

5-8-2015

# Portfolio Choice with Life Annuities under Probability Distortion

Wenyuan Zheng

University of Connecticut - Storrs, [wenyuan.zheng@uconn.edu](mailto:wenyuan.zheng@uconn.edu)

Follow this and additional works at: <https://opencommons.uconn.edu/dissertations>

---

## Recommended Citation

Zheng, Wenyuan, "Portfolio Choice with Life Annuities under Probability Distortion" (2015). *Doctoral Dissertations*. 750.  
<https://opencommons.uconn.edu/dissertations/750>

# Portfolio Choice with Life Annuities under Probability Distortion

Wenyuan Zheng, Ph.D.

University of Connecticut, 2015

## ABSTRACT

Retirement planning has attracted considerable attentions from retirees, finance industry and the government. Its significance lies in the protection against individual longevity risk. As a result, optimal portfolio selection problem in a market with annuities has been extensively studied. Yarri (1965) [35] first suggested that individuals with no bequest motive should annuitize all her savings. However, the volume of the voluntary annuity purchases is much lower than predicted by such model, which is the so-called annuity puzzle. In this dissertation, I aim to explore the annuity puzzle from the behavioral economics point of view. Particularly, one major finding from the behavioral experiments is that people always overestimate small-probability events and underestimate large-probability events.

By introducing the probability distortion on the perceived stock price process, I revisit the dynamic optimal portfolio selection model in a financial market with a riskless bond, a risky asset, and commutable life annuities. In particular, the portfolio problem is studied under both a reverse S-shaped probability distortion function and a convex probability distortion function.

I find that with a constant relative risk aversion utility function, the reverse S-shaped probability distortion function brings more fear of large losses than the hope of large gains. In order to address the fear, people tend to buy more annuities, invest more including margin purchases and consume more. On the other side, the convex probability distortion function essentially increases the perceived stock price drift, which results in more investment on the stock market, more consumption and less demand in the annuity industry. It provides a plausible explanation for the annuity puzzle.

# Portfolio Choice with Life Annuities under Probability Distortion

Wenyuan Zheng

M.S. Actuarial Science, Nankai University, Tianjin, China, 2010

B.S. Insurance, Southwestern University of Finance and Economics, Chengdu,  
China, 2007

A Dissertation

Submitted in Partial Fulfillment of the

Requirements for the Degree of

Doctor of Philosophy

at the

University of Connecticut

2015

Copyright by

Wenyuan Zheng

2015

# APPROVAL PAGE

Doctor of Philosophy Dissertation

## Portfolio Choice with Life Annuities under Probability Distortion

Presented by

Wenyuan Zheng, B.S. Insurance, M.S. Actuarial Science

Major Advisor

---

Brian M. Hartman

Associate Advisor

---

James G. Bridgeman

Associate Advisor

---

Guojun Gan

University of Connecticut

2015

## ACKNOWLEDGMENTS

First, I am extremely thankful to my advisor, James Bridgeman. It is him who knew my motivation for actuarial research and selected me in this Ph.D. program. In the first two Ph.D. years he always provided me with instructive suggestions on math courses and English exam. It was my honor to do research under him afterwards. His insight and reasoning skills have been instrumental to my dissertation work. I always enjoyed our discussions, which indeed made the problem clearer and my understanding better. He also encouraged me to choose my own interested topic. I really appreciate the way he trained my research skills. In addition to the research, I also learned life lessons such as how to work more efficiently and how to better communicate with others, which are important for my future career. Being his student is one of the best experiences in my Ph.D. life.

Second, I definitely want to thank Brian Hartman for being my advisor. He always loved to share his insights on my research. And he was extremely supportive and helpful throughout my job application process. He patiently went me through every step during my preparation, which indeed built my confidence in this process. I am incredibly grateful to have his valuable guidance helping me pursue my career as a teacher and a researcher. It was also a great experience being his teaching assistant for three semesters.

Another special thanks go to Guojun Gan for serving on my dissertation committee and sharing opinions on my research. He also provided me with useful guidance when I started writing the thesis.

I want to thank Jeyaraj Vadiveloo and Emiliano Valdez for offering me opportunities in some actuarial projects. It provided me a chance to have the access to real-world problems, which was one important part in actuarial studies.

I would like to extend my gratitude to some faculty members in our Math department. I benefited a lot from their teaching, which built my foundation in mathematics for the future researches. They are Patrick Mckenna, Vadim Olshevsky, Yung Choi, Iddo Ben-Ari and Jeffrey Connors.

Another thanks go to my teaching assistant instructors. They helped me develop teaching skills and provided me with valuable suggestions on how to deal with teaching-related issues. They are Steven Pon, Robert Heffernan and Jeffrey Tollefson.

I also want to thank my classmates Rozita Ramli and Shujuan Huang. We shared the ups and downs in the Ph.D. life together. I am grateful to be with you for five years.

I want to express my gratitude to the Society of Actuaries for honoring me as the Hickman Scholar. Their financial support indeed helped me focus more on my actuarial research.



Most importantly, I want to thank my beloved parents and husband. They are always by my side whenever I have difficulties. It is their love encourages me to always be positive towards life. Without their unconditional support, I cannot survive my Ph.D. life here.

# Contents

<b>Ch. 1. Portfolio Choice with Life Annuities under a Reverse S-Shaped Probability Distortion Function</b>	<b>1</b>
1.1 Introduction . . . . .	1
1.2 Problem Formulation . . . . .	7
1.2.1 Probability Distortion Function . . . . .	10
1.2.2 Weibull Distribution . . . . .	17
1.2.3 Stochastic Control Problem . . . . .	21
1.3 Model . . . . .	23
1.3.1 Financial Market and Policyholders . . . . .	23
1.3.2 Wealth Dynamics and Value Function . . . . .	26
1.3.3 Numerical Method . . . . .	33
1.3.4 Parameters . . . . .	35
1.4 Results . . . . .	37
1.4.1 Stochastic Process for the Distorted Probability . . . . .	37
1.4.2 Results on Portfolio Selections with a Distorted Probability . .	40
1.4.3 Sensitivity Analysis . . . . .	51
1.5 Conclusions . . . . .	65
<b>Ch. 2. Portfolio Choice with Life Annuities under a Convex Probability Distortion Function</b>	<b>69</b>
2.1 Introduction . . . . .	69
2.2 Problem Formulation . . . . .	71
2.3 Model . . . . .	72
2.4 Results . . . . .	77
2.5 Numerical Illustration . . . . .	80
2.6 Conclusion . . . . .	86

**Bibliography**

# Chapter 1

## Portfolio Choice with Life Annuities under a Reverse S-Shaped Probability Distortion Function

### 1.1 Introduction

Retirement planning has attracted considerable attentions from retirees, insurance industry and the government. For retirees, it is a lifetime project which has a huge impact on their post-retirement living standard. Particularly, it helps them to protect against the risk of outliving their assets. The retirement planning is not only important for individuals, but also critical for insurance companies. Due to the product innovation the financial markets have seen over the past two decades, annuitants' behavior is much more complicated than ever before. Thus understanding policyholders' behavior is crucial in a way that affects almost all aspects of insurance company

operations, especially product design, pricing and reserving decisions. From the government's point of view, an overall well-managed retirement planning indicates a robust social security system.

Among the financial instruments, insurance product is designed to diversify the mortality-related risk based on the Central Limit Theory. Thus it is an indispensable component in the retirement planning. In this work, I focus on individuals' portfolio selection decisions, particularly their annuitization strategy.

Since 1960's, the optimal portfolio selection problem in a market with life annuities has been extensively studied. The seminal paper by Yari (1965) [35] suggested that individuals with no bequest motive should annuitize all her savings. However, it is not the case in reality, which is the so-called "annuity puzzle". The goal of my work is to interpret the annuity puzzle, from a behavioral economics point of view.

This work merges a variety of distinct strands in the annuity puzzle, behavioral economics literature and dynamic asset allocation framework. First, as the development of the financial markets, retirement planning has become an increasingly complicated issue. Scholars have started to investigate the optimal annuitization strategy from 1960's. The seminal paper by Yari (1965) [35] suggested that individuals with no bequest motive should hold all their assets in mortality-contingent annuities since they stochastically dominate the payout from traditional investment classes. However, it is not the case in reality. Empirical studies were conducted on both the defined contribution plans (DC) and the defined benefit plans (DB). For defined contribution plans, Schaus (2005) [27] reported that only 6% of the participants

elected an annuity when it was available, using data from a survey of 450 large 401(k) plans. Butler and Teppa (2007) [4] studied the annuitization decision within 10 Swiss pension plans between 1996 and 2006, and found that 54% chose the annuity over a lump sum of money. A caveat here is that the annuity was the default option in most Swiss plans. Generally speaking, the annuitization rate of the DC plans is relatively lower than that of the DB plans because only few DC plans offer the annuity option. For defined benefit plans, Mottola and Utkus (2007) [25] analyzed the payout choices in a Fortune 500 company and found that only 27% retirees selected the annuity. Bernartzi et al. (2011) [1] studied more than 103,000 payout decisions from 112 different DB plans between 2002 and 2008, and concluded that 49% selected annuities over the lump sum, again with annuities as the default option or easier-to-accept option.

Thus the annuity puzzle has been the subject of much research in public economics and insurance literature. Explanations include the high administrative loads which are embedded in annuity prices (among others, see Friedman and Warshawsky (1990) [10], Mitchell et al. (1997) [23]). Bequest motive is another substantial reason. It reduces the attractiveness among seniors, since the wealth allocated to annuities cannot be bequeathed upon death (see Hurd (1989) [13]). Liquidity preference is also an explanation for the annuity puzzle, because medical expenditure shock occurs sometimes (see Turra (2004) [31]). Because of the adverse selection, the annuity is usually not fairly priced, especially for those healthy annuitants (see Finkelstein et al. (2004) [8] and Cutler et al. (2008) [5]).

The goal of this work is to interpret the annuity puzzle by the behavioral economics. The textbook theory assumes that individuals are perfectly rational. How-

ever, human decision-making violations of rationality are being identified, and the strong effect of psychological biases are being studied. Behavioral economics develops as the marriage of classical economic theory and psychology, examining the effects of emotional and social factors on individual and institutional economic decisions. There has been a burgeoning interest in applying it to model the investors'/ consumers' behavior on the financial markets. And recently it has started to gain more interest across the life insurance industry to better understand the policyholders' behavior. It is especially efficient to explain the annuity puzzle. One well-explored reason is the mental accounting. Wei-Yin Hu and Jason Scott (2007) [12] argued that retirees consider purchasing annuities as a gamble. When policyholders decide whether to annuitize, they wonder if they will live long enough to make their initial payment back. Another explanation for the annuity puzzle is framing. Shlomo Benartzi, Alessandro Previtero and Richard Thaler (2011) [1] conducted a survey to test the attractiveness of an annuity under a consumption frame and an investment frame respectively. The results showed that 70% chose to annuitize under the consumption frame whereas only 21% annuitized under the investment frame, which implied the superiority of a consumption frame for annuity. Other explanations include J. Mark Iwry and John Turner (2009) [14] on default decisions and David Laibson (1997) [18] on hyperbolic discounting.

The assumption that people overestimate small-probability events and underestimate large-probability events (whether of good outcomes or bad ones) is characterized by a reverse S-shaped distortion function applied to the decumulative probability function. Due to the introduction of such probability distortion function, the optimal asset allocation model is not a concave maximization problem anymore. While

most works focus on empirical and numerical studies (see among others, Benartzi and Thaler (1993) [2], Shefrin and Statman (2000) [28], De Giorgi and Post (2008) [7]), Bernard and Ghossoub (2009) [3] have carried out analytical treatment on single-period portfolio choice models and obtained closed-form solutions for a number of interesting cases. Importantly, Jin and Zhou (2008) [15] first analytically developed new approaches in driving the optimal solution by proposing a quantile formulation.

I extend the Jin and Zhou (2008) [15] model by considering an optimization problem of lifetime utility, rather than one-time utility of the terminal wealth. As a result, the quantile formulation is not applicable in my model. To get around this, my model will apply a technical tool developed by Young and Zariphopoulou (2000) [36] to derive the perceived stochastic diffusion process for a risky asset under a distorted probability. The methodology comes from theories of stochastic control, stochastic differential games, and non-linear partial differential equations. In order to apply this method, an explicit inverse function of the probability distortion is required. While no previously developed reverse S-shaped probability distortion functions have an explicit inverse function (Kahneman and Tversky (1992) [33], Tversky and Fox (1995) [32] and Prelec (1998) [26]), I propose a new probability distortion function that does have one. This function characterizes a family of reverse S-shaped functions exhibiting various curvatures and elevations. More importantly, its explicit inverse function enables me to derive the perceived stochastic process for a risky asset under the distorted probability. Employing the new distortion function, the diffusion process of the perceived stock price has the hazard function of the stock price distribution embedded in it. In order to obtain a closed form for the diffusion process of the perceived stock prices, I approximate the true stock price process as a Weibull



distribution (which has a closed form hazard function) locally rather than the more usual lognormal distribution (which does not). In sum, I develop a technical tool to derive an explicit stochastic differential equation for the distorted stock price process by proposing a new distortion function and modeling the true stock price locally as a Weibull distribution.

This work lies in the classical Merton (1971) [19] optimal asset allocation and consumption framework. Moshe A. Milevsky and Virginia R. Young (2007) [21] first realistically incorporated mortality-contingent payout annuities within this framework. Individuals were modeled to maximize their lifetime utility. Moshe A. Milevsky, et al. (2007) [20] followed with the goal of minimizing the ruin probability. To investigate the policyholders' lapse behavior, Wang and Young (2012) [34] modeled the portfolio selection problem with commutable life annuities, single premium immediate annuities with a proportional surrender charge. These two works were done with the goal of maximizing the lifetime utility, and minimizing the ruin probability respectively.

Finally, my work is related to the literature on the optimal portfolio selection under proportional transaction costs. Since the surrender charge is proportional to the actuarial present value of the annuity, there is a proportional transaction cost associated with surrendering annuities. I assume there is no corresponding transaction cost in buying annuities. Davis and Norman (1990) [6] proved that, under a constant relative risk aversion utility function, the optimal investment strategy with transaction costs is one of the singular and impulse control within a "wedge" bounded by two rays. The ranges of these two critical ratios of bond-to-stock were derived by Shreve and Soner (1994) [29]. My work finds a similar wealth/annuity boundary empirically

for the annuitization strategy.

Interestingly, with a constant relative risk aversion utility function, the reverse S-shaped probability distortion function brings more fear of large losses than the hope of large gains. In order to address the fear, people tend to buy more annuities, invest more including margin purchases and consume more.

The remainder of this chapter is organized as follows. In section 2, I introduce the technical tools to overcome the difficulty of solving a non-concave maximization problem. I then present my model in terms of the wealth dynamics and the its value function in section 3. In section 4, numerical solutions on the annuitization, investment and consumption strategies are obtained, and the sensitivity analysis are conducted. Finally, section 5 concludes and briefly discusses future researches.

## 1.2 Problem Formulation

Given the amount of wealth and annuities, my goal is to identify the optimal annuitization strategy which maximizes the lifetime utility. In this research, the task is completed in two steps. First, fixing the amount of annuities, I explicitly derive the instantaneous investment and consumption strategies for different levels of wealth. As a byproduct, I also obtain the lifetime utility for every given endowment of wealth and annuities, assuming no purchases or surrender of annuities. Second, I obtain the short-term (or long-term) annuitization strategy by comparing lifetime utilities in a certain way and attaining the greatest one through buying one annuity (or annuities), surrendering an existing annuity (or annuities) or doing nothing.

Specifically, now I briefly go through the separate piece of work to attain my goal. First, with the aim to derive an optimal portfolio choice reflecting individuals' perception towards the real world, I incorporate a reverse S-shaped probability distortion into the stock price distribution, which is proposed by behavioral economics. More specifically, the probability distortion represents the fact that people always overestimate the small-probability events and underestimate large-probability events. Due to the introduction of such probability distortion function, the optimal asset allocation model is not a concave maximization problem anymore. I approach this problem under a stochastic process setting, which allows for more flexibilities to address the probability distortion. I apply a technical tool developed by Young and Zariphopoulou (2000) [36] to derive the stochastic diffusion process for a distorted probability.

In order to apply this method, an explicit inverse function of the probability distortion function is required. While no previously developed reverse S-shaped probability distortion functions have an explicit inverse function, I propose a new probability distortion function.

Employing the new distortion function, the diffusion process of the perceived stock price has the hazard function of the stock price distribution embedded in it. Therefore, I approximate the stock price locally as a Weibull process in order to obtain a closed form hazard function.

Since I work with the stochastic process and I aim to maximize the lifetime utility,

my model is essentially a stochastic control problem. Based on my model formulation, I first write out the wealth dynamics and value function. Applying the stochastic control theory, the Hamilton-Jacobi-Bellman (HJB) partial differential equation can be obtained. Assuming the amount of annuity is fixed, instantaneous investment and consumption strategies are thus explicitly derived.

By numerically solving the HJB equation, I have the lifetime utility for every grid point on the wealth-annuity plane. The short-term annuitization strategy will be obtained as follows. Given the individual is endowed with  $A_n$  annuities and  $W_m$  wealth now, she then compares  $U(A_n, W_m)$ ,  $U(A_{n+1}, W_{m-1})$  and  $U(A_{n-1}, W_{m+1})$ , and moves to the point with the greatest lifetime utility. For instance, if  $U(A_{n+1}, W_{m-1})$  is the largest one, the individual will buy one more annuity.

The long-term annuitization strategy will be obtained as follows. For people who are considering buying more annuities, they list all the lifetime utilities with the same total endowment of wealth and annuities and locate the largest one. Then people will keep on buying annuities until they reach the point with the largest lifetime utility. Similarly, when they are considering surrendering some existing annuities, with the surrender charge taken into account they list all the lifetime utilities with an equivalent total endowment of wealth and annuities and locate the largest one. Then people will keep on surrendering annuities until they reach the point with the largest lifetime utility.

Now I am going to show this work piece by piece at one time.

### 1.2.1 Probability Distortion Function

In order to solve the non-concave maximization problem under a stochastic process framework, I propose a new probability distortion function and approximate the stock price process locally as a Weibull process. In this section, I first introduce this new probability distortion function and its corresponding properties. Then I derive the stochastic diffusion equation for the distorted stock price process using the techniques proposed by Young and Zariphopoulou (2000) [36].

Due to the introduction of a reverse S-shaped probability distortion function, the optimal asset allocation model is not a concave maximization problem anymore. Jin and Zhou (2008) [15] first derived an explicit solution by proposing a quantile formulation. Since my model differs from theirs by considering an optimization problem of lifetime utility rather than one-time utility of the terminal wealth, I tried to combine the quantile formulation and the path integral method to solve the lifetime non-concave maximization problem. However, the complexity of embedded infinitely many quantile variables prevent me from pursuing in this way. As an alternative, I approach this problem under a stochastic process setting, which allows for more flexibilities to address the probability distortion. I apply a technical tool developed by Young and Zariphopoulou (2000) [36] to derive the stochastic diffusion process for a distorted probability. The methodology comes from theories of stochastic control, stochastic differential games, and the non-linear partial differential equations.

I start with a brief review of the fundamental results on the stochastic differential equation for a general distorted probability. See Young and Zariphopoulou (2000)

[36] for the derivations.

**Theorem 1.2.1.** *Assume that the original differential equation for a stochastic process is*

$$dX_s = b(X_s, s)ds + \sigma(X_s, s)dB_s$$

*And the conditional decumulative probability function is*

$$u(x, t; y, T) = Pr(X_T > y | X_t = x)$$

*After applying a probability distortion on this decumulative probability function, we have a distorted cumulative distribution function  $F(x, t) = d(u(x, t))$ , the new stochastic differential equation for the distorted probability is*

$$dX_s = f(X_s, s)ds + \sigma(X_s, s)dB_s \tag{1.2.1}$$

*where*

$$f(X_s, s) = b(X_s, s) + \sigma^2(X_s, s)F_x \frac{\frac{\partial^2 d^{-1}(F)}{\partial F^2}}{\frac{\partial d^{-1}(F)}{\partial F}} \tag{1.2.2}$$

*Proof.* Young and Zariphopoulou assume that the stock price process  $X_s$  solves the following stochastic differential equation

$$dX_s = b(X_s, s)ds + \sigma(X_s, s)dB_s, X_0 = x, x \in \mathcal{R}$$

for  $s \in [0, T]$ .

Below is a conditional decumulative probability function

$$u(x, t; y, T) = Pr(X_T > y | X_t = x)$$

in which  $y$  is a fixed parameter and  $T$  is a fixed horizon.

The corresponding Feynman-Kac partial differential equation is

$$u_t + \frac{1}{2}\sigma^2(x, t)u_{xx} + b(x, t)u_x = 0$$

with a boundary condition  $u(x, T; y, T) = \begin{cases} 1 & \text{if } x > y \\ 0 & \text{if } x \leq y \end{cases}$ .

Young and Zariphopoulou now consider a general distortion function  $d : [0, 1] \rightarrow [0, 1]$  on a decumulative probability function:

$$F(x, t) = d(u(x, t)) \tag{1.2.3}$$

By the Ito's formula,  $F$  solves the following stochastic differential equation:

$$F_t + \frac{1}{2}\sigma^2(x, t)F_{xx} + b(x, t)F_x = -\frac{1}{2}\sigma^2(x, t)F_x^2 \frac{[d^{-1}(F)]''}{[d^{-1}(F)]'} \tag{1.2.4}$$

By change of variables, Young and Zariphopoulou show that the above differential

equation can be rewritten as

$$F_t + \frac{1}{2}\sigma^2(x, t)F_{xx} + b(x, t)F_x + \frac{1}{2}\sigma^2(x, t) \min_{\alpha=(\alpha_1, \alpha_2)} \max_{|z| \leq 1} [-\alpha_1 F_x - \frac{1}{4}\alpha_1^2 z L F + \frac{1}{4}\alpha_1^2 (H(\alpha_2) + L\alpha_2 z)] = 0 \quad (1.2.5)$$

in which  $\alpha = (\alpha_1, \alpha_2)$  and  $z$  are the control variables,  $H$  is a Lipschitz function and  $L$  is its Lipschitz constant.

Then they proceed to use the stochastic differential games theory to show that the above differential equation is the Hamilton-Jacobian-Bellman equation of the following stochastic control problem:

$$dX_s = f(x_s, \alpha_s, z_s, s)ds + \sigma(X_s, s)dB_s \quad (1.2.6)$$

with the initial condition  $X_t = x, x \in \mathcal{R}$

The value function is:

$$J(x, t; \alpha, z) = E\left[\int_t^T h(X_s, \alpha_s, z_s, u_s) e^{-\int_t^s c(X_\tau, \alpha_\tau, z_\tau, u_\tau) d\tau} ds + \Phi(X_T)\right] \quad (1.2.7)$$

The function  $c(X, \alpha, z, u)$  and  $h(X, \alpha, z, u)$  and  $\Phi(x)$  represent the discount factor, the running cost and the terminal penalty function respectively, where



$$f(x, \alpha, z, t) = b(x, t) + \sigma^2(x, t) F_x \frac{[d^{-1}(F)]''}{[d^{-1}(F)]'} \quad (1.2.8)$$

$$h(x, \alpha, z, t) = \frac{1}{8} \sigma^2(x, t) \alpha_1^2 (H(\alpha_2) + L \alpha_2 z) \quad (1.2.9)$$

$$c(x, \alpha, z, t) = \frac{1}{8} \sigma^2(x, t) \alpha_1^2 z L \quad (1.2.10)$$

$$\Phi(x) = \mathbf{1}_{\{x > y\}} \quad (1.2.11)$$

□

**Remark 1.2.2.** Inspecting (1.2.8), we see that an explicit inverse function of the probability distortion function is required.

While no previously developed reverse S-shaped probability distortion functions have an explicit inverse function (see Kahneman and Tversky (1992) [33], Tversky and Fox (1995) [32] and Prelec (1998) [26]), I propose a new probability distortion function:

$$d(u) = 1 - \frac{1}{1 - \delta \cdot \ln u} \quad (1.2.12)$$

with distortion parameter  $\delta > 0$

**Proposition 1.2.3.** *The stochastic differential equation of the distorted stock price process is*

$$dX_s = f(X_s, s)ds + \sigma(X_s, s)dB_s$$

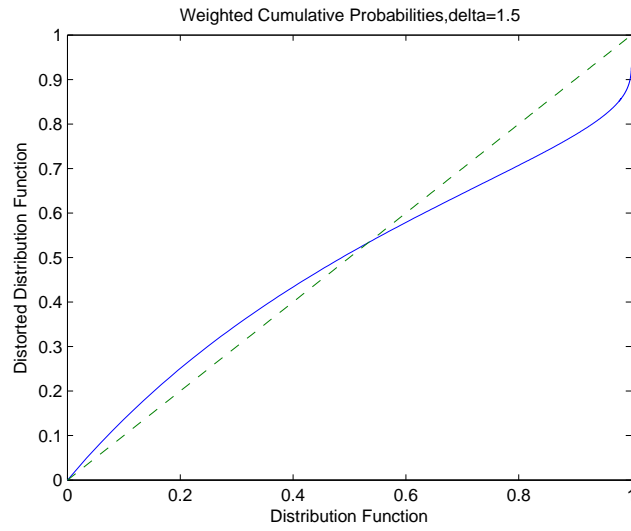


FIGURE 1.2.1: New probability distortion function with distortion parameter  $\delta = 1.5$

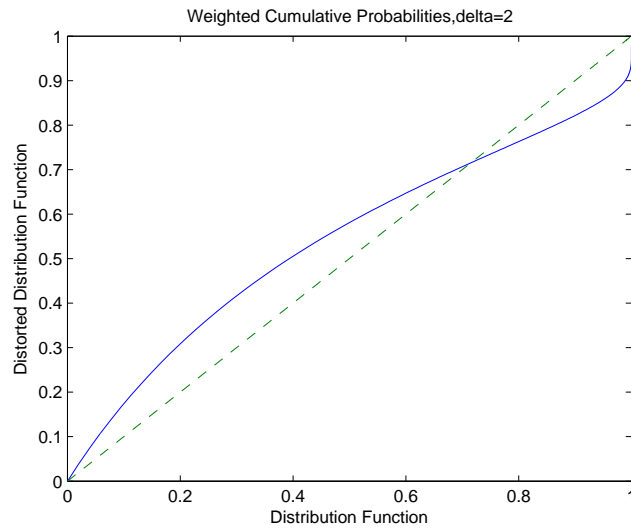


FIGURE 1.2.2: New probability distortion function with distortion parameter  $\delta = 2$

where

$$f(X_s, s) = b(X_s, s) - \frac{1}{2}\sigma^2(X_s, s)\left\{2\frac{u_x(x, s)}{u(x, s)}[-1 + 2\delta(1 - d(u(x, s)))]\right\} \quad (1.2.13)$$

*Proof.* Since

$$F(x, t) = d(u(x, t)) = 1 - \frac{1}{1 - \delta \ln(u(x, t))}$$

$$F_x = -\frac{\delta}{u}(1 - F)^2 u_x$$

The inverse function:

$$u = e^{\frac{F}{F\delta - \delta}}$$

Then

$$\frac{\partial d^{-1}(F)}{\partial F} = e^{\frac{F}{F\delta - \delta}} \frac{-\delta}{(F\delta - \delta)^2}$$

and

$$\frac{\partial^2 d^{-1}(F)}{\partial F^2} = e^{\frac{F}{F\delta - \delta}} \frac{-\delta}{(F\delta - \delta)^2} \left[ \frac{-\delta}{(F\delta - \delta)^2} - \frac{2\delta}{F\delta - \delta} \right]$$

Therefore,

$$F_x \frac{\frac{\partial^2 d^{-1}(F)}{\partial F^2}}{\frac{\partial d^{-1}(F)}{\partial F}} = -\left\{ \frac{u_x}{u} [-1 + 2\delta(1 - d(u))] \right\}$$

Thus,

$$f(X_s, s) = b(X_s, s) - \frac{1}{2}\sigma^2(X_s, s)\left\{2\frac{u_x(x, s)}{u(x, s)}[-1 + 2\delta(1 - d(u(x, s)))]\right\} \quad (1.2.14)$$

□

**Remark 1.2.4.** Employing the new probability distortion function, I derive the stock price process which involves a negative hazard function  $\frac{u_x(x, s)}{u(x, s)}$ . In order to obtain an

explicit stochastic differential equation for the distorted stock price process, a closed form hazard function is required.

**Proposition 1.2.5.** *The reverse S-shaped probability distortion function is concave firstly, and convex beyond. The inflection point of this probability distortion function is*

$$p = 1 - \exp^{\frac{1-2\delta}{\delta}} \quad (1.2.15)$$

*Proof.* Since

$$d'(p) = F(x, t) = 1 - \frac{1}{1 - \delta \ln(u)} = 1 - \frac{1}{1 - \delta \ln(1 - p)}$$

thus,

$$d'(p)' = \left(\frac{1}{1 - \delta \ln(1 - p)}\right)^2 \frac{\delta}{1 - p}$$

and

$$d'(p)'' = \left(\frac{1}{1 - \delta \ln(1 - p)}\right)^2 \frac{\delta}{(1 - p)^2} \left(\frac{-2\delta}{1 - \delta \ln(1 - p)} + 1\right)$$

Let  $d'(p)'' = 0$ , we obtain

$$p = 1 - \exp^{\frac{1-2\delta}{\delta}}$$

□

## 1.2.2 Weibull Distribution

While the lognormal distribution does not have a closed form hazard function, the Weibull distribution does.

The probability density function of a Weibull distribution is

$$f(x) = \frac{\beta x^{\beta-1}}{\sigma^\beta} e^{-(\frac{x}{\sigma})^\beta} \quad (1.2.16)$$

with a cumulative probability function

$$F(x) = 1 - e^{-(\frac{x}{\sigma})^\beta} \quad (1.2.17)$$

and a hazard function:

$$h(x) = \frac{\beta x^{\beta-1}}{\sigma^\beta} \quad (1.2.18)$$

By using a Weibull distribution locally to approximate the stock price, we also benefit from obtaining a simplified distortion function:

$$1 - d(u(x, s)) = \frac{1}{1 + \delta(\frac{x}{\sigma})^\beta} \quad (1.2.19)$$

Greg Hertzler (2003) [11] systematically derived the stochastic differential equations with Beta family distributions as stable or semi-stable solutions. Following his work, I define a stochastic differential equation with our Weibull distribution as a stable solution. In particular, as the geometric Brownian motion has an increasing drift with respect to the stock price, this process is designed to have a positive drift.

Given  $F(y, t|x, s)$  and  $f(y, t|x, s)$ , Hertzler seeks to find the corresponding stochastic differential equation  $dy = g(y, t)dt + h(y, t)dB_t$ . Functions  $g$  and  $h$  are chosen to satisfy the Kolmogorov forward and backward equations:

$$\frac{\partial f}{\partial t} + \frac{\partial[g(y, t)f]}{\partial y} - \frac{\partial^2[h(y, t)^2 f]}{2\partial y^2} = 0 \quad (1.2.20)$$

$$\frac{\partial f}{\partial s} + g(x, s)\frac{\partial f}{\partial x} + h(x, s)^2\frac{\partial^2 f}{2\partial x^2} = 0 \quad (1.2.21)$$

Hertzler proceeds to solve for  $g$  and  $h$  to generate a desired distribution as semi-stable solution for the stochastic differential equation. By contrast, I assume that the Weibull distribution is the stable solution for a strongly stationary process. That is,  $dy = g(y)dt + h(y)dB_t$ . After integrating the forward equation with respect to  $y$ , the drift part is:

$$g = \frac{1}{2} \frac{\partial[h^2 f]}{\partial x} \frac{1}{f} \quad (1.2.22)$$

In order to guarantee a positive drift term, I define:

$$h(x) = (2e^{(\frac{x}{\sigma})^\beta} x^{\gamma-\beta+1} \frac{\sigma^\beta}{b\beta})^{\frac{1}{2}} \quad (1.2.23)$$

Then,

$$h^2 f = 2 \frac{y^\gamma}{b}$$

Plug in (1.2.22), therefore

$$g(x) = \frac{1}{b} \gamma \frac{\sigma^\beta}{\beta} x^{\gamma-\beta} e^{(\frac{x}{\sigma})^\beta} \quad (1.2.24)$$

Thus, Weibull process's stochastic differential equation is

$$dx = \left[ \frac{1}{b} \gamma \frac{\sigma^\beta}{\beta} x^{\gamma-\beta} e^{(\frac{x}{\sigma})^\beta} \right] ds + \left( 2e^{(\frac{x}{\sigma})^\beta} x^{\gamma-\beta+1} \frac{\sigma^\beta}{b\beta} \right)^{\frac{1}{2}} dB_s \quad (1.2.25)$$

with the shape parameter  $\beta$ , the scale parameters  $\sigma$ ,  $\gamma$  and  $b$ .

**Proposition 1.2.6.** *A stochastic process is a Weibull process with shape parameter  $\beta$  and scale parameters  $\sigma$ ,  $\gamma$  and  $b$ . After applying a probability distortion function  $F(x, t) = 1 - \frac{1}{1-\delta \ln(u)}$ , the stochastic differential equation for the distorted Weibull process is:*

$$dx = \left[ \frac{1}{b} \gamma \frac{\sigma^\beta}{\beta} x^{\gamma-\beta} e^{(\frac{x}{\sigma})^\beta} + \frac{2}{b} x^\gamma e^{(\frac{x}{\sigma})^\beta} \left( -1 + \frac{2\delta}{1 + \delta (\frac{x}{\sigma})^\beta} \right) \right] ds + \left( 2e^{(\frac{x}{\sigma})^\beta} x^{\gamma-\beta+1} \frac{\sigma^\beta}{b\beta} \right)^{\frac{1}{2}} dB_s \quad (1.2.26)$$

*Proof.* Employing the new distortion function, I obtain that

$$dX_s = \left\{ b(X_s, s) - \frac{1}{2} \sigma^2(X_s, s) \left\{ 2 \frac{u_x(x, s)}{u(x, s)} [-1 + 2\delta(1 - d(u(x, s)))] \right\} \right\} ds + \sigma(X_s, s) dB_s \quad (1.2.27)$$

Now I consider a process with Weibull distribution as a stable solution with

$$b(X_s, s) = \frac{1}{b} \gamma \frac{\sigma^\beta}{\beta} x^{\gamma-\beta} e^{(\frac{x}{\sigma})^\beta} \quad (1.2.28)$$

$$\sigma^2(X_s, s) = 2e^{(\frac{x}{\sigma})^\beta} x^{\gamma-\beta+1} \frac{\sigma^\beta}{b\beta} \quad (1.2.29)$$

$$\frac{u_x(x, s)}{u(x, s)} = -\frac{\beta}{\sigma^\beta} x^{\beta-1} \quad (1.2.30)$$

$$1 - d(u(x, s)) = \frac{1}{1 + \delta (\frac{x}{\sigma})^\beta} \quad (1.2.31)$$

Plug in (1.2.27)

Therefore,

$$dx = \left[ \frac{1}{b} \gamma \frac{\sigma^\beta}{\beta} x^{\gamma-\beta} e^{(\frac{x}{\sigma})^\beta} + \frac{2}{b} x^\gamma e^{(\frac{x}{\sigma})^\beta} \left( -1 + \frac{2\delta}{1 + \delta(\frac{x}{\sigma})^\beta} \right) \right] ds + \left( 2e^{(\frac{x}{\sigma})^\beta} x^{\gamma-\beta+1} \frac{\sigma^\beta}{b\beta} \right)^{\frac{1}{2}} dB_s \quad (1.2.32)$$

□

### 1.2.3 Stochastic Control Problem

Since I work with the stochastic process and I aim to maximize the lifetime utility, my model is essentially a stochastic control problem. In this part, I seek to briefly introduce its framework and the Hamilton-Jacobi-Bellman (HJB) partial differential equation.

The following diffusion process will be considered:

$$dX_s = f(s, X_s, u_s)ds + \sigma(s, X_s, u_s)dB_s, X_t = x_t \quad (1.2.33)$$

The state  $\{X_s\}_{s \geq t}$  depends on the control process  $\{u_s\}_{s \geq t}$

The goal of a stochastic control problem is to control the diffusion to behave in a certain way, such that the cost or value functional is minimized or maximized. Define the value functional:

$$V(t, x) = \sup_u E \left[ \int_t^\tau L(s, X_s, u_s) ds + \Psi(\tau, X_\tau) | X_t = x \right] \quad (1.2.34)$$



In order to derive an equation for  $V$ , I first introduce the Bellman's dynamic programming principle:

$$V(t, x) = \sup_u E\left[\int_{t+h}^{\tau} L(s, X_s, u_s)ds + V(t+h, X_{t+h}) \mid X_t = x\right] \quad (1.2.35)$$

The intuition behind this Bellman's dynamic programming principle is that the maximal benefit on  $[t, \tau]$  is achieved when running optimally in  $[t, t+h]$  and then continue optimally in  $[t+h, \tau]$  with  $X(t+h)$  as an initial value. This principle is used to derive the HJB equation.

Let the control process be constant  $u(s) = v$  for  $s \in [t, t+h]$ . By Fubini's Theorem and Ito's Formula, I obtain:

$$0 = V_t(t, x) + \sup_v [V_x(t, x)f(t, x, v) + \frac{1}{2}\sigma^2(t, x, v)V_{xx}(t, x) + L(t, x, v)] \quad (1.2.36)$$

Interested readers may refer to the book *Controlled Markov Processes and Viscosity Solutions*, Fleming and Soner (2006) [9] for details.

Now I rewrite the dynamic programming equation in terms of the Hamiltonian:

$$\mathcal{H}(t, x, V_x, V_{xx}) = \inf_v [-f(t, x, v)V_x - \frac{1}{2}V_{xx}\sigma^2(t, x, v) - L(t, x, v)] \quad (1.2.37)$$

Then the dynamic programming equation takes the form of the HJB equation, satis-

fyng:

$$-V_t + \mathcal{H}(t, x, V_x, V_{xx}) = 0 \quad (1.2.38)$$

$$V(T, x) = \Psi(x) \quad (1.2.39)$$

## 1.3 Model

In this section, I first introduce the model settings in terms of the financial market and the policyholders. Then I discuss the problem formulation with wealth dynamics and value function, and I derive the Hamilton-Jacobi-Bellman equation by stochastic control theory. Finally I show my numerical method to solve the partial differential equation.

### 1.3.1 Financial Market and Policyholders

On the financial market, an individual can deposit money in a bank account with a risk-free interest rate  $r$ . Besides, she can invest on a risky assets that we can approximate locally by a stochastic process whose stable solution is a Weibull distribution.

$$dx = \left[ \frac{1}{b} \gamma \frac{\sigma^\beta}{\beta} x^{\gamma-\beta} e^{(\frac{x}{\sigma})^\beta} \right] ds + \left( 2e^{(\frac{x}{\sigma})^\beta} x^{\gamma-\beta+1} \frac{\sigma^\beta}{b\beta} \right)^{\frac{1}{2}} dB_s \quad (1.3.1)$$

To examine the effect of the probability distortion on policyholders' behavior, I consider a stochastic process in this work whose stable solution is an distorted Weibull distribution. This distorted stochastic process is:

$$dx = [\frac{1}{b}\gamma\frac{\sigma^\beta}{\beta}x^{\gamma-\beta}e^{(\frac{x}{\sigma})^\beta} + \frac{2}{b}x^\gamma e^{(\frac{x}{\sigma})^\beta}(-1 + \frac{2\delta}{1 + \delta(\frac{x}{\sigma})^\beta})]ds + (2e^{(\frac{x}{\sigma})^\beta}x^{\gamma-\beta+1}\frac{\sigma^\beta}{b\beta})^{\frac{1}{2}}dB_s \quad (1.3.2)$$

$\beta$  is the shape parameter,  $\sigma$  is the scale parameter, and  $\gamma$  is another scale parameter which determines the shape of the process volatility.

In order to investigate policyholders' annuitization strategy, I consider an unrestricted life annuity market. More specifically, I assume commutable life annuities, of which the policyholder can purchase or surrender any amount at any point of time.

I consider an individual with a future lifetime followed by a random variable  $\tau_d$ . I assume that  $\tau_d$  follows an exponential distribution with the parameter  $\lambda$ .  $\lambda$  is the force of mortality.

The price of a life annuity which pays \$1 per year continuously until the individual dies is given by

$$\bar{a} = \int_0^\infty e^{-rs}e^{-\lambda s}ds = \frac{1}{r + \lambda} \quad (1.3.3)$$

Due to the commutability of the life annuities, retirees can surrender any portion of the existing annuity income in exchange for the money from the insurers. The surrender charge is proportional to the price of life annuities. That is, the surrender value \$1 of the annuity income is  $(1 - p)\bar{a}$  with  $0 \leq p \leq 1$ , in which  $p$  is the proportional surrender charge.

In my model, there are two separate accounts: wealth account and annuity ac-

count. The wealth account consists of both bonds and investment in stocks; whereas the annuity account includes the life annuities only. Within the wealth account, the retiree can invest on the risky asset with money borrowed from the riskless asset at a risk-free interest rate  $r$ . The proceeds from annuities and the money from surrendering annuities are added to the wealth account. On the other hand, in order to buy more annuities, retirees withdraw money from the wealth account. In particular, to prevent the retiree from borrowing against her future annuity income, the wealth account is required to be non-negative. This assumption is reasonable since annuities stop paying proceeds once the retiree dies, so we do not allow her to die with a negative wealth account. To keep the wealth account from being negative, I assume that the retiree will surrender the existing annuity income as needed.

Following Yaari (1965) [35], I also consider a retiree without a bequest motive. In other words, the utility only comes from her consumption. Normally she will first consume the proceeds from the annuities. If the proceeds is not enough to cover her consumption, she will withdraw money from the wealth account; if the proceeds is more than what she consumes, she will add the money to the wealth account.

In this work, I assume that the retiree is risk averse and is characterized by a constant relative risk aversion (CRRA) utility function, which is consistent with other portfolio selection literatures. That is,

$$u(c) = \frac{c^{1-\alpha}}{1-\alpha}$$

$\alpha > 0$  and  $\alpha \neq 1$

in which  $\alpha$  is the risk aversion parameter.

### 1.3.2 Wealth Dynamics and Value Function

In this work, I assume that the retiree consumes at a rate of  $c_s$  at time  $s$ .  $\pi_s$  represents the amount of money invested in the stock market at time  $s$ .  $A_s$  is the amount of life annuity income at time  $s$ . The initial endowment at time  $t$  in the wealth account and annuity account are  $W$  and  $A$  respectively.

I first derive the investment and consumption strategies by assuming that the amount in annuity account stays the same at time  $s$ . Therefore, the change in the wealth account consists of four parts: the interest from the bond, the return from the stock, the proceeds from the annuities, and the consumption.

Following the assumption that the stock price follows a process with Weibull distribution as a stable solution, the wealth dynamics is:

$$\begin{aligned} dW_s = & \{r(W_s - \pi_s) + \frac{\pi_s}{X_s} [\frac{1}{b} \gamma \frac{\sigma^\beta}{\beta} X_s^{\gamma-\beta} e^{(\frac{X_s}{\sigma})^\beta} + \frac{2}{b} X_s^\gamma e^{(\frac{X_s}{\sigma})^\beta} (-1 + \frac{2\delta}{1 + \delta(\frac{X_s}{\sigma})^\beta})] - c_s + A_s\} ds \\ & + \frac{\pi_s}{X_s} (2e^{(\frac{X_s}{\sigma})^\beta} X_s^{\gamma-\beta+1} \frac{\sigma^\beta}{\beta})^{\frac{1}{2}} dB_s \end{aligned} \tag{1.3.4}$$

with the initial endowments  $A_t = A \geq 0$  and  $W_t = W \geq 0$ .

The control processes are the corresponding investment strategy  $\{\pi_s\}_{s \geq t}$  and con-

sumption strategy  $\{c_s\}_{s \geq t}$ .

In this model, the individual seeks to maximize her lifetime utility. The value function is:

$$U(A_t, W_t) = \sup_{\pi_s, c_s} \mathbb{E} \left[ \int_t^{\tau_d} e^{-rs} u(c_s) ds \mid A_t = A, W_t = W \right] \quad (1.3.5)$$

Recall that an individual's future lifetime follows an exponential distribution with the parameter  $\lambda$ .

Therefore,

$$U(A_t, W_t) = \sup_{\pi_s, c_s} \mathbb{E} \left[ \int_t^{\infty} e^{-(r+\lambda)s} u(c_s) ds \mid A_t = A, W_t = W \right] \quad (1.3.6)$$

Note here for the future work, I would like to examine the effect from mortality distortion on the policyholders' behavior by distorting  $\lambda$ .

The above wealth dynamics and value function form a typical stochastic control model with control processes  $\{\pi_s\}_{s \geq t}$  and  $\{c_s\}_{s \geq t}$ . Following Bellman's dynamic programming principle, we have:

$$U(A_t, W_t) = \sup_{\pi_s, c_s} \mathbb{E} \left[ \int_t^{t+h} e^{-(r+\lambda)s} u(c_s) ds + U(W_h, A_h) \mid A_t = A, W_t = W \right] \quad (1.3.7)$$

That means, to maximize the lifetime utility, the individual should maximize her instant utility at that moment. Thus I obtain the optimal investment and consumption strategies based on the endowment at the current time. Similarly, the optimal strategies at the next instant time depend on the endowment at that point of time. In this

work, I aim to derive the instantaneous investment, consumption and annuitization strategies to maximize the lifetime utility.

As I described earlier, the amount of annuity is fixed in this step and I will discuss the annuitization strategy in the next step. Thus the lifetime utility here can be considered as a function of only one variable  $W$ .

By the stochastic control theory, I derive the corresponding Hamilton-Jacobi-Bellman equation as shown below.

First, the Hamiltonian is:

$$\begin{aligned}
 \mathcal{H}(t, w, U_w, U_{ww}) = \inf_{\pi_t, c_t} \{ & -\left\{ \frac{\pi_t}{X_t} \left[ \gamma \frac{\sigma^\beta}{\beta} X_t^{\gamma-\beta} e^{b(\frac{X_t}{\sigma})^\beta} + 2bX_t^\gamma e^{b(\frac{X_t}{\sigma})^\beta} \left( -1 + \frac{2\delta}{1 + \delta b(\frac{X_t}{\sigma})^\beta} \right) \right] \right. \\
 & + r(W_t - \pi_t) - c_t + A_t \} U_w \\
 & \left. - \frac{1}{2} \left( \frac{\pi_t}{X_s} \right)^2 \left[ 2 \frac{\sigma^\beta}{\beta} \exp^{b(\frac{X_t}{\sigma})^\beta} X_t^{\gamma-\beta+1} \right] U_{ww} - \frac{c_t^{1-\alpha}}{1-\alpha} \right\}
 \end{aligned} \tag{1.3.8}$$

And

$$U_t = -(r + \lambda)U \tag{1.3.9}$$

Therefore, the Hamilton-Jacobi-Bellman equation is:

$$\begin{aligned}
(r + \lambda)U &= (rW_t + A_t)U_w \\
&+ \max_{\pi_t} \{ [-r\pi_t + \pi_t [\frac{1}{b}\gamma\frac{\sigma^\beta}{\beta}X_t^{\gamma-\beta-1}e^{(\frac{X_t}{\sigma})^\beta} + \frac{2}{b}X_t^{\gamma-1}e^{(\frac{X_t}{\sigma})^\beta}(-1 + \frac{2\delta}{1+\delta(\frac{X_t}{\sigma})^\beta})]]U_w \\
&+ \pi_t^2 [\frac{\sigma^\beta}{b\beta}e^{(\frac{X_t}{\sigma})^\beta}X_t^{\gamma-\beta-1}]U_{ww} \} \\
&+ \max_{c_t} (\frac{c_t^{1-\alpha}}{1-\alpha} - c_t U_w)
\end{aligned} \tag{1.3.10}$$

Maximizing the portion of the HJB equation involving investment and consumption strategies, the investment and consumption strategies can be explicitly obtained:

$$\pi_t = - \frac{[\frac{1}{b}\gamma\frac{\sigma^\beta}{\beta}X_t^{\gamma-\beta-1}e^{(\frac{X_t}{\sigma})^\beta} + \frac{2}{b}X_t^{\gamma-1}e^{(\frac{X_t}{\sigma})^\beta}(-1 + \frac{2\delta}{1+\delta(\frac{X_t}{\sigma})^\beta}) - r]U_w}{2X_t^{\gamma-\beta-1}\frac{\sigma^\beta}{b\beta}e^{(\frac{X_t}{\sigma})^\beta}U_{ww}} \tag{1.3.11}$$

$$c_t = U_w^{-\frac{1}{\alpha}} \tag{1.3.12}$$

Therefore, the simplified HJB equation is:

$$\begin{aligned}
(r + \lambda)U &= (rW_t + A_t)U_w - \frac{[\frac{1}{b}\gamma\frac{\sigma^\beta}{\beta}X_t^{\gamma-\beta-1}e^{(\frac{X_t}{\sigma})^\beta} + \frac{2}{b}X_t^{\gamma-1}e^{(\frac{X_t}{\sigma})^\beta}(-1 + \frac{2\delta}{1+\delta(\frac{X_t}{\sigma})^\beta}) - r]^2 U_w^2}{4X_t^{\gamma-\beta-1}\frac{\sigma^\beta}{b\beta}e^{(\frac{X_t}{\sigma})^\beta}U_{ww}} \\
&+ U_w^{-\frac{1}{\alpha}+1} \frac{\alpha}{1-\alpha}
\end{aligned} \tag{1.3.13}$$

In this work, since the partial differential equation is not explicitly solvable, I refer to a numerical method which will be discussed in more details later. The key idea underlying this algorithm is a discretization of the wealth-annuity space. On a



wealth-annuity plane, we have points  $(A_n, W_m)$ ,  $n = 0, 1, 2, \dots$  and  $m = 0, 1, 2, \dots$ . The distance between  $W_m$  and  $W_{m+1}$  is  $\bar{a}$  (the price of a life annuity which pays \$1 per year continuously) multiplies the distance between  $A_n$  and  $A_{n+1}$ , because  $\bar{a}$  is needed in exchange for one more annuity. Then I compare the lifetime utilities among three points  $U(A_{n+1}, W_{m-1})$ ,  $U(A_n, W_m)$  and  $U(A_{n-1}, W_{m+1})$ , with the last one adjusted by the surrender charge. Then the individual will move to the point which has the largest lifetime utility. It then provides us her annuitization strategy.

For the sake of simplicity, I demonstrate the short-term annuitization strategy when there is no surrender charge  $p = 0$ . If the distance between  $W_m$  and  $W_{m+1}$  is  $d$ , then the distance between  $A_n$  and  $A_{n+1}$  is  $\frac{d}{\bar{a}}$ . The annuitization strategy is obtained as follows. An individual with an endowment of  $A_n$  and  $W_m$ :

if  $U(A_{n-1}, W_{m+1})$  is the largest one, she will surrender  $\frac{d}{\bar{a}}$  existing annuities in exchange for  $\$d$ .

if  $U(A_n, W_m)$  is the largest one, she will neither buy more annuities nor surrender any existing annuities;

if  $U(A_{n+1}, W_{m-1})$  is the largest one, she will use  $\$d$  from the wealth account to buy  $\frac{d}{\bar{a}}$  annuities.

The case with a surrender charge is demonstrated in Figure 1.3.1.

In sum, I obtain the instantaneous investment, consumption and annuitization

Behavior	Utility		
Buy	$U(A+4, W-50)$		
Do nothing		$U(A, W)$	
Surrender			$U(A-4, W+40)$

FIGURE 1.3.1: Demonstration of a short-term annuitization strategy

strategies based on the endowment at the current time. Once these strategies are obtained from our stochastic control system, the retirees perform them simultaneously. After that, they move to a new point  $(A_{n'}, W_{m'})$  on the grid, and the strategies for the next point of time will be obtained based on the new endowment  $A_{n'}$  and  $W_{m'}$ .

In addition, the long-term annuitization strategy will be obtained as follows. For people who are considering buying more annuities, I list all the lifetime utilities with the same total endowment of wealth and annuities, and locate the largest one. Then people will keep on buying annuities until they reach the point with the largest lifetime utility. Different from the short-term annuitization strategy, it could be over multiple grids. Similarly, when they are considering surrendering some existing annuities, with the surrender charge taken into account I list all the lifetime utilities with an equivalent total endowment of wealth and annuities, and locate the largest one. Then people will keep on surrendering annuities until they reach the point with the largest lifetime utility. The calculation is demonstrated in the following two graphs respectively:

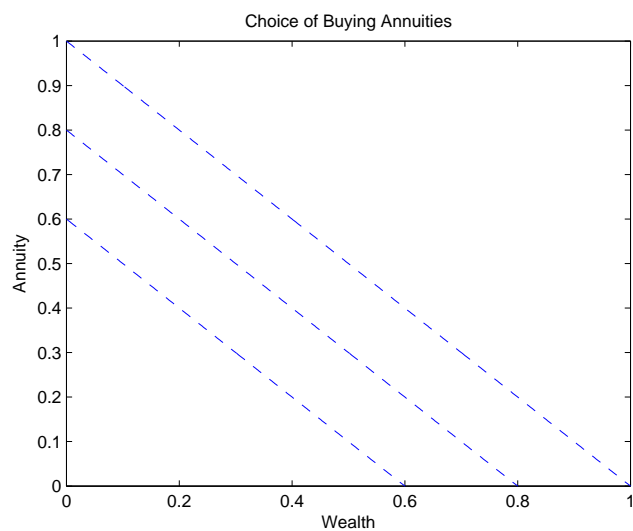


FIGURE 1.3.2: Demonstration of a long-term annuitization strategy

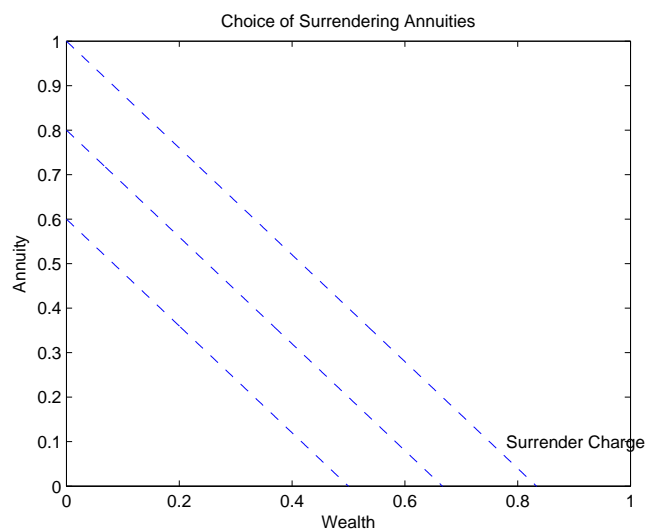


FIGURE 1.3.3: Demonstration of a long-term annuitization strategy

### 1.3.3 Numerical Method

Since the above partial differential equation is not explicitly solvable, I refer to a numerical method. Basically I apply a forward finite difference method formulated as below:

$$U_w(A_i, W_{j+1}) = U_w(A_i, W_j) + [W_{j+1} - W_j] \cdot U_{ww}(A_i, W_{j+1}) \quad (1.3.14)$$

$$U(A_i, W_{j+1}) = U(A_i, W_j) + [W_{j+1} - W_j] \cdot U_w(A_i, W_j) \quad (1.3.15)$$

Using this numerical algorithm, we have to have  $U(A_i, W_1)$ ,  $U(A_i, W_2)$  and  $U_w(A_i, W_1)$  for all  $i = 1, 2, 3, \dots$  to start with. Here I have three assumptions about the boundary conditions.

Assumption 1: When  $A_1$  is relatively small and  $W = 0$ , the individual will consume all the proceeds from the annuity.

That is,

$$\bar{U}(A_1 = A, W = 0) = \int_0^\infty e^{-(r+\lambda)s} \frac{A^{1-\alpha}}{1-\alpha} ds = \frac{A^{1-\alpha}}{(1-\alpha)(r+\lambda)} \quad (1.3.16)$$

Assumption 2: When  $A_1$  is relatively small and  $W$  is also small, the individual will spend all the wealth to buy annuities and consume all the proceeds from the annuity.

$$\bar{U}(A_1 = A, W = W) = \int_0^\infty e^{-(r+\lambda)s} \frac{(A + \frac{W}{a})^{1-\alpha}}{1-\alpha} ds = \frac{(A + \frac{W}{a})^{1-\alpha}}{(1-\alpha)(r+\lambda)} \quad (1.3.17)$$

Therefore,

$$\begin{aligned}\bar{U}_w(A_1 = A, W = 0) &= \lim_{W \rightarrow 0} \frac{\bar{U}(A_1 = A, W = W) - \bar{U}(A_1 = A, W = 0)}{W} \\ &= \lim_{W \rightarrow 0} \frac{\frac{(A + \frac{W}{a})^{1-\alpha}}{(1-\alpha)(r+\lambda)} - \frac{A^{1-\alpha}}{(1-\alpha)(r+\lambda)}}{W} = A^{-\alpha}\end{aligned}\quad (1.3.18)$$

Assumption 3: When  $A_i = A$  and  $W = 0$ , its lifetime utility equals to the lifetime utility of surrendering one annuity.

$$\begin{aligned}\bar{U}(A + \Delta A, 0) &= \bar{U}(A, (1-p)\bar{a}\Delta A) \\ &= \bar{U}(A, 0) + (1-p)\bar{a}\Delta A \bar{U}_w(A, 0) \\ &= \bar{U}(A, 0) + (1-p)\bar{a}\Delta A A^{-\alpha}\end{aligned}\quad (1.3.19)$$

$$\frac{\bar{U}(A + \Delta A, 0) - \bar{U}(A, 0)}{\Delta A} = (1-p)\bar{a}\Delta A A^{-\alpha} \quad (1.3.20)$$

$$\frac{\partial \bar{U}(A, 0)}{\partial A} = (1-p)\bar{a}\Delta A A^{-\alpha} \quad (1.3.21)$$

Therefore,

$$\int_A^{A_1} \frac{\partial \bar{U}(A, 0)}{\partial A} dA = (1-p)\bar{a} \int_A^{A_1} A^{-\alpha} dA \quad (1.3.22)$$

$$\bar{U}(A_1, 0) - \bar{U}(A, 0) = (1-p)\bar{a} \left[ \frac{A_1^{1-\alpha}}{1-\alpha} - \frac{A^{1-\alpha}}{1-\alpha} \right] \quad (1.3.23)$$

$$\bar{U}(A_1, 0) = \frac{A^{1-\alpha}}{(1-\alpha)(r+\lambda)} + (1-p)\bar{a} \left[ \frac{A_1^{1-\alpha}}{1-\alpha} - \frac{A^{1-\alpha}}{1-\alpha} \right] \quad (1.3.24)$$

Note that when we compare the surrendered lifetime utility, we have to adjust it with the surrender charge. That is,

$$U(A_i, W_{j+1}) = U(A_i, W_j) + (1 - p)[W_{j+1} - W_j] \cdot U_w(A_i, W_j)$$

### 1.3.4 Parameters

In order to approximate the geometric Brownian motion with  $\mu = 0.08$  and  $\sigma = 0.2$ , I first use a Maximum Likelihood Estimation method (MLE). Specifically, I simulate  $n = 10,000$  numbers from the lognormal distribution, then estimate Weibull parameters using MLE. Because

$$f(x) = \frac{\beta x^{\beta-1}}{\sigma^\beta} e^{-(\frac{x}{\sigma})^\beta}$$

Therefore

$$L = \prod_{i=1}^n \frac{\beta x_i^{\beta-1}}{\sigma^\beta} e^{-(\frac{x_i}{\sigma})^\beta} \quad (1.3.25)$$

$$\ln L = n \ln \beta + (\beta - 1) \sum_{i=1}^n \ln x_i - n \beta \ln \sigma - \sum_{i=1}^n \left(\frac{x_i}{\sigma}\right)^\beta \quad (1.3.26)$$

In order to maximize  $\ln L$ , I take its derivative with respect to  $\beta$  and  $\sigma$  respectively, and let them equal to 0. Then I have:

$$\beta = \frac{n}{\frac{\sum_{i=1}^n x_i^\beta \ln x_i}{\sum_{i=1}^n x_i^\beta} n - \sum_{i=1}^n \ln x_i} \quad (1.3.27)$$

$$\sigma = \left( \frac{\sum_{i=1}^n x_i^\beta}{n} \right)^{\frac{1}{\beta}} \quad (1.3.28)$$

In my case,

$$\beta = 5.02$$

$$\sigma = 117.5977$$

Next, I estimate parameters  $\gamma$  and  $b$ . Since all the rest of my analyses will be constrained at stock price  $x = 100$  locally, I estimate the Weibull process parameters in order to approximate the geometric Brownian motion behavior at  $x = 100$ . Two criteria are set up:

Let

$$g(x)/x = 0.08 \tag{1.3.29}$$

That is,

$$\frac{1}{b}\gamma\frac{\sigma^\beta}{\beta}x^{\gamma-\beta-1}e^{(\frac{x}{\sigma})^\beta} = 0.08$$

And

$$\frac{d(g(x)/x)}{dx} = 0 \tag{1.3.30}$$

That is,

$$\gamma - \beta - 1 + \frac{x^\beta\beta}{\sigma^\beta} = 0$$

Then

$$\gamma = 3.7951$$

$$b = 12930236.8218$$

For the distortion parameter  $\delta$ , the distortion effect signifies as  $\delta$  becomes larger. I choose  $\delta = 1.5$  to represent a moderate distortion.

## 1.4 Results

### 1.4.1 Stochastic Process for the Distorted Probability

The stochastic differential equation with Weibull distribution as a stable solution is:

$$dx = \left[ \frac{1}{b} \gamma \frac{\sigma^\beta}{\beta} x^{\gamma-\beta} e^{\left(\frac{x}{\sigma}\right)^\beta} \right] ds + \left( 2e^{\left(\frac{x}{\sigma}\right)^\beta} x^{\gamma-\beta+1} \frac{\sigma^\beta}{b\beta} \right)^{\frac{1}{2}} dB_s \quad (1.4.1)$$

The stochastic differential equation with distorted Weibull distribution as a stable solution is:

$$dx = \left[ \frac{1}{b} \gamma \frac{\sigma^\beta}{\beta} x^{\gamma-\beta} e^{\left(\frac{x}{\sigma}\right)^\beta} + \frac{2}{b} x^\gamma e^{\left(\frac{x}{\sigma}\right)^\beta} \left( -1 + \frac{2\delta}{1 + \delta \left(\frac{x}{\sigma}\right)^\beta} \right) \right] ds + \left( 2e^{\left(\frac{x}{\sigma}\right)^\beta} x^{\gamma-\beta+1} \frac{\sigma^\beta}{b\beta} \right)^{\frac{1}{2}} dB_s \quad (1.4.2)$$

As shown in the above stochastic differential equations, the probability distortion changes the drift part of the stochastic process by  $\frac{2}{b} x^\gamma e^{\left(\frac{x}{\sigma}\right)^\beta} \left( -1 + \frac{2\delta}{1 + \delta \left(\frac{x}{\sigma}\right)^\beta} \right) ds$  and the volatility part remains the same.

With regard to the stochastic process with distorted Weibull distribution as a stable solution, the drift part is larger at first and eventually drops down. The critical point is solved by

$$-1 + \frac{2\delta}{1 + \delta \left(\frac{X_s}{\sigma}\right)^\beta} = 0 \quad (1.4.3)$$



that is,

$$X_s = \sigma \left( \frac{2\delta - 1}{\delta} \right)^{\frac{1}{\beta}} \quad (1.4.4)$$

When the stock price  $X_s > \sigma \left( \frac{2\delta - 1}{\delta} \right)^{\frac{1}{\beta}}$ , the drift of distorted stochastic process is greater than that of undistorted process; while  $X_s < \sigma \left( \frac{2\delta - 1}{\delta} \right)^{\frac{1}{\beta}}$ , the drift part falls below the undistorted one.

By my formulation, the drift part of undistorted Weibull process always stays positive, which is a good representation of the real world stock market. After distorting the probability, individuals may assume a negative drift of the stock price process. That is because, the probability distortion assigns more weights on the two extremes. Compared to the drift of undistorted stochastic process, the distorted one starts with a higher drift and ends up with a lower drift, which greatly augments the probabilities at the two ends. The comparison is shown in figures 1.4.1, 1.4.2 and 1.4.3.

Moreover, the probability distortion essentially characterizes the fact that people psychologically assume a large gain when the stock price is low and assume a large loss when the stock price is relatively high. That is because, when the probability distortion assigns more weights on the two extremes, the drift part tries to drag the stochastic process back to the middle in order to guarantee a strong stationary process.

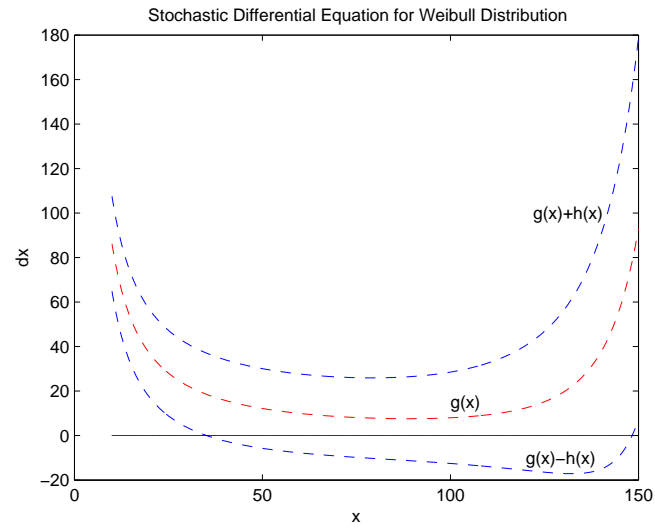


FIGURE 1.4.1: Stochastic process for Weibull process

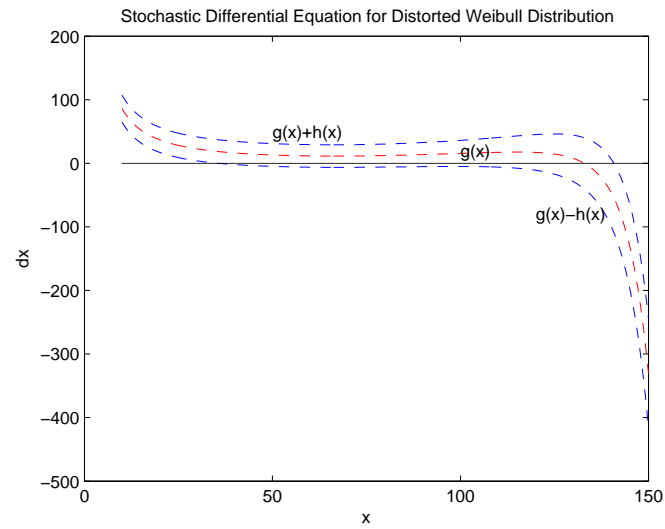


FIGURE 1.4.2: Stochastic process for distorted Weibull process

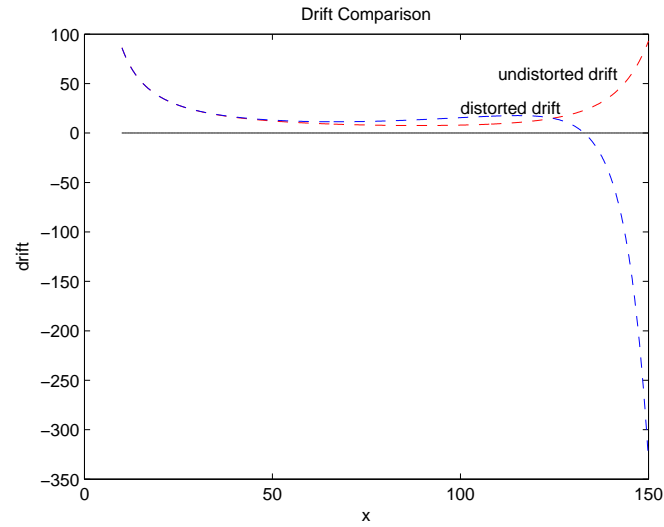


FIGURE 1.4.3: Comparison of the drift part between undistorted and distorted Weibull processes

### 1.4.2 Results on Portfolio Selections with a Distorted Probability

The goal of this part is to compare the portfolio choice between people with and without the probability distortion respectively. I start by investigating the annuitization, investment and consumption strategies based on the benchmark parameters. The I conduct sensitivity analyses and illustrate how the underlying parameters affect individuals' portfolio choice.

In my benchmark analysis, the value of the parameters are:

$$\beta = 5.02$$

$$\sigma = 117.5977$$

$$\gamma = 3.7951$$

$$b = 12930236.8218$$

$$r = 0.04$$

$$\lambda = 0.04$$

$$p = 0.25$$

$$\alpha = 2$$

$$\delta = 1.5$$

### **Annuitization strategy**

Now I want to compare the annuitization strategies between people without and with the probability distortion.

I will show both the short-term and long-term annuitization strategies in the following graphs. From a short-term perspective, the blue  $+$  indicates that people will spend  $\bar{a}$  to buy one more annuity; the green  $\circ$  implies that people will neither buy one more annuity nor surrender one existing annuity; the red  $\times$  means they will surrender one existing annuity and redeem  $\bar{a}(1 - p)$  back to the wealth account. In a long-term run, when people are considering buying more annuities, they will keep on buying annuities until they reach the black  $\diamond$ . When they are considering surrendering existing annuities, they will continue selling annuities until they reach the yellow  $*$ . Here  $p$  is the surrender charge.

On a wealth-annuity grid, wealth is indicated on the horizontal axis from \$0 to \$0.2, and the annuity is on the vertical axis from \$0.5 to \$0.515. It is worth noting that over every small grid, the length of wealth is  $\bar{a}$  times of that of annuity, where  $\bar{a}$  is the price of a life annuity paying \$1 per year continuously ( $\bar{a} = 12.5$  in my case). This formulation is based on the fact that it costs  $\bar{a}$  to buy one more annuity, for

the purpose of showing the annuitization strategy.

Figure 1.4.4 shows the annuitization strategy for people without the probability distortion. According to this graph, in a short-term run, people could surrender one existing annuity, neither surrender nor buy one annuity, or buy one more annuity based on their wealth level. When wealth is zero, individuals will surrender one existing annuity to keep the wealth non-negative. When the wealth is relatively small, people will neither surrender nor buy one annuity. When the wealth exceeds certain level (\$0.04 in my case), individuals will buy one more annuity. That is because, given enough wealth to cover their living expenses, they do not have an urgent need for liquidity and profitability. Therefore, the annuity turns out to be the best investment which is both safe and profitable. For a long-time scale, if they are considering surrendering some annuities, they tend to surrender only one more annuity to keep the wealth account non-negative. If they are considering buying more annuities, they will keep a certain level of wealth (\$0.035 in my case), and spend all the rest money on annuities. Interestingly, the reason that people tend to firstly keep some wealth is that when the wealth is small, the penalty from consuming little is considerably large. Therefore, the individual has to keep some money for consumption.

Figure 1.4.5 shows the annuitization strategy for people without the probability distortion, while the annuity endowment is relatively large (starting from  $A = 2$ ). We can see a bigger "do nothing" area, which is exactly consistent with Wang and Young's (2012) [34] results. Note that there is some noise in this figure from the numerical solution of the differential equation.

Figure 1.4.6 displays the annuitization strategy for people with the probability distortion. It follows a similar pattern for both short-term annuitization strategy and the long-term one. However, people with a probability distortion on the perceived stock price distribution tend to start buying more annuities at a lower wealth level. The reason is that although more probabilities are assigned to both low stock price and high stock price, the fact that we use a risk-averse utility function implies that people are more afraid of large losses than the hope of large gains. To address the fear, individuals tend to buy more annuities, in order to guarantee certain level of consumptions. Figure 1.4.7 also shows the annuitization strategy for people with the probability distortion while the annuity endowment is relatively large (starting from  $A = 2$ ). There is also a bigger area of "do nothing", which is consistent with Wang and Young's (2012) [34] result.

Comparison on the annuitization strategies between people without and with the probability distortion shows that the probability distortion changes those annuitization decision who are endowed with a relatively small amount of wealth. It changes them from doing nothing to starting buying more annuities. This result provides valuable implications for insurance companies, helping them identify the potential customers more accurately by their endowed wealth and annuities.

### **Investment strategy**

Maximizing the portion of the HJB equation dependent upon the investment strategy gives the investment strategy explicitly:

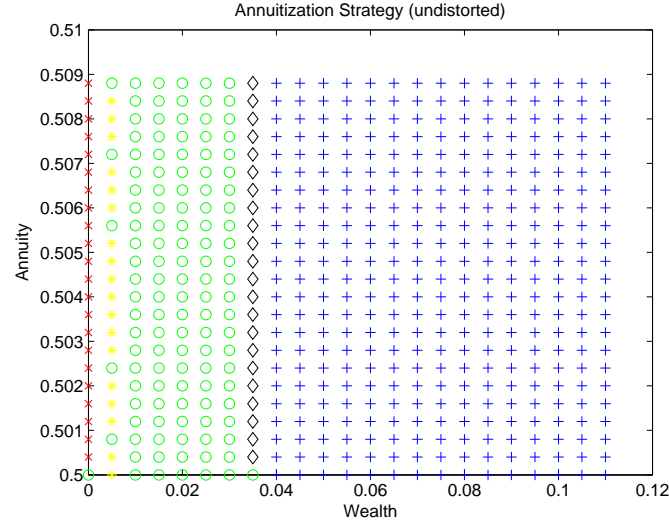


FIGURE 1.4.4: This graph illustrates the annuitization strategy for people without the probability distortion, annuity starting from  $A = 0.5$ . We set  $r = 0.04$ ,  $\lambda = 0.04$ ,  $\alpha = 2$ ,  $p = 0.25$ .  $+$ : buy (short-term),  $o$ : do nothing (short-term),  $\times$ : surrender (short-term),  $\diamond$ : stop buying (long-term),  $*$ : stop surrendering (long-term)

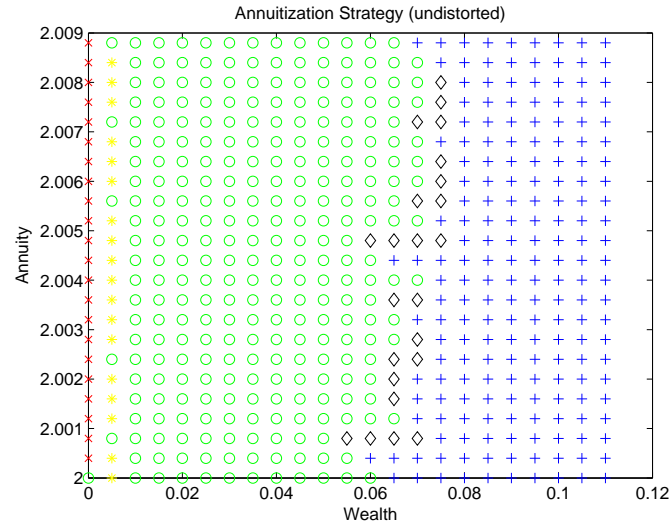


FIGURE 1.4.5: This graph illustrates the annuitization strategy for people without the probability distortion, annuity starting from  $A = 2$ . We set  $r = 0.04$ ,  $\lambda = 0.04$ ,  $\alpha = 2$ ,  $p = 0.25$ .  $+$ : buy (short-term),  $o$ : do nothing (short-term),  $\times$ : surrender (short-term),  $\diamond$ : stop buying (long-term),  $*$ : stop surrendering (long-term)

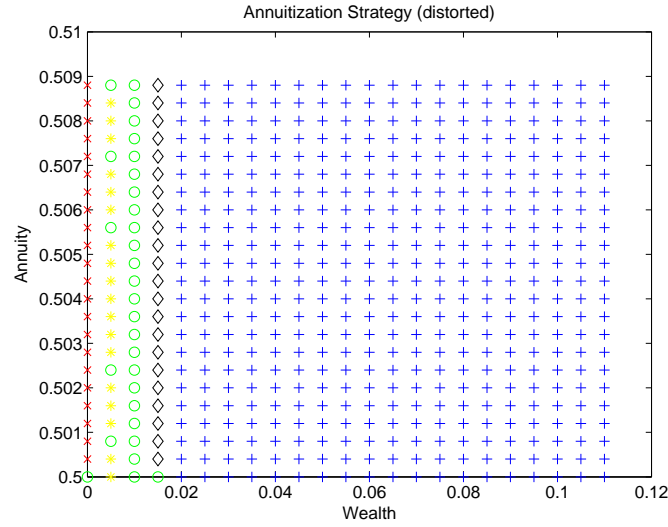


FIGURE 1.4.6: This graph illustrates the annuitization strategy for people with the probability distortion, annuity starting from  $A = 0.5$ . We set  $r = 0.04$ ,  $\lambda = 0.04$ ,  $\alpha = 2$ ,  $p = 0.25$ ,  $\delta = 1.5$ .  $+$ : buy (short-term),  $o$ : do nothing (short-term),  $\times$ : surrender (short-term),  $\diamond$ : stop buying (long-term),  $*$ : stop surrendering (long-term)

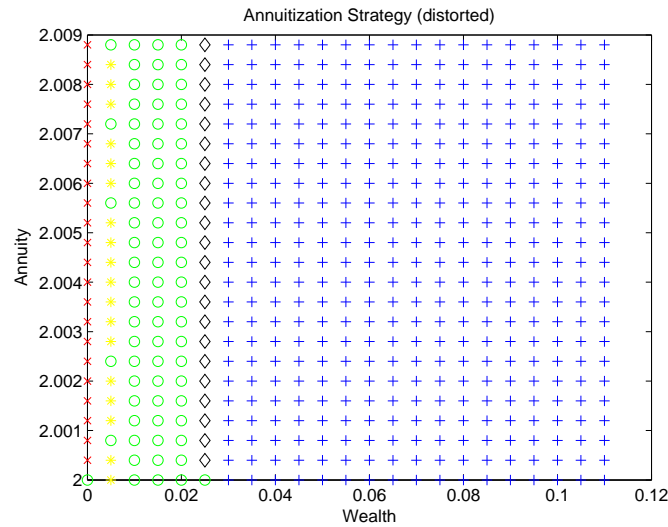


FIGURE 1.4.7: This graph illustrates the annuitization strategy for people with the probability distortion, annuity starting from  $A = 2$ . We set  $r = 0.04$ ,  $\lambda = 0.04$ ,  $\alpha = 2$ ,  $p = 0.25$ ,  $\delta = 1.5$ .  $+$ : buy (short-term),  $o$ : do nothing (short-term),  $\times$ : surrender (short-term),  $\diamond$ : stop buying (long-term),  $*$ : stop surrendering (long-term)



$$\pi_t = - \frac{[\frac{1}{b}\gamma\frac{\sigma^\beta}{\beta}X_t^{\gamma-\beta-1}e^{(\frac{X_t}{\sigma})^\beta} + \frac{2}{b}X_t^{\gamma-1}e^{(\frac{X_t}{\sigma})^\beta}(-1 + \frac{2\delta}{1+\delta(\frac{X_t}{\sigma})^\beta}) - r]U_w}{2X_t^{\gamma-\beta-1}\frac{\sigma^\beta}{b\beta}e^{(\frac{X_t}{\sigma})^\beta}U_{ww}} \quad (1.4.5)$$

Figure 1.4.8 compares the investment strategies between people without and with the probability distortion. First, it shows that investment in the stock market increases with respect to both the wealth endowment and annuity endowment. That is because, with a larger profitability in the stock market, investment is a good source for more future consumptions once there are more endowments.

Secondly, figure 1.4.8 also indicates that people with a probability distortion tend to invest more in the stock market than people without a probability distortion. Since individuals with a probability distortion will buy more annuities, which largely protect them from little consumption, they are more motivated to invest in risky assets for more future consumptions.

Now let us consider a long-term investment strategy. When people follow a long-term annuitization strategy, (that is, they will keep a certain level of wealth and spend the rest of all the money to buy annuities), they move to a point with lower wealth and higher annuity. Accordingly, as shown on figure 1.4.8, when they move to a lower wealth and higher annuity point, they will invest less in the stock market. That is because, the wealth are used to buy annuities instead of risky investment.

Moreover, figure 1.4.9 shows the investment proportion with respect to its current wealth level. That is,  $\frac{\text{investment}}{\text{wealth}}$  is on the y-axis. We can see that the proportion is larger than 1 when the wealth is relatively small. It indicates that people invest on

margin. According to our assumption, the drift part of the stock price process is \$8 at  $x = 100$ , which is much larger than the risk-free interest (which is \$4) gained from the bank account. Therefore, people will borrow money from the bank and invest it in the stock market. And the drift part of the distorted stock price process is even higher than that of the undistorted one. Hence, it is reasonable for investors to invest on margin.

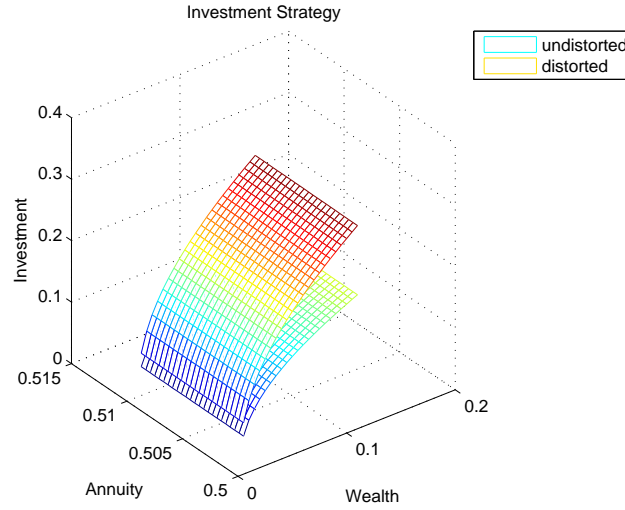


FIGURE 1.4.8: This graph compares the investment strategy for people without and without the probability distortion, annuity starting from  $A = 0.5$ . We set  $r = 0.04$ ,  $\lambda = 0.04$ ,  $\alpha = 2$ ,  $p = 0.25$ ,  $\delta = 1.5$

### Consumption strategy

Maximizing the portion of the HJB equation involving consumption strategy, the consumption strategy is obtained explicitly:

$$c_t = U_w^{-\frac{1}{\alpha}} \quad (1.4.6)$$

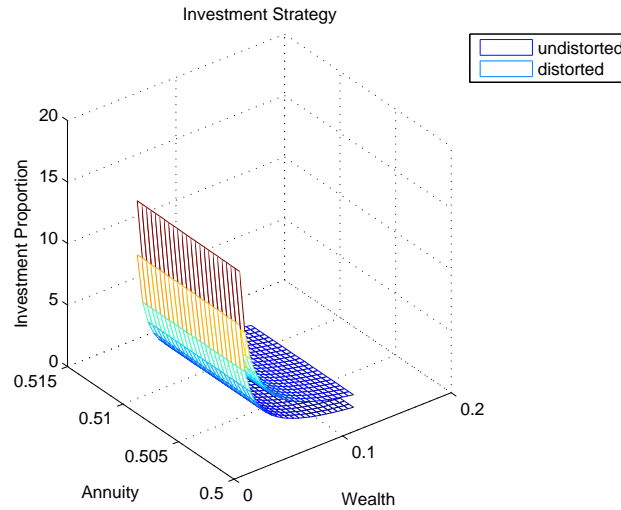


FIGURE 1.4.9: This graph compares  $\frac{\text{investment}}{\text{wealth}}$  for people without and with the probability distortion, annuity starting from  $A = 0.5$ . We set  $r = 0.04$ ,  $\lambda = 0.04$ ,  $\alpha = 2$ ,  $p = 0.25$ ,  $\delta = 1.5$

First, figure 1.4.11 shows that individuals will consume more as both their endowed wealth and annuities increase. Moreover, it increases faster with respect to the wealth endowment. For instance, people without the probability distortion increase their consumption rate from \$0.5393 to \$1.0026 along the wealth axis, and only from \$0.5393 to \$0.5490 on the annuity axis.

Secondly, figure 1.4.11 also indicates that people with a probability distortion tend to consume more than people without a probability distortion. The reason is that the consumption is supported mostly by the proceeds from the life annuities. Therefore, more annuities leads to more consumptions.

In order to reach the conclusion that consumption is supported mostly by the proceeds from the life annuities, I list the primary characteristics of the three finan-

	Safety	Liquidity	Profitability
Bond	✓	✓	
Stock		✓	✓
Annuity	✓		✓

FIGURE 1.4.10: Comparison of investment instruments

cial instruments in my model in figure 1.4.10. First, to formulate a solid retirement plan, safety is their first priority. Thus the stock investment is not a stable source to support the current consumption. Next, a comparison between the bond and the annuity suggests that the annuity always delivers higher proceeds than the risk-free interest. For instance under our continuous setting, the risk-free interest rate is  $r$  and the future lifetime follows an exponential distribution with a parameter  $\lambda$ . Thus the price of the bond is  $\frac{1}{r}$ , whereas that of the annuity is  $\frac{1}{r+\lambda}$ . Given the individual is alive, she will receive  $(\lambda \cdot \text{Principal})$  more compared to the interest gained from the bank account every year, if all the principal is used to purchase annuities. In this sense, annuity is the best candidate to support the current consumption. As I described earlier in the problem formulation, the retiree will first consume all the proceeds from the annuities. If the proceeds is not enough to cover her consumption, she will withdraw money from the wealth account; if the proceeds is more than what she consumes, she will add the money to the wealth account. In sum, my analysis suggests that the consumption level largely depends on her annuitization strategy.

From the annuitization strategy, we know that the individual with a probability distortion will buy more annuities, which largely protect them from little consump-

tion. In order to maximize the lifetime utility from consumption, people will consume more now, which is demonstrated by the above analysis.

In terms of a long-term consumption strategy, people would choose to consume less now in order to consume more in the future. When people follow a long-term annuitization strategy, (that is, they will keep a certain level of wealth and spend the rest of all the money to buy annuities), they move to a point with lower wealth and higher annuity. Accordingly, as shown on figure 1.4.11, when they move to a lower wealth and higher annuity point, they will consume less at that moment. That is because, the wealth are used to buy annuities for more future consumptions instead of immediate consumption of the wealth.

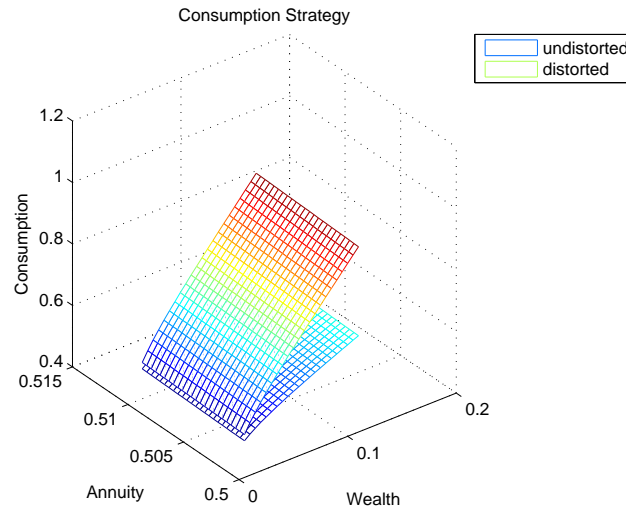


FIGURE 1.4.11: This graph compares the consumption strategy for people without and without the probability distortion, annuity starting from  $A = 0.5$ . We set  $r = 0.04$ ,  $\lambda = 0.04$ ,  $\alpha = 2$ ,  $p = 0.25$ ,  $\delta = 1.5$

### 1.4.3 Sensitivity Analysis

#### Distortion parameter

From figures 1.4.12 and 1.4.13, the probability distortion with  $\delta = 2$  assigns more probabilities at the low extreme than that with  $\delta = 1.5$ , since the function has a greater slope at the low price side. It means that people are more afraid of the large losses than the hope of large gains. To address the fear, individuals tend to buy more annuities. As annuities guarantee certain level of consumption, people are more likely to invest more and consume more, which follows the previous analysis.

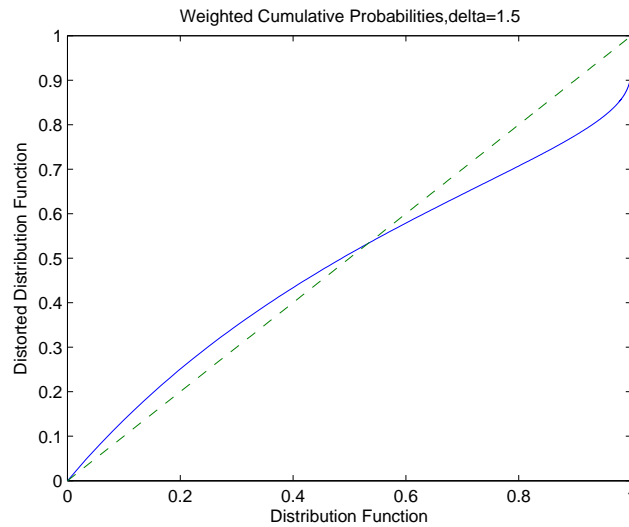


FIGURE 1.4.12: Probability distortion function with distortion parameter  $\delta = 1.5$

Now I also would like to compare the portfolio choice between people with the distortion parameter  $\delta = 1.5$  and  $\delta = 0.9$ . Interestingly, from figure 1.4.19 we can see that this probability distortion can be considered as a "quasi-convex" distortion function. Since I will discuss the portfolio choice under a convex probability distor-

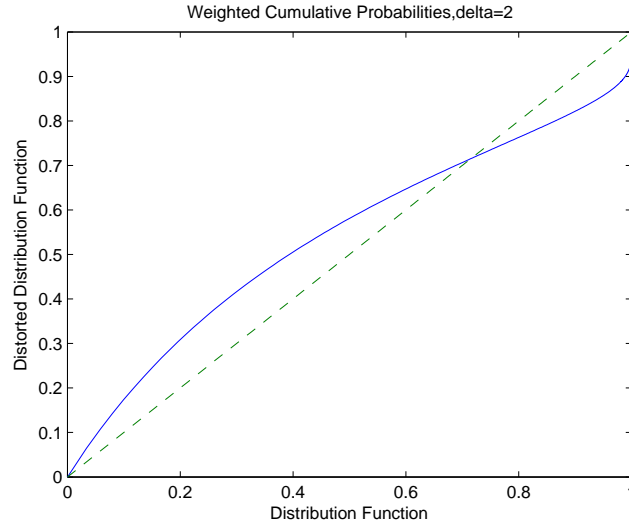


FIGURE 1.4.13: Probability distortion function with distortion parameter  $\delta = 2$

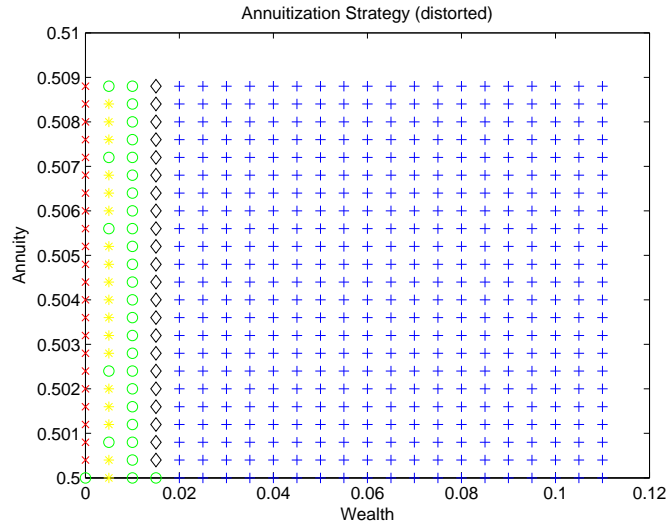


FIGURE 1.4.14: This graph illustrates the annuitization strategy for people with the probability distortion, annuity starting from  $A = 0.5$ . We set  $r = 0.04$ ,  $\lambda = 0.04$ ,  $\alpha = 2$ ,  $p = 0.25$ ,  $\delta = 1.5$ .  $+$ : buy (short-term),  $o$ : do nothing (short-term),  $\times$ : surrender (short-term),  $\diamond$ : stop buying (long-term),  $*$ : stop surrendering (long-term)

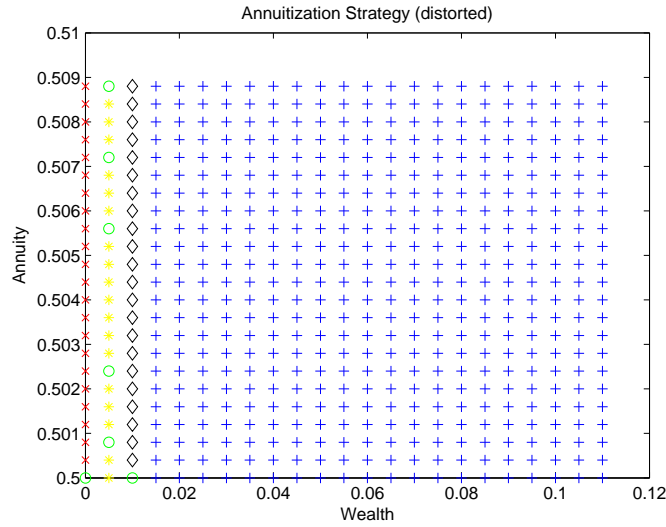


FIGURE 1.4.15: This graph illustrates the annuitization strategy for people with the probability distortion, annuity starting from  $A = 0.5$ . We set  $r = 0.04$ ,  $\lambda = 0.04$ ,  $\alpha = 2$ ,  $p = 0.25$ ,  $\delta = 2$ .  $+$ : buy (short-term),  $o$ : do nothing (short-term),  $x$ : surrender (short-term),  $d$ : stop buying (long-term),  $*$ : stop surrendering (long-term)

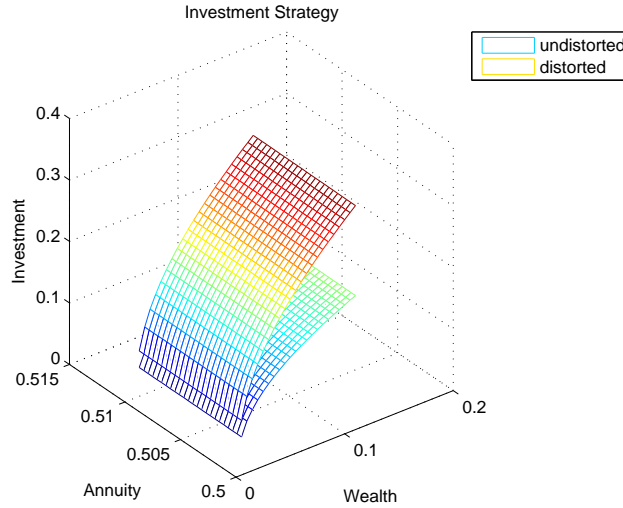


FIGURE 1.4.16: This graph compares the investment strategy for people without and without the probability distortion, annuity starting from  $A = 0.5$ . We set  $r = 0.04$ ,  $\lambda = 0.04$ ,  $\alpha = 2$ ,  $p = 0.25$ ,  $\delta = 2$



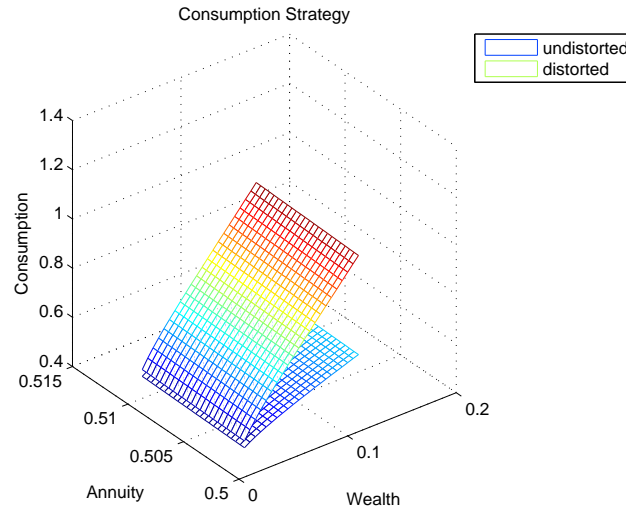


FIGURE 1.4.17: This graph compares the consumption strategy for people without and without the probability distortion, annuity starting from  $A = 0.5$ . We set  $r = 0.04$ ,  $\lambda = 0.04$ ,  $\alpha = 2$ ,  $p = 0.25$ ,  $\delta = 2$

tion in chapter 2, this demonstration could be an lead-in analysis towards chapter 2.

From figure 1.4.19, the probability distortion with  $\delta = 0.9$  assigns more probabilities at the high extreme than that with  $\delta = 1.5$ , since the function has a greater slope at the high price side. It means that people are more optimistic about large gains. Accordingly, individuals tend to buy less annuities, invest less and consume less. In this sense, this "quasi-convex" probability distortion provides us a good explanation for the annuity puzzle.

### Surrender charge $p$

In this part, I discuss the impact from changing the surrender charge from 0.25 to 0.2. As the cost of surrendering annuities decreases, people would like to buy more

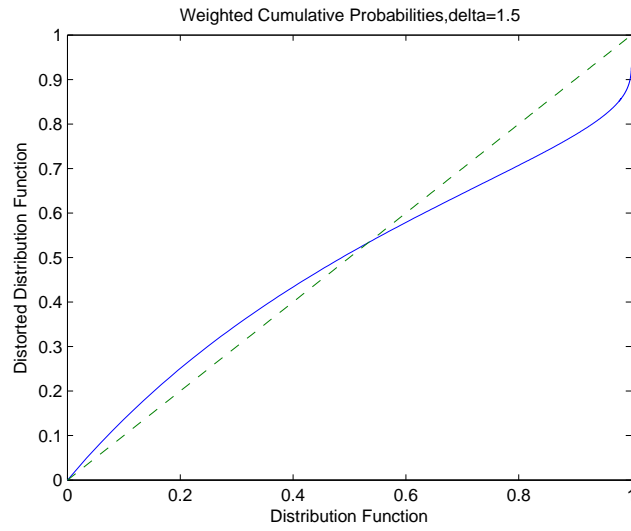


FIGURE 1.4.18: Probability distortion function with distortion parameter  $\delta = 1.5$

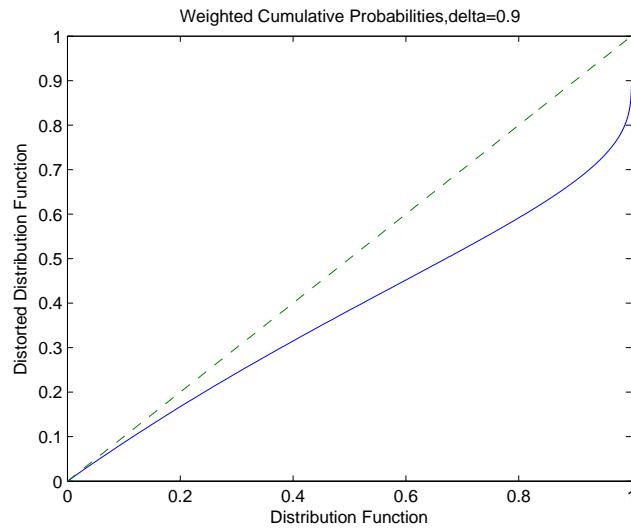


FIGURE 1.4.19: Probability distortion function with distortion parameter  $\delta = 0.9$

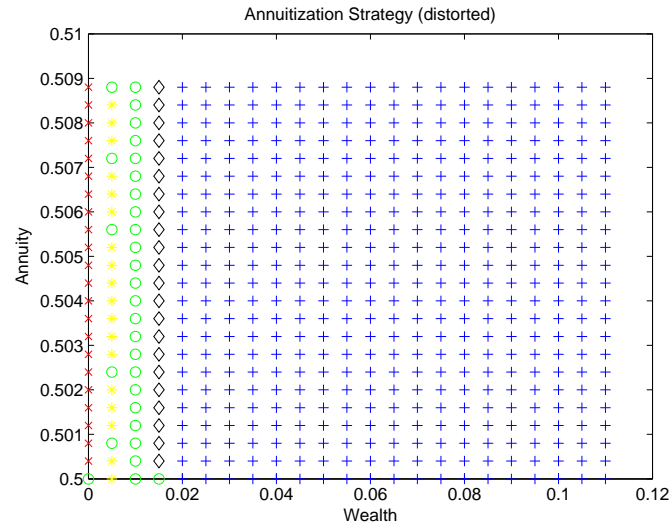


FIGURE 1.4.20: This graph illustrates the annuitization strategy for people with the probability distortion, annuity starting from  $A = 0.5$ . We set  $r = 0.04$ ,  $\lambda = 0.04$ ,  $\alpha = 2$ ,  $p = 0.25$ ,  $\delta = 1.5$ . +: buy (short-term), o: do nothing (short-term), x: surrender (short-term), d: stop buying (long-term), \*: stop surrendering (long-term)

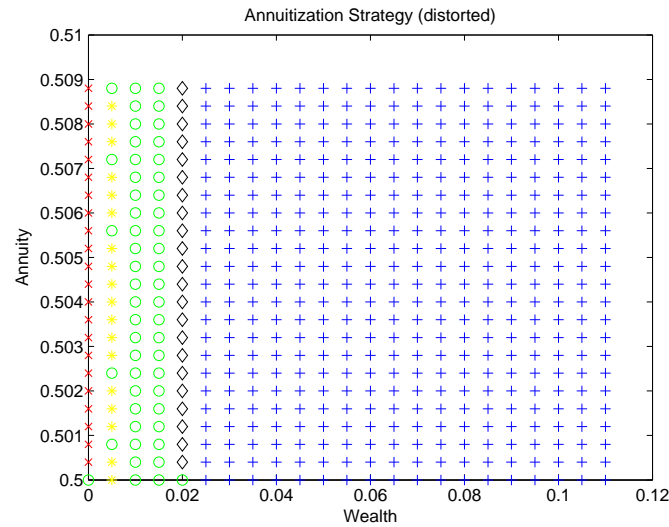


FIGURE 1.4.21: This graph illustrates the annuitization strategy for people with the probability distortion, annuity starting from  $A = 0.5$ . We set  $r = 0.04$ ,  $\lambda = 0.04$ ,  $\alpha = 2$ ,  $p = 0.25$ ,  $\delta = 0.9$ . +: buy (short-term), o: do nothing (short-term), x: surrender (short-term), d: stop buying (long-term), \*: stop surrendering (long-term)

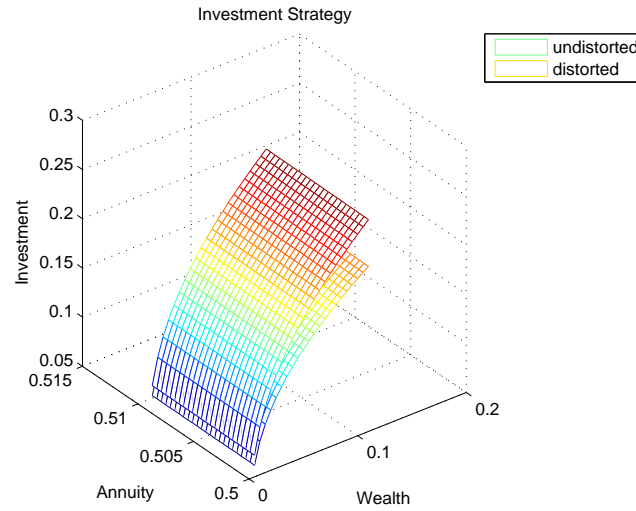


FIGURE 1.4.22: This graph compares the investment strategy for people without and without the probability distortion, annuity starting from  $A = 0.5$ . We set  $r = 0.04$ ,  $\lambda = 0.04$ ,  $\alpha = 2$ ,  $p = 0.25$ ,  $\delta = 0.9$

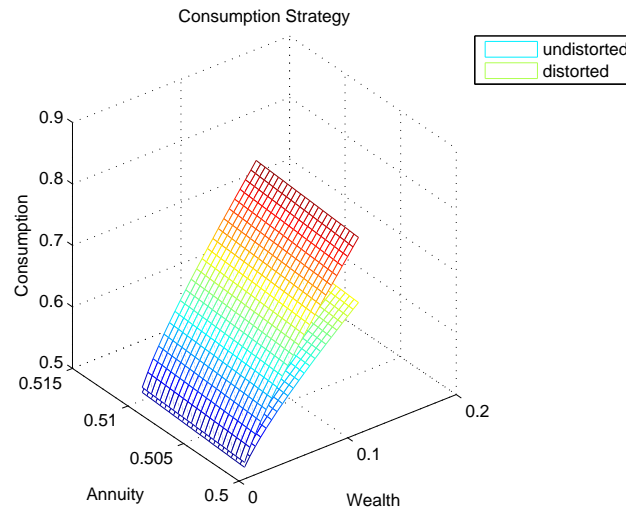


FIGURE 1.4.23: This graph compares the consumption strategy for people without and without the probability distortion, annuity starting from  $A = 0.5$ . We set  $r = 0.04$ ,  $\lambda = 0.04$ ,  $\alpha = 2$ ,  $p = 0.25$ ,  $\delta = 0.9$

annuities. People with the probability distortion are sensitive towards the change of surrender charge, too. Besides, because the surrender charge does not change both the drift term and volatility part of the stock price process, it does not have a significant impact on the investment and consumption in my case.

Figure 1.4.25 and figure 1.4.27 display the annuitization strategies for people without and with the probability distortion respectively, when the surrender charge  $p = 0.2$ . I find that there are more points of "buy more annuities", which changes from "do nothing". As a result, the decrease of surrender charge changes those annuitization strategy most whose wealth endowment is relatively small.

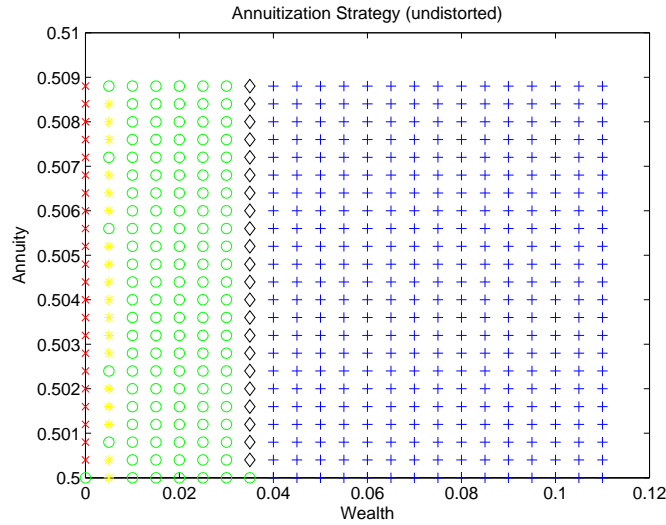


FIGURE 1.4.24: This graph illustrates the annuitization strategy for people without the probability distortion, annuity starting from  $A = 0.5$ . We set  $r = 0.04$ ,  $\lambda = 0.04$ ,  $\alpha = 2$ ,  $p = 0.25$ . +: buy (short-term), o: do nothing (short-term), x: surrender (short-term),  $\diamond$ : stop buying (long-term), \*: stop surrendering (long-term)

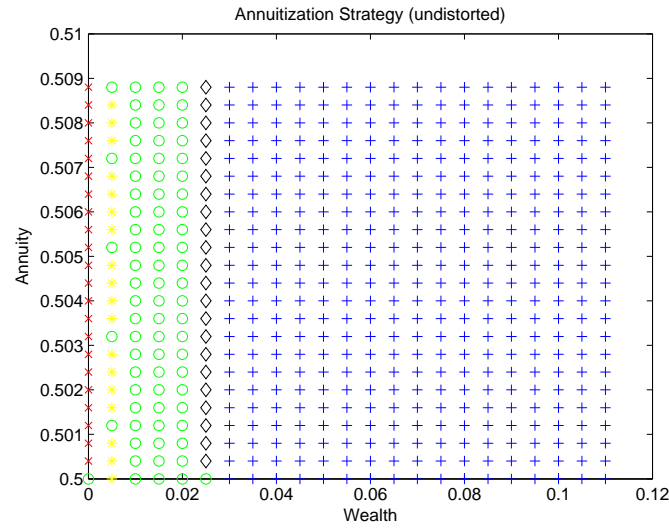


FIGURE 1.4.25: This graph illustrates the annuitization strategy for people without the probability distortion, annuity starting from  $A = 0.5$ . We set  $r = 0.04$ ,  $\lambda = 0.04$ ,  $\alpha = 2$ ,  $p = 0.2$ .  $+$ : buy (short-term),  $o$ : do nothing (short-term),  $\times$ : surrender (short-term),  $\diamond$ : stop buying (long-term),  $*$ : stop surrendering (long-term)

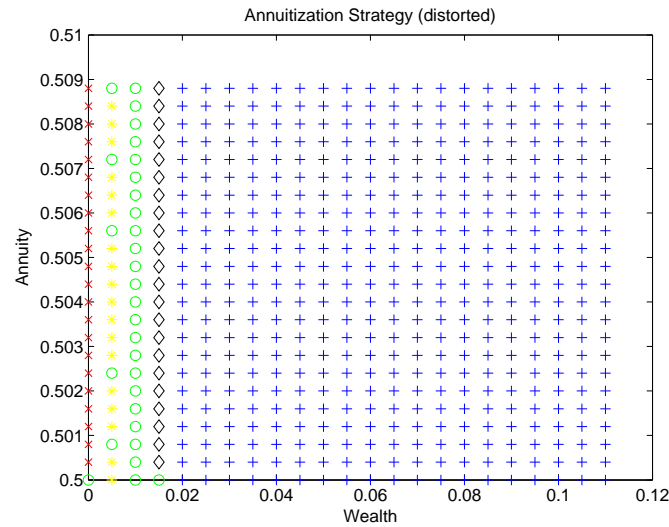


FIGURE 1.4.26: This graph illustrates the annuitization strategy for people with the probability distortion, annuity starting from  $A = 0.5$ . We set  $r = 0.04$ ,  $\lambda = 0.04$ ,  $\alpha = 2$ ,  $p = 0.25$ ,  $\delta = 1.5$ .  $+$ : buy (short-term),  $o$ : do nothing (short-term),  $\times$ : surrender (short-term),  $\diamond$ : stop buying (long-term),  $*$ : stop surrendering (long-term)

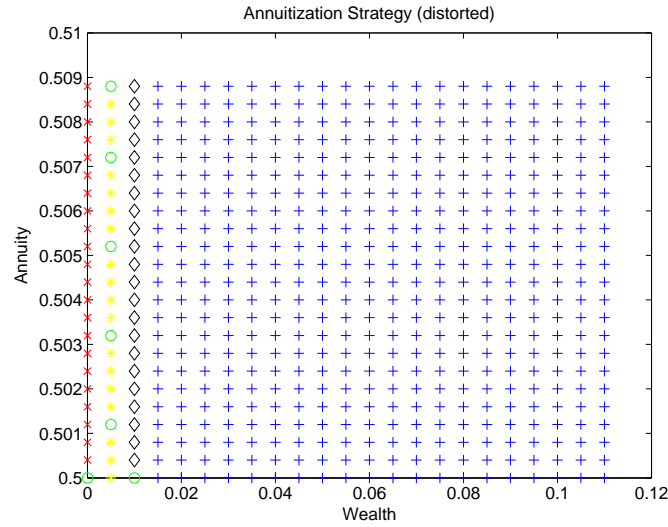


FIGURE 1.4.27: This graph illustrates the annuitization strategy for people with the probability distortion, annuity starting from  $A = 0.5$ . We set  $r = 0.04$ ,  $\lambda = 0.04$ ,  $\alpha = 2$ ,  $p = 0.2$ ,  $\delta = 1.5$ .  $+$ : buy (short-term),  $o$ : do nothing (short-term),  $x$ : surrender (short-term),  $d$ : stop buying (long-term),  $*$ : stop surrendering (long-term)

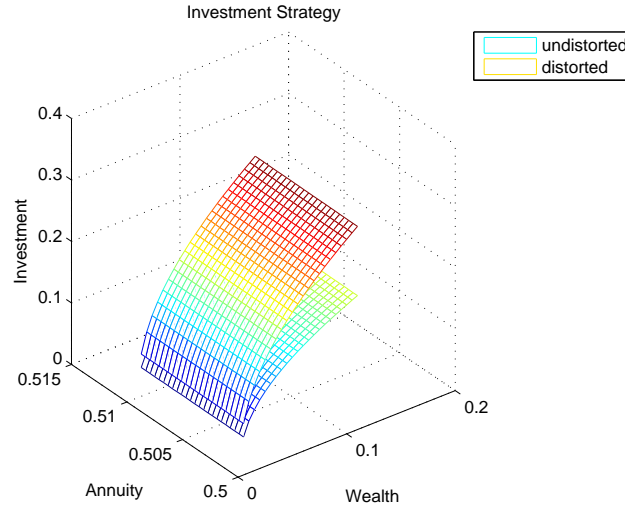


FIGURE 1.4.28: This graph compares the investment strategy for people without and without the probability distortion, annuity starting from  $A = 0.5$ . We set  $r = 0.04$ ,  $\lambda = 0.04$ ,  $\alpha = 2$ ,  $p = 0.2$ ,  $\delta = 1.5$

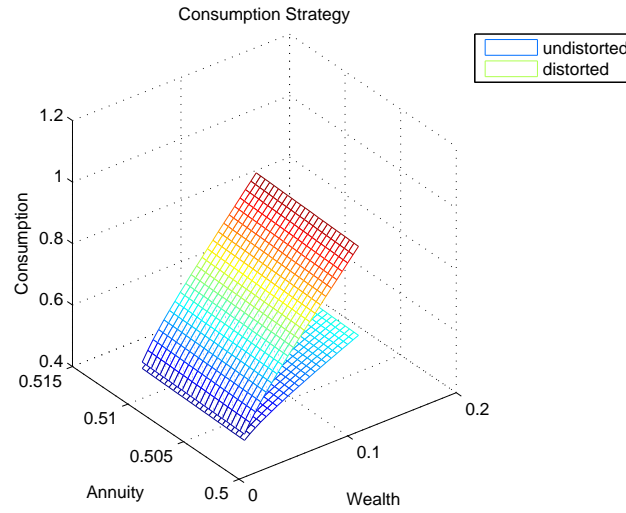


FIGURE 1.4.29: This graph compares the consumption strategy for people without and without the probability distortion, annuity starting from  $A = 0.5$ . We set  $r = 0.04$ ,  $\lambda = 0.04$ ,  $\alpha = 2$ ,  $p = 0.2$ ,  $\delta = 1.5$

#### Risk aversion parameter: $\alpha$

Finally, I investigate the impact on the portfolio choice from changing the risk aversion parameter  $\alpha$ . The constant relative risk aversion  $\alpha$  measures people's risk attitude. The one with a higher value of  $\alpha$  is considered more risk averse. By increasing  $\alpha$  from 2 to 2.5, we assume a policyholder who are more risk averse.

For a more risk averse utility function, the utility from consumption drops much faster once the consumption is below certain level. Therefore, to protect the consumption from declining below that level, people would like to buy more annuities. This conclusion is founded by the analysis from previous part that the consumption is supported mostly by the proceeds from the life annuities. For the distorted case, people will also buy more annuities, which follows the same reasoning.



With regard to the investment strategy, more risk-averse investors choose to invest less. That is because, as the investment will eventually turn into consumption, the individuals are less likely to take much risk in the stock market in order to protect against little consumption.

Regarding the consumption strategy, we don't see dramatic change in this case. Even though people buy more annuities to provide a thicker cushion for consumption, it does not necessarily mean that they would consume more. Hence it is reasonable for them to have almost the same consumption level.

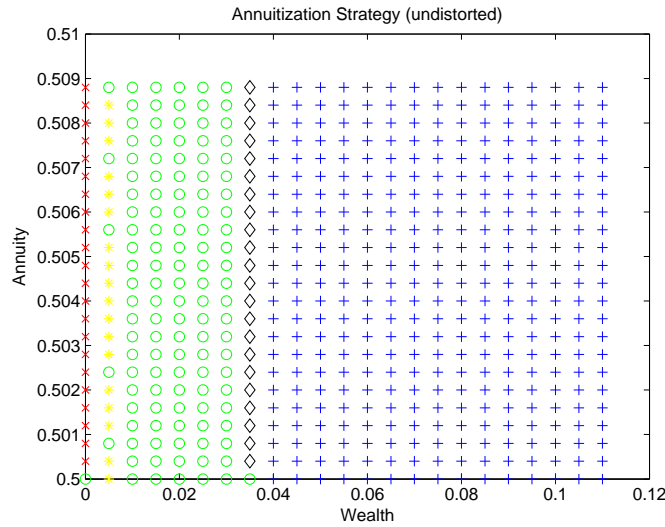


FIGURE 1.4.30: This graph illustrates the annuitization strategy for people without the probability distortion, annuity starting from  $A = 0.5$ . We set  $r = 0.04$ ,  $\lambda = 0.04$ ,  $\alpha = 2$ ,  $p = 0.25$ . +: buy (short-term), o: do nothing (short-term), x: surrender (short-term), ◇: stop buying (long-term), \*: stop surrendering (long-term)

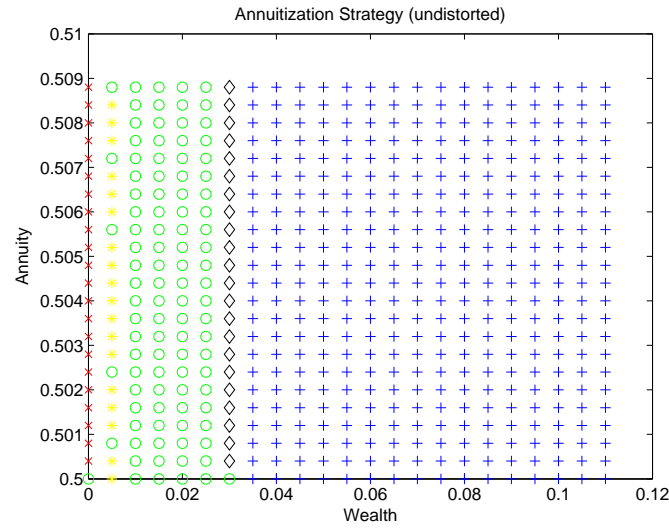


FIGURE 1.4.31: This graph illustrates the annuitization strategy for people without the probability distortion, annuity starting from  $A = 0.5$ . We set  $r = 0.04$ ,  $\lambda = 0.04$ ,  $\alpha = 2.5$ ,  $p = 0.25$ . +: buy (short-term), o: do nothing (short-term), x: surrender (short-term),  $\diamond$ : stop buying (long-term), \*: stop surrendering (long-term)

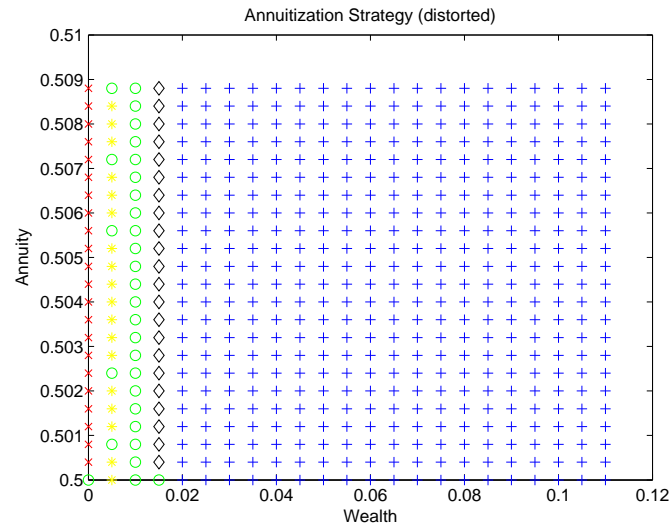


FIGURE 1.4.32: This graph illustrates the annuitization strategy for people with the probability distortion, annuity starting from  $A = 0.5$ . We set  $r = 0.04$ ,  $\lambda = 0.04$ ,  $\alpha = 2$ ,  $p = 0.25$ ,  $\delta = 1.5$ . +: buy (short-term), o: do nothing (short-term), x: surrender (short-term),  $\diamond$ : stop buying (long-term), \*: stop surrendering (long-term)

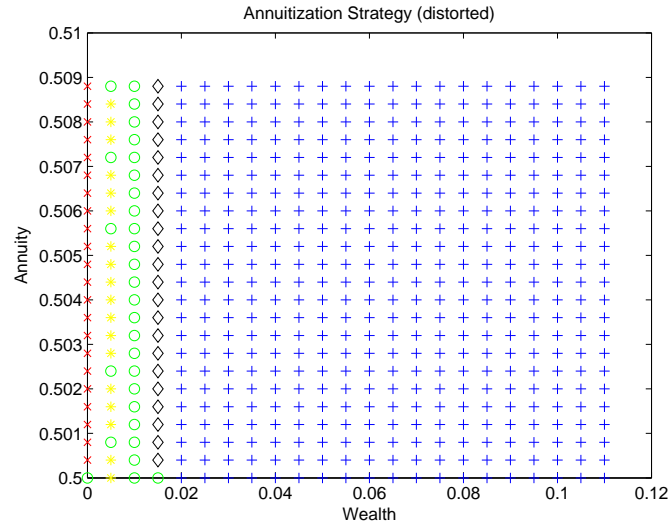


FIGURE 1.4.33: This graph illustrates the annuitization strategy for people with the probability distortion, annuity starting from  $A = 0.5$ . We set  $r = 0.04$ ,  $\lambda = 0.04$ ,  $\alpha = 2.5$ ,  $p = 0.25$ ,  $\delta = 1.5$ .  $+$ : buy (short-term),  $o$ : do nothing (short-term),  $x$ : surrender (short-term),  $d$ : stop buying (long-term),  $*$ : stop surrendering (long-term)

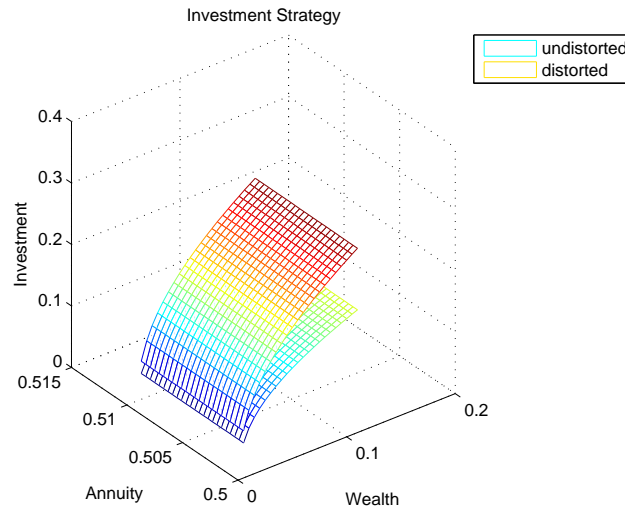


FIGURE 1.4.34: This graph compares the investment strategy for people without and without the probability distortion, annuity starting from  $A = 0.5$ . We set  $r = 0.04$ ,  $\lambda = 0.04$ ,  $\alpha = 2.5$ ,  $p = 0.25$ ,  $\delta = 1.5$

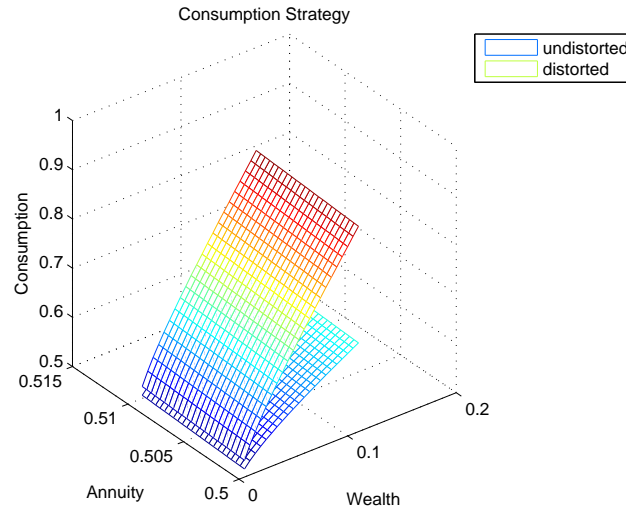


FIGURE 1.4.35: This graph compares the consumption strategy for people without and without the probability distortion, annuity starting from  $A = 0.5$ . We set  $r = 0.04$ ,  $\lambda = 0.04$ ,  $\alpha = 2.5$ ,  $p = 0.25$ ,  $\delta = 1.5$

## 1.5 Conclusions

This chapter focuses on the annuitization strategy for an individual retirement plan. The goal is to shed light on the annuity puzzle from a behavioral economics point of view. Empirical studies from behavioral economics show that people tend to overestimate small-probability events and underestimate large-probability events. In this chapter, I consider an individual with a reverse S-shaped probability distortion and investigate its impact on her portfolio choice. I incorporate the probability distortion into a stochastic portfolio selection framework, with commutable life annuities.

I summarize the short-term annuitization strategy as follows. When the wealth is zero, individuals will surrender one existing annuity to keep the wealth non-negative. When the wealth is relatively small, people will neither surrender nor buy one annuity. When the wealth exceeds certain level, individuals will buy one more annuity.

For the long-term annuitization strategy, if they are considering surrendering some annuities, they tend to surrender only one more annuity to keep the wealth account non-negative. If they are considering buying more annuities, they will keep a certain level of wealth, and spend all the rest money on annuities.

I find that people with a probability distortion on the perceived stock price distribution tend to start buying more annuities at a lower wealth level. The reason is that although more probabilities are assigned to both low stock price and high stock price, the fact that we use a risk averse utility function implies that people are more afraid of large losses than the hope of large gains. To address the fear, individuals tend to buy more annuities, in order to guarantee certain level of consumption.

With regard to the investment strategy, first it increases with respect to both the wealth endowment and annuity endowment. Moreover, when the wealth is relatively small, people will invest on margin. We also show that people with a probability distortion tend to invest more in the stock market than people without a probability distortion. That is because, the individual with a probability distortion will buy more annuities, which largely protect them from little consumption, they are more motivated to invest in the risky assets for more future consumptions.

Regarding the consumption strategy, first I analyzed that the consumption is supported mostly by the proceeds from life annuities. Following this analysis, I find that in order to maximize the lifetime utility from consumption, people will consume more now. That is because, more annuities (bought by people with a probability distortion) provide them more support for current consumption.

Technically, in order to incorporate a reverse S-shaped probability distortion function into the stochastic portfolio selection framework, three contributions are made. First, I propose a new distortion function which has an explicit inverse function, required to derive the underlying stochastic differential equation. After employing this new function, a closed form hazard function is desired. Thus I model the stock price process locally as a Weibull distribution instead of the usual lognormal distribution. Finally, based on the Kolmogorov forward and backward equation, I define a corresponding stochastic process which always has a positive drift locally.

This work not only provides the optimal portfolio selection for individuals, but also offer insights on marketing and contract design for insurance companies. I illustrate that the probability distortion changes the annuitization strategy, especially for people who are endowed with a relatively small amount of wealth. Specifically, it tends to expand the range of wealth in which they choose to buy annuities. This finding helps the insurance company to formulate a more accurate marketing strategy.

There are several natural extensions under the optimal life-cycle model that I will pursue in the near future. First, in this work I simplify the analysis by assuming an exponential distribution for the mortality rate, which means people in different ages have a same future lifetime. In the next work, I will assume the future lifetime follows a more realistic mortality table, Gompertz distribution for instance. Second, I assume a strong stationary Weibull process in this work. I plan to consider only a stationary Weibull process, which may better characterize the stock price process. In other words, a stationary Weibull process will evolve over time. Finally, I find

different boundary conditions for this optimal control model lead to different behavior patterns. I plan to explore other boundary conditions, and investigate how they affect the portfolio choice.

In addition, I will extend the current framework to the annuitization strategy on an integrated insurance package instead of a stand-alone product, which includes life insurance, variable annuities, long-term care and etc.. In order to delve into the policyholders' lapse behavior, I plan to incorporate different premium payment options such as a deferred immediate annuity and a series of level single premium purchases over time. In making these explorations, I aim to further scrutinize the potential implications of the probability distortion on the optimal portfolio selection with life annuities.

## Chapter 2

# Portfolio Choice with Life Annuities under a Convex Probability Distortion Function

### 2.1 Introduction

In the first chapter, I investigated individuals' portfolio selection under a reverse S-shaped probability distortion function. In this chapter, I simplify the analysis by working with a convex probability distortion function. According to Slovic et al. (1997) [30], people do not use insurance to protect against rare large losses which they tend to neglect. In other words, people underestimate all uncertain loss events, not only the 'average' ones, whenever they have the incentive to buy insurance. Mathematically, policyholders have a convex probability distortion.

The convex probability distortion function is shown below:



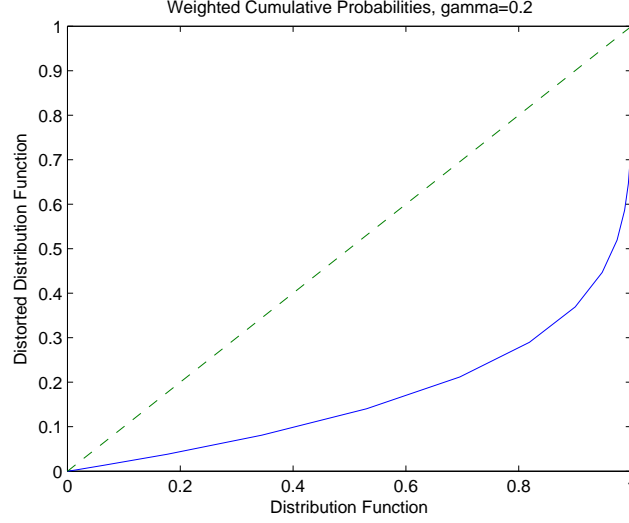


FIGURE 2.1.1: Convex probability distortion function with  $\gamma = 0.2$

In this chapter, I also work under the stochastic portfolio selection framework with commutable life annuities. By introducing a convex probability distortion function, I revisit Wang and Young's (2012) [34] framework in order to examine its impact on the annuitization, investment and consumption strategies. Following Young and Zariphopoulou (2000)'s [36] technique, the stochastic differential equation for the distorted stock price process can be explicitly derived.

I find that a convex probability distortion function essentially increases the perceived stock price drift, which results in more investment on the stock market, more consumption and less demand in the annuity industry. Correspondingly, consumption increases as individuals hold a higher expectation on the return from investment.

Simply put, the convex probability distortion provides a plausible explanation for

the annuity puzzle. In other words, an individual with a convex probability distortion put more weights on those large gain events. As a result, she will invest more, consume more and buy less annuities. On the other hand, an individual with a reverse S-shaped probability distortion, puts more weights on both the large gain and large loss events, and less weights in the middle. With a constant risk averse utility function, this reverse S-shaped distortion function actually brings more fear of large losses than the hope of large gains. As a result, she tends to buy more annuities to address the fear. This completes my analysis on the portfolio choice with life annuities under probability distortion, with both the reverse S-shaped case and a convex one.

## 2.2 Problem Formulation

In this section, I aim to derive the stochastic differential equation for the distorted stock price process under a convex probability distortion. By applying the stochastic control theory, it can be explicitly derived as in Young and Zariphopoulou (2000) [36].

Assume that the stock price is a Geometric Brownian Motion:

$$dX_s = \mu X_s ds + \sigma X_s dB_s \quad (2.2.1)$$

Applying Ito's formula, we have  $\ln(X_s) \sim N(\ln(X_0) + (\mu - \frac{1}{2}\sigma^2)s, \sigma^2 s)$

Thus, the conditional decumulative probability function is

$$u(x, t; y, T) = \Phi\left(\frac{\ln X_t + (\mu - \frac{1}{2}\sigma^2)(T - t) - \ln X_T}{\sigma\sqrt{T - t}}\right) \quad (2.2.2)$$

and

$$u_x(x, t; y, T) = \phi\left(\frac{\ln X_t + (\mu - \frac{1}{2}\sigma^2)(T - t) - \ln X_T}{\sigma\sqrt{T - t}}\right) \frac{1}{\sigma\sqrt{T - t}} \frac{1}{x} \quad (2.2.3)$$

The convex probability distortion is defined as

$$v(x, t; y, T) = u(x, t; y, T)^\gamma \quad (2.2.4)$$

where  $0 < \gamma < 1$ ,  $x$  is the stock price at time  $t$  and  $y$  is the stock price at time  $T$ .

Following Young and Zariphopoulou's (2000) [36] work, the distorted stochastic differential equation is

$$dX_s = [\mu X_s + \frac{(1 - \gamma)\sigma X_s}{\sqrt{T - t}} \frac{\phi(\frac{\ln X_t + (\mu - \frac{1}{2}\sigma^2)(T - t) - \ln X_T}{\sigma\sqrt{T - t}})}{\Phi(\frac{\ln X_t + (\mu - \frac{1}{2}\sigma^2)(T - t) - \ln X_T}{\sigma\sqrt{T - t}})}] ds + \sigma X_s dB_s \quad (2.2.5)$$

Compared with the stochastic process of the undistorted stock price, the distorted one only differs in the drift part. More specifically, it has one more term

$$\frac{(1 - \gamma)\sigma X_s}{\sqrt{T - t}} \frac{\phi(\frac{\ln X_t + (\mu - \frac{1}{2}\sigma^2)(T - t) - \ln X_T}{\sigma\sqrt{T - t}})}{\Phi(\frac{\ln X_t + (\mu - \frac{1}{2}\sigma^2)(T - t) - \ln X_T}{\sigma\sqrt{T - t}})} ds \quad (2.2.6)$$

## 2.3 Model

In this section, I first describe the model settings in terms of the financial market and the policyholders. Then I introduce the problem formulation with the wealth dynamics and value function. Finally, I derive the Hamilton-Jacobi-Bellman equation and obtain a semi-explicit solution as in Wang and Young (2012) [34].

In my model, three financial products are on the market: a bond, a stock and commutable life annuities. A bond earns a risk-free interest rate  $r$ . The stock price follows a lognormal distribution with the stochastic differential equation

$$dX_s = \mu X_s ds + \sigma X_s dB_s$$

In order to investigate the effect of a convex probability distortion following Young and Zariphopoulou (2000) [36], I derived its stochastic differential equation as follows:

$$dX_s = [\mu X_s + \frac{(1-\gamma)\sigma X_s}{\sqrt{T-t}} \frac{\phi(\frac{\ln X_t + (\mu - \frac{1}{2}\sigma^2)(T-t) - \ln X_T}{\sigma\sqrt{T-t}})}{\Phi(\frac{\ln X_t + (\mu - \frac{1}{2}\sigma^2)(T-t) - \ln X_T}{\sigma\sqrt{T-t}})}] ds + \sigma X_s dB_s \quad (2.3.1)$$

where  $0 < \gamma < 1$  is the distortion parameter,  $x$  is the stock price at time  $t$  and  $y$  is the stock price at time  $T$ .

The rest of the model settings are the same as described in the previous chapter.

In this work, I assume that the retiree consumes at a rate of  $c_s$  at time  $s$ .  $\pi_s$  represents the amount of money invested in the stock market at time  $s$ .  $P_s$  denotes the cumulative amount of annuity income purchased on or before time  $s$ , and  $S_s$  represents the cumulative amount of annuity income surrendered on or before time  $s$ .  $A_s = P_s - S_s$  is the cumulative amount of the life annuities income at time  $s$ . The initial endowments in the wealth account and annuity account are  $W$  and  $A$  respectively.

The change in the wealth account comes from six parts: the interest from the

bond, the return from the stock, the proceeds from the annuities, the consumption, the spending on purchasing annuities and proceeds from surrendering annuities.

Following the assumption that the stock price follows a distorted lognormal distribution, the wealth dynamics is:

$$\begin{aligned}
dW_s &= r(W_s - \pi_s)ds + \frac{\pi_s}{X_s}dX_s + (-c_s + A_s)ds - \bar{a}dP_s + (1-p)\bar{a}dS_s \\
&= [rW_s + \pi_s(\mu + \frac{(1-\gamma)\sigma}{\sqrt{T-t}} \frac{\phi(\frac{\ln X_t + (\mu - \frac{1}{2}\sigma^2)(T-t) - \ln X_T}{\sigma\sqrt{T-t}})}{\Phi(\frac{\ln X_t + (\mu - \frac{1}{2}\sigma^2)(T-t) - \ln X_T}{\sigma\sqrt{T-t}})} - r) + (-c_s + A_s)]ds \\
&\quad - \bar{a}dP_s + (1-p)\bar{a}dS_s + \sigma\pi_s dB_s
\end{aligned} \tag{2.3.2}$$

The annuity dynamics is given below:

$$dA_s = dP_s - dS_s \tag{2.3.3}$$

where  $\{\pi_s\}$ ,  $\{c_s\}$ ,  $\{P_s\}$  and  $\{S_s\}$  are control processes.

In my model, individuals seek to maximize their lifetime utility. That is,

$$U(A, W) = \sup_{\pi_s, c_s, P_s, A_s} \mathbb{E}[\int_0^{\tau_d} e^{-rs} u(c_s) ds | A_0 = A, W_0 = W] \tag{2.3.4}$$

In the literature on the proportional transaction costs, Davis and Norman (1990) [6] and Shreve and Soner (1994) [29] show that for individuals with a constant relative risk aversion preference, the optimal investment strategy is one of the singular and impulse control within a "wedge" bounded by two rays in wealth-investment

space. Following the same line of analysis, Milevsky and Young (2007) [21] prove that the annuitization strategy barrier is also a ray emanating from the origin in the wealth-annuity space. Let the critical wealth-to-annuity ratio be  $z_0$ . If  $\frac{W}{A} > z_0$ , the individual will purchase annuity income to raise her annuity income and reduce her wealth such that  $\frac{W'}{A'} = z_0$ . Furthermore, when  $\frac{W}{A} \leq z_0$ , she purchases annuity income to keep the ratio of wealth-to-annuity no greater than  $z_0$ . The individual neither buy nor surrender annuity income if  $0 < \frac{W}{A} < z_0$ . On the boundary  $W = 0$ , Milevsky and Young hypothesize that she will surrender existing annuity income in the force of negative investment return fluctuations to keep her wealth non-negative.

Accordingly, Milevsky and Young divide the wealth-annuity space into two parts:  $\mathcal{R}_1 = \{(W, A) : 0 \leq \frac{W}{A} \leq z_0\}$  and  $\mathcal{R}_2 = \{(W, A) : \frac{W}{A} > z_0\}$ . Once  $\frac{W}{A} > z_0$ , the individual will purchase

$$\Delta A = \frac{W - z_0 A}{z_0 + \bar{a}} \quad (2.3.5)$$

such that  $(W - \bar{a} \Delta A, A + \Delta A) \in \partial \mathcal{R}_1$ , and

$$U(A, W) = U(A + \Delta A, W - \bar{a} \Delta A) \quad (2.3.6)$$

Now I consider the case when  $0 \leq \frac{W}{A} \leq z_0$ . As discussed before, people neither buy

nor surrender any annuities in this area. In this case, the wealth dynamics is:

$$\begin{aligned}
dW_s &= r(W_s - \pi_s)ds + \frac{\pi_s}{X_s}dX_s + (-c_s + A_s)ds \\
&= [rW_s + \pi_s(\mu + \frac{(1-\delta)\sigma}{\sqrt{T-t}} \frac{\phi(\frac{\ln X_t + (\mu - \frac{1}{2}\sigma^2)(T-t) - \ln X_T}{\sigma\sqrt{T-t}})}{\Phi(\frac{\ln X_t + (\mu - \frac{1}{2}\sigma^2)(T-t) - \ln X_T}{\sigma\sqrt{T-t}})} - r) + (-c_s + A_s)]ds + \sigma\pi_s dB_s
\end{aligned} \tag{2.3.7}$$

along with the value function:

$$U(A, W) = \sup_{\pi_s, c_s} \mathbb{E} \left[ \int_0^{\tau_d} e^{-rs} u(c_s) ds \mid A_0 = A, W_0 = W \right] \tag{2.3.8}$$

By the stochastic control theory as in chapter 1, I derive the corresponding Hamilton-Jacobi-Bellman equation as shown below:

$$\begin{aligned}
(r + \lambda)U &= (rw + A)U_w + \max_{\pi_s} \left[ \left( \pi_s \left( \mu + \frac{(1-\delta)\sigma}{\sqrt{T-t}} \frac{\phi(\frac{\ln X_t + (\mu - \frac{1}{2}\sigma^2)(T-t) - \ln X_T}{\sigma\sqrt{T-t}})}{\Phi(\frac{\ln X_t + (\mu - \frac{1}{2}\sigma^2)(T-t) - \ln X_T}{\sigma\sqrt{T-t}})} - r \right) \right) U_w + \frac{1}{2} \sigma^2 \pi^2 U_{ww} \right] \\
&\quad + \max_{c \geq 0} \left( \frac{c^{1-\gamma}}{1-\gamma} - cU_w \right)
\end{aligned} \tag{2.3.9}$$

The corresponding boundary conditions are:

$$U_A(0, A) = (1-p)\bar{a}U_w(0, A) \tag{2.3.10}$$

$$U_A(z_0 A, A) = \bar{a}U_w(z_0 A, A) \tag{2.3.11}$$

$$U_{Aw}(z_0 A, A) = \bar{a}U_{ww}(z_0 A, A) \tag{2.3.12}$$

The boundary conditions for this partial differential equation follow an analysis

from our problem formulation. The first condition comes from the assumption that to keep the wealth account non-negative, the retiree surrenders an existing portion of her annuities in the face of negative investment return fluctuations once the wealth is 0. The second condition follows Wang and Young (2012) [34]. The last boundary condition is a smooth fit condition, also following Wang and Young (2012) [34].

## 2.4 Results

Following Wang and Young's (2012) [34] work, I solve this stochastic control problem semi-explicitly by performing a dimension reduction followed by linearization via the Legendre transform.

I list the main steps below. Interested readers may refer to Wang and Young (2012) [34].

Step 1: Dimension reduction.

The value function  $U$  is homogeneous of degree  $1 - \gamma$  with respect to wealth  $W$  and annuity  $A$  due to the homogeneity property of the constant relative risk aversion utility function. More specifically,  $U(\alpha A, \alpha W) = \alpha^{1-\gamma} U(A, W)$  for  $\alpha > 0$ . Define  $V(z) = U(z, 1)$ , thus we have

$$U(A, W) = A^{1-\gamma} U\left(\frac{W}{A}, 1\right) = A^{1-\gamma} V(z) \quad (2.4.1)$$



Accordingly,

$$U_w(A, W) = A^{-\gamma} V_z(z) \quad (2.4.2)$$

$$U_{ww}(A, W) = A^{-\gamma-1} V_{zz}(z) \quad (2.4.3)$$

$$U_A(A, W) = A^{-\gamma} [(1 - \gamma)V(z) - zV_z(z)] \quad (2.4.4)$$

$$U_{Aw}(A, W) = -A^{-\gamma-1} [\gamma V_z(z) + zV_{zz}(z)] \quad (2.4.5)$$

Step 2: Linearization.

In order to convert the above non-linear ordinary differential equation (ODE) into a linear one, Wang and Young perform a Legendre transform.

Define

$$\hat{V}(y) = \max_{z \geq 0} [V(z) - yz] \quad (2.4.6)$$

That is, for a given  $y$ , a  $z^*$  maximizes  $V(z) - yz$ .

It follows:

$$z^* = -\hat{V}_y(y) \quad (2.4.7)$$

$$V(z^*) = y\hat{V}_y(y) + \hat{V}(y) \quad (2.4.8)$$

$$V_z(z^*) = y \quad (2.4.9)$$

$$V_{zz}(z^*) = -\frac{1}{\hat{V}_{yy}(y)} \quad (2.4.10)$$

Moreover, Wang and Young define

$$y_s = V_z(0)$$

$$y_b = V_z(z_0)$$

The Legendre transformation gives us a linear ODE:

$$\hat{V}(y) = D_1 y^{B_1} + D_2 y^{B_2} + \frac{y}{r} + C y^{\frac{\gamma-1}{\gamma}} \quad (2.4.11)$$

in which

$$m = \frac{1}{2} \left( \frac{\mu + \frac{(1-\gamma)\sigma}{\sqrt{T-t}} \frac{\phi\left(\frac{\ln X_t + (\mu - \frac{1}{2}\sigma^2)(T-t) - \ln X_T}{\sigma\sqrt{T-t}}\right)}{\Phi\left(\frac{\ln X_t + (\mu - \frac{1}{2}\sigma^2)(T-t) - \ln X_T}{\sigma\sqrt{T-t}}\right)} - r}{\sigma} \right)^2 \quad (2.4.12)$$

$$B_1 = \frac{1}{2m} [(m - \lambda) + \sqrt{(m - \lambda)^2 + 4m(r + \lambda)}] \quad (2.4.13)$$

$$B_2 = \frac{1}{2m} [(m - \lambda) - \sqrt{(m - \lambda)^2 + 4m(r + \lambda)}] \quad (2.4.14)$$

$$C = \frac{\gamma}{1-\gamma} \left[ r + \frac{\lambda}{\gamma} - m \frac{1-\gamma}{\gamma^2} \right]^{-1} \quad (2.4.15)$$

$$D_1 = -\frac{\lambda}{r(r + \lambda)} \frac{1 - B_2}{B_1 - B_2} \frac{1}{1 + \gamma(B_1 - 1)} y_b^{1-B_1} \quad (2.4.16)$$

$$D_2 = -\frac{\lambda}{r(r + \lambda)} \frac{1 - B_1}{B_1 - B_2} \frac{1}{1 + \gamma(B_2 - 1)} y_b^{1-B_2} \quad (2.4.17)$$

$$(2.4.18)$$

Now define  $x = \frac{y_s}{y_b}$ , then  $y_s$ ,  $y_b$  and  $z_0$  can be implicitly solved by

$$\begin{aligned} \frac{1-B_2}{B_1-B_2}x^{B_1-1} + \frac{B_1-1}{B_1-B_2}x^{B_2-1} &= 1 + \frac{pr}{\lambda} \quad (2.4.19) \\ \frac{1-\gamma}{\gamma}Cy_s^{-\frac{1}{\gamma}} &= -\frac{\lambda}{r(r+\lambda)} \left[ \frac{B_1(1-B_2)}{B_1-B_2} \frac{x^{B_1-1}}{1+\gamma(B_1-1)} + \frac{B_2(B_1-1)}{B_1-B_2} \frac{x^{B_2-1}}{1+\gamma(B_2-1)} \right] \quad (2.4.20) \\ y_b &= \frac{y_s}{x} \quad (2.4.21) \\ z_0 &= \frac{1-\gamma}{\gamma}C[y_b^{-\frac{1}{\gamma}} + \frac{\lambda}{r(r+\lambda)}\frac{1}{\gamma}(r+\lambda - m\frac{1-\gamma}{\gamma})] \quad (2.4.22) \end{aligned}$$

## 2.5 Numerical Illustration

I now provide numerical examples to illustrate the semi-analytical results in the previous section. Following Wang and Young's work, I use the parameter values listed below:

$$\lambda = 0.04$$

$$r = 0.04$$

$$\mu = 0.08$$

$$\sigma = 0.2$$

$$T - t = 25$$

$$X_t = 10$$

$$X_T = 100$$

First I investigate the effects of surrender charge and distortion parameter on the annuitization strategy. As discussed in Wang and Young's paper,  $z_0 = \frac{W}{A}$  is an indicator of a policyholder's willingness to purchase annuities. A higher value of  $z_0$  implies a weaker incentive to buy more annuities. In figure 2.5.1, I list  $z_0$  under different

surrender charges  $p$  and power distortion parameters  $\gamma$ . I observe that  $z_0$  decreases with respect to  $\gamma$  for a fixed value  $p$  whereas it increases with respect to  $p$  for a fixed  $\gamma$ . In other words, a lower value  $\gamma$  leads to a weaker inclination to buy annuities. More precisely, the policyholder who put more probability weights on the large gains will buy less annuities. It is reasonable as she tends to invest more on the stock market. On another side, higher transaction fees  $p$  will reduce the willingness to buy more annuities, which is consistent with the analysis in Wang and Young's (2012) [34] work.

For example, assuming  $p = 0.2$ ,  $\alpha = 2.5$  and  $\gamma = 0.2$ . Here is a policyholder with \$50 wealth and \$25 annuities. Based on our calculation, she would buy 0.4787 annuities to reach a critical ratio of  $\frac{W}{A} = 1.7276$  without any probability distortion. She would neither buy nor surrender any annuities with a power distortion parameter  $\gamma = 0.2$ , because the critical ratio  $2 < 5.5189$ .

Figure 2.5.2 compares the investment strategies for a policyholder without and with the probability distortion. As shown on the graph, the investment increases with respect to both endowed wealth and annuities. In particular, an individual who expects a heavier probability for large gains will invest more than people who hold an objective view about the real-world probabilities. That is because they are more optimistic towards the stock market performance.

Figure 2.5.3 displays their consumption strategies for people without and with the probability distortion. I observe that people tend to consume more if they view a promising stock market. Since they expect more gains from the stock market in the future, they would like to consume more now to maximize the lifetime utility.

As shown by the critical wealth-to-annuity ratio  $z_0$ , the investment strategy and consumption strategy, an individual with a convex probability distortion choose to invest more in the stock market, consume more and buy less life annuities. It essentially explains the annuity puzzle from the behavioral economics point of view. The reason why people stop buying annuities at a lower level is because too much investment leaves no money to buy annuities. More specifically, stock market investment exceeds the total wealth in most part of the wealth-annuity plane, which already requires them to borrow from the bank to keep the wealth account non-negative. The behavior of borrowing from bank essentially reduces the incentive to buy more annuities. To keep a balance of both wealth and annuity account, purchasing more annuities leads to borrowing more from the bank account, and vice versa. It eventually eliminates the profitability from buying life annuities. For example, as in my model settings,  $r = 0.04$  and  $\lambda = 0.04$ , it gives a bond price \$25 and an annuity price \$12.5. That means, the return from life annuities is substantially two times that from the bond. However, if we borrow money at a rate  $r = 0.04$  to buy an annuity, it leaves us the same return as buying bonds. Moreover, we lose the liquidity by purchasing more annuities. Hence, it leaves us with no reason to buy more annuities.

Now we examine the effect of the risk aversion parameter on the annuitization strategy, investment strategy and consumption strategy. By decreasing  $\gamma$  from 2.5 to 1.5, we now consider a policyholder who are less risk averse. Compared with the portfolio selection with  $\gamma = 2.5$ , she will invest more and buy less annuities. Interestingly, the consumption strategy displays a mixed effect from both the investment and annuitization level. More specifically, she will consume less if she is endowed

p	$\gamma = 0.2$	$\gamma = 0.5$	$\gamma = 0.8$	$\gamma = 1$
0.01	1.0959	0.8604	0.6320	0.4838
0.02	1.5783	1.2260	0.8894	0.6734
0.04	2.2884	1.7502	1.2461	0.9280
0.08	3.3423	2.4961	1.7277	1.2548
0.1	3.7800	2.7946	1.9109	1.3726
0.2	5.5189	3.9063	2.5312	1.7276
0.3	6.7727	4.6121	2.8398	1.8362

FIGURE 2.5.1:  $z_0$  and  $\alpha = 2.5$  for various level of  $p$  and  $\gamma$

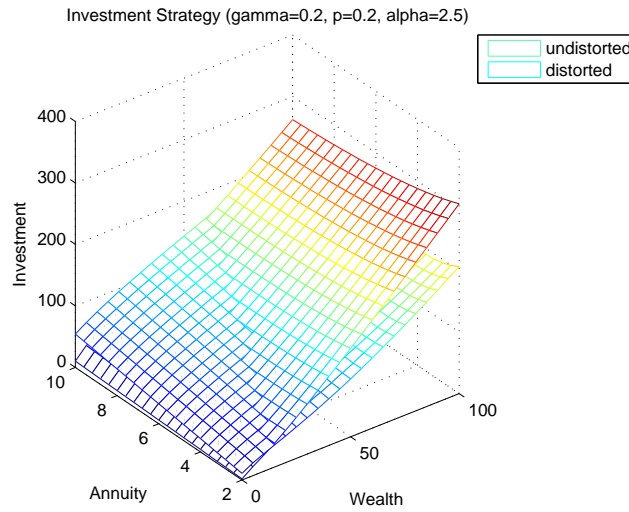


FIGURE 2.5.2: This graph compares the investment strategy without and with the probability distortion. We set  $r = 0.04$ ,  $\lambda = 0.04$ ,  $\mu = 0.08$ ,  $\sigma = 0.2$ ,  $\alpha = 2.5$ ,  $p = 0.2$ , and  $\gamma = 0.2$ .

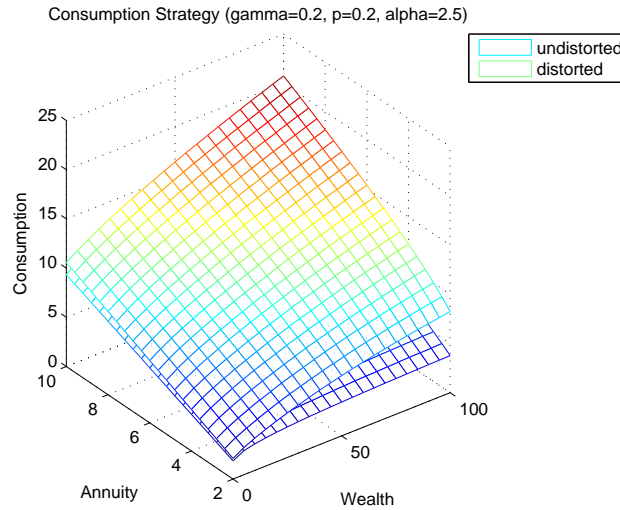


FIGURE 2.5.3: This graph compares the consumption strategy without and with the probability distortion. We set  $r = 0.04$ ,  $\lambda = 0.04$ ,  $\mu = 0.08$ ,  $\sigma = 0.2$ ,  $\alpha = 2.5$ ,  $p = 0.2$ , and  $\gamma = 0.2$ .

with a low wealth; she choose to consume more under a high endowed wealth. That is because, the consumption level depends on both the investment expectation and the annuity level. A higher expectation on the future gains will naturally motivates more consumptions. However, since the consumption mainly comes from the proceeds from annuities, a lower level of annuities reduces the consumption desire. As a result, the effect from annuity dominates that from investment when the endowed wealth is small, and vice versa. Simply put, a less risk averse individual decides to consume more only if the current wealth level is large enough to support it.

The analysis from a convex probability distortion is consistent with that from a reverse S-shaped distortion.

p	$\gamma = 0.2$	$\gamma = 0.5$	$\gamma = 0.8$	$\gamma = 1$
0.01	1.8812	1.4681	1.0719	0.8173
0.02	2.7451	2.1135	1.5200	1.1443
0.04	4.0589	3.0631	2.1531	1.5904
0.08	6.1062	4.4677	3.0332	2.1760
0.1	6.9931	5.0488	3.3764	2.3914
0.2	10.7507	7.3236	4.5823	3.0593
0.3	13.7331	8.8804	5.2166	3.2723

FIGURE 2.5.4:  $z_0$  and  $\alpha = 1.5$  for various level of  $p$  and  $\gamma$

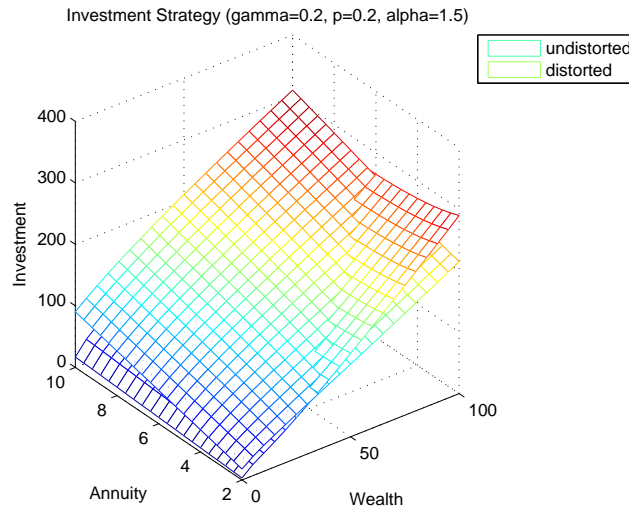


FIGURE 2.5.5: This graph compares the investment strategy without and with the probability distortion. We set  $r = 0.04$ ,  $\lambda = 0.04$ ,  $\mu = 0.08$ ,  $\sigma = 0.2$ ,  $\alpha = 1.5$ ,  $p = 0.2$ , and  $\gamma = 0.2$ .



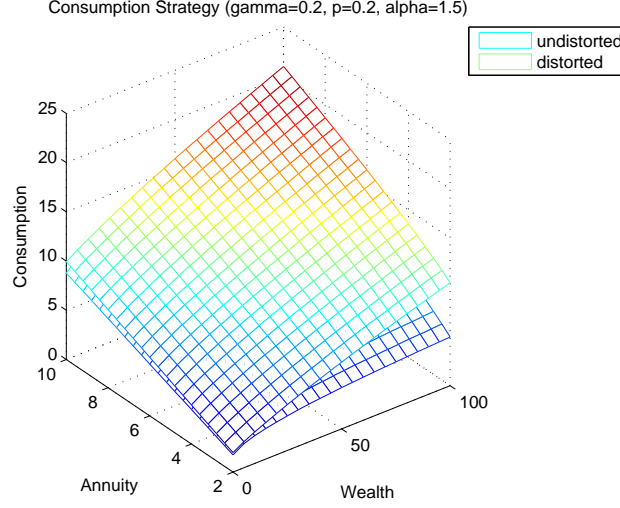


FIGURE 2.5.6: This graph compares the consumption strategy without and with the probability distortion. We set  $r = 0.04$ ,  $\lambda = 0.04$ ,  $\mu = 0.08$ ,  $\sigma = 0.2$ ,  $\alpha = 1.5$ ,  $p = 0.2$ , and  $\gamma = 0.2$ .

## 2.6 Conclusion

A convex probability distortion function essentially raises the drift part of the stock price process, which results in more investment in the stock market and less demand on the annuity industry. Correspondingly, consumption increases as individuals hold a higher expectation on the return from investment.

Notably, I derive a semi-analytical solution for annuitization, investment and consumption strategies under the convex probability distortion. Among them, annuitization strategy is represented by a wealth-to-annuity critical ratio  $z_0$ . As discussed in Wang and Young's paper,  $z_0$  is an indicator of a policyholder's willingness to purchase annuities. A higher value of  $z_0$  implies a weaker incentive to buy more annuities. As the drift part of the stock price process is raised up by the convex probability distortion, the critical ratio increases over the whole wealth-annuity space. That is because,

too much investment leaves no money to buy annuities. More precisely, purchasing an annuity by borrowing money from the bank leads to a much lower rate of return. Moreover, we lose the liquidity by purchasing more annuities. Hence, it leaves us with no reason to buy more annuities.

Sensitivity analysis show that for a less risk averse individual, she will invest more and buy less annuities. She decides to consume more only if the current wealth level is large enough to support it. More specifically, she will consume less if she is endowed with a low wealth; she choose to consume more under a high endowed wealth. That is because, the consumption level depends on both the investment expectation and the annuity level. A higher expectation on the future gain will naturally motivates a more consumption. However, since the consumption mainly comes from the proceed from annuities, a lower level of annuities reduces the consumption desire. Thus, the consumption strategy displays a mixed effect from both the investment and annuitization level.

The introduction of convex probability distortion into the stochastic portfolio choice model with life annuities essentially provides a plausible explanation for the annuity puzzle from the behavioral economics point of view.

# Bibliography

- [1] Shlomo Benartzi, Alessandro Previtero, and Richard H Thaler, *Annuity puzzles*, The Journal of Economic Perspectives (2011), 143–164.
- [2] Shlomo Benartzi and Richard H Thaler, *Myopic loss aversion and the equity premium puzzle*, Tech. report, National Bureau of Economic Research, 1993.
- [3] Carole Bernard and Mario Ghossoub, *Static portfolio choice under cumulative prospect theory*, Mathematics and financial economics **2** (2010), no. 4, 277–306.
- [4] Monika Bütler and Federica Teppa, *The choice between an annuity and a lump sum: Results from swiss pension funds*, Journal of Public Economics **91** (2007), no. 10, 1944–1966.
- [5] David M Cutler, Amy Finkelstein, and Kathleen McGarry, *Preference heterogeneity and insurance markets: Explaining a puzzle of insurance*, Tech. report, National Bureau of Economic Research, 2008.
- [6] Mark HA Davis and Andrew R Norman, *Portfolio selection with transaction costs*, Mathematics of Operations Research **15** (1990), no. 4, 676–713.

- [7] Enrico De Giorgi and Thierry Post, *Second-order stochastic dominance, reward-risk portfolio selection, and the capm*, Journal of Financial and Quantitative Analysis **43** (2008), no. 02, 525–546.
- [8] Amy Finkelstein and James Poterba, *Adverse selection in insurance markets: Policyholder evidence from the uk annuity market*, Journal of Political Economy **112** (2004), no. 1, 183–208.
- [9] Wendell H Fleming and Halil Mete Soner, *Controlled markov processes and viscosity solutions*, vol. 25, Springer Science & Business Media, 2006.
- [10] Benjamin M Friedman and Mark J Warshawsky, *The cost of annuities: Implications for saving behavior and bequests*, The Quarterly Journal of Economics (1990), 135–154.
- [11] Greg Hertzler et al., *A stochastic differential equation for modeling the classical probability distributions*, 2003 Conference (47th), February 12-14, 2003, Fremantle, Australia, no. 57891, Australian Agricultural and Resource Economics Society, 2003.
- [12] Wei-Yin Hu and Jason S Scott, *Behavioral obstacles in the annuity market*, Financial Analysts Journal **63** (2007), no. 6, 71–82.
- [13] Michael D Hurd, *Mortality risk and bequests*, Econometrica: Journal of the Econometric Society (1989), 779–813.
- [14] J Mark Iwry, John Andrew Turner, and Pew Charitable Trusts, *Automatic annuitization: New behavioral strategies for expanding lifetime income in 401 (k)s*, Retirement Security Project, 2009.

- [15] Hanqing Jin and Xun Yu Zhou, *Behavioral portfolio selection in continuous time*, Mathematical Finance **18** (2008), no. 3, 385–426.
- [16] Daniel Kahneman and Amos Tversky, *Prospect theory: An analysis of decision under risk*, Econometrica: Journal of the Econometric Society (1979), 263–291.
- [17] FSA Kate Li, FSA Louis Lombardi, ASA Lucian Lombardi, FSA Marianne Purushotham, and Anand Rao, *Modeling of policyholder behavior for life insurance and annuity products*, (2014).
- [18] David Laibson, *Golden eggs and hyperbolic discounting*, The Quarterly Journal of Economics (1997), 443–477.
- [19] Robert C Merton, *Lifetime portfolio selection under uncertainty: The continuous-time case*, The review of Economics and Statistics (1969), 247–257.
- [20] Moshe A Milevsky, Kristen S Moore, and Virginia R Young, *Asset allocation and annuity-purchase strategies to minimize the probability of financial ruin*, Mathematical Finance **16** (2006), no. 4, 647–671.
- [21] Moshe A Milevsky and Virginia R Young, *Annuitization and asset allocation*, Journal of Economic Dynamics and Control **31** (2007), no. 9, 3138–3177.
- [22] Alan Mills, *Complexity science: An introduction (and invitation) for actuaries*, Society Of Actuaries Health Section (Version 1) (2010).
- [23] Olivia S Mitchell, James M Poterba, and Mark J Warshawsky, *New evidence on the money’s worth of individual annuities*, Tech. report, National bureau of economic research, 1997.

- [24] Oskar Morgenstern and John Von Neumann, *Theory of games and economic behavior*, (1953).
- [25] Gary R Mottola and Stephen P Utkus, *Lump sum or annuity? an analysis of choice in db pension payouts*, Vanguard Center for Retirement Research **30** (2007), no. 89, 2.
- [26] Drazen Prelec, *The probability weighting function*, *Econometrica* (1998), 497–527.
- [27] S Schaus, *Annuities make a comeback*, *Journal of Pension Benefits: Issues in Administration* **12** (2005), no. 4, 34–38.
- [28] Hersh Shefrin and Meir Statman, *Behavioral portfolio theory*, *Journal of financial and quantitative analysis* **35** (2000), no. 02, 127–151.
- [29] Steven E Shreve and H Mete Soner, *Optimal investment and consumption with transaction costs*, *The Annals of Applied Probability* (1994), 609–692.
- [30] Paul Slovic, Baruch Fischhoff, Sarah Lichtenstein, Bernard Corrigan, and Barbara Combs, *Preference for insuring against probable small losses: Insurance implications*, *Journal of Risk and Insurance* (1977), 237–258.
- [31] Cassio Turra and Olivia S Mitchell, *The impact of health status and out-of-pocket medical expenditures on annuity valuation*, (2004).
- [32] Amos Tversky and Craig R Fox, *Weighing risk and uncertainty.*, *Psychological review* **102** (1995), no. 2, 269.

- [33] Amos Tversky and Daniel Kahneman, *Advances in prospect theory: Cumulative representation of uncertainty*, Journal of Risk and uncertainty **5** (1992), no. 4, 297–323.
- [34] Ting Wang and Virginia R Young, *Maximizing the utility of consumption with commutable life annuities*, Insurance: Mathematics and Economics **51** (2012), no. 2, 352–369.
- [35] Menahem E Yaari, *Uncertain lifetime, life insurance, and the theory of the consumer*, The Review of Economic Studies (1965), 137–150.
- [36] Virginia R Young and Thaleia Zariphopoulou, *Computation of distorted probabilities for diffusion processes via stochastic control methods*, Insurance: Mathematics and Economics **27** (2000), no. 1, 1–18.