical capital. The first-order conditions of this problem are given by

$$\rho_t = \phi K_t^{\phi-1} L_t^{1-\phi} \quad \text{and} \quad w_t = (1 - \phi) K_t^\phi L_t^{-\phi}.$$

### 4.2.2 Depositors

Consider a depositor who is born at time  $t$ . Let  $c_{y,t}$  and  $c_{o,t+1}$  denote his consumption in the young age and old age, respectively. His preferences over  $(c_{y,t}, c_{o,t+1})$  are represented by

$$U(c_{y,t}, c_{o,t+1}) = \frac{c_{y,t}^{1-\sigma}}{1-\sigma} + \beta \frac{c_{o,t+1}^{1-\sigma}}{1-\sigma}, \quad (4.1)$$

where  $\beta \in (0, 1)$  is the subjective discount factor and  $\sigma > 0$  is the inverse of intertemporal elasticity of substitution (IES). The depositor's labor income when young ( $w_t$ ) is allocated

between consumption and deposit holdings ( $d_t$ ). The gross return from bank deposit is deterministic and is denoted by  $R_{t+1}$ .

Taking  $w_t$  and  $R_{t+1}$  as given, the depositor's problem is to choose an allocation  $(c_{y,t}, c_{o,t+1}, d_t)$  so as to maximize his lifetime utility in (4.1), subject to the budget constraints:  $c_{y,t} + d_t = w_t$  and  $c_{o,t+1} = R_{t+1}d_t$ . The solution of this problem is standard and is given by

$$c_{y,t} = \frac{w_t}{1 + \beta^{\frac{1}{\sigma}} R_{t+1}^{\frac{1}{\sigma}-1}}, \quad \text{and} \quad d_t = \left[ \frac{\beta^{\frac{1}{\sigma}} R_{t+1}^{\frac{1}{\sigma}-1}}{1 + \beta^{\frac{1}{\sigma}} R_{t+1}^{\frac{1}{\sigma}-1}} \right] w_t \equiv \Sigma(R_{t+1}) w_t, \quad (4.2)$$

where  $\Sigma(R_{t+1})$  is the depositor's personal saving rate. If the depositor's IES is greater than (or less than) unity, then the personal saving rate is strictly increasing (or strictly decreasing) in  $R_{t+1}$ . In the knife-edge case where  $\sigma = 1$  (i.e., logarithmic utility), the depositor's personal saving rate is independent of the return from deposit. Since all the depositors within the same cohort are identical, the aggregate supply of deposit is given by  $D_t \equiv (1 - \alpha) N_t d_t$ . It follows that the aggregate supply curve of bank deposit is upward sloping (or downward sloping) when the IES is greater than (or less than) one.

#### 4.2.3 Entrepreneurs

In each period  $t$ , each young entrepreneur has access to a risky investment project. The entrepreneur can use both internal funding (i.e., his own savings  $s_t$ ) and external financing (i.e., borrowing  $b_t$ ) to fund the project. The total amount of investment is denoted by  $I_t = s_t + b_t$ . By investing  $I_t \geq 0$  units of final goods at time  $t$ , the project will generate  $zI_t$  units of physical capital at time  $t+1$ , where  $z$  is an idiosyncratic productivity shock. The random variable  $z$  is drawn from the interval  $[0, \bar{z}]$  according to the distribution  $G(z)$ , where  $G : [0, \bar{z}] \rightarrow [0, 1]$  is a twice continuously differentiable, strictly increasing function. The productivity shock is assumed to be independent across entrepreneurs. At



the beginning of time  $t + 1$ , the value of  $z$  is privately and costlessly observed by the entrepreneur. All other agents (including the external financier) will have to incur a cost in order to observe this value. The entrepreneur then rents the physical capital to the firms in the final-good sector at a rate  $\rho_{t+1}$ . Thus, the gross return from investment is  $\tilde{\rho}_{t+1}zI_t$  units of final goods at time  $t + 1$ , where  $\tilde{\rho}_{t+1} \equiv (1 - \delta + \rho_{t+1})$  and  $\delta \in (0, 1)$  is the depreciation rate of physical capital.

#### 4.2.4 Financial Intermediation

All the borrowing and lending activities are carried out through financial intermediaries or banks.<sup>1</sup> The total number of banks in this economy is denoted by  $M$ , which is a positive integer. Each bank accepts deposits from the depositors and provides loans to a large number of entrepreneurs. By lending to a large number of entrepreneurs, the bank can diversify away the idiosyncratic risk associated with the investment projects. Thus, the bank can offer a riskless return ( $R_{t+1}$ ) to the depositors. Similar to Matutes and Vives (2000), Allen and Gale (2004) and Boyd, De Nicolò and Jalal (2009), we assume that there is imperfect competition in the deposit market. The loan market, on the other hand, is assumed to be perfectly competitive. Besides lending to businesses, each bank can also choose to lend to other banks in the interbank loan market. The gross return from interbank loan between time  $t$  and time  $t + 1$  is denoted by  $\mathcal{R}_{t+1}$ . In the following subsections, we first describe and analyze the lending operations of an individual bank, then we turn to the imperfect competition in the deposit market.

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<sup>1</sup>One justification for this assumption is that the financial intermediaries have a cost advantage in monitoring loan contracts over individual depositors. See Diamond (1984) and Williamson (1986) for a formal analysis of delegated monitoring under a perfectly competitive credit market.

#### 4.2.4.1 Loan Contracts

Any entrepreneur who is in need of external finance will have to negotiate a loan contract with a bank. The bank has perfect knowledge about the distribution of the productivity shocks, but it cannot directly observe the realized values of the shock. Thus, it has to rely on the entrepreneur's report when collecting repayment. This asymmetry of information provides an incentive for the entrepreneur to misreport the productivity level in order to lower repayment. As a countermeasure, the bank can verify or audit the accuracy of the report by incurring a cost. Following Khan (2001), we assume that the costs of verification are proportional to the output of the investment project. Intuitively, this means projects with high value also tend to be more complex, and are thus more costly to appraise and monitor. For a project with gross return  $\tilde{\rho}_{t+1}z(s_t + b_t)$ , the costs of verification are given by  $\lambda\tilde{\rho}_{t+1}z(s_t + b_t)$ , where  $\lambda \in (0, 1)$ .

A loan contract in this context is characterized by three things: (i) the amount of borrowing  $b_t \geq 0$ , (ii) a repayment schedule  $Q_t : [0, \bar{z}] \rightarrow \mathbb{R}_+$  which specifies the amount of repayment in each possible state, and (iii) the circumstances under which auditing would occur. These circumstances are summarized by a subset  $A_t$  in the state space  $[0, \bar{z}]$ . The loan contract will have the following properties: First, the repayment must be affordable by the entrepreneur. This means the repayment in any given state cannot exceed the output of the investment project, i.e.,

$$Q_t(z) \leq \tilde{\rho}_{t+1}z(s_t + b_t), \quad \text{for all } z \in [0, \bar{z}]. \quad (4.3)$$

Second, the contract will induce the entrepreneur to report the true value of the productivity shock. Third, any optimal contract must give the bank an expected return that is

no less than the return from the interbank loan market  $\mathcal{R}_{t+1}$ , so that

$$\int_0^{\bar{z}} Q_t(z) dG(z) - \lambda \tilde{\rho}_{t+1}(s_t + b_t) \int_{A_t} z dG(z) \geq \mathcal{R}_{t+1} b_t, \quad (4.4)$$

where  $\lambda \tilde{\rho}_{t+1}(s_t + b_t) \int_{A_t} z dG(z)$  is the expected cost of verification. Equation (4.4) is also referred to as the bank's participation constraint. Finally, an optimal contract is one that maximizes the entrepreneur's expected return after repayment. Thus, an optimal contract is one that solves the following maximization problem:

$$W_t(s_t) \equiv \max_{Q_t(\cdot), A_t, b_t} \left\{ \int_0^{\bar{z}} [\tilde{\rho}_{t+1} z (s_t + b_t) - Q_t(z)] dG(z) \right\} \quad (4.5)$$

subject to (4.3) and (4.4).

Since the work of Gale and Hellwig (1985) and Williamson (1986, 1987), it is well-known that the optimal loan contract under costly state verification will take the form of a standard debt contract.<sup>2</sup> The specifics of this type of contract are as follows: In order to curb the agency problem, the bank will always choose to audit in some states. Specifically, auditing will occur if the reported productivity is lower than a certain threshold, denoted by  $\hat{z}_t$ . Thus, the verification region  $A_t$  can be represented by  $A_t = [0, \hat{z}_t]$ . Once auditing happens, the bank will effectively take over the investment project and retain a fraction  $(1 - \lambda)$  of the project return. If the reported productivity is greater than the threshold, then the amount of repayment is independent of the reported state. The repayment schedule under the optimal loan contract can be represented by<sup>3</sup>

$$Q_t(z) = \begin{cases} \tilde{\rho}_{t+1}(s_t + b_t) z, & \text{for } z \in [0, \hat{z}_t], \\ \tilde{\rho}_{t+1}(s_t + b_t) \hat{z}_t \equiv q_t, & \text{for } z \in (\hat{z}_t, \bar{z}]. \end{cases} \quad (4.6)$$

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<sup>2</sup>The contracting problem in the current study, however, has some features that are not present in these original work (e.g., the use of collateral and endogenous leverage). Thus, for the sake of completeness, we provide a detailed characterization of the optimal contract in Appendix A.

<sup>3</sup>Alternatively, one can interpret  $A_t$  as the states under which the entrepreneur will declare bankruptcy. Specifically, the entrepreneur is called bankrupt if the realized productivity is too low (i.e., lower than  $\hat{z}_t$ ) so that he cannot afford the fixed repayment  $q_t$  specified in the loan contract.

Note that in the event of auditing, the entrepreneur's own savings  $s_t$  will be confiscated by the bank. Thus,  $s_t$  also serves as a collateral for the loan. Under an optimal contract, the bank's participation constraint must be binding. This means the bank's expected return from business lending must be equated to the outside return  $\mathcal{R}_{t+1}$ . Using (4.6) and the bank's participation constraint, we can derive an expression for the size of borrowing, which is

$$b_t = \left[ \frac{J(\hat{z}_t) \tilde{\rho}_{t+1}}{\mathcal{R}_{t+1} - J(\hat{z}_t) \tilde{\rho}_{t+1}} \right] s_t, \quad (4.7)$$

where

$$J(x) \equiv (1 - \lambda) \int_0^x z dG(z) + x [1 - G(x)]. \quad (4.8)$$

Finally, using (4.6) we can rewrite the maximization problem in (4.5) as

$$W(s_t) \equiv \max_{\hat{z}_t \in [0, \bar{z}]} \left\{ \tilde{\rho}_{t+1} (s_t + b_t) \left[ \int_{\hat{z}_t}^{\bar{z}} (z - \hat{z}_t) dG(z) \right] \right\}, \quad (4.9)$$

subject to (4.7). The optimal contract problem is now boiled down to the choice of a single variable  $\hat{z}_t$ . This problem, however, may not be concave. This means it is possible to have multiple solutions to this problem which will then give rise to a multiplicity of optimal loan contract. To avoid this, we impose the following additional condition on the distribution function  $G(\cdot)$ .<sup>4</sup>

### Assumption A1

For any  $z \in [0, \bar{z}]$ ,  $G'(z) + zG''(z) > 0$ .

Our first proposition provides a set of conditions under which a unique optimal loan contract with interior threshold value, i.e.,  $\hat{z}_t \in (0, \bar{z})$ , exists. We focus on contracts

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<sup>4</sup>This assumption is satisfied by any distribution function of the following form:  $G(z) = (z/\bar{z})^\theta$ , for all  $z \in [0, \bar{z}]$ , with  $\theta > 0$ . A uniform distribution over the range  $[0, \bar{z}]$  corresponds to the case when  $\theta = 1$ .

with interior threshold value for the following reasons: If  $\hat{z}_t = 0$  then auditing will never occur. It follows from (4.6) and (4.7) that both borrowing and repayment are zero, i.e.,  $b_t = 0$  and  $Q_t(\cdot) \equiv 0$ . Intuitively, this means the bank will not lend to the entrepreneur if there is no chance to monitor the project return. On the other hand, a contract with  $\hat{z}_t = \bar{z}$  means that the bank will monitor the investment project in all possible states. It follows from (4.6) that all the realized output of the project will be forfeited by the bank, leaving the entrepreneur with zero payoff. Since the entrepreneur can always choose to deposit their savings in the bank, such a contract will not be accepted by any rational entrepreneur. An equilibrium with either one of these contract will thus leave no role for financial intermediation. The conditions in Proposition 4.2.1 ensure that these extreme and uninteresting cases will not occur. The proofs of this and other propositions can be found in Appendix B.

**Proposition 4.2.1.** Suppose  $\mathcal{R}_{t+1} > 0$ ,  $\tilde{\rho}_{t+1} > 0$  and  $s_t > 0$ . Suppose Assumption A1 is satisfied. Then a unique optimal loan contract with interior threshold value  $\hat{z}_t \in (0, \bar{z})$  exists if and only if  $(1 - \lambda) E(z) < \mathcal{R}_{t+1} / \tilde{\rho}_{t+1} < E(z)$ . The threshold value  $\hat{z}_t$  is uniquely determined by

$$\mathcal{R}_{t+1} = H(\hat{z}_t) \tilde{\rho}_{t+1}, \quad (4.10)$$

where  $H : [0, \bar{z}] \rightarrow \mathbb{R}_+$  is defined by

$$H(x) = J(x) + \frac{J'(x) \left[ \int_x^{\bar{z}} (z - x) dG(z) \right]}{1 - G(x)}. \quad (4.11)$$

According to (4.10), the cutoff level for auditing is determined by four factors: (i) the bank's outside return  $\mathcal{R}_{t+1}$ , (ii) the gross return from physical capital (net of depreciation)  $\tilde{\rho}_{t+1}$ , (iii) the costs of verification as captured by  $\lambda$ , and (iv) the distribution of the

productivity shocks. Substituting (4.10) into (4.7) gives

$$b_t = \left[ \frac{J(\hat{z}_t)}{H(\hat{z}_t) - J(\hat{z}_t)} \right] s_t. \quad (4.12)$$

This equation shows that the entrepreneur's financial leverage (defined as the ratio between total borrowing and his own asset) under the optimal contract is endogenously determined by (i) the threshold value  $\hat{z}_t$ , (ii) the costs of verification as captured by  $\lambda$ , and (iii) the distribution of the productivity shocks. Our next proposition explains how changes in  $\mathcal{R}_{t+1}$ ,  $\tilde{\rho}_{t+1}$  and  $\lambda$  would affect  $\hat{z}_t$  and the entrepreneur's leverage.

**Proposition 4.2.2.** Suppose the conditions in Proposition 4.2.1 are satisfied.

- (i) Holding other things constant, an increase in the ratio  $\mathcal{R}_{t+1}/\tilde{\rho}_{t+1}$  will lower the cutoff value  $\hat{z}_t$  and the entrepreneur's financial leverage.
- (ii) Holding other things constant, an increase in  $\lambda$  will lower the cutoff value  $\hat{z}_t$  and the entrepreneur's financial leverage.

The intuitions of these results are as follows. Holding other things constant, a higher value of  $\mathcal{R}_{t+1}$  means that the bank now faces a greater opportunity cost of business lending. Thus, it will provide fewer loans to the entrepreneur. Fewer lending also means that the agency problem involved in the loan contract is now alleviated. As a result, the bank will choose to monitor the investment project less frequently (i.e., a lower value of  $\hat{z}_t$ ) in order to cut back on the verification costs. A similar mechanism is in place when there is either a decline in  $\tilde{\rho}_{t+1}$  or an increase in  $\lambda$ . The former lowers the expected return from the investment project, while the latter directly raises the costs of verification. Both of these changes will discourage the bank from lending to the entrepreneur.

#### 4.2.4.2 Bank Competition for Deposits

Following Allen and Gale (2004) and Boyd, De Nicolò and Jalal (2009), we assume that banks engage in Cournot competition in the deposit market.<sup>5</sup> Specifically, in each period, each bank chooses the amount of deposits that it would accept, taking as given the choices made by its rivals and the depositors' saving decisions. The bank then uses the deposits to make business loans to the entrepreneurs.

Let  $\zeta_{i,t}$  be the amount of deposits chosen by bank  $i \in \{1, 2, \dots, M\}$  at time  $t$ . The total amount of deposits chosen by all  $M$  banks at time  $t$  is thus  $\sum_{i=1}^M \zeta_{i,t}$ . Given these choices and the depositors' decision rules in (4.2), the deposit market at time  $t$  clears when

$$\sum_{i=1}^M \zeta_{i,t} = (1 - \alpha) N_t \left[ \frac{\beta^{\frac{1}{\sigma}} R_{t+1}^{\frac{1}{\sigma}-1}}{1 + \beta^{\frac{1}{\sigma}} R_{t+1}^{\frac{1}{\sigma}-1}} \right] w_t.$$

This equation implicitly defines a relationship between  $\sum_{i=1}^M \zeta_{i,t}$  and the deposit return that clears the deposit market. Formally, this relationship is given by

$$R_{t+1} \equiv \Gamma_t \left( \sum_{i=1}^M \zeta_{i,t} \right) = \beta^{\frac{1}{\sigma-1}} \left[ \frac{(1 - \alpha) N_t w_t}{\sum_{i=1}^M \zeta_{i,t}} - 1 \right]^{\frac{\sigma}{\sigma-1}}.$$

The function  $\Gamma_t(\cdot)$  is often referred to as the inverse supply function of deposits. Note that this function is strictly increasing if and only if the depositor's IES is greater than unity (i.e.,  $\sigma < 1$ ). In the following analysis, we will restrict our attention to the case when the inverse supply function is strictly increasing.

An individual bank's profit maximization problem at time  $t$  is given by

$$\max_{\zeta_{i,t}} \left\{ \mathcal{R}_{t+1} \zeta_{i,t} - \Gamma_t \left( \sum_{i=1}^M \zeta_{i,t} \right) \zeta_{i,t} \right\}, \quad (4.13)$$

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<sup>5</sup>In Section 3, we consider an extended model with Cournot competition in both the loan market and the deposit market.

subject to  $\zeta_{i,t} \geq 0$ , where  $\mathcal{R}_{t+1}$  is the bank's return from the optimal loan contract. The first-order necessary condition for this problem is

$$\mathcal{R}_{t+1} - \Gamma_t \left( \sum_{i=1}^M \zeta_{i,t} \right) \leq \Gamma'_t \left( \sum_{i=1}^M \zeta_{i,t} \right) \zeta_{i,t}, \quad (4.14)$$

with equality holds if  $\zeta_{i,t} > 0$ .<sup>6</sup> Since the banks in this economy are identical in all regards, it is natural to consider a symmetric equilibrium. Specifically, a symmetric Cournot equilibrium with interior solution is one in which all banks choose to accept the same positive amount of deposits in every period, i.e.,  $\zeta_{i,t} = \bar{\zeta}_t > 0$  for all  $t$ . The quantity  $\bar{\zeta}_t$  and the deposit return  $R_{t+1}$  in this type of equilibrium are determined by

$$R_{t+1} = \beta^{\frac{1}{\sigma-1}} \left[ \frac{(1-\alpha) N_t w_t}{M \bar{\zeta}_t} - 1 \right]^{\frac{\sigma}{\sigma-1}}, \quad (4.15)$$

$$\mathcal{R}_{t+1} - \Gamma_t (M \bar{\zeta}_t) = \Gamma'_t (M \bar{\zeta}_t) \bar{\zeta}_t. \quad (4.16)$$

#### 4.2.5 Equilibrium

Given the total number of banks  $M \geq 1$  and the initial value of capital  $K_0 > 0$ , an equilibrium of this economy consists of sequences of allocations for the depositors  $\{c_{y,t}, c_{o,t+1}, d_t\}_{t=0}^{\infty}$ , allocations for the entrepreneurs  $\{s_t, \tilde{c}_{o,t+1}(z)\}_{t=0}^{\infty}$ , aggregate inputs in final good production  $\{K_t, L_t\}_{t=0}^{\infty}$ , factor prices  $\{w_t, \rho_t\}_{t=0}^{\infty}$ , loan contracts  $\{Q_t(\cdot), b_t, \hat{z}_t\}_{t=0}^{\infty}$ , and other financial market variables  $\{\mathcal{R}_{t+1}, R_{t+1}, \bar{\zeta}_t\}_{t=0}^{\infty}$  such that the following conditions are satisfied for all  $t \geq 0$ ,

- (i) Given  $w_t$  and  $R_{t+1}$ , the allocation  $\{c_{y,t}, c_{o,t+1}, d_t\}$  is optimal for the depositors in generation  $t$ .

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<sup>6</sup>As is evident from (4.13), a bank will choose to have  $\zeta_{i,t} > 0$  if and only if the return that it can obtain from business lending ( $\mathcal{R}_{t+1}$ ) is no less than the return that it offers to the depositors ( $R_{t+1}$ ). According to (4.14), this can happen only when  $\Gamma_t(\cdot)$  is increasing.



- (ii) All young entrepreneurs will invest their labor income in the investment project, i.e.,  $s_t = w_t$ . Their state-contingent consumption in the old age is determined by the difference between the project return and the repayment, i.e.,  $\tilde{c}_{o,t+1}(z) = \tilde{\rho}_{t+1}(s_t + b_t) \max\{(z - \hat{z}_t), 0\}$ , for all  $z \in [0, \bar{z}]$ .
- (iii) Given  $w_t$  and  $\rho_t$ , the inputs  $\{K_t, L_t\}$  solve the final-good producer's problem at time  $t$ .
- (iv) Given  $\mathcal{R}_{t+1}$  and  $\tilde{\rho}_{t+1}$ ,  $\{Q_t(\cdot), b_t, \hat{z}_t\}$  is the optimal loan contract, i.e., (4.6), (4.10) and (4.12) are satisfied.
- (v) Given  $w_t$  and  $\mathcal{R}_{t+1}$ , the deposit market variables  $(\bar{\zeta}_t, R_{t+1})$  are determined in a symmetric Cournot equilibrium, i.e., (4.15) and (4.16) are satisfied.
- (vi) All markets clear, so that  $L_t = N_t$ ,  $\alpha N_t d_t = M \bar{\zeta}_t$  and

$$K_{t+1} = \alpha N_t \int_0^{\bar{z}} (s_t + b_t) z dG(z) = \alpha N_t (s_t + b_t) E(z). \quad (4.17)$$

Equation (4.17) states that aggregate capital at time  $t+1$  is formed by aggregating the output of all the investment projects. Define  $k_t \equiv K_t/N_t$ . Using  $w_t = (1 - \phi) k_t^\phi$ ,  $s_t = w_t$  and (4.12), we can obtain

$$k_{t+1} = \frac{\alpha (1 - \phi) E(z)}{1 + n} \left[ \frac{H(\hat{z}_t)}{H(\hat{z}_t) - J(\hat{z}_t)} \right] k_t^\phi. \quad (4.18)$$

This equation shows how the provisions in the optimal loan contract will affect the accumulation of physical capital. In equilibrium, all the deposits received by the banks will be lent to the entrepreneurs as business loans, so that

$$(1 - \alpha) N_t \left[ \frac{\beta^{\frac{1}{\sigma}} R_{t+1}^{\frac{1}{\sigma} - 1}}{1 + \beta^{\frac{1}{\sigma}} R_{t+1}^{\frac{1}{\sigma} - 1}} \right] w_t = \alpha N_t b_t. \quad (4.19)$$

#### 4.2.5.1 Stationary Equilibrium

A stationary equilibrium of this economy can be summarized by a set of positive real numbers  $(k^*, \hat{z}^*, \mathcal{R}^*, R^*)$ , which represent the per-worker amount of capital, the cutoff level for auditing, the return from interbank loan, and the deposit return in a steady state, respectively. These values are completely characterized by the following equations

$$k^* = \frac{\alpha(1-\phi)E(z)}{1+n} \left[ \frac{H(\hat{z}^*)}{H(\hat{z}^*) - J(\hat{z}^*)} \right] (k^*)^\phi, \quad (4.20)$$

$$\frac{\beta^{\frac{1}{\sigma}} (R^*)^{\frac{1}{\sigma}-1}}{1 + \beta^{\frac{1}{\sigma}} (R^*)^{\frac{1}{\sigma}-1}} = \frac{\alpha}{1-\alpha} \left[ \frac{J(\hat{z}^*)}{H(\hat{z}^*) - J(\hat{z}^*)} \right], \quad (4.21)$$

$$\mathcal{R}^* = \left[ 1 - \delta + \phi(k^*)^{\phi-1} \right] H(\hat{z}^*), \quad (4.22)$$

$$\mathcal{R}^* = R^* \left\{ 1 + \frac{\sigma}{(1-\sigma)M} \left[ 1 + \beta^{\frac{1}{\sigma}} (R^*)^{\frac{1}{\sigma}-1} \right] \right\}. \quad (4.23)$$

Equations (4.20)-(4.22) are the steady-state version of (4.18), (4.19) and (4.10), respectively. The formal derivation of (4.23) can be found in Appendix B. Proposition 4.2.3 provides the conditions under which a unique stationary equilibrium exists.

**Proposition 4.2.3.** Suppose Assumption A1 is satisfied. Suppose  $\sigma < 1$  so that the inverse supply curve of deposits is strictly increasing. Then a unique stationary equilibrium of this economy exists for any  $M \geq 1$ .

We now consider the effects of an increase in  $M$  on the stationary equilibrium. To put this in context, consider two economies that are identical in all aspects, except the number of banks in their financial market. Let  $M_j \geq 1$  be the number of banks in economy  $j \in \{1, 2\}$  and let  $(k_j^*, \hat{z}_j^*, \mathcal{R}_j^*, R_j^*)$  be the unique stationary equilibrium in this economy. Our next proposition provides a comparison of these two equilibria.

**Proposition 4.2.4.** Suppose the conditions in Proposition 4.2.3 are satisfied. Then  $M_1 > M_2$  implies  $k_1^* > k_2^*$ ,  $\hat{z}_1^* > \hat{z}_2^*$ ,  $\mathcal{R}_1^* < \mathcal{R}_2^*$  and  $R_1^* > R_2^*$ . In addition, the entrepreneur's financial leverage is higher in the economy with more banks.

The interpretation of this result is as follows. An increase in  $M$  means that there is now more competition in the deposit market. This lowers the market power enjoyed by each bank and reduces their ability to extract profits from their deposit operations. Specifically, the intensified competition will drive up the deposit return ( $R^*$ ) and lower the bank's outside option ( $\mathcal{R}^*$ ). In the limit where  $M$  is infinite, the two returns will be exactly identical, i.e.,  $\mathcal{R}^* = R^*$ , and the zero-profit condition for the banks will prevail. In this case, both the loan market and the deposit market are perfectly competitive, and the choices made by a single bank will have no effect on the market outcomes. An increase in the competition for deposits will also affect the lending operations of the banks. In particular, the decline in  $\mathcal{R}^*$  means that the banks now face a lower opportunity cost of business lending. This raises the entrepreneurs' financial leverage, and increases the investment in physical capital. But at the same time, the increase in borrowing will also aggravate the agency problem involved in the loan contract. As a result, the banks will monitor the investment projects more frequently (i.e., an increase in  $\hat{z}^*$ ).

### 4.3 An Extended Model

We now extend the baseline model to allow for Cournot competition among banks in both the loan market and the deposit market. Let  $\psi_{i,t}$  and  $\zeta_{i,t}$  be the amount of loans and deposits chosen by bank  $i \in \{1, 2, \dots, M\}$  at time  $t$ . Similar to Section 2.4.2, the inverse

supply function of deposits is given by

$$R_{t+1} \equiv \Gamma_t \left( \sum_{i=1}^M \zeta_{i,t} \right) = \beta^{\frac{1}{\sigma-1}} \left[ \frac{(1-\alpha) N_t w_t}{\sum_{i=1}^M \zeta_{i,t}} - 1 \right]^{\frac{\sigma}{\sigma-1}}.$$

We again focus on the case when  $\Gamma_t(\cdot)$  is strictly increasing, i.e., when  $\sigma < 1$ .

The deposits received by the banks will be used to provide loans to the entrepreneurs. The optimal loan contract is determined as follows: As before, a loan contract between an entrepreneur and one of the banks, say bank  $i$ , will have to specify (i) the amount of borrowing  $b_{i,t} \geq 0$ , (ii) a repayment schedule  $Q_{i,t} : [0, \bar{z}] \rightarrow \mathbb{R}_+$ , and (iii) a set of states under which auditing would occur,  $A_{i,t}$ . The repayment in all possible states must be affordable by the entrepreneur, i.e.,

$$Q_{i,t}(z) \leq \tilde{\rho}_{t+1}(s_t + b_{i,t})z, \quad \text{for all } z \in [0, \bar{z}].$$

The loan contract will guarantee the bank an expected return that is no less than  $\tilde{\mathcal{R}}_{t+1}$  for each unit of borrowing, so that

$$\int_0^{\bar{z}} Q_{i,t}(z) dG(z) - \lambda \tilde{\rho}_{t+1}(s_t + b_{i,t}) \int_{A_{i,t}} z dG(z) \geq \tilde{\mathcal{R}}_{t+1} b_{i,t}.$$

We will refer to  $\tilde{\mathcal{R}}_{t+1}$  as the loan return to the bank. Finally, competition in the loan market means that all banks will ask for the same return and that the contract must be one that maximizes the entrepreneur's expected return after repayment.

Following the same steps as in Section 2.4.1, we can show that the optimal loan contract will again take the form of a standard debt contract with a unique cutoff value  $\hat{z}_{i,t}$ . Under the conditions in Proposition 4.2.1, this value is determined by

$$\tilde{\mathcal{R}}_{t+1} = H(\hat{z}_{i,t}) \tilde{\rho}_{t+1}, \tag{4.24}$$

and the amount of borrowing is

$$b_{i,t} = \left[ \frac{J(\hat{z}_{i,t})}{H(\hat{z}_{i,t}) - J(\hat{z}_{i,t})} \right] s_t. \tag{4.25}$$

Equations (4.24) and (4.25) have two important implications: First, when facing the same value of  $(\tilde{\mathcal{R}}_{t+1}, \tilde{\rho}_{t+1}, s_t)$ , all the banks will draft the same contract with the entrepreneurs, i.e.,  $\hat{z}_{i,t} = \hat{z}_t$  and  $b_{i,t} = b_t$  for all  $i \in \{1, 2, \dots, M\}$ . Second, the above equations implicitly define a relationship between the demand for loans by an individual entrepreneur and the loan return to the banks. It is straightforward to show that the demand for loans is strictly decreasing in  $\tilde{\mathcal{R}}_{t+1}$ .

Given the banks' choices  $(\psi_{1,t}, \dots, \psi_{M,t})$  and the entrepreneur's demand for loans, the loan market at time  $t$  clears when

$$\sum_{i=1}^M \psi_{i,t} = \left[ \frac{J(\hat{z}_t)}{H(\hat{z}_t) - J(\hat{z}_t)} \right] \alpha N_t s_t.$$

Using this and (4.24), we can define (again implicitly) the inverse demand function for loans,

$$\tilde{\mathcal{R}}_{t+1} = \Omega_t \left( \sum_{i=1}^M \psi_{i,t} \right).$$

The inverse demand function is continuously differentiable and strictly decreasing, i.e.,  $\Omega'_t(\cdot) < 0$ .

An individual bank's profit maximization problem is now given by

$$\max_{\psi_{i,t}, \zeta_{i,t}} \left\{ \Omega_t \left( \sum_{i=1}^M \psi_{i,t} \right) \psi_{i,t} - \Gamma_t \left( \sum_{i=1}^M \zeta_{i,t} \right) \zeta_{i,t} \right\}$$

subject to the capacity constraint:  $\zeta_{i,t} \geq \psi_{i,t} \geq 0$ . This constraint states that both loans and deposits must be non-negative and that the banks cannot lend out more than the amount of deposits received. Since the total liabilities owed by each bank, i.e.,  $\Gamma_t \left( \sum_{i=1}^M \zeta_{i,t} \right) \zeta_{i,t}$ , is strictly increasing  $\zeta_{i,t}$ , it is never optimal for the bank to have an excess of deposits, i.e.,  $\zeta_{i,t} > \psi_{i,t}$ . Thus, the capacity constraint is always binding. Any

solution of the bank's problem must satisfy the first-order necessary condition

$$\Omega_t \left( \sum_{i=1}^M \psi_{i,t} \right) + \Omega'_t \left( \sum_{i=1}^M \psi_{i,t} \right) \psi_{i,t} = \Gamma_t \left( \sum_{i=1}^M \zeta_{i,t} \right) + \Gamma'_t \left( \sum_{i=1}^M \zeta_{i,t} \right) \zeta_{i,t}. \quad (4.26)$$

The second term on both sides of (4.26) represents the bank's market power in the loan market and the deposit market. Specifically, by choosing the quantity of loans and deposits, the bank can affect the loan return and the deposit return that clears these markets.

In a symmetric Cournot equilibrium, all banks choose the same amount of loans and deposits in every period, i.e.,  $\psi_{i,t} = \bar{\psi}_t$  and  $\zeta_{i,t} = \bar{\zeta}_t$  for all  $t$ . Such an equilibrium is completely characterized by

$$R_{t+1} = \beta^{\frac{1}{\sigma-1}} \left[ \frac{(1-\alpha) N_t w_t}{M \bar{\zeta}_t} - 1 \right]^{\frac{\sigma}{\sigma-1}},$$

$$\tilde{\mathcal{R}}_{t+1} + \Omega'_t (M \bar{\psi}_t) \bar{\psi}_t = R_{t+1} + \Gamma'_t (M \bar{\zeta}_t) \bar{\zeta}_t,$$

and  $\bar{\psi}_t = \bar{\zeta}_t$ .

The other parts of the economy are the same as in the baseline model and are thus not repeated. A stationary equilibrium of the extended model can be summarized by a set of positive real numbers  $(k^*, \hat{z}^*, \tilde{\mathcal{R}}^*, R^*)$ , which solves the following equations:

$$k^* = \frac{\alpha(1-\phi) E(z)}{1+n} \left[ \frac{H(\hat{z}^*)}{H(\hat{z}^*) - J(\hat{z}^*)} \right] (k^*)^\phi, \quad (4.27)$$

$$\frac{\beta^{\frac{1}{\sigma}} (R^*)^{\frac{1}{\sigma}-1}}{1 + \beta^{\frac{1}{\sigma}} (R^*)^{\frac{1}{\sigma}-1}} = \frac{\alpha}{1-\alpha} \left[ \frac{J(\hat{z}^*)}{H(\hat{z}^*) - J(\hat{z}^*)} \right], \quad (4.28)$$

$$\tilde{\mathcal{R}}^* = \left[ 1 - \delta + \phi (k^*)^{\phi-1} \right] H(\hat{z}^*), \quad (4.29)$$

$$\tilde{\mathcal{R}}^* \left[ 1 + \frac{1}{M} \frac{H'(\hat{z}^*)}{H(\hat{z}^*)} \frac{\mathcal{L}(\hat{z}^*)}{\mathcal{L}'(\hat{z}^*)} \right] = R^* \left\{ 1 + \frac{\sigma}{(1-\sigma)M} \left[ 1 + \beta^{\frac{1}{\sigma}} (R^*)^{\frac{1}{\sigma}-1} \right] \right\}, \quad (4.30)$$

where  $\mathcal{L}(\hat{z}^*) \equiv J(\hat{z}^*) / [H(\hat{z}^*) - J(\hat{z}^*)]$  is the entrepreneur's financial leverage under the optimal contract. Note that (4.27)-(4.29) are essentially the same as (4.20)-(4.22) in the baseline model. Equation (4.30) is a modification of (4.23), which takes into account the

banks' market power in the loan market. A formal derivation of this equation can be found in Appendix B.

#### 4.4 Numerical Examples

We now provide some numerical examples to illustrate the effects of IES  $(1/\sigma)$ , the costs of verification (as captured by  $\lambda$ ), the intensity of bank competition (as captured by  $M$ ) and the share of entrepreneurs ( $\alpha$ ) on the steady-state values  $(k^*, \hat{z}^*, \mathcal{R}^*, R^*)$ . Suppose one model period takes 30 years. Set the annual subjective discount factor to 0.9750 and the annual employment growth rate to 1.6%. This latter is based on the average annual growth rate of US employment over the period 1953-2008. Then we have  $\beta = (0.9750)^{30} = 0.4678$  and  $n = (1.0160)^{30} - 1 = 0.6099$ . We also set  $\phi = 0.33$  so that capital's share of income is about one-third, and the annual depreciation rate of physical capital to 5% so that  $\delta = 1 - (1 - 0.05)^{30} = 0.78$ . The idiosyncratic productivity shock is assumed to be uniformly distributed over the interval  $[0, 10]$ . We then solve for the steady states under different combinations of  $(\lambda, M, \sigma, \alpha)$  in the following examples. In the benchmark scenario, as represented by the blue solid line in each figure, we set  $M = 10$ ,  $\sigma = 0.75$  and  $\alpha = 0.05$ . Both the baseline model and the extended model are solved using these parameter values. It turns out that the numerical results obtained from these models are almost identical. This happens because the only difference between the steady state of the two models is the relation between  $\mathcal{R}$  and  $R$ , [see equation (4.23) and (4.30)] and the numerical value of  $\left[1 + \frac{1}{M} \frac{H'(\hat{z}^*)}{H(\hat{z}^*)} \frac{\mathcal{L}(\hat{z}^*)}{\mathcal{L}'(\hat{z}^*)}\right]$  in equation (4.30) is very close to one under the chosen parameter values. Hence, equations (4.23) and (4.30) are essentially identical. For this reason, we only report the results from the baseline model.

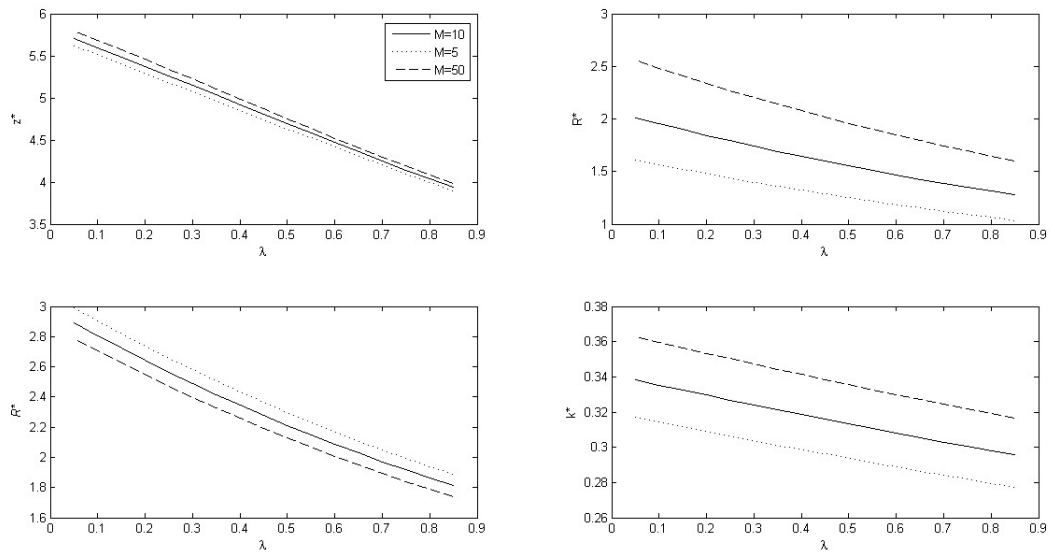


Figure 4.1: Changing the Intensity of Bank Competition.

Figure 1 shows the value of  $(k^*, \hat{z}^*, \mathcal{R}^*, R^*)$  under different combinations of  $\lambda$  and  $M$ . Holding  $M$  constant, all four variables decrease as the verification cost  $\lambda$  grows larger. A higher value of  $\lambda$  means that the banks now face a greater cost of business lending. This will discourage them from lending to the entrepreneurs. Fewer lending means that the information asymmetry problem involved in the loan contract is now less severe. Hence, the bank will choose to audit the investment project less frequently and the threshold value  $\hat{z}^*$  declines. On the other hand, with less lending banks desire less deposit and thus pay less return to savers. Another consequence is that a larger wedge is generated between the capital return and that received by banks, since increase in monitoring difficulty has granted the entrepreneurs more advantage in the borrowing. As for the effect of bank competition, it is shown that more competition (i.e., an increase in  $M$ ) induces more saving, hence more capital accumulation. Meanwhile, the deposit return is driven up and the overall interest rate charged to the entrepreneurs is driven down. These results are consistent with the predictions of Proposition 4.2.4.



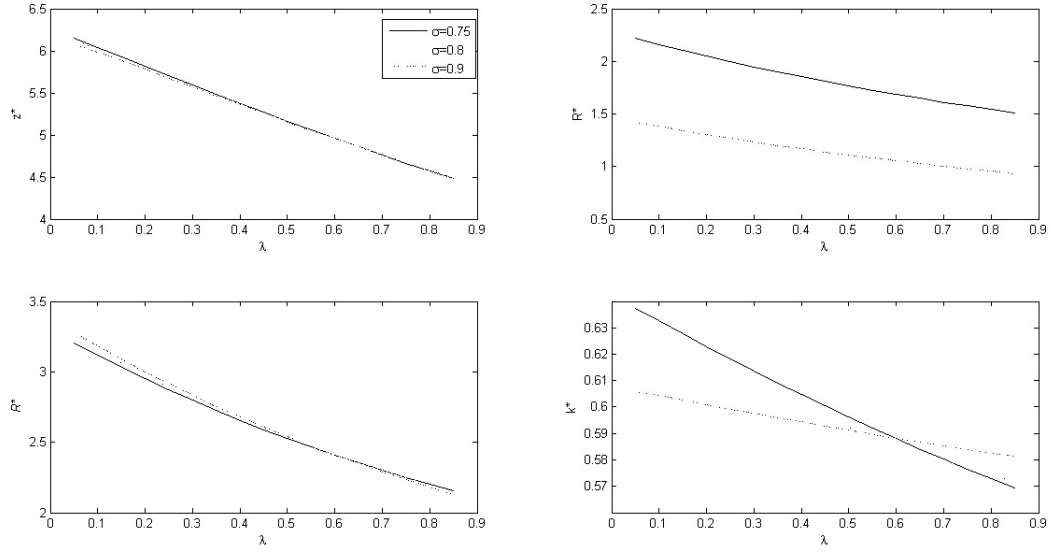


Figure 4.2: Changing the Depositor's IES.

Figure 2 shows the steady state values under different combinations of  $\lambda$  and  $\sigma$ . These results depend crucially on the effects of  $\sigma$  on the depositor's saving rate  $\Sigma(\sigma, R)$ . Straightforward differentiation yields

$$\frac{\partial \Sigma(\sigma, R)}{\partial \sigma} = -\frac{\left(\beta^{\frac{1}{\sigma}} R^{\frac{1}{\sigma}-1}\right) \ln(\beta R)}{\sigma^2 \left(1 + \beta^{\frac{1}{\sigma}} R^{\frac{1}{\sigma}-1}\right)^2} \gtrless 0 \quad \text{iff} \quad \beta R \lessgtr 1.$$

Thus, when the deposit return  $R^*$  is high (as in the case when  $\lambda$  is low), an increase in  $\sigma$  will lower the depositors' willingness to save. As a result, fewer resources will be invested in capital accumulation and  $k^*$  decreases as a result. The opposite is true when the deposit return  $R^*$  is low. In this case, the depositors will save more when  $\sigma$  increases. These changes, however, have very little impact on the loan contract. In particular, the threshold value  $\hat{z}^*$ , the loan return  $\mathcal{R}^*$  and the entrepreneur's financial leverage (not shown here) are not sensitive to changes in  $\sigma$ . Another thing worth mentioning is that the interest rate gap  $(\mathcal{R}^* - R^*)$  increases ubiquitously as  $\sigma$  increases. Intuitively, an increase in  $\sigma$  (which is equivalent to a decrease in IES) means that the supply of deposit is now

less responsible to changes in  $R^*$ . This inelastic supply enables the banks to offer a lower deposit return and extract more profits by widening the gap between  $\mathcal{R}^*$  and  $R^*$ .

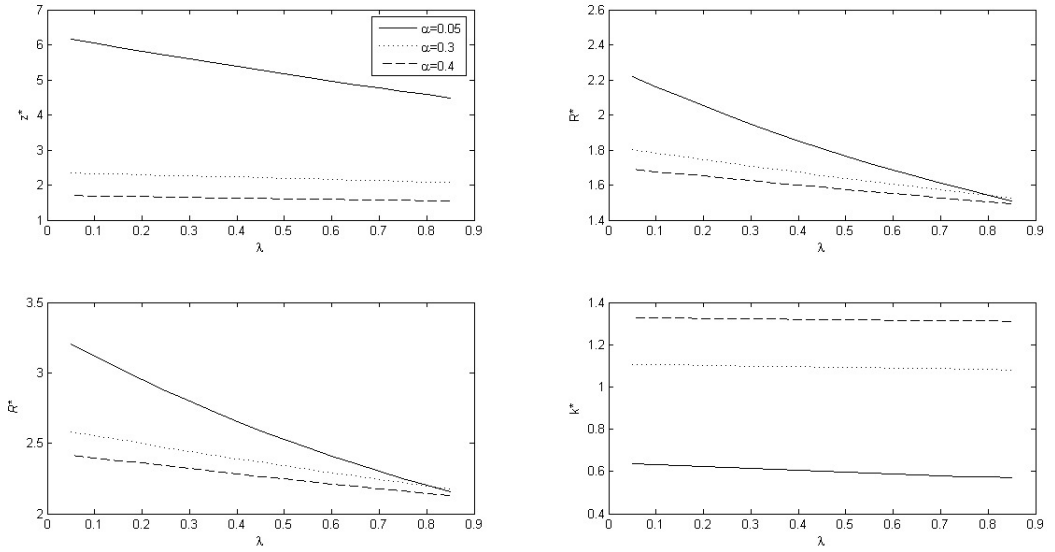


Figure 4.3: Changing the Population Share of Entrepreneurs.

Figure 3 depicts the steady state values under different  $(\lambda - \alpha)$  schemes. The effects of verification cost are consistent with the first two examples. In general, an increase in the number of entrepreneurs will push up the steady state capital and drives down both  $R^*$  and  $\mathcal{R}^*$ . The positive relation between capital and the share of entrepreneurs is due to the fact that given the same wage rate and factor price, risk neutral entrepreneurs save more than risk adverse depositors. Thus when the economy has a larger share of entrepreneurs, the total saving (eventually in the form of capital) will increase too. It can also be considered from the fact that all the capital is produced through entrepreneur's investment projects. When their share reduces, there are fewer channel to transfer savings into capital. The outcome of  $R^*$  and  $\mathcal{R}^*$  can be derived from equation (4.21). According to this equation, individual's deposit supply curve is not affected by  $\alpha$ . But an increase in  $\alpha$  will affect the demand for deposits in two opposite ways. First, the value of  $\alpha/(1 - \alpha)$  increases as

the population share of entrepreneurs increases. In words, this means each depositor now faces a large number of entrepreneurs asking for loans. This is a positive drive of demand. Second, an increase in  $\alpha$  also indirectly reduces each entrepreneur's desire for loans and the leverage ratio, since the economy now has more capital accumulation, thus less capital return. This further gives a larger  $\mathcal{R}/\tilde{\rho}$  ratio and consequently, a lower threshold  $\hat{z}^*$  and lower leverage ratio. In this numerical example, the negative effect on demand dominates and we see a decrease in  $R^*$  when the number of entrepreneurs increases.

#### 4.5 Conclusion

This paper has presented a dynamic general equilibrium model to examine the implication of bank competition on capital accumulation. The relationship between financial development and the real economy in the presence of CSV problem has been largely explored in the macroeconomic literature. The financial market in these works, however, is either perfect competitive or monopolistic. A less extreme market structure has not been thoroughly examined yet. On the other hand, bank size and distribution are well studied in the finance research, but most of the works are abstract from consideration of the aggregate economy. Our paper fills in this gap in the literature by combining the two lines of research.

In this model, financial market consists of Cournot competitive banks who intermediate credit between savers and entrepreneurs and have power in both the saving and the loan market. By affecting the deposit rate as well as the price of loans, the banks control the volume of saving and borrowing, which jointly determine the capital level. In addition, due to the asymmetric information problem, the monitoring intensity is positively associated with leverage ratio. Therefore, when the financial market becomes less concentrated,

a higher volume of credits will be issued to entrepreneurs, which leads to more capital investment. Meanwhile, banks demand more active monitoring that will aggravate the inefficiency. Within this framework, we also analyze how the severity of asymmetry, the share of entrepreneurs or the intertemporal elasticity of substitution affect the capital accumulation. The results show a negative, positive and mixing effect respectively.

\*

# Appendix A

## Chapter 3

### I: Mathematical Derivations

#### Post-Crash Equilibrium

In this section, we provide a detailed characterization of a post-crash equilibrium. Since the consumer's problem in the post-crash economy is standard, the derivations of (4.3)-(4.5) are omitted. The dynamical system in (4.13)-(4.14) can be derived as follows. In equilibrium, the market wage rate and the gross return from physical capital are determined by  $\hat{w}_t = (1 - \alpha) \hat{K}_t^\alpha \hat{L}_t^{1-\alpha}$  and  $\hat{R}_t = \alpha \hat{K}_t^{\alpha-1} \hat{L}_t^{1-\alpha}$ , respectively. Using these, we can obtain

$$\hat{w}_t \hat{l}_t = \frac{1 - \alpha}{\alpha} \hat{R}_t \hat{k}_t, \quad (\text{A.1})$$

$$\hat{w}_t = (1 - \alpha) \left( \frac{\alpha}{\hat{R}_t} \right)^{\frac{\alpha}{1-\alpha}}, \quad (\text{A.2})$$

$$\hat{l}_t = \left( \frac{\hat{R}_t}{\alpha} \right)^{\frac{1}{1-\alpha}} \hat{k}_t. \quad (\text{A.3})$$

where  $\widehat{k}_t \equiv \widehat{K}_t/N_t$  and  $\widehat{l}_t \equiv \widehat{L}_t/N_t$ . Then we can rewrite the capital market clearing condition as

$$(1+n)\widehat{k}_{t+1} = \left[ \frac{\beta^{\frac{1}{\sigma}} \left( \widehat{R}_{t+1} \right)^{\frac{1}{\sigma}-1}}{1 + \beta^{\frac{1}{\sigma}} \left( \widehat{R}_{t+1} \right)^{\frac{1}{\sigma}-1}} \right] \widehat{w}_t \widehat{l}_t \equiv \Sigma \left( \widehat{R}_{t+1} \right) \widehat{w}_t \widehat{l}_t.$$

Substituting (A.2) into the above expression gives (4.13). Next, substituting (A.2) and (A.3) into (4.4) gives

$$\begin{aligned} \left( \frac{\widehat{R}_t}{\alpha} \right)^{\frac{1}{1-\alpha}} \widehat{k}_t &= A^{-\frac{1}{\sigma+\psi}} \left[ 1 + \beta^{\frac{1}{\sigma}} \left( \widehat{R}_{t+1} \right)^{\frac{1}{\sigma}-1} \right]^{\frac{\sigma}{\sigma+\psi}} \left[ (1-\alpha) \left( \frac{\alpha}{\widehat{R}_t} \right)^{\frac{\alpha}{1-\alpha}} \right]^{\frac{1-\sigma}{\sigma+\psi}} \\ \Rightarrow \left( \frac{\widehat{R}_t}{\alpha} \right)^{\eta} \widehat{k}_t &= \left[ \frac{(1-\alpha)^{1-\sigma}}{A} \right]^{\frac{1}{\sigma+\psi}} \left[ 1 + \beta^{\frac{1}{\sigma}} \left( \widehat{R}_{t+1} \right)^{\frac{1}{\sigma}-1} \right]^{\frac{\sigma}{\sigma+\psi}}, \end{aligned} \quad (\text{A.4})$$

where

$$\begin{aligned} \eta &\equiv \frac{1}{1-\alpha} + \frac{\alpha}{1-\alpha} \frac{1-\sigma}{\sigma+\psi} = \frac{\psi + \alpha + \sigma(1-\alpha)}{(1-\alpha)(\sigma+\psi)} > 0, \\ \eta - 1 &= \frac{\alpha}{1-\alpha} \frac{1+\psi}{\sigma+\psi} > 0, \end{aligned}$$

for any  $\sigma > 0$ . Equation (4.14) can be obtained by rearranging terms in (A.4).

### **Local Analysis**

We now explore the local stability property of the unique bubbleless steady state under different values of  $\sigma$ . To achieve this, we consider a linearized version of the dynamical system in (4.13)-(4.14). First, taking logarithms of both sides of these equations gives

$$\begin{aligned} \ln \widehat{k}_{t+1} - \ln \Sigma \left( \widehat{R}_{t+1} \right) &= \ln \left[ \frac{1-\alpha}{\alpha(1+n)} \right] + \ln \widehat{R}_t + \ln \widehat{k}_t, \\ \ln \left\{ \alpha^{\eta} \left[ \frac{(1-\alpha)^{1-\sigma}}{A} \right]^{\frac{1}{\sigma+\psi}} \right\} &+ \frac{\sigma}{\sigma+\psi} \ln \left( 1 + \beta^{\frac{1}{\sigma}} \widehat{R}_{t+1}^{\frac{1}{\sigma}-1} \right) = \eta \ln \widehat{R}_t + \ln \widehat{k}_t. \end{aligned}$$

Next, taking the first-order Taylor expansion of these equations around  $(\widehat{k}^*, \widehat{R}^*)$  gives

$$\widehat{\mathbf{k}}_{t+1} - \frac{\widehat{R}^* \Sigma' \left( \widehat{R}^* \right)}{\Sigma \left( \widehat{R}^* \right)} \widehat{\mathbf{R}}_{t+1} = \widehat{\mathbf{k}}_t + \widehat{\mathbf{R}}_t,$$

$$\frac{1-\sigma}{\sigma+\psi} \left[ \frac{\beta^{\frac{1}{\sigma}} (\hat{R}^*)^{\frac{1}{\sigma}-1}}{1 + \beta^{\frac{1}{\sigma}} (\hat{R}^*)^{\frac{1}{\sigma}-1}} \right] \hat{\mathbf{R}}_{t+1} = \hat{\mathbf{k}}_t + \eta \hat{\mathbf{R}}_t,$$

where  $\hat{\mathbf{k}}_t \equiv (\hat{k}_t - \hat{k}^*) / \hat{k}^*$  and  $\hat{\mathbf{R}}_t \equiv (\hat{R}_t - \hat{R}^*) / \hat{R}^*$  represent the percentage deviations of  $\hat{k}_t$  and  $\hat{R}_t$  from their steady-state values. Finally, rewrite the linearized system in matrix form

$$\underbrace{\begin{bmatrix} 1 & b_{12} \\ 0 & b_{22} \end{bmatrix}}_B \begin{bmatrix} \hat{\mathbf{k}}_{t+1} \\ \hat{\mathbf{R}}_{t+1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & \eta \end{bmatrix} \begin{bmatrix} \hat{\mathbf{k}}_t \\ \hat{\mathbf{R}}_t \end{bmatrix}, \quad (\text{A.5})$$

where

$$b_{12} = -\frac{\hat{R}^* \Sigma'(\hat{R}^*)}{\Sigma(\hat{R}^*)} = \left(1 - \frac{1}{\sigma}\right) \left[1 + \beta^{\frac{1}{\sigma}} (\hat{R}^*)^{\frac{1}{\sigma}-1}\right]^{-1},$$

$$b_{22} = \frac{1-\sigma}{\sigma+\psi} \left[ \frac{\beta^{\frac{1}{\sigma}} (\hat{R}^*)^{\frac{1}{\sigma}-1}}{1 + \beta^{\frac{1}{\sigma}} (\hat{R}^*)^{\frac{1}{\sigma}-1}} \right].$$

The inverse of the matrix  $B$  is given by

$$B^{-1} = \frac{1}{b_{22}} \begin{bmatrix} b_{22} & -b_{12} \\ 0 & 1 \end{bmatrix}.$$

Using this, we can rewrite (A.5) as

$$\begin{bmatrix} \hat{\mathbf{k}}_{t+1} \\ \hat{\mathbf{R}}_{t+1} \end{bmatrix} = \underbrace{\frac{1}{b_{22}} \begin{bmatrix} b_{22} - b_{12} & b_{22} - \eta b_{12} \\ 1 & \eta \end{bmatrix}}_J \begin{bmatrix} \hat{\mathbf{k}}_t \\ \hat{\mathbf{R}}_t \end{bmatrix}, \quad (\text{A.6})$$

where  $J$  is the Jacobian matrix of the linearized system. Let  $\rho_1$  and  $\rho_2$  be the characteristic roots of the linearized system. These can be obtained by solving

$$\Xi(\rho) \equiv \rho^2 - \left(1 - \frac{b_{12}}{b_{22}} + \frac{\eta}{b_{22}}\right) \rho + \frac{\eta - 1}{b_{22}} = 0.$$

If  $\sigma < 1$ , then we have  $b_{12} < 0$  and  $b_{22} > 0$  which imply

$$\Xi(\rho) > 0, \quad \text{for all } \rho < 0,$$

$$\Xi(0) = \frac{\eta - 1}{b_{22}} > 0, \quad \text{as } \eta > 1,$$

$$\Xi(1) \equiv 1 - \left(1 - \frac{b_{12}}{b_{22}} + \frac{\eta}{b_{22}}\right) + \frac{\eta - 1}{b_{22}} = \frac{b_{12} - 1}{b_{22}} < 0.$$

The last two inequalities ensure that one of the characteristic roots can be found within the interval of  $(0, 1)$ . This rules out the possibility of complex roots. Since  $\Xi(\rho) > 0$  for all  $\rho \leq 0$ , both  $\rho_1$  and  $\rho_2$  must be strictly positive. Finally, if both  $\rho_1$  and  $\rho_2$  are within the interval of  $(0, 1]$ , then we should have  $\Xi(1) \geq 0$  instead. Thus, the second root must be greater than one. This proves that the system in (A.6) is saddle-path stable within the neighborhood of the bubbleless steady state when  $\sigma < 1$ . Proposition 4.2.2 strengthens this result by showing that this steady state is globally saddle-path stable when  $\sigma < 1$ .

If  $\sigma > 1$ , then we have  $b_{12} \in (0, 1)$  and  $b_{22} < 0$  which imply  $\Xi(0) < 0 < \Xi(1)$ . Hence, one of the characteristic roots must lie within the interval of  $(0, 1)$ . Since the product of roots  $\Xi(0)$  is strictly negative, the second characteristic root must be strictly negative. If  $\Xi(-1) > 0$ , then the second root must lie within the interval of  $(-1, 0)$ . In this case, the linearized system has two stable roots which means the bubbleless steady state is a sink. If  $\Xi(-1) < 0$ , then the absolute magnitude of the second root is greater than one. In this case, the bubbleless steady state is again saddle-path stable. The value of  $\Xi(-1)$  is determined by

$$\Xi(-1) = 2 - \underbrace{\frac{b_{12}}{b_{22}}}_{(+)} + \underbrace{\frac{2\eta - 1}{b_{22}}}_{(-)}.$$

Unfortunately, the sign of this expression cannot be readily determined. Hence, the local stability property of the post-crash equilibrium is ambiguous when  $\sigma > 1$ .



## Bubbly Equilibrium

In this section, we will provide a detailed characterization of the consumer's problem in the pre-crash economy, and present the derivation of (3.18)-(4.20). Substituting (4.6) and (4.7) into the consumer's expected lifetime utility gives

$$\mathcal{L} = \frac{(w_t l_t - s_t - p_t m_t)^{1-\sigma}}{1-\sigma} - A \frac{l_t^{1+\psi}}{1+\psi} + \beta \left[ \frac{q (R_{t+1} s_t + p_{t+1} m_t)^{1-\sigma} + (1-q) (\hat{R}_{t+1} s_t)^{1-\sigma}}{1-\sigma} \right].$$

The first-order conditions with respect to  $s_t$ ,  $m_t$  and  $l_t$  are, respectively, given by

$$(w_t l_t - s_t - p_t m_t)^{-\sigma} = \beta \left[ q R_{t+1} (R_{t+1} s_t + p_{t+1} m_t)^{-\sigma} + (1-q) \hat{R}_{t+1} (\hat{R}_{t+1} s_t)^{-\sigma} \right], \quad (\text{A.7})$$

$$(w_t l_t - s_t - p_t m_t)^{-\sigma} = \beta q \left( \frac{p_{t+1}}{p_t} \right) (R_{t+1} s_t + p_{t+1} m_t)^{-\sigma}, \quad (\text{A.8})$$

$$A l_t^\psi = w_t (w_t l_t - s_t - p_t m_t)^{-\sigma}. \quad (\text{A.9})$$

Here we only focus on interior solutions of  $m_t$ . Define  $\pi_{t+1} \equiv p_{t+1}/p_t$ . Combining (A.1) and (A.8) gives

$$\begin{aligned} q \pi_{t+1} (R_{t+1} s_t + p_{t+1} m_t)^{-\sigma} &= q R_{t+1} (R_{t+1} s_t + p_{t+1} m_t)^{-\sigma} + (1-q) \hat{R}_{t+1} (\hat{R}_{t+1} s_t)^{-\sigma}, \\ \Rightarrow q (\pi_{t+1} - R_{t+1}) (R_{t+1} s_t + p_{t+1} m_t)^{-\sigma} &= (1-q) \hat{R}_{t+1} (\hat{R}_{t+1} s_t)^{-\sigma}, \\ \Rightarrow R_{t+1} s_t + p_{t+1} m_t &= \underbrace{\left[ \frac{q (\pi_{t+1} - R_{t+1})}{(1-q) \hat{R}_{t+1}} \right]^{\frac{1}{\sigma}}}_{\Omega_{t+1}} (\hat{R}_{t+1} s_t), \end{aligned} \quad (\text{A.10})$$

$$\begin{aligned} \Rightarrow m_t &= \frac{1}{p_{t+1}} (\Omega_{t+1} \hat{R}_{t+1} - R_{t+1}) s_t, \\ \Rightarrow s_t + p_t m_t &= \left[ 1 + \frac{R_{t+1}}{\pi_{t+1}} (\Lambda_{t+1} - 1) \right] s_t, \end{aligned} \quad (\text{A.11})$$

where  $\Lambda_{t+1} \equiv \Omega_{t+1} \hat{R}_{t+1} / R_{t+1}$ . Using (A.8) and (A.10), we can get

$$R_{t+1} s_t + p_{t+1} m_t = (\beta q \pi_{t+1})^{\frac{1}{\sigma}} (w_t l_t - s_t - p_t m_t) = \Omega_{t+1} \hat{R}_{t+1} s_t,$$

$$\Rightarrow s_t = \left\{ \frac{(\beta q \pi_{t+1})^{\frac{1}{\sigma}}}{\Omega_{t+1} \widehat{R}_{t+1} + (\beta q \pi_{t+1})^{\frac{1}{\sigma}} \left[ 1 + \frac{R_{t+1}}{\pi_{t+1}} (\Lambda_{t+1} - 1) \right]} \right\} w_t l_t. \quad (\text{A.12})$$

Using this and (A.6), we can obtain

$$c_{y,t} = w_t l_t - (s_t + p_t m_t) = \left\{ \frac{\Omega_{t+1} \widehat{R}_{t+1}}{\Omega_{t+1} \widehat{R}_{t+1} + (\beta q \pi_{t+1})^{\frac{1}{\sigma}} \left[ 1 + \frac{R_{t+1}}{\pi_{t+1}} (\Lambda_{t+1} - 1) \right]} \right\} w_t l_t. \quad (\text{A.13})$$

Substituting this into (A.9) and rearranging terms give

$$A l_t^{\psi+\sigma} = (w_t)^{1-\sigma} \left\{ \frac{\Omega_{t+1} \widehat{R}_{t+1} + (\beta q \pi_{t+1})^{\frac{1}{\sigma}} \left[ 1 + \frac{R_{t+1}}{\pi_{t+1}} (\Lambda_{t+1} - 1) \right]}{\Omega_{t+1} \widehat{R}_{t+1}} \right\}^{\sigma}. \quad (\text{A.14})$$

These equations characterize the optimal choice of  $c_{y,t}$ ,  $l_t$ ,  $s_t$  and  $m_t$  before the crash.

We now provide the derivation of (3.18)-(4.20). In equilibrium, the market for physical capital clears when

$$\begin{aligned} (1+n) k_{t+1} = s_t &= \left\{ \frac{(\beta q \pi_{t+1})^{\frac{1}{\sigma}}}{\Omega_{t+1} \widehat{R}_{t+1} + (\beta q \pi_{t+1})^{\frac{1}{\sigma}} \left[ 1 + \frac{R_{t+1}}{\pi_{t+1}} (\Lambda_{t+1} - 1) \right]} \right\} w_t l_t \\ \Rightarrow (1+n) k_{t+1} &= \left\{ \frac{(\beta q \pi_{t+1})^{\frac{1}{\sigma}}}{\Omega_{t+1} \widehat{R}_{t+1} + (\beta q \pi_{t+1})^{\frac{1}{\sigma}} \left[ 1 + \frac{R_{t+1}}{\pi_{t+1}} (\Lambda_{t+1} - 1) \right]} \right\} \left( \frac{1-\alpha}{\alpha} \right) R_t k_t. \end{aligned} \quad (\text{A.15})$$

The second line uses the fact that  $\alpha w_t l_t = (1-\alpha) R_t k_t$ . Combining (A.7) and (A.9) gives

$$A l_t^{\psi+\sigma} = (w_t)^{1-\sigma} \left\{ \frac{(\beta q \pi_{t+1})^{\frac{1}{\sigma}}}{\Omega_{t+1} \widehat{R}_{t+1}} \left[ \frac{1-\alpha}{\alpha (1+n)} \right] \frac{R_t k_t}{k_{t+1}} \right\}^{\sigma}. \quad (\text{A.16})$$

Upon setting  $k_{t+1} = k_t = k^*$ ,  $R_t = R_{t+1} = R^*$ ,  $\widehat{R}_{t+1} = \widehat{R}_0^*$  and  $\pi_{t+1} = 1+n$ , equation (A.9) becomes

$$1+n = \left\{ \frac{[\beta q (1+n)]^{\frac{1}{\sigma}}}{\Omega^* \widehat{R}_0^* + [\beta q (1+n)]^{\frac{1}{\sigma}} \left[ 1 + \frac{R^*}{1+n} (\Lambda^* - 1) \right]} \right\} \left( \frac{1-\alpha}{\alpha} \right) R^*, \quad (\text{A.17})$$

where  $\Lambda^* = \Omega^* \widehat{R}_0^* / R^*$ . Rearranging terms in this equation gives

$$1 + \left[ 1 + (\beta q)^{-\frac{1}{\sigma}} (1+n)^{1-\frac{1}{\sigma}} \right] \left( \frac{\Omega^* \widehat{R}_0^*}{1+n} \right) = \frac{1}{\alpha} \frac{R^*}{1+n}$$

$$\Rightarrow 1 + \left[ 1 + (\beta q)^{-\frac{1}{\sigma}} (1+n)^{1-\frac{1}{\sigma}} \right] \left( \frac{q}{1-q} \right)^{\frac{1}{\sigma}} \left( \frac{\widehat{R}_0^*}{1+n} \right)^{1-\frac{1}{\sigma}} \left( 1 - \frac{R^*}{1+n} \right)^{\frac{1}{\sigma}} = \frac{1}{\alpha} \frac{R^*}{1+n},$$

which is equation (3.18) in the text. Similarly, after substituting the stationarity conditions into (A.11), we can obtain

$$A(l^*)^{\psi+\sigma} = (w^*)^{1-\sigma} \left\{ \frac{[\beta q(1+n)]^{\frac{1}{\sigma}}}{\Omega^* \widehat{R}_0^*} \left( \frac{1-\alpha}{\alpha} \right) \frac{R^*}{1+n} \right\}^{\sigma}.$$

Equation (4.18) follows immediately from this equation. Equations (4.17) and (4.19) can be obtained from (4.2). Finally, equation (4.20) can be obtained from (4.8).

Define  $\theta^* \equiv R^*/(1+n)$ . Then we can rewrite (3.18) as

$$\Psi(\theta^*) \equiv 1 + \left[ 1 + (\beta q)^{-\frac{1}{\sigma}} (1+n)^{1-\frac{1}{\sigma}} \right] \left( \frac{q}{1-q} \right)^{\frac{1}{\sigma}} \left( \frac{\widehat{R}_0^*}{1+n} \right)^{1-\frac{1}{\sigma}} (1-\theta^*)^{\frac{1}{\sigma}} = \frac{\theta^*}{\alpha}. \quad (\text{A.18})$$

For any  $\widehat{R}_0^* > 0$  and  $\sigma > 0$ ,  $\Psi : [0, 1] \rightarrow \mathbb{R}_+$  is a strictly decreasing function that satisfies  $\Psi(0) > 0$  and  $\Psi(1) = 1 < 1/\alpha$ . Meanwhile, the right-hand side of the above equation is a straight line that passes through the origin and  $1/\alpha$  (when  $\theta^* = 1$ ). Thus, for any  $\widehat{R}_0^* > 0$  and  $\sigma > 0$ , there exists a unique  $\theta^* \in (0, 1)$  that solves (A.12). Once  $\theta^*$  is determined, the value of  $\{k^*, w^*, l^*, a^*\}$  can be uniquely determined using (4.17)-(4.20).

### ***Propensity to Consume When Young***

Using (A.13), we can get

$$\begin{aligned} \frac{c_y^*}{w^* l^*} &= \frac{\Omega^* \widehat{R}_0^*}{\Omega^* \widehat{R}_0^* + \beta^{\frac{1}{\sigma}} [q(1+n)]^{\frac{1}{\sigma}} \left[ 1 + \frac{R^*}{1+n} (\Lambda^* - 1) \right]} \\ &= \left\{ 1 + \beta^{\frac{1}{\sigma}} \frac{[q(1+n)]^{\frac{1}{\sigma}}}{\Omega^* \widehat{R}_0^*} \left[ 1 + \frac{R^*}{1+n} (\Lambda^* - 1) \right] \right\}^{-1} \equiv \left[ 1 + \beta^{\frac{1}{\sigma}} (\rho^*)^{\frac{1}{\sigma}-1} \right]^{-1}, \end{aligned}$$

where  $\rho^*$  is the certainty equivalent return defined in the text. An alternative expression for the propensity to consume can be obtained as follows. First, rewrite the above expression

as

$$\frac{c_y^*}{w^* l^*} = \frac{\Omega^* \hat{R}_0^*}{[\beta q (1+n)]^{\frac{1}{\sigma}}} \left\{ \frac{[\beta q (1+n)]^{\frac{1}{\sigma}}}{\Omega^* \hat{R}_0^* + [\beta q (1+n)]^{\frac{1}{\sigma}} \left[ 1 + \frac{R^*}{1+n} (\Lambda^* - 1) \right]} \right\}. \quad (\text{A.19})$$

Using (A.13), we can obtain

$$\frac{[\beta q (1+n)]^{\frac{1}{\sigma}}}{\Omega^* \hat{R}_0^* + [\beta q (1+n)]^{\frac{1}{\sigma}} \left[ 1 + \frac{R^*}{1+n} (\Lambda^* - 1) \right]} = \frac{\alpha (1+n)}{1-\alpha} \frac{1}{R^*}.$$

Substituting this into (A.10) gives

$$\frac{c_y^*}{w^* l^*} = \frac{\Omega^* \hat{R}_0^*}{[\beta q (1+n)]^{\frac{1}{\sigma}}} \left[ \frac{\alpha (1+n)}{1-\alpha} \frac{1}{R^*} \right].$$

On the other hand, in the bubbleless steady state, we have

$$\frac{\hat{c}_y^*}{\hat{w}^* \hat{l}^*} = \left[ 1 + \beta^{\frac{1}{\sigma}} \left( \hat{R}^* \right)^{\frac{1}{\sigma}-1} \right]^{-1} = \frac{\alpha (1+n)}{1-\alpha} \left( \beta \hat{R}^* \right)^{-\frac{1}{\sigma}}.$$

The second equality follows from (4.15). Hence, we have

$$\frac{\hat{c}_y^*}{\hat{w}^* \hat{l}^*} > \frac{c_y^*}{w^* l^*} \quad \Leftrightarrow \quad \left( \hat{R}^* \right)^{-\frac{1}{\sigma}} > \frac{\Omega^* \hat{R}_0^*}{[q (1+n)]^{\frac{1}{\sigma}} R^*} \quad \Leftrightarrow \quad \left[ \frac{q (1+n)}{\hat{R}^*} \right]^{\frac{1}{\sigma}} > \frac{\Omega^* \hat{R}_0^*}{R^*}.$$

## II: Proofs

### Proof of Proposition 4.2.1

In any bubbleless steady state, we have  $\widehat{k}_{t+1} = \widehat{k}_t = \widehat{k}^*$  and  $\widehat{R}_{t+1} = \widehat{R}_t = \widehat{R}^*$  for all  $t$ .

Substituting these into (4.5) and rearranging terms gives

$$\Gamma(\widehat{R}^*) \equiv \frac{\beta^{\frac{1}{\sigma}} (\widehat{R}^*)^{\frac{1}{\sigma}}}{1 + \beta^{\frac{1}{\sigma}} (\widehat{R}^*)^{\frac{1}{\sigma}-1}} = \frac{(1+n)\alpha}{1-\alpha}. \quad (\text{A.20})$$

Substituting the steady state conditions into (4.14) and rearranging terms gives (4.16).

Note that the function  $\Gamma : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  defined in (A.20) is continuously differentiable and satisfies  $\Gamma(0) = 0$ . Straightforward differentiation gives

$$\Gamma'(\widehat{R}) = \frac{\beta^{\frac{1}{\sigma}} \widehat{R}^{\frac{1}{\sigma}-1} \left( \frac{1}{\sigma} + \beta^{\frac{1}{\sigma}} \widehat{R}^{\frac{1}{\sigma}-1} \right)}{\left( 1 + \beta^{\frac{1}{\sigma}} \widehat{R}^{\frac{1}{\sigma}-1} \right)^2} > 0, \quad \text{for any } \sigma > 0.$$

Hence, there exists a unique value of  $\widehat{R}^* > 0$  that solves (A.20). Using (4.16), one can obtain a unique value of  $\widehat{k}^* > 0$ . This establishes the existence and uniqueness of bubbleless steady state.

### Proof of Proposition 4.2.2

First, consider the case when  $\sigma = 1$ . Equations (4.13) and (4.14) now become

$$\widehat{k}_{t+1} = \frac{1-\alpha}{\alpha(1+n)} \left( \frac{\beta}{1+\beta} \right) \widehat{R}_t \widehat{k}_t, \quad \text{and} \quad \widehat{R}_t^{\frac{1}{1-\alpha}} \widehat{k}_t = \alpha^{\frac{1}{1-\alpha}} \left( \frac{1+\beta}{A} \right)^{\frac{1}{1+\psi}}. \quad (\text{A.21})$$

Combining the two gives

$$\widehat{k}_{t+1} = \frac{\beta(1-\alpha)}{(1+\beta)(1+n)} \left( \frac{1+\beta}{A} \right)^{\frac{1-\alpha}{1+\psi}} \widehat{k}_t^\alpha.$$

Since  $\alpha \in (0, 1)$ , there exists a unique non-trivial steady state  $\widehat{k}^* > 0$  which is globally stable. The second equation in (A.21) can be rewritten as

$$\widehat{R}_t = \alpha \left( \frac{1+\beta}{A} \right)^{\frac{1-\alpha}{1+\psi}} (\widehat{k}_t)^{\alpha-1} \equiv \Phi(\widehat{k}_t),$$

where  $\Phi(\cdot)$  is a strictly decreasing function.

Next, consider the case when  $\sigma < 1$ . To prove that the bubbleless steady state is globally saddle-path stable, we will use the same “phase diagram” approach as in Tirole (1985) and Weil (1987). To start, define a function  $\mathcal{F} : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  according to

$$\mathcal{F}(R) = \alpha^\eta \left[ \frac{(1-\alpha)^{1-\sigma}}{A} \right]^{\frac{1}{\sigma+\psi}} \left( 1 + \beta^{\frac{1}{\sigma}} R^{\frac{1}{\sigma}-1} \right)^{\frac{\sigma}{\sigma+\psi}} R^{-\eta}. \quad (\text{A.22})$$

Note that the unique bubbleless steady state must satisfy  $\hat{k}^* = \mathcal{F}(\hat{R}^*)$ . Taking the logarithm of both sides of (A.22) and differentiating the resultant expression with respect to  $R$  gives

$$\frac{R\mathcal{F}'(R)}{\mathcal{F}(R)} = \frac{1-\sigma}{\sigma+\psi} \left( \frac{\beta^{\frac{1}{\sigma}} R^{\frac{1}{\sigma}-1}}{1 + \beta^{\frac{1}{\sigma}} R^{\frac{1}{\sigma}-1}} - \tilde{\eta} \right) = \frac{1-\sigma}{\sigma+\psi} [\Sigma(R) - \tilde{\eta}],$$

where  $\tilde{\eta} \equiv (\sigma + \psi)\eta / (1 - \sigma)$  and  $\Sigma(\cdot)$  is the function defined in (4.5). There are two possible scenarios: (i)  $\tilde{\eta} \geq 1$  and (ii)  $\tilde{\eta} < 1$ . Since  $\Sigma(\cdot)$  is strictly increasing and bounded above by one, in the first scenario we have  $\mathcal{F}'(R) < 0$  for all  $R \geq 0$ ,  $\lim_{R \rightarrow 0} \mathcal{F}(R) = +\infty$  and  $\lim_{R \rightarrow \infty} \mathcal{F}(R) = 0$ . In the second scenario,  $\mathcal{F}(\cdot)$  is a U-shaped function. Figures B1 and B2 provide a graphical illustration of these two scenarios. In both diagrams, the function  $\mathcal{F}(\cdot)$  and the vertical line representing  $R = \hat{R}^*$  divide the  $(R, k)$ -space into four quadrants:

$$Q_1 \equiv \left\{ (R, k) : k \leq \mathcal{F}(R), R \leq \hat{R}^*, \text{ and } (R, k) \neq (\hat{R}^*, \hat{k}^*) \right\},$$

$$Q_2 \equiv \left\{ (R, k) : k > \mathcal{F}(R) \text{ and } R < \hat{R}^* \right\},$$

$$Q_3 \equiv \left\{ (R, k) : k \geq \mathcal{F}(R), R \geq \hat{R}^*, \text{ and } (R, k) \neq (\hat{R}^*, \hat{k}^*) \right\},$$

$$Q_4 \equiv \left\{ (R, k) : k < \mathcal{F}(R) \text{ and } R > \hat{R}^* \right\}.$$

The rest of the proof is divided into a number of intermediate steps. These steps are valid both when  $\tilde{\eta} \geq 1$  and when  $\tilde{\eta} < 1$ .

### Step 1

For any initial value  $\left(\widehat{R}_T, \widehat{k}_T\right) > 0$ , there exists a unique sequence  $\left\{\widehat{R}_{T+1}, \widehat{k}_{T+1}, \widehat{R}_{T+2}, \widehat{k}_{T+2}, \dots\right\}$  that solves the dynamical system in (4.13)-(4.14). Whether this is part of a non-stationary bubbleless equilibrium depends on the location of  $\left(\widehat{R}_T, \widehat{k}_T\right)$  on the  $(R, k)$ -space. A solution  $\left\{\widehat{R}_{T+1}, \widehat{k}_{T+1}, \widehat{R}_{T+2}, \widehat{k}_{T+2}, \dots\right\}$  is said to originate from  $Q_n$  if  $\left(\widehat{R}_T, \widehat{k}_T\right) \in Q_n$ , for  $n \in \{1, 2, 3, 4\}$ . In the first step of the proof, it is shown that any solution that originates from  $Q_1$  or  $Q_3$  cannot be part of a bubbleless equilibrium.

Suppose  $\left(\widehat{R}_t, \widehat{k}_t\right)$  is in  $Q_1$  for some  $t \geq T$ . This means either (i)  $\widehat{k}_t < \mathcal{F}\left(\widehat{R}_t\right)$  and  $\widehat{R}_t \leq \widehat{R}^*$ , or (ii)  $\widehat{k}_t = \mathcal{F}\left(\widehat{R}_t\right)$  and  $\widehat{R}_t < \widehat{R}^*$ . First consider the case when  $\widehat{k}_t < \mathcal{F}\left(\widehat{R}_t\right)$  and  $\widehat{R}_t \leq \widehat{R}^*$ . Using (4.14), we can obtain

$$\begin{aligned} \widehat{R}_t^\eta \widehat{k}_t &= \alpha^\eta \left[ \frac{(1-\alpha)^{1-\sigma}}{A} \right]^{\frac{1}{\sigma+\psi}} \left[ 1 + \beta^{\frac{1}{\sigma}} \left( \widehat{R}_{t+1} \right)^{\frac{1}{\sigma}-1} \right]^{\frac{\sigma}{\sigma+\psi}} \\ &< \alpha^\eta \left[ \frac{(1-\alpha)^{1-\sigma}}{A} \right]^{\frac{1}{\sigma+\psi}} \left[ 1 + \beta^{\frac{1}{\sigma}} \left( \widehat{R}_t \right)^{\frac{1}{\sigma}-1} \right]^{\frac{\sigma}{\sigma+\psi}}, \end{aligned}$$

which implies  $\widehat{R}_{t+1} < \widehat{R}_t \leq \widehat{R}^*$ . Recall that the function  $\Sigma(\cdot)$  defined in (4.5) is strictly increasing when  $\sigma < 1$ . Then it follows from (4.13) that

$$\begin{aligned} \widehat{k}_{t+1} &= \frac{1-\alpha}{\alpha(1+n)} \Sigma\left(\widehat{R}_{t+1}\right) \widehat{R}_t \widehat{k}_t \\ &< \frac{1-\alpha}{\alpha(1+n)} \Sigma\left(\widehat{R}^*\right) \widehat{R}_t \widehat{k}_t \leq \frac{1-\alpha}{\alpha(1+n)} \Sigma\left(\widehat{R}^*\right) \widehat{R}^* \widehat{k}_t = \widehat{k}_t. \end{aligned}$$

The last equality follows from equation (4.15). This result implies  $\widehat{k}_{t+1} < \widehat{k}_t < \mathcal{F}\left(\widehat{R}_t\right) < \mathcal{F}\left(\widehat{R}_{t+1}\right)$ . Next, consider the case when  $\widehat{k}_t = \mathcal{F}\left(\widehat{R}_t\right)$  and  $\widehat{R}_t < \widehat{R}^*$ . Equation (4.14) and  $\widehat{k}_t = \mathcal{F}\left(\widehat{R}_t\right)$  together imply  $\widehat{R}_{t+1} = \widehat{R}_t < \widehat{R}^*$ . This, together with (4.13), implies  $\widehat{k}_{t+1} < \widehat{k}_t < \mathcal{F}\left(\widehat{R}_t\right) = \mathcal{F}\left(\widehat{R}_{t+1}\right)$ . This proves the following: Any solution that originates from  $Q_1$  is a strictly decreasing sequence and is confined in  $Q_1$ , i.e.,  $\left(\widehat{R}_t, \widehat{k}_t\right) \in Q_1$  for all

$t \geq T$ . Since both  $\widehat{k}_t$  and  $\widehat{R}_t$  are strictly decreasing over time, in the long run we will have either  $\widehat{k}_t = 0$  or  $\widehat{R}_t = 0$ , which cannot happen in equilibrium.

Using a similar argument, we can show that any solution that originates from  $Q_3$  is a strictly increasing sequence and is confined in  $Q_3$ . Using the young consumer's budget constraint and the capital market clearing condition, we can obtain the following condition

$$\widehat{s}_t = \frac{\widehat{k}_{t+1}}{1+n} < \widehat{w}_t \widehat{l}_t \leq \widehat{w}_t = (1-\alpha) \left( \frac{\alpha}{\widehat{R}_t} \right)^{\frac{\alpha}{1-\alpha}}.$$

Obviously, this will be violated at some point if both  $\widehat{k}_t$  and  $\widehat{R}_t$  are strictly increasing over time. Hence, any solution that originates from  $Q_3$  cannot be part of a bubbleless equilibrium.

## Step 2

We now show that any solution that originates from  $Q_2$  will never enter  $Q_4$ , i.e.,  $(\widehat{R}_T, \widehat{k}_T) \in Q_2$  implies  $(\widehat{R}_t, \widehat{k}_t) \notin Q_4$ , for all  $t > T$ ; likewise, any solution that originates from  $Q_4$  will never enter  $Q_2$ .

Suppose  $(\widehat{R}_t, \widehat{k}_t)$  is in  $Q_2$  for some  $t \geq T$ . Then we have

$$\begin{aligned} \widehat{R}_t^\eta \widehat{k}_t &= \alpha^\eta \left[ \frac{(1-\alpha)^{1-\sigma}}{A} \right]^{\frac{1}{\sigma+\psi}} \left[ 1 + \beta^{\frac{1}{\sigma}} \left( \widehat{R}_{t+1} \right)^{\frac{1}{\sigma}-1} \right]^{\frac{\sigma}{\sigma+\psi}} \\ &> \alpha^\eta \left[ \frac{(1-\alpha)^{1-\sigma}}{A} \right]^{\frac{1}{\sigma+\psi}} \left[ 1 + \beta^{\frac{1}{\sigma}} \left( \widehat{R}_t \right)^{\frac{1}{\sigma}-1} \right]^{\frac{\sigma}{\sigma+\psi}}, \end{aligned}$$



which implies  $\widehat{R}_{t+1} > \widehat{R}_t$ . Suppose the contrary that  $(\widehat{R}_{t+1}, \widehat{k}_{t+1})$  is in  $Q_4$ , so that  $\widehat{R}_{t+1} > \widehat{R}^* > \widehat{R}_t$  and  $\widehat{k}_{t+1} < \mathcal{F}(\widehat{R}_{t+1})$ . Then, using (4.13) we can get

$$\begin{aligned} \widehat{R}_{t+1}\widehat{k}_{t+1} &= \frac{1-\alpha}{\alpha(1+n)} \left[ \frac{\beta^{\frac{1}{\sigma}} (\widehat{R}_{t+1})^{\frac{1}{\sigma}}}{1 + \beta^{\frac{1}{\sigma}} (\widehat{R}_{t+1})^{\frac{1}{\sigma}-1}} \right] \widehat{R}_t \widehat{k}_t \\ &> \frac{1-\alpha}{\alpha(1+n)} \left[ \frac{\beta^{\frac{1}{\sigma}} (\widehat{R}^*)^{\frac{1}{\sigma}}}{1 + \beta^{\frac{1}{\sigma}} (\widehat{R}^*)^{\frac{1}{\sigma}-1}} \right] \widehat{R}_t \widehat{k}_t = \widehat{R}_t \widehat{k}_t. \end{aligned} \quad (\text{A.23})$$

The second line uses the fact that  $\Sigma(\cdot)$  is strictly increasing and  $\widehat{R}_{t+1} > \widehat{R}^*$ . The last equality follows from the steady-state condition in (4.15). Since  $\eta > 1$ , we also have  $\widehat{R}_{t+1}^{\eta-1} > \widehat{R}_t^{\eta-1}$ . This, together with (4.14) and (A.23), implies

$$\begin{aligned} \widehat{R}_{t+1}^\eta \widehat{k}_{t+1} &> \widehat{R}_t^\eta \widehat{k}_t = \alpha^\eta \left[ \frac{(1-\alpha)^{1-\sigma}}{A} \right]^{\frac{1}{\sigma+\psi}} \left[ 1 + \beta^{\frac{1}{\sigma}} (\widehat{R}_{t+1})^{\frac{1}{\sigma}-1} \right]^{\frac{\sigma}{\sigma+\psi}} \\ &\Rightarrow \widehat{k}_{t+1} > \mathcal{F}(\widehat{R}_{t+1}), \end{aligned}$$

which gives rise to a contradiction. Hence, any solution that originates from  $Q_2$  will never enter  $Q_4$ . Using similar arguments, we can show that any solution that originates from  $Q_4$  will never enter  $Q_2$ .

### Step 3

Consider a solution that originates from  $Q_2$ . As shown in Step 2,  $(\widehat{R}_T, \widehat{k}_T) \in Q_2$  implies  $\widehat{R}_{T+1} > \widehat{R}_T$ . If  $\widehat{R}_{T+1} \geq \widehat{R}^*$ , then the economy is in  $Q_3$  at time  $T+1$  and by the results in Step 1, we know that  $\widehat{R}_t$  will diverge to infinity in the long run. If  $\widehat{R}_{T+1} < \widehat{R}^*$ ,

then using (4.13) we can obtain

$$\begin{aligned}\widehat{k}_{T+1} &= \frac{1-\alpha}{\alpha(1+n)} \left[ \frac{\beta^{\frac{1}{\sigma}} \left( \widehat{R}_{T+1} \right)^{\frac{1}{\sigma}-1}}{1 + \beta^{\frac{1}{\sigma}} \left( \widehat{R}_{T+1} \right)^{\frac{1}{\sigma}-1}} \right] \widehat{R}_T \widehat{k}_T \\ &< \frac{1-\alpha}{\alpha(1+n)} \left[ \frac{\beta^{\frac{1}{\sigma}} \left( \widehat{R}^* \right)^{\frac{1}{\sigma}}}{1 + \beta^{\frac{1}{\sigma}} \left( \widehat{R}^* \right)^{\frac{1}{\sigma}-1}} \right] \widehat{k}_T = \widehat{k}_T.\end{aligned}$$

There are two possible scenarios: First, if  $\widehat{R}_{T+1} < \widehat{R}^*$  and  $\widehat{k}_{T+1} \leq \mathcal{F}(\widehat{R}_{T+1})$ , then the economy is in  $Q_1$  at time  $T+1$ . By the results in Step 1, we know that all subsequent values of  $\widehat{R}_t$  will be strictly less than  $\widehat{R}^*$ . Second, if  $\widehat{R}_{T+1} < \widehat{R}^*$  and  $\mathcal{F}(\widehat{R}_{T+1}) < \widehat{k}_{T+1}$ , then that means the economy remains in  $Q_2$  at time  $T+1$ . In addition, we have  $\widehat{R}_{T+1} > \widehat{R}_T$  and  $\widehat{k}_T > \widehat{k}_{T+1}$  which means the economy is now getting closer to the steady state  $(\widehat{R}^*, \widehat{k}^*)$ . Thus, any solution that originates from  $Q_2$  has three possible fates: (i) It will enter  $Q_3$  at some point and  $\widehat{R}_t$  will then diverge to infinity. (ii) It will enter  $Q_1$  at some point and  $\widehat{R}_t$  will be strictly less than  $\widehat{R}^*$  afterward. (iii) It will converge to the bubbleless steady state. For reasons explained above, the first two types of solutions cannot be part of an equilibrium. Hence, a solution originating from  $Q_2$  is an equilibrium path only if it converges to the steady state  $(\widehat{R}^*, \widehat{k}^*)$ . The above argument also shows that, along the convergent path,  $\widehat{k}_t$  is decreasing towards  $\widehat{k}^*$  while  $\widehat{R}_t$  is increasing towards  $\widehat{R}^*$ .

Using a similar argument, we can show that any solution originating from  $Q_4$  is an equilibrium path only if it converges to the steady state  $(\widehat{R}^*, \widehat{k}^*)$ , and that along the convergent path,  $\widehat{k}_t$  is increasing towards  $\widehat{k}^*$  while  $\widehat{R}_t$  is decreasing towards  $\widehat{R}^*$ .

#### Step 4

We now establish the uniqueness of saddle path. Fix  $\widehat{k}_T > 0$ . Suppose the contrary that there exists two saddle paths, denoted by  $\{\widehat{R}'_t, \widehat{k}'_t\}_{t=T}^{\infty}$  and  $\{\widehat{R}''_t, \widehat{k}''_t\}_{t=T}^{\infty}$ , with  $\widehat{k}'_T =$

$\widehat{k}_T'' = \widehat{k}_T$  and  $\widehat{R}_T' > \widehat{R}_T'' > 0$ . By the results in Step 3, we know that  $\lim_{t \rightarrow \infty} \widehat{R}_t' = \lim_{t \rightarrow \infty} \widehat{R}_t'' = \widehat{R}^*$ .

Substituting  $\widehat{k}_T' = \widehat{k}_T''$  and  $\widehat{R}_T' > \widehat{R}_T''$  into (4.14) gives

$$\left( \frac{\widehat{R}_T'}{\widehat{R}_T''} \right)^\eta = \left[ \frac{1 + \beta^{\frac{1}{\sigma}} \left( \widehat{R}_{T+1}' \right)^{\frac{1}{\sigma}-1}}{1 + \beta^{\frac{1}{\sigma}} \left( \widehat{R}_{T+1}'' \right)^{\frac{1}{\sigma}-1}} \right]^{\frac{\sigma}{\sigma+\psi}} > 1,$$

which implies  $\widehat{R}_{T+1}' > \widehat{R}_{T+1}'' > 0$ . Using (4.13), we can get

$$\frac{\widehat{k}_{T+1}'}{\widehat{k}_{T+1}''} = \frac{\Sigma \left( \widehat{R}_{T+1}' \right) \widehat{R}_T'}{\Sigma \left( \widehat{R}_{T+1}'' \right) \widehat{R}_T''} > 1.$$

Using (4.14) again, but now for  $t = T + 1$ , gives

$$\left( \frac{\widehat{R}_{T+1}'}{\widehat{R}_{T+1}''} \right)^\eta \left( \frac{\widehat{k}_{T+1}'}{\widehat{k}_{T+1}''} \right) = \left[ \frac{1 + \beta^{\frac{1}{\sigma}} \left( \widehat{R}_{T+2}' \right)^{\frac{1}{\sigma}-1}}{1 + \beta^{\frac{1}{\sigma}} \left( \widehat{R}_{T+2}'' \right)^{\frac{1}{\sigma}-1}} \right]^{\frac{\sigma}{\sigma+\psi}} > 1,$$

which implies  $\widehat{R}_{T+2}' > \widehat{R}_{T+2}''$ . By an induction argument, we can show that  $\widehat{R}_{T+j}' > \widehat{R}_{T+j}''$  implies  $\widehat{k}_{T+j}' > \widehat{k}_{T+j}''$ , and  $\widehat{R}_{T+j+1}' > \widehat{R}_{T+j+1}''$ , for all  $j \geq 1$ . The last result contradicts  $\lim_{t \rightarrow \infty} \widehat{R}_t' = \lim_{t \rightarrow \infty} \widehat{R}_t'' = \widehat{R}^*$ . Hence, we can rule out the possibility of multiple saddle paths.

In sum, we have shown that any equilibrium path that originates from a given value of  $\widehat{k}_T > 0$  must be unique and converge to the bubbleless steady state. Hence, the dynamical system in (4.13)-(4.14) is globally saddle-path stable. The one-to-one relationship between  $\widehat{R}_T$  and  $\widehat{k}_T$  can be captured by a function  $\Phi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ . Since the saddle path is downward sloping in the  $(R, k)$ -space,  $\Phi(\cdot)$  must be strictly decreasing. This completes the proof of Proposition 4.2.2.

### Proof of Proposition 4.2.3

In the post-crash economy, optimal labor supply is determined by (4.4). Setting  $\sigma = 1$  gives  $\widehat{l}_t = \left( \frac{1+\beta}{A} \right)^{\frac{1}{1+\psi}}$  for all  $t$ . In the pre-crash economy, optimal labor supply is determined

by

$$Al_t^{\psi+\sigma} = (w_t)^{1-\sigma} \left\{ \frac{\Omega_{t+1} \widehat{R}_{t+1} + (\beta q \pi_{t+1})^{\frac{1}{\sigma}} \left[ 1 + \frac{R_{t+1}}{\pi_{t+1}} (\Lambda_{t+1} - 1) \right]}{\Omega_{t+1} \widehat{R}_{t+1}} \right\}^{\sigma},$$

which is equation (A.7) in Appendix A, where

$$\Omega_{t+1} \equiv \left[ \frac{q(\pi_{t+1} - R_{t+1})}{(1-q)\widehat{R}_{t+1}} \right]^{\frac{1}{\sigma}} \quad \text{and} \quad \Lambda_{t+1} \equiv \frac{\Omega_{t+1} \widehat{R}_{t+1}}{R_{t+1}}.$$

When  $\sigma = 1$ , the right-hand side (RHS) of the above equation becomes

$$\begin{aligned} RHS &= 1 + \left( \Omega_{t+1} \widehat{R}_{t+1} \right)^{-1} (\beta q \pi_{t+1}) \left[ 1 + \frac{R_{t+1}}{\pi_{t+1}} (\Lambda_{t+1} - 1) \right] \\ &= 1 + \frac{\beta(1-q)}{\pi_{t+1} - R_{t+1}} \left( \pi_{t+1} + \Omega_{t+1} \widehat{R}_{t+1} - R_{t+1} \right) \\ &= 1 + \frac{\beta(1-q)}{\pi_{t+1} - R_{t+1}} \left[ \pi_{t+1} - R_{t+1} + \frac{q(\pi_{t+1} - R_{t+1})}{1-q} \right] = 1 + \beta. \end{aligned}$$

Hence, we have  $Al_t^{\psi+1} = 1 + \beta$  for all  $t$ . The desired result follows immediately from this expression. This completes the proof of Proposition 4.2.3.

#### Proof of Proposition 4.2.4

The main ideas of the proof are as follows. In any conditional bubbly steady state, we have  $a^* > 0$  which is equivalent to  $\Lambda^* > 1$ . This, together with  $\sigma < 1$  and  $R^* \leq \widehat{R}^*$ , implies two things:  $k^* > \widehat{k}^*$  and  $\widehat{R}_0^* \equiv \Phi(k^*) > \widehat{R}^*$ . But as we have seen in Proposition 4.2.2, these two results cannot be both true which means we have reached a contradiction. Hence, it must be the case that  $R^* > \widehat{R}^*$ .

The main task of the proof is to verify the following two claims:

#### Claim #1

*Suppose  $\sigma < 1$  and  $\Lambda^* > 1$ . Then  $R^* \leq \widehat{R}^*$  implies  $l^* > \widehat{l}^*$  and  $k^* > \widehat{k}^*$ .*

**Claim #2**

Suppose  $\sigma < 1$  and  $\Lambda^* > 1$ . Then  $R^* \leq \widehat{R}^*$  implies  $\widehat{R}^* < \widehat{R}_0^*$ .

**Proof of Claim #1**

Suppose  $R^* \leq \widehat{R}^*$  and  $l^* > \widehat{l}^*$  are both true. Then using (4.19), we can get

$$k^* = l^* \left( \frac{\alpha}{R^*} \right)^{\frac{1}{1-\alpha}} > \widehat{l}^* \left( \frac{\alpha}{\widehat{R}^*} \right)^{\frac{1}{1-\alpha}} = \widehat{k}^*.$$

Hence, it suffice to show that  $R^* \leq \widehat{R}^*$  implies  $l^* > \widehat{l}^*$ .

When evaluated in a recurring bubbly equilibrium, equation (A.7) becomes

$$\begin{aligned} A(l^*)^{\psi+\sigma} &= (w^*)^{1-\sigma} \left\{ 1 + \frac{[\beta q(1+n)]^{\frac{1}{\sigma}}}{\Omega^* \widehat{R}_0^*} \left[ 1 + \frac{R^*}{1+n} (\Lambda^* - 1) \right] \right\}^{\sigma} \\ &= \left[ (1-\alpha) \left( \frac{\alpha}{R^*} \right)^{\frac{\alpha}{1-\alpha}} \right]^{1-\sigma} \left\{ 1 + \frac{[\beta q(1+n)]^{\frac{1}{\sigma}}}{\Omega^* \widehat{R}_0^*} \left[ 1 + \frac{R^*}{1+n} (\Lambda^* - 1) \right] \right\}^{\sigma}. \end{aligned}$$

On the other hand, the value of  $\widehat{l}^*$  in the bubbleless steady state is determined by

$$A(\widehat{l}^*)^{\psi+\sigma} = \left[ (1-\alpha) \left( \frac{\alpha}{\widehat{R}^*} \right)^{\frac{\alpha}{1-\alpha}} \right]^{1-\sigma} \left[ 1 + \beta^{\frac{1}{\sigma}} (\widehat{R}^*)^{\frac{1}{\sigma}-1} \right]^{\sigma}.$$

Combining the two gives

$$\left( \frac{l^*}{\widehat{l}^*} \right)^{\psi+\sigma} = \left( \frac{\widehat{R}^*}{R^*} \right)^{\frac{\alpha(1-\sigma)}{1-\alpha}} \left\{ \frac{1 + \frac{[\beta q(1+n)]^{\frac{1}{\sigma}}}{\Omega^* \widehat{R}_0^*} \left[ 1 + \frac{R^*}{1+n} (\Lambda^* - 1) \right]}{1 + \beta^{\frac{1}{\sigma}} (\widehat{R}^*)^{\frac{1}{\sigma}-1}} \right\}^{\sigma}.$$

Since  $\sigma < 1$  and  $R^* \leq \widehat{R}^*$ , we have  $(\widehat{R}^*/R^*)^{\frac{\alpha(1-\sigma)}{1-\alpha}} \geq 1$ . Thus, it suffice to show that

$$\frac{[\beta q(1+n)]^{\frac{1}{\sigma}}}{\Omega^* \widehat{R}_0^*} > \beta^{\frac{1}{\sigma}} (\widehat{R}^*)^{\frac{1}{\sigma}-1}.$$

Define  $\theta^* \equiv R^*/(1+n)$ ,  $\widehat{\theta}_0^* \equiv \widehat{R}_0^*/(1+n)$  and  $\widehat{\theta}^* \equiv \widehat{R}^*/(1+n)$ . As shown in Proposition 4.2.1, the value of  $\widehat{R}^*$  is determined by (4.15), which can be rewritten as

$$\left( \frac{1-\alpha}{\alpha} \right) \widehat{\theta}^* = 1 + \beta^{-\frac{1}{\sigma}} (1+n)^{1-\frac{1}{\sigma}} (\widehat{\theta}^*)^{1-\frac{1}{\sigma}}. \quad (\text{A.24})$$

On the other hand, the relationship between  $R^*$  and  $\widehat{R}_0^*$  is characterized by (3.18), which is derived from (A.13) in Appendix A. The latter can be rewritten as

$$\begin{aligned} \left(\frac{1-\alpha}{\alpha}\right)\theta^* &= 1 + \theta^*(\Lambda^* - 1) + [\beta q(1+n)]^{-\frac{1}{\sigma}} \Omega^* \widehat{R}_0^* \\ &= 1 + \theta^*(\Lambda^* - 1) + \beta^{-\frac{1}{\sigma}} q^{-\frac{1}{\sigma}} (1+n)^{1-\frac{1}{\sigma}} \Omega^* \widehat{\theta}_0^*. \end{aligned} \quad (\text{A.25})$$

Combining (A.24) and (A.25) gives

$$\left(\frac{1-\alpha}{\alpha}\right)(\theta^* - \widehat{\theta}^*) = \theta^*(\Lambda^* - 1) + \beta^{-\frac{1}{\sigma}} (1+n)^{1-\frac{1}{\sigma}} \left[ q^{-\frac{1}{\sigma}} \Omega^* \widehat{\theta}_0^* - (\widehat{\theta}^*)^{1-\frac{1}{\sigma}} \right]. \quad (\text{A.26})$$

Under the conditions  $\sigma < 1$ ,  $\Lambda^* > 1$  and  $R^* \leq \widehat{R}^*$  (i.e.,  $\theta^* \leq \widehat{\theta}^*$ ), we can get

$$\begin{aligned} q^{-\frac{1}{\sigma}} \Omega^* \widehat{\theta}_0^* &< (\widehat{\theta}^*)^{1-\frac{1}{\sigma}} \\ \Leftrightarrow q^{-\frac{1}{\sigma}} \Omega^* \left( \frac{\widehat{R}_0^*}{1+n} \right) &< \left( \frac{\widehat{R}^*}{1+n} \right)^{1-\frac{1}{\sigma}} \\ \Leftrightarrow q^{-\frac{1}{\sigma}} \Omega^* \widehat{R}_0^* &< (\widehat{R}^*)^{1-\frac{1}{\sigma}} (1+n)^{\frac{1}{\sigma}} \\ \Leftrightarrow (\widehat{R}^*)^{\frac{1}{\sigma}-1} &< \frac{[q(1+n)]^{\frac{1}{\sigma}}}{\Omega^* \widehat{R}_0^*}. \end{aligned} \quad (\text{A.27})$$

This establishes Claim #1.

## Proof of Claim #2

First, note that  $\Lambda^* > 1$  is true if and only if

$$\begin{aligned} q(1+n) &> \left[ q + (1-q) \left( \frac{\widehat{R}_0^*}{\widehat{R}^*} \right)^{1-\sigma} \right] R^* \\ \Leftrightarrow \frac{q(1-\theta^*)}{\theta^*} &> (1-q) \left( \frac{\widehat{\theta}_0^*}{\widehat{\theta}^*} \right)^{1-\sigma}. \end{aligned} \quad (\text{A.28})$$

Next, rewrite (A.27) as

$$q^{-\frac{1}{\sigma}} \left[ \frac{q(1-\theta^*)}{(1-q)\widehat{\theta}_0^*} \right]^{\frac{1}{\sigma}} \widehat{\theta}_0^* < (\widehat{\theta}^*)^{1-\frac{1}{\sigma}}$$

$$\Leftrightarrow 1 - \theta^* < (1 - q) \left( \frac{\widehat{\theta}_0^*}{\widehat{\theta}^*} \right)^{1-\sigma}. \quad (\text{A.29})$$

Using (A.28) and (A.29), and the assumptions of  $\theta^* \leq \widehat{\theta}^*$  and  $\sigma < 1$ , we can get

$$\frac{q(1 - \theta^*)}{\theta^*} > (1 - q) \left( \frac{\widehat{\theta}_0^*}{\theta^*} \right)^{1-\sigma} \geq (1 - q) \left( \frac{\widehat{\theta}_0^*}{\widehat{\theta}^*} \right)^{1-\sigma} > 1 - \theta^*,$$

which implies  $q > \theta^*$ . Using (A.29) and  $q > \theta^*$ , we can get

$$(1 - q) \left( \frac{\widehat{\theta}_0^*}{\widehat{\theta}^*} \right)^{1-\sigma} > 1 - \theta^* > 1 - q \Rightarrow \widehat{\theta}_0^* > \widehat{\theta}^*.$$

This establishes Claim #2.

# Appendix B

## Chapter 4

### I: Characterizing the Optimal Loan Contract

Throughout this section, we will assume that  $\mathcal{R}_{t+1} > 0$ ,  $\tilde{\rho}_{t+1} > 0$  and  $s_t > 0$ . We begin by stating some of the basic properties of an optimal contract.

#### Property 1

Let  $\tilde{z}(z)$  be the reported value of productivity when the true value is  $z$ . If verification does not occur, i.e.,  $\tilde{z}(z) \notin A_t$ , then the entrepreneur will always choose to report a value that minimizes his repayment. Thus, the repayment is a constant whenever  $\tilde{z}(z) \notin A_t$ . By the truthful reporting property, we have  $\tilde{z}(z) = z$  for all  $z \in [0, \bar{z}]$ . Hence,  $Q_t(z) = q_t$  whenever  $z \notin A_t$ .

#### Property 2

The repayment schedule  $Q_t(z)$  is bounded above by  $q_t$  for all  $z \in A_t$ . Suppose the contrary that  $Q_t(\hat{z}) > q_t$ , for some  $\hat{z} \in A_t$ . Then an entrepreneur with true state  $\hat{z} \in A_t$



can lower his repayment to  $q_t$  by reporting a value in  $A_t$ , which contradicts the truthful reporting property.

### Property 3

The bank's participation constraint must hold with equality under any optimal contract. Otherwise, it is possible to increase the entrepreneur's expected payoff by lowering the repayment in some states without violating the bank's participation constraint.

We now establish the debt structure of the optimal contract. Specifically, we want to show that the repayment schedule under any optimal contract with  $b_t > 0$  will take the form in (4.6). Let  $\{Q'_t(\cdot), b'_t, q'_t\}$  be an optimal contract with borrowing  $b'_t > 0$ . Define  $A'_t = \{z \in [0, \bar{z}] : Q'_t(z) \leq q'_t\}$  and  $B'_t = [0, \bar{z}] - A'_t$ . In particular,  $Q'_t(\cdot)$  and  $q'_t$  are chosen such that the bank's participation constraint holds with equality, i.e.,

$$\int_{A'_t} [Q'_t(z) - \lambda \tilde{\rho}_{t+1}(s_t + b'_t)z] dG(z) + q'_t \int_{B'_t} dG(z) = R_{t+1}b'_t. \quad (\text{A.1})$$

By the affordability condition, we have  $Q'_t(z) \leq \tilde{\rho}_{t+1}(s_t + b'_t)z$  for all  $z \in [0, \bar{z}]$ . Suppose the contrary that strict inequality holds for some  $z \in A'_t$ . Specifically, define the set  $\mathcal{S}_t$  according to

$$\mathcal{S}_t = \{z \in A'_t : Q'_t(z) < \tilde{\rho}_{t+1}(s_t + b'_t)z\}$$

and suppose  $\mathcal{S}_t$  has positive mass under the distribution  $G(\cdot)$ . Fix  $b'_t > 0$  and consider another contract  $\{Q''_t(z), b'_t, q''_t\}$  with  $Q''_t(z) = \tilde{\rho}_{t+1}z(s_t + b'_t)$  for  $z \in A''_t$ , and

$$A''_t = \left[0, \frac{q''_t}{\tilde{\rho}_{t+1}(s_t + b'_t)}\right].$$

The quantity  $q''_t$  is chosen so that the bank's participation constraint is binding, i.e.,

$$\int_{A''_t} [Q''_t(z) - \lambda \tilde{\rho}_{t+1}(s_t + b'_t)z] dG(z) + q''_t \int_{B''_t} dG(z) = R_{t+1}b'_t, \quad (\text{A.2})$$

where  $B_t'' = [0, \bar{z}] - A_t''$ . We now show that there exists at least one such  $q_t''$  in the interval  $(0, q_t')$ .

First define a function  $\Phi_t(x)$  according to

$$\Phi_t(x) \equiv \int_{A(x)} [Q_t''(z) - \lambda \tilde{\rho}_{t+1}(s_t + b_t') z] dG(z) + x \int_{B_t(x)} dG(z),$$

where

$$A(x) = \left[0, \frac{x}{\tilde{\rho}_{t+1}(s_t + b_t')}\right] \quad \text{and} \quad B(x) = [0, \bar{z}] - A(x).$$

Since  $A(0)$  is an empty set, it follows that  $\Phi_t(0) = 0 < R_{t+1}b_t'$ . If  $x = q_t'$ , then  $A(x) = A_t'$ .

Since  $Q_t''(z) > Q_t'(z)$  for all  $z \in \mathcal{S}_t$ , we have

$$\begin{aligned} \Phi_t(q_t') &= \int_{A_t'} [Q_t''(z) - \lambda \tilde{\rho}_{t+1}(s_t + b_t') z] dG(z) + q_t' \int_{B_t'} dG(z) \\ &> \int_{A_t'} [Q_t'(z) - \lambda \tilde{\rho}_{t+1}(s_t + b_t') z] dG(z) + q_t' \int_{B_t'} dG(z) = R_{t+1}b_t'. \end{aligned}$$

Since  $\Phi_t(\cdot)$  is a continuous function, by the intermediate value theorem, there exists at least one  $q_t'' \in (0, q_t')$  that solves (A.2).

For any  $z \in A_t''$ , we have  $Q_t'(z) \leq Q_t''(z) = \tilde{\rho}_{t+1}z(s_t + b_t') \leq q_t'' < q_t'$ . Hence any  $z \in A_t''$  also belongs to  $A_t'$ . This proves that  $A_t''$  is a proper subset of  $A_t'$ . Under  $\{Q_t'(z), b_t', q_t'\}$ , the entrepreneur's expected utility is given by

$$\int_0^{\bar{z}} \tilde{\rho}_{t+1}(s_t + b_t') z dG(z) - \int_{A_t'} Q_t'(z) dG(z) - q_t' \int_{B_t'} dG(z).$$

Under  $\{Q_t''(z), b_t', q_t''\}$ , his expected utility is

$$\int_0^{\bar{z}} \tilde{\rho}_{t+1}(s_t + b_t') z dG(z) - \int_{A_t''} Q_t''(z) dG(z) - q_t'' \int_{B_t''} dG(z).$$

The difference between the two is

$$\lambda \tilde{\rho}_{t+1}(s_t + b_t') \left[ \int_{A_t''} z dG(z) - \int_{A_t'} z dG(z) \right].$$

Since  $z$  is a non-negative random variable and  $A_t''$  is a proper subset of  $A_t'$ , we have  $\int_{A_t''} z dG(z) < \int_{A_t'} z dG(z)$ . This means the entrepreneur strictly prefers  $\{Q_t''(z), b_t', q_t''\}$  to  $\{Q_t'(z), b_t', q_t'\}$ , which contradicts the assumption that  $\{Q_t'(z), b_t', q_t'\}$  is optimal. Thus, the repayment schedule under any optimal contract with  $b_t > 0$  will take the form in (4.6) and there exists a unique threshold level  $\hat{z}_t \in [0, \bar{z}]$  such that verification occurs whenever  $z \leq \hat{z}_t$ , i.e.,  $A_t = [0, \hat{z}_t]$ . Using these results, we can rewrite the bank's participation constraint as

$$\begin{aligned} \tilde{\rho}_{t+1}(s_t + b_t) \left\{ (1 - \lambda) \int_0^{\hat{z}_t} z dG(z) + \hat{z}_t [1 - G(\hat{z}_t)] \right\} &= \mathcal{R}_{t+1} b_t \\ \Rightarrow J(\hat{z}_t) \tilde{\rho}_{t+1}(s_t + b_t) &= \mathcal{R}_{t+1} b_t, \end{aligned} \quad (\text{A.3})$$

where  $J(\cdot)$  is defined in the text. Likewise, we can also rewrite the entrepreneur's expected payoff as

$$\tilde{\rho}_{t+1}(s_t + b_t) \left[ \int_{\hat{z}_t}^{\bar{z}} (z - \hat{z}_t) dG(z) \right]. \quad (\text{A.4})$$

Thus, the optimal contract problem involves choosing a threshold level  $\hat{z}_t$  so as to maximize the expression in (A.4) subject to the bank's participation constraint in (A.3).

Finally, we will derive the expression of borrowing in (4.7). Since  $J(0) = 0$  and  $J(\cdot)$  is a continuous function, the following set  $\{x \in [0, \bar{z}] : \mathcal{R}_{t+1} > J(x) \tilde{\rho}_{t+1}\}$  must be non-empty. We now show that any optimal threshold value  $\hat{z}_t$  must be chosen from this set. Suppose the contrary that  $\mathcal{R}_{t+1} \leq J(\hat{z}_t) \tilde{\rho}_{t+1}$ . Then the bank's participation constraint implies  $\mathcal{R}_{t+1}(s_t + b_t) \leq J(\hat{z}_t) \tilde{\rho}_{t+1}(s_t + b_t) = \mathcal{R}_{t+1} b_t$ , which contradicts  $b_t > 0$ . Equation (4.7) can be obtained by rearranging the terms in (A.3).

## II: Proofs and Derivations

### Properties of $J(\cdot)$ and $H(\cdot)$

Much of the results in this study depend crucially on the properties of two auxiliary functions  $J(\cdot)$  and  $H(\cdot)$ , defined in (4.8) and (4.11), respectively. Thus, before we prove the main results of this paper, we first describe the main properties of these functions.

The function  $J(\cdot)$  is twice continuously differentiable, and has the following properties:

$J(0) = 0$ ,  $J(\bar{z}) = (1 - \lambda)E(\bar{z}) > 0$ , and

$$J'(x) = 1 - G(x) - \lambda x G'(x), \quad (\text{A.5})$$

where  $J'(0) = 1$  and  $J'(\bar{z}) = -\lambda \bar{z} G'(\bar{z}) < 0$ . These properties imply that  $J(\cdot)$  is a non-monotonic function. In particular, the second-order derivative of  $J(\cdot)$  is given by

$$\begin{aligned} J''(x) &= -[(1 + \lambda)G'(x) + \lambda x G''(x)] \\ &= -\{G'(x) + \lambda[G'(x) + x G''(x)]\}. \end{aligned} \quad (\text{A.6})$$

Since  $G(\cdot)$  is assumed to be strictly increasing, we have  $G'(x) > 0$  for all  $x \in [0, \bar{z}]$ . By Assumption A1, we also have  $G'(x) + x G''(x) > 0$ , for all  $x \in [0, \bar{z}]$ . Hence,  $J''(x) < 0$  for all  $x \in [0, \bar{z}]$ . This also implies that there exists a unique  $z_m$  such that  $J(z_m) > J(z)$  for all other  $z \in [0, \bar{z}]$ .

Next, we consider the function  $H(\cdot)$ . Since  $J(0) = 0$ , and  $J'(0) = 1$ , we have  $H(0) = E(0) > 0$ . We can also show that  $H(\bar{z}) = J(\bar{z}) = (1 - \lambda)E(\bar{z})$ . To see this, note that

$$\begin{aligned} \lim_{x \rightarrow \bar{z}} \left\{ \frac{J'(x) \left[ \int_x^{\bar{z}} (z - x) dG(z) \right]}{1 - G(x)} \right\} &= J'(\bar{z}) \lim_{x \rightarrow \bar{z}} \left[ \frac{\int_x^{\bar{z}} z dG(z)}{1 - G(x)} - x \right] \\ &= J'(\bar{z}) \lim_{x \rightarrow \bar{z}} \left[ \frac{-x G'(x)}{-G'(x)} - x \right] = 0. \end{aligned}$$

The second equality is obtained by using L'Hospital's rule. Hence, we have  $H(\bar{z}) = J(\bar{z})$ .

Assumption A1 implies that  $H(\cdot)$  is a strictly decreasing function. To see this, consider

the first-order derivative of  $H(x)$ ,

$$H'(x) = \frac{\left[ \int_x^{\bar{z}} (z-x) dG(z) \right]}{[1-G(x)]^2} \{ J''(x) [1-G(x)] + J'(x) G'(x) \}.$$

Using (A.5) and (A.6), we can get

$$J''(x) [1-G(x)] + J'(x) G'(x) = -\lambda [G'(x) + xG''(x)] [1-G(x)] - \lambda x [G'(x)]^2,$$

which is strictly negative as  $G'(x) + xG''(x) > 0$ , for all  $x \in [0, \bar{z}]$ . Since  $J'(z_m) = 0$ , we have  $H(z_m) = J(z_m)$ . It is straightforward to show that  $H(z) \geq J(z)$  if and only if  $z \leq z_m$ . All these properties are summarized in Figure A1.

#### Proof of Proposition 4.2.1

Substituting (4.7) into the objective function in (4.9) gives

$$\hat{z}_t \equiv \arg \max_{x \in [0, \bar{z}]} \left\{ \frac{\tilde{\rho}_{t+1} \left[ \int_x^{\bar{z}} (z-x) dG(z) \right]}{\mathcal{R}_{t+1} - J(x) \tilde{\rho}_{t+1}} \mathcal{R}_{t+1} s_t \right\}.$$

The first-order necessary condition for this problem is given by

$$- [1-G(x)] + \frac{\tilde{\rho}_{t+1} \left[ \int_x^{\bar{z}} (z-x) dG(z) \right] J'(x)}{\mathcal{R}_{t+1} - J(x) \tilde{\rho}_{t+1}} \begin{cases} > 0 & \text{if } \hat{z}_t = \bar{z}, \\ = 0 & \text{if } \hat{z}_t \in [0, \bar{z}], \\ < 0 & \text{if } \hat{z}_t = 0. \end{cases}$$

Using (4.11), we can simplify this to become

$$H(\hat{z}_t) \tilde{\rho}_{t+1} - \mathcal{R}_{t+1} \begin{cases} > 0 & \text{if } \hat{z}_t = \bar{z}, \\ = 0 & \text{if } \hat{z}_t \in [0, \bar{z}], \\ < 0 & \text{if } \hat{z}_t = 0. \end{cases} \quad (\text{A.7})$$

Under Assumption A1, the function  $H(\cdot)$  is strictly decreasing from  $H(0) = E(z)$  to  $H(\bar{z}) = (1-\lambda)E(z)$ . Thus, a unique interior threshold value exists if and only if

$(1 - \lambda) E(z) = H(\bar{z}) < \mathcal{R}_{t+1}/\tilde{\rho}_{t+1} < H(0) = E(z)$ . Finally, we show that the second-order condition for maximization is satisfied at this unique value. Consider the function

$$\Gamma_t(x) \equiv \frac{\left[ \int_x^{\bar{z}} (z - x) dG(z) \right]}{\mathcal{R}_{t+1} - J(x) \tilde{\rho}_{t+1}}.$$

Straightforward differentiation yields

$$\begin{aligned} \Gamma'_t(x) &= \frac{-[1 - G(x)]}{\mathcal{R}_{t+1} - J(x) \tilde{\rho}_{t+1}} + \frac{J'(x) \tilde{\rho}_{t+1} \left[ \int_x^{\bar{z}} (z - x) dG(z) \right]}{[\mathcal{R}_{t+1} - J(x) \tilde{\rho}_{t+1}]^2}, \\ \Gamma''_t(x) &= \frac{G'(x)}{\mathcal{R}_{t+1} - J(x) \tilde{\rho}_{t+1}} - \frac{2J'(x) \tilde{\rho}_{t+1} [1 - G(x)]}{[\mathcal{R}_{t+1} - J(x) \tilde{\rho}_{t+1}]^2} + \frac{J''(x) \tilde{\rho}_{t+1} \left[ \int_x^{\bar{z}} (z - x) dG(z) \right]}{[\mathcal{R}_{t+1} - J(x) \tilde{\rho}_{t+1}]^2} \\ &\quad + \frac{2[J'(x) \tilde{\rho}_{t+1}]^2 \left[ \int_x^{\bar{z}} (z - x) dG(z) \right]}{[\mathcal{R}_{t+1} - J(x) \tilde{\rho}_{t+1}]^3}. \end{aligned} \quad (\text{A.8})$$

When these derivatives are evaluated at the unique solution of  $\mathcal{R}_{t+1} = H(\hat{z}_t) \tilde{\rho}_{t+1}$ , we have

$$\frac{[1 - G(\hat{z}_t)]}{\mathcal{R}_{t+1} - J(\hat{z}_t) \tilde{\rho}_{t+1}} = \frac{J'(\hat{z}_t) \tilde{\rho}_{t+1} \left[ \int_{\hat{z}_t}^{\bar{z}} (z - \hat{z}_t) dG(z) \right]}{[\mathcal{R}_{t+1} - J(\hat{z}_t) \tilde{\rho}_{t+1}]^2}.$$

This means the second and the fourth expressions in (A.8) can be canceled out. Hence,

we can get

$$\Gamma''_t(\hat{z}_t) = \frac{1}{\mathcal{R}_{t+1} - J(\hat{z}_t) \tilde{\rho}_{t+1}} \left\{ G'(\hat{z}_t) + \frac{J''(\hat{z}_t) \tilde{\rho}_{t+1} \left[ \int_{\hat{z}_t}^{\bar{z}} (z - \hat{z}_t) dG(z) \right]}{\mathcal{R}_{t+1} - J(\hat{z}_t) \tilde{\rho}_{t+1}} \right\}.$$

Using (4.11) and  $\mathcal{R}_{t+1} = H(\hat{z}_t) \tilde{\rho}_{t+1}$ , we can get

$$\begin{aligned} \mathcal{R}_{t+1} - J(\hat{z}_t) \tilde{\rho}_{t+1} &= [H(\hat{z}_t) - J(\hat{z}_t)] \tilde{\rho}_{t+1} = \frac{J'(\hat{z}_t) \tilde{\rho}_{t+1} \left[ \int_{\hat{z}_t}^{\bar{z}} (z - \hat{z}_t) dG(z) \right]}{1 - G(\hat{z}_t)} \\ &\Rightarrow \frac{J''(\hat{z}_t) \tilde{\rho}_{t+1} \left[ \int_{\hat{z}_t}^{\bar{z}} (z - \hat{z}_t) dG(z) \right]}{\mathcal{R}_{t+1} - J(\hat{z}_t) \tilde{\rho}_{t+1}} = \frac{J''(\hat{z}_t) [1 - G(\hat{z}_t)]}{J'(\hat{z}_t)}. \end{aligned}$$

Using this, we can obtain

$$\Gamma''_t(\hat{z}_t) = \frac{J''(\hat{z}_t) [1 - G(\hat{z}_t)] + G'(\hat{z}_t) J'(\hat{z}_t)}{[\mathcal{R}_{t+1} - J(\hat{z}_t) \tilde{\rho}_{t+1}] J'(\hat{z}_t)}.$$

Note that we have come across the expression  $\{J''(x)[1 - G(x)] + G'(x)J'(x)\}$  when evaluating the derivative of  $H(\cdot)$ . In particular, we have shown that this expression is strictly negative for any  $x \in [0, \bar{z}]$  under Assumption A1. In addition,  $J'(\hat{z}_t) > 0$  for any  $\hat{z}_t > 0$ . Hence, we have  $\Gamma_t''(\hat{z}_t) < 0$  meaning that the second-order condition for maximization is satisfied. This completes the proof of Proposition 4.2.1.

### **Proof of Proposition 4.2.2**

#### **Part (i)**

Since  $H(\cdot)$  is strictly decreasing under Assumption A1, it follows immediately from (4.10) that an increase in  $\mathcal{R}_{t+1}/\tilde{\rho}_{t+1}$  will lower the threshold value  $\hat{z}_t$ . Next, let  $\mathcal{L}(\hat{z}_t)$  be the entrepreneur's financial leverage under the optimal contract, i.e.,

$$\mathcal{L}(\hat{z}_t) = \frac{J(\hat{z}_t)}{H(\hat{z}_t) - J(\hat{z}_t)}.$$

Straightforward differentiation yields

$$\mathcal{L}'(\hat{z}_t) = \frac{J'(\hat{z}_t)H(\hat{z}_t) - J(\hat{z}_t)H'(\hat{z}_t)}{[H(\hat{z}_t) - J(\hat{z}_t)]^2} > 0.$$

The above expression is strictly positive because (i) the function  $H(\cdot)$  is strictly decreasing under Assumption A1, and (ii)  $H(\hat{z}_t) > J(\hat{z}_t)$  if and only if  $\hat{z}_t < z_m$ , which means  $J'(\hat{z}_t) > 0$ .

**Part (ii)**

In order to highlight the dependence on  $\lambda$ , we will rewrite (4.10) as  $\mathcal{R}_{t+1} = H(\hat{z}_t; \lambda) \tilde{\rho}_{t+1}$  and the financial leverage as  $\mathcal{L}(\hat{z}_t; \lambda)$ . Define  $H_z(z; \lambda)$  and  $H_\lambda(z; \lambda)$  as the partial derivative of  $H(\cdot)$  with respect to  $z$  and  $\lambda$ . Similar define  $\mathcal{L}_z(z; \lambda)$  and  $\mathcal{L}_\lambda(z; \lambda)$ . Under Assumption A1,  $H_z(z; \lambda) < 0$  for all  $z \in [0, \bar{z}]$ . Straightforward differentiation yields

$$H_\lambda(z; \lambda) = - \int_0^z x dG(x) - z G'(z) < 0, \quad \text{for all } z \in [0, \bar{z}].$$

Hence, we have

$$\frac{d\hat{z}_t}{d\lambda} = - \frac{H_\lambda(\hat{z}_t; \lambda)}{H_z(\hat{z}_t; \lambda)} < 0.$$

This means an increase in  $\lambda$  will lower the threshold level of auditing. Next, we turn to the effects on  $\mathcal{L}(\hat{z}_t; \lambda)$  which are given by

$$\begin{aligned} \frac{d\mathcal{L}(\hat{z}_t; \lambda)}{d\lambda} &= \mathcal{L}_z(\hat{z}_t; \lambda) \frac{d\hat{z}_t}{d\lambda} + \mathcal{L}_\lambda(\hat{z}_t; \lambda) \\ &= \frac{H(\hat{z}_t; \lambda)}{[H(\hat{z}_t; \lambda) - J(\hat{z}_t; \lambda)]^2} \left[ -J_z(\hat{z}_t; \lambda) \frac{H_\lambda(\hat{z}_t; \lambda)}{H_z(\hat{z}_t; \lambda)} + J_\lambda(\hat{z}_t; \lambda) \right]. \end{aligned}$$

The above expressions is negative because: (i)  $J_z(\hat{z}_t; \lambda) > 0$ , (ii)  $-H_\lambda(\hat{z}_t; \lambda)/H_z(\hat{z}_t; \lambda) < 0$ , and (iii)  $J_\lambda(\hat{z}_t; \lambda) = - \int_0^{\hat{z}_t} x dG(x) < 0$ . This completes the proof of Proposition 4.2.2.

**Derivation of Equation (4.23)**

Recall the definition of the inverse supply function of deposits,

$$\Gamma_t \left( \sum_{i=1}^M \zeta_{i,t} \right) = \beta^{\frac{1}{\sigma-1}} \left[ \frac{(1-\alpha) N_t w_t}{\sum_{i=1}^M \zeta_{i,t}} - 1 \right]^{\frac{\sigma}{\sigma-1}}.$$

Straightforward differentiation yields

$$\Gamma'_t \left( \sum_{i=1}^M \zeta_{i,t} \right) = \beta^{\frac{1}{\sigma-1}} \left( \frac{\sigma}{1-\sigma} \right) \left[ \frac{(1-\alpha) N_t w_t}{\sum_{i=1}^M \zeta_{i,t}} - 1 \right]^{\frac{1}{\sigma-1}} \frac{(1-\alpha) N_t w_t}{\left( \sum_{i=1}^M \zeta_{i,t} \right)^2}. \quad (\text{A.9})$$



In equilibrium, we have

$$\begin{aligned} \sum_{i=1}^M \zeta_{i,t} &= M\bar{\zeta}_t = (1-\alpha) N_t d_t = (1-\alpha) N_t \left[ \frac{\beta^{\frac{1}{\sigma}} R_{t+1}^{\frac{1}{\sigma}-1}}{1 + \beta^{\frac{1}{\sigma}} R_{t+1}^{\frac{1}{\sigma}-1}} \right] w_t \\ &\Rightarrow \frac{(1-\alpha) N_t w_t}{\sum_{i=1}^M \zeta_{i,t}} = \frac{1 + \beta^{\frac{1}{\sigma}} R_{t+1}^{\frac{1}{\sigma}-1}}{\beta^{\frac{1}{\sigma}} R_{t+1}^{\frac{1}{\sigma}-1}}. \end{aligned}$$

Substituting these into (A.9) gives

$$\Gamma'_t(M\bar{\zeta}_t) = \frac{\sigma}{1-\sigma} \left[ R_{t+1} \left( 1 + \beta^{\frac{1}{\sigma}} R_{t+1}^{\frac{1}{\sigma}-1} \right) \right] \frac{1}{M\bar{\zeta}_t}.$$

Using this, we can rewrite (4.16) as

$$\mathcal{R}_{t+1} = R_{t+1} \left[ 1 + \frac{\sigma}{(1-\sigma)M} \left( 1 + \beta^{\frac{1}{\sigma}} R_{t+1}^{\frac{1}{\sigma}-1} \right) \right].$$

Equation (4.23) is the steady-state version of this equation.

### Proof of Proposition 4.2.3

First, rewrite (4.20) as

$$\frac{1+n}{\alpha(1-\phi)E(z)} [H(\hat{z}^*) - J(\hat{z}^*)] = H(\hat{z}^*) (k^*)^{\phi-1}.$$

Substituting this into (4.22) gives

$$\mathcal{R}^* = \left[ \frac{\phi(1+n)}{\alpha(1-\phi)E(z)} + 1 - \delta \right] H(\hat{z}^*) - \left[ \frac{\phi(1+n)}{\alpha(1-\phi)E(z)} \right] J(\hat{z}^*) \equiv \Delta(\hat{z}^*). \quad (\text{A.10})$$

Recall that under Assumption A1, the function  $J(\cdot)$  is strictly concave. Thus, there exists a unique value  $z_m \in (0, \bar{z})$  such that  $H(z_m) = J(z_m) > J(z)$  for all other  $z$ . Since any interior threshold value  $\hat{z}_t$  would imply  $J'(\hat{z}_t) > 0$ , we can focus on the range  $[0, z_m]$ . The newly defined function  $\Delta : [0, z_m] \rightarrow \mathbb{R}$  has the following properties:

$$\Delta(0) = \left[ \frac{(1+n)\phi}{\alpha(1-\phi)E(z)} + 1 - \delta \right] H(0) = \frac{(1+n)\phi}{\alpha(1-\phi)} + (1-\delta)E(z) > 0,$$

$$\Delta(z_m) = (1 - \delta) J(z_m),$$

$$\Delta'(\hat{z}) = \left[ \frac{(1+n)\phi}{\alpha(1-\phi)E(z)} + 1 - \delta \right] H'(\hat{z}) - \left[ \frac{(1+n)\phi}{\alpha(1-\phi)E(z)} \right] J'(\hat{z}).$$

Since  $H'(\hat{z}) < 0$  for all  $\hat{z} \in [0, \bar{z}]$  and  $J'(\hat{z}) \geq 0$  for all  $z \in [0, z_m]$ , we have  $\Delta'(\hat{z}) < 0$  over the range  $[0, z_m]$ .

Next, using (4.21) we can obtain

$$\beta^{\frac{1}{\sigma}} (R^*)^{\frac{1}{\sigma}-1} = \frac{\alpha J(\hat{z}^*)}{(1-\alpha)H(\hat{z}^*) - J(\hat{z}^*)}$$

which in turn implies

$$1 + \beta^{\frac{1}{\sigma}} (R^*)^{\frac{1}{\sigma}-1} = \frac{(1-\alpha)[H(\hat{z}^*) - J(\hat{z}^*)]}{(1-\alpha)H(\hat{z}^*) - J(\hat{z}^*)} \equiv \Psi(\hat{z}^*), \quad (\text{A.11})$$

and

$$R^* = \left( \frac{\beta}{\alpha^\sigma} \right)^{\frac{1}{\sigma-1}} \left[ (1-\alpha) \frac{H(\hat{z}^*)}{J(\hat{z}^*)} - 1 \right]^{\frac{\sigma}{\sigma-1}} \equiv \Lambda(\hat{z}^*). \quad (\text{A.12})$$

Using these auxiliary functions, we can rewrite (4.23) as

$$\mathcal{R}^* = \Lambda(\hat{z}^*) \left[ 1 + \frac{\sigma}{(1-\sigma)M} \Psi(\hat{z}^*) \right] \equiv \Theta(\hat{z}^*). \quad (\text{A.13})$$

Note that both  $\Psi(\cdot)$  and  $\Lambda(\cdot)$  are positive only for values of  $z$  that satisfy  $(1-\alpha)H(z) > J(z)$ . Since  $H(0) > J(0) = 0$ ,  $H(z_m) = J(z_m) = J_{\max}$ , and  $H'(\hat{z}) < 0 \leq J'(\hat{z})$  for all  $z \in [0, z_m]$ , there exists a unique value, denoted by  $z_e \in (0, z_m)$ , such that  $(1-\alpha)H(z_e) = J(z_e)$ , and  $(1-\alpha)H(z) > J(z)$ , for all  $z \in [0, z_e]$ . Thus, both of these functions are defined over the interval  $[0, z_e]$ . Since

$$\lim_{z \rightarrow 0} \left[ (1-\alpha) \frac{H(z)}{J(z)} - 1 \right] \rightarrow +\infty \quad \text{and} \quad \lim_{z \rightarrow z_e} \left[ (1-\alpha) \frac{H(z)}{J(z)} - 1 \right] \rightarrow 0,$$

we have

$$\lim_{z \rightarrow 0} \Psi(z) = 0 \quad \text{and} \quad \lim_{z \rightarrow z_e} \Psi(z) = +\infty,$$

$$\lim_{z \rightarrow 0} \Lambda(z) = 0 \quad \text{and} \quad \lim_{z \rightarrow z_e} \Lambda(z) = +\infty, \quad \text{when } \sigma < 1.$$

Finally, we want to show that both  $\Psi(\cdot)$  and  $\Lambda(\cdot)$  are strictly increasing functions. This property depends crucially on the shape of  $H(\cdot)/J(\cdot)$ . Using (4.11), we can get

$$\frac{H(x)}{J(x)} = 1 + \frac{J'(x) \left[ \int_x^{\bar{z}} (z-x) dG(z) \right]}{J(x) [1 - G(x)]}.$$

Straightforward differentiation gives

$$\begin{aligned} \frac{d}{dx} \left[ \frac{H(x)}{J(x)} \right] &= \frac{J''(x) \left[ \int_x^{\bar{z}} (z-x) dG(z) \right]}{J(x) [1 - G(x)]} - \underbrace{\frac{J'(x)}{J(x)}}_{(+)} - \underbrace{\frac{[J'(x)]^2 \left[ \int_x^{\bar{z}} (z-x) dG(z) \right]}{[J(x)]^2 [1 - G(x)]}}_{(+)} \\ &\quad + \frac{J'(x) \left[ \int_x^{\bar{z}} (z-x) dG(z) \right]}{J(x) [1 - G(x)]^2} G'(x). \end{aligned}$$

The first and the last term can be combined to become

$$\begin{aligned} &\frac{J''(x) \left[ \int_x^{\bar{z}} (z-x) dG(z) \right]}{J(x) [1 - G(x)]} + \frac{J'(x) \left[ \int_x^{\bar{z}} (z-x) dG(z) \right]}{J(x) [1 - G(x)]^2} G'(x) \\ &= \frac{\left[ \int_x^{\bar{z}} (z-x) dG(z) \right]}{J(x) [1 - G(x)]^2} \{ J''(x) [1 - G(x)] + J'(x) G'(x) \} < 0. \end{aligned}$$

The expression inside the curly brackets is strictly negative under Assumption A1. Hence,  $H(x)/J(x)$  is strictly decreasing over the range  $(0, z_e)$ . It follows immediately from (A.12) that  $\Lambda(\cdot)$  is strictly increasing when  $\sigma < 1$ . Next, consider  $\Psi(\cdot)$  which can be expressed as

$$\Psi(x) = \frac{(1 - \alpha) \left[ \frac{H(x)}{J(x)} - 1 \right]}{(1 - \alpha) \frac{H(x)}{J(x)} - 1}.$$

The first-order derivative of this function is given by

$$\Psi'(x) = \frac{-\alpha(1 - \alpha)}{\left[ (1 - \alpha) \frac{H(x)}{J(x)} - 1 \right]^2} \frac{d}{dx} \left[ \frac{H(x)}{J(x)} \right] > 0.$$

These results together imply that  $\Theta(\cdot)$  is a strictly increasing that approaches zero as  $z$  tends to 0, and is infinitely large when  $z$  is close to  $z_e$ .

We are now ready to establish the existence and uniqueness result. Any steady-state  $\widehat{z}^*$  is a solution of the equation  $\Delta(\widehat{z}^*) = \Theta(\widehat{z}^*)$ . As we have shown above, when  $\sigma < 1$ , we have

$$\Delta(0) > \Theta(0) = 0 \quad \text{and} \quad \Delta(z_e) < \Theta(z_e) = +\infty.$$

Since both  $\Delta(\cdot)$  and  $\Theta(\cdot)$  are continuous functions, by the intermediate value theorem, the equation  $\Delta(\widehat{z}^*) = \Theta(\widehat{z}^*)$  has at least one solution within the interval  $(0, z_e)$ . Since  $\Delta(\cdot)$  is strictly decreasing and  $\Theta(\cdot)$  is strictly increasing, there exists at most one such solution. Once the value of  $\widehat{z}^*$  is determined, the value of  $R^*$ ,  $\mathcal{R}^*$  and  $k^*$  can be uniquely determined using (A.12), (A.13) and (4.20). This completes the proof of Proposition 4.2.3.

#### Proof of Proposition 4.2.4

The proof of this result is built upon the proof of Proposition 4.2.3. Recall that the threshold value for auditing ( $\widehat{z}^*$ ) is uniquely determined by

$$\Delta(\widehat{z}^*) = \Lambda(\widehat{z}^*) \left[ 1 + \frac{\sigma}{(1-\sigma)M} \Psi(\widehat{z}^*) \right] \equiv \Theta(\widehat{z}^*; M).$$

The notation  $\Theta(\widehat{z}^*; M)$  highlights the dependence of this function on  $M$ . Note that none of the auxiliary functions,  $\Delta(\cdot)$ ,  $\Lambda(\cdot)$  and  $\Psi(\cdot)$ , are affected by  $M$ . From the above equation, it is obviously that  $\Theta(z; M_1) < \Theta(z; M_2)$  for all  $z \in (0, z_e)$  when  $M_1 > M_2$ . This in turn implies  $\widehat{z}_1^* > \widehat{z}_2^*$ . A graphical illustration of this result is shown in Figure A2.

Since  $\Delta(\cdot)$  is strictly decreasing, it follows from (A.10) that  $\mathcal{R}_1^* = \Delta(\widehat{z}_1^*) < \Delta(\widehat{z}_2^*) = \mathcal{R}_2^*$ . Similarly, since  $\Lambda(\cdot)$  is strictly increasing, it follows from (A.12) that  $R_1^* = \Lambda(\widehat{z}_1^*) > \Lambda(\widehat{z}_2^*) = R_2^*$ . Next, rewrite (4.20) as

$$k_j^* = \left[ \frac{\alpha(1-\phi)E(z)}{1+n} \right]^{\frac{1}{1-\phi}} \left\{ 1 + \left[ \frac{H(\widehat{z}_j^*)}{J(\widehat{z}_j^*)} - 1 \right]^{-1} \right\}^{\frac{1}{1-\phi}}.$$

Since  $H(x)/J(x)$  is strictly decreasing for all  $x \in (0, z_e)$ , it follows from the above equation that  $k_1^* > k_2^*$ . Finally, we compare the entrepreneur's financial leverage in these two economies. As shown in the proof of Proposition 4.2.2, financial leverage  $\mathcal{L}(\hat{z}_t)$  is strictly increasing in  $\hat{z}_t$ . Thus, we have  $\mathcal{L}(\hat{z}_1^*) > \mathcal{L}(\hat{z}_2^*)$ . This completes the proof of Proposition 4.2.4.

### Derivation of Equation (4.30)

The right side of equation (4.30) is the same as that of (4.23), thus here we will focus on the left side of the equation. According to (4.25), an entrepreneur's demand for loans is given by  $b_t = \mathcal{L}(\hat{z}_t) s_t$ , where  $\mathcal{L}(\hat{z}_t)$  is the financial leverage. The total demand for loans is thus given by

$$\mathcal{B}_t = \alpha N_t b_t = \mathcal{L}(\hat{z}_t) \alpha N_t w_t, \quad (\text{A.14})$$

after imposing  $s_t = w_t$ . The threshold value  $\hat{z}_t$  is determined by (4.24), which implies

$$d\hat{z}_t = \left[ \frac{1}{\tilde{\rho}_{t+1} H'(\hat{z}_t)} \right] d\tilde{\mathcal{R}}_{t+1}.$$

Totally differentiate (A.14) with respect to  $\mathcal{B}_t$  and  $\hat{z}_t$  gives

$$d\mathcal{B}_t = \alpha N_t w_t \mathcal{L}'(\hat{z}_t) d\hat{z}_t.$$

Combining these two expressions gives

$$\frac{d\tilde{\mathcal{R}}_{t+1}}{d\mathcal{B}_t} = \Omega'_t(\mathcal{B}_t) = \frac{\tilde{\rho}_{t+1}}{\alpha N_t w_t} \frac{H'(\hat{z}_t)}{\mathcal{L}'(\hat{z}_t)} = \frac{\tilde{\mathcal{R}}_{t+1}}{\alpha N_t w_t} \frac{H'(\hat{z}_t)}{H(\hat{z}_t)} \frac{1}{\mathcal{L}'(\hat{z}_t)}.$$

The last equality follows from (4.24). Thus, in equilibrium, we have

$$\Omega_t(M\bar{\psi}_t) + \Omega'_t(M\bar{\psi}_t) \bar{\psi}_t = \tilde{\mathcal{R}}_{t+1} + \left[ \frac{\tilde{\mathcal{R}}_{t+1}}{\alpha N_t w_t} \frac{H'(\hat{z}_t)}{H(\hat{z}_t)} \frac{1}{\mathcal{L}'(\hat{z}_t)} \right] \left[ \frac{1}{M} \mathcal{L}(\hat{z}_t) \alpha N_t w_t \right],$$

where we have used the loan market clearing condition:  $M\bar{\psi}_t = \mathcal{L}(\hat{z}_t)\alpha N_t w_t$ . This expression can be simplified to become

$$\tilde{\mathcal{R}}_{t+1} \left[ 1 + \frac{1}{M} \frac{H'(\hat{z}_t)}{H(\hat{z}_t)} \frac{\mathcal{L}(\hat{z}_t)}{\mathcal{L}'(\hat{z}_t)} \right],$$

which is the left-hand side of (4.30).

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