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Traveling Fronts to Reaction Diffusion Equations with Fractional Laplacians

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ABSTRACT

We consider the traveling fronts of the reaction diffusion equation:

$$u_t + (-\Delta)^s u = f(u), \quad \text{in } \mathbb{R}\times\mathbb{R},$$

for $f \in C^1(\mathbb{R})$. Namely, the solution to the following equation:

$$\begin{cases} (-\Delta)^s u(x) + cu'(x) = f(u(x)), & \forall x \in \mathbb{R} \\ \lim\limits_{x\to-\infty} u(x) = 0, \quad \lim\limits_{x\to\infty} u(x) = 1 \end{cases} \tag{0.0.1}$$

where c is the speed of propagation of the front and the operator $(-\Delta)^s$ denotes the fractional power of the Laplacian in one dimension with $0 < s < 1$. Recall the fractional Laplacian is defined as follows:

$$(-\Delta)^s u(x) = C_{1,s}(\text{P.V.}) \int_{\mathbb{R}} \frac{u(x)-u(y)}{|x-y|^{1+2s}} dy,$$

where (P.V.) stands for Cauchy principal value and $C_{1,s} = \dfrac{2^{2s} s\Gamma((1+2s)/2)}{\pi^{1/2}\Gamma(1-s)}$.

We show the nonexistence of traveling fronts in the combustion model with fractional Laplacian $(-\Delta)^s$ when $s \in (0, 1/2]$. Our method can be used to give a direct and simple proof of the nonexistence of traveling fronts for the usual Fisher-KPP nonlinearity. Also we prove the existence and nonexistence of traveling waves solutions for different ranges of the fractional power s for the generalized Fisher-KPP type model. When considering the Allen-Cahn type nonlinearity, we show the approach of the solution to the traveling front for a large range of initial value problems.

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Contents

Chapter 1

Introduction

1.1 Reaction Diffusion Equations

Reaction diffusion equations are considered in many areas of natural science and engineering. The classical application is to the population genetics which is formulated by R.A. Fisher in 1937. In his application, a population of diploid individuals distributed on a planar habitat is considered. Suppose the gene at a specific locus in a specific chromosome pair occurs in two allele forms denoted by a and A. The population is thus divided into three classes: homo-zygotes which have genotypes aa and AA, and hetero-zygotes which have genotype aA. Then the relative density of the allele A $u(x,t)$ at the point x of the habitat at time t will be close to the solution of

$$\frac{\partial u}{\partial t} - \Delta u = f(u), \text{ in } (0, +\infty) \times \mathbb{R}^n \tag{1.1.1}$$

with

$$f(u) = u(1-u)\{(\tau_1 - \tau_2) - (\tau_1 - 2\tau_2 + \tau_3)u\}.$$

where τ_1, τ_2 and τ_3 are death rates for AA, aA and aa respectively.

In combustion theory, some flame propagation problems also lead to the form (1.1.1) where $f \in C^1[0,1]$, $f(0) = f(1) = 0$ and

$$f(u) = f(1) = 0, \ \forall u \in [0, \theta], \ f(u) > 0, \ \forall u \in (\theta, 1), \ f'(1) < 0.$$

where $\theta > 0$ is the ignition temperature.

More generally, we will consider $f \in C^1([0,1])$, $f(0) = f(1) = 0$ and we will consider the reaction diffusion equation:

$$u_t + Lu = f(u) \quad \text{in } (0, \infty) \times \mathbb{R} \tag{1.1.2}$$

where L is the infinitesimal generator of Levy processes. According to the Levy-Kintchine formula, they have the general form

$$\begin{gathered} Lu(x) = \mathrm{tr}A(x) \cdot D^2 u + b(x) \cdot \nabla u + c(x)u + d(x) + \\ \int_{\mathbb{R}^n} (u(x+y) - u(x) - y \cdot \nabla u(x) 1_{b_1}(y)) dc_x(y) \end{gathered}$$

where $A(x)$ is a nonegative matrix for all x, and c_x is a nonnegative measure for all x satisfying

$$\int_{\mathbb{R}^n} \min(y^2, 1) dc_x(y) < +\infty.$$

The above definition is very general. The simplest of all is the fractional Laplacian. The fractional Laplacian $(-\Delta)^s$ is a classical operator which gives the standard Laplacian when $s = 1$.

1.2 Fractional Laplacians

One way to define the fractional Laplacian is to consider the fractional Laplacian as a pseudo-differential operator.

We will assume the Fourier transform be: For any $f \in L^1(\mathbb{R}^n)$, then

$$\hat{f}(x) = \int_{\mathbf{R}^n} f(y) e^{-2\pi i x \cdot y}\, dy, \quad \text{for all } y \in \mathbf{R}.$$

Recall that if $f \in \mathcal{S}(\mathbf{R}^n)$, then $-\Delta f \in \mathcal{S}(\mathbf{R}^n)$, $\widehat{(-\Delta) f} \in \mathcal{S}(\mathbf{R}^n)$, and

$$\widehat{(-\Delta) f}(x) = (2\pi |x|)^2 \hat{f}(x), \quad \text{for all } x \in \mathbf{R}^n.$$

By induction, for any $k \in \mathbf{N}$, we can get

$$\widehat{(-\Delta)^k f}(x) = (2\pi |x|)^{2k} \hat{f}(x), \quad \text{for all } x \in \mathbf{R}^n.$$

For any $s \in \mathbf{R}$, the pseudo-differential operator $(-\Delta)^s$ is formally defined as: For any $f \in \mathcal{S}(\mathbf{R}^n)$, we have

$$\widehat{(-\Delta)^s f}(x) = (2\pi |x|)^{2s} \hat{f}(x), \quad \text{for all } x \in \mathbf{R}^n. \tag{1.2.1}$$

This formula is simple to understand and it is useful for problems in the whole space. On the other hand, it is hard to obtain local estimates from it.

Fractional Laplacian can also be defined as the generator of $\alpha-$stable Levy process. More precisely, if X_y is the isotropic $\alpha-$stable Levy process starting at zero and f is a smooth function, then

$$(-\Delta)^{\alpha/2} f(x) = \lim_{h\to 0} \frac{1}{h}\ \mathrm{E}[f(x) - f(x + X_h)].$$

One can also think of $(-\Delta)^s$ as a singular integral given by the following theorem.

Theorem 1.2.1. *Let $0 < s < 1$, and $f \in \mathcal{S}(\mathbf{R}^n)$, then we have*

$$(-\Delta)^s f(x) = C(n,s)\cdot \text{P.V.} \int_{\mathbf{R}^n} \frac{f(x) - f(y)}{|x-y|^{n+2s}}\, dy < \infty, \quad \text{for all } x \in \mathbf{R}^n. \quad (1.2.2)$$

Where $C(n,s) = \dfrac{s2^{2s}\Gamma\left(\frac{n+2s}{2}\right)}{\pi^{\frac{n}{2}}\Gamma(1-s)}$.

The above formula is most useful to study the local proper ties of equations involving the fractional Laplacian. It will be the definition we will use in later chapters. Now we would like to derive an equivalent expression of the fractional Laplacian based on the singular integral definition. We apply the formula in (1.2.2), we can get

$$\begin{aligned}
(-\Delta)^s f(x) &= C(n,s)\cdot \text{P.V.} \int_{\mathbf{R}^n} \frac{f(x) - f(y)}{|x-y|^{n+2s}}\, dy \\
&= C(n,s)\cdot \lim_{\epsilon\searrow 0,\ R\nearrow\infty} \int_{\epsilon<|y-x|<R} \frac{f(x) - f(y)}{|x-y|^{n+2s}}\, dy \\
&= C(n,s)\cdot \lim_{\epsilon\searrow 0,\ R\nearrow\infty} \int_{\epsilon<|z|<R} \frac{f(x) - f(x+z)}{|z|^{n+2s}}\, dz \quad \text{Let } z = y - x \\
&= C(n,s)\cdot \text{P.V.} \int_{\mathbf{R}^n} \frac{f(x) - f(x+y)}{|y|^{n+2s}}\, dy \qquad (1.2.3) \\
&< \infty, \quad \text{for all } x \in \mathbf{R}^n.
\end{aligned}$$

Besides, we can get

$$\begin{aligned}
(-\Delta)^s f(x) &= C(n,s)\cdot \text{P.V.} \int_{\mathbf{R}^n} \frac{f(x)-f(y)}{|x-y|^{n+2s}}\, dy \\
&= C(n,s)\cdot \lim_{\epsilon\searrow 0,\ R\nearrow\infty} \int_{\epsilon<|y-x|<R} \frac{f(x)-f(y)}{|x-y|^{n+2s}}\, dy \\
&= C(n,s)\cdot \lim_{\epsilon\searrow 0,\ R\nearrow\infty} \int_{\epsilon<|z|<R} \frac{f(x)-f(x-z)}{|z|^{n+2s}}\, dz \quad \text{Let } z=x-y \\
&= C(n,s)\cdot \text{P.V.} \int_{\mathbf{R}^n} \frac{f(x)-f(x-y)}{|y|^{n+2s}}\, dy \qquad (1.2.4)\\
&< \infty, \quad \text{for all } x\in\mathbf{R}^n.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
(-\Delta)^s f(x) &= -\frac{C(n,s)}{2}\cdot \text{P.V.} \int_{\mathbf{R}^n} \frac{f(x+y)+f(x-y)-2f(x)}{|y|^{n+2s}}\, dy \\
&= D(n,s)\cdot \text{P.V.} \int_{\mathbf{R}^n} \frac{f(x+y)+f(x-y)-2f(x)}{|y|^{n+2s}}\, dy. \qquad (1.2.5)
\end{aligned}$$

Where

$$D(n,s) = -\frac{C(n,s)}{2} = -\frac{s2^{2s-1}\Gamma\left(\frac{n+2s}{2}\right)}{\pi^{\frac{n}{2}}\Gamma(1-s)}.$$

This particular expression shows that the fractional Laplacian enjoys the following monotonicity property: if u has a global maximum at x, then $(-\Delta)^s u(x) \geq 0$, with equality only if u is constant. From this monotonicity, a comparison principle can be derived for equations involving the fractional Laplacian.

1.3 Traveling wave solutions to reaction diffusion equations

A traveling wave solution is a solution of the reaction diffusion equation which is defined as $u \in C^2(\mathbb{R}^2_+)$ such that

$$u(x,t) = \phi(x+ct),\ \forall (x,t) \in \mathbb{R}^2_+.$$

where c is the speed of the traveling wave solution. ϕ is called the profile function. When the traveling wave solution is monotone and bounded, we call it a traveling front.

Traveling waves arise in many applied problems, see for example [15], [22], [4], [3], [16], [6], [7], [19] and references therein. In [28], for instance, traveling wave solutions are used to describe the propagation of impulse in nerve fibers. Various different kinds of waves can often be observed in chemical reactions, see for example, [27].

Define the traveling wave coordinates: $z = x + ct$. For the classical reaction diffusion equation with Laplacian

$$u_t + (-\Delta)u = f(u), \quad \text{in } \mathbb{R} \times \mathbb{R},$$

for $f \in C^1(\mathbb{R})$. By substitution, the profile of the traveling fronts to the above equation will satisfy

$$\begin{cases} -\phi''(z) + c\phi'(z) = f(\phi), \quad \forall z \in \mathbb{R} \\ \phi'(z) > 0, \quad \forall z \in \mathbb{R} \\ \lim\limits_{z\to-\infty} \phi(z) = 0, \quad \lim\limits_{z\to\infty} \phi(z) = 1 \end{cases} \tag{1.3.1}$$

Similarly, for the reaction diffusion equation (1.1.2) where $L = (-\Delta)^s$, we know that the profile of the traveling front will satisfy the following PDE:

$$\begin{cases} (-\Delta)^s \phi(z) + c\phi'(z) = f(\phi(z)), \quad \forall z \in \mathbb{R} \\ \lim_{z \to -\infty} \phi(z) = 0, \quad \lim_{z \to \infty} \phi(z) = 1 \end{cases} \tag{1.3.2}$$

Notice that traveling fronts are translation invariant. So as for the uniqueness, we will always consider the uniqueness up to translation.

As for the reaction nonlinearity f, there are three cases which are of particular interests in this thesis:

- Fisher-KPP Model:

$$f(0) = f(1) = 0,\ f(u) > 0,\ \forall u \in (0,1), f'(0) > 0,\ f'(1) < 0. \tag{1.3.3}$$

- Combustive Model:

$$f(u) = f(1) = 0,\ \forall u \in [0,\theta]\ f(u) > 0,\ \forall u \in (\theta, 1),\ f'(1) < 0. \tag{1.3.4}$$

- Bistable Model:

$$f(0) = f(1) = 0,\ f'(0) < 0,\ f'(1) < 0. \tag{1.3.5}$$

When $A = -\Delta$, the classical Laplacian case, it is well-known that there exists a traveling front u for any speed c larger than or equal to some minimum speed c_0 in the Fisher-KPP equation. And it has been shown that the front propagation speed could be very fast depending on initial values. Fisher-KPP equation with a fractional Laplacian displays a very different behavior, due to the super diffusion process involved. It was discovered numerically in [13], [14], [24], [25] that the front propagation can accelerate exponentially in time. Also it has been rigorously studied and proved in [12] and [8]. Since a traveling front propagates linearly in t, it is an immediate consequence that there is no traveling fronts.

In the bistable equation, it was shown that both in the Laplacian case and the fractional Laplacian case, there will exists a unique pair (u, c) to (1.1.2) for all $0 < s < 1$. In the fractional Laplacian case, Gui and Zhao [20] applied the method of continuity to get the uniform bound of the speed in terms of potential. And their result indicate that there

In the combustion equation with the laplacian, it is known that there exists a unique pair (u, c). As for the equation with the fractional Laplacian it has recently been discovered in [26] that when $\frac{1}{2} < s < 1$, there exists a unique pair (u, c).

1.4 Previous results of the existence of traveling wave solutions

In Chapter 2, we study the existence of the traveling wave solutions of the reaction diffusion equations with fractional Laplacians in the combustion and the generalized Fisher-KPP models.

To provide the setting for our result, in this section we recall important results on some closely related questions. For comparison purpose, for each type of the three nonlinear reaction term, we give the existence of traveling wave solutions to the equations with the Laplacian and the existence results to the equations when Laplacian is replaced by the fractional Laplacian with a brief review of the method involved.

The traveling wave solution to the classical reaction diffusion equation with Laplacian is referred to the following:

$$\begin{cases} -\phi''(z) + c\phi'(z) = f(\phi), \quad \forall z \in \mathbb{R} \\ \phi'(z) > 0, \quad \forall z \in \mathbb{R} \\ \lim_{z\to-\infty} \phi(z) = 0, \quad \lim_{z\to\infty} \phi(z) = 1 \end{cases} \tag{1.4.1}$$

While the traveling wave solution to the reaction diffusion equation with the fractional Laplacian is given by

$$\begin{cases} (-\Delta)^s\phi(z) + c\phi'(z) = f(\phi(z)), \quad \forall z \in \mathbb{R} \\ \lim_{z\to-\infty} \phi(z) = 0, \quad \lim_{z\to\infty} \phi(z) = 1 \end{cases} \tag{1.4.2}$$

When $A = -\Delta$, the classical Laplacian case, it is well-known that there exists a traveling front u for any speed c larger than or equal to some minimum speed c_0 in the Fisher-KPP equation. And it has been shown that the front propagation speed could be very fast depending on initial values. Fisher-KPP equation with a fractional Laplacian displays a very different behavior, due to the super diffusion process involved. It was discovered numerically in [**?**], [**?**] that the front propagation can accelerate exponentially in time. Since a traveling front propagates linearly in t, it is an immediate consequence that there is no traveling fronts.

For the bistable model, sometimes known as the Allen-Cahn equation, there exists a unique traveling wave solution. An interesting observation is the relationship between the double well potential and the speed of the traveling wave solution. When the bistable nonlinearity is balanced, i.e., the associated double well potential has two wells with equal depths, a traveling wave solution with one spatial variable for the classical Allen-Cahn equation is indeed a standing wave or a layer solution, i.e., the speed c must be zero. If the Laplacian is replaced by the fractional Laplacian $(-\Delta)^s$, it is shown in [10] for $s = 1/2$ and [8], [9] for $s \in (0,1)$ that a standing wave solution exists.

For the unbalanced equation, it was shown that both in the Laplacian case and the fractional Laplacian case, there will exists a unique pair (u, c) to (1.4.2) for all $0 < s < 1$. In the fractioanl Laplacian case, Gui and Zhao [20] applied the method of continuity to get the uniform bound of the speed in terms of potential.

In the combustion equation with the Laplacian, it is known that there exists a unique pair (u, c). As for the equation with the fractional Laplacian it has recently been discovered in [26] that when $\frac{1}{2} < s < 1$, there exists a unique pair (u, c).

J. Roquejoffre, A. Mellet and Y. Sire [26] studied the combustion model. They have shown that when $s \in (1/2, 1)$ and f satisfies (1.3.4), there exists unique (c, u) with $c > 0$ to (1.4.2).

Chapter 2

Traveling fronts to the reaction diffusion equations

2.1 Combustion and Fisher-KPP models

By a compactness argument, we know that if (1.4.2) has a solution $u(x)$ then

$$\lim_{|x|\to\infty} u'(x) = 0 \quad \text{and} \quad f(0) = f(1) = 0 \tag{2.1.1}$$

Multiply $u'(x)$ on both sides in (1.4.2) and integrate over $\mathbb{R}$, we can get the Hamiltonian identity as in [20]:

$$c \int_{\mathbb{R}} |u'(x)|^2 \, dx = \int_0^1 f(u) \, du \tag{2.1.2}$$

The nonlinear reaction term we consider here is the following.

There exists some $\theta \in (0,1)$ such that $f \in C^1(\mathbb{R})$ satisfies

$$\int_0^1 f(u)\, du > 0, \qquad \text{and} \qquad f'(u) \geq 0, \ \ \forall u \in (0, \theta]. \tag{2.1.3}$$

In this section, we assume $0 < s \leq 1/2$ and $f \in C^1(\mathbb{R})$ satisfies condition (2.1.9). Assume that (c, u) is a solution to (1.4.2). By (2.1.2) and (2.1.9), we have $c > 0$.

Let $\overline{u}$ be the s-harmonic extension of u on $\mathbb{R}^2_+$, that is,

$$\overline{u}(x,y) = P_y * u(x), \quad \forall (x,y) \in \mathbb{R}^2_+. \tag{2.1.4}$$

with

$$P_y(x) = \frac{a_s y^{2s}}{[y^2 + x^2]^{\frac{1+2s}{2}}}, \ \ \forall (x,y) \in \mathbb{R}^2_+ \quad \text{and} \quad a_s = \frac{\Gamma\left(\frac{1+2s}{2}\right)}{\pi^{\frac{1}{2}}\Gamma(s)}. \tag{2.1.5}$$

Let $v(x,y) = \overline{u}_x(x,y) = P_y * u'(x)$ for all $(x,y) \in \mathbb{R}^2_+$, L. Caffarelli and L. Silvestre[11] proved that v satisfies

$$\begin{cases} \operatorname{div}[y^{1-2s}\nabla v(x,y)] = 0, \quad \forall (x,y) \in \mathbb{R}^2_+, \\ \lim\limits_{y \searrow 0} -d_s y^{1-2s} v_y(x,y) = (-\Delta)^s u'(x), \quad \forall x \in \mathbb{R}, \\ v(x,0) = u'(x), \quad \forall x \in \mathbb{R}. \end{cases}$$

where $d_s = \dfrac{2^{1-2s}\Gamma(1-s)}{\Gamma(s)}$.

By the standard maximal principle arguments, it is easy to see that $u'(x) > 0$ for all $x \in \mathbb{R}$ and $\lim\limits_{|x| \to \infty} u'(x) = 0$ (see, e.g., [26] and [20]). Then we know that

$$v(x,y) > 0, \ \ \forall (x,y) \in \overline{\mathbb{R}^2_+}, \qquad \text{and} \qquad \lim_{|(x,y)| \to \infty} v(x,y) = 0.$$

By (1.4.2), without loss of generality, we can assume $u(-1) = \theta$. Since $u'(x) > 0$ for all $x \in \mathbb{R}$, we have $f'(u(x)) \geq 0$ for all $x \leq -1$. In summary, we know that v satisfies

$$\begin{cases} \operatorname{div}[y^{1-2s}\nabla v(x,y)] = 0, \quad \forall (x,y) \in \mathbb{R}^2_+, \\ \lim\limits_{y\searrow 0} -d_s y^{1-2s} v_y(x,y) + cv_x(x,0) = f'(u(x))u'(x) \geq 0, \quad \forall x \leq -1, \\ v(x,y) > 0, \ \ \forall (x,y) \in \overline{\mathbb{R}^2_+} \qquad \text{and} \qquad \lim\limits_{|(x,y)|\to\infty} v(x,y) = 0. \end{cases} \tag{2.1.6}$$

Define the auxiliary function

$$\varphi(x,y) = \frac{y^{2s}}{[x^2+y^2]^{\frac{1+2s}{2}}} + \frac{sd_s}{c} \cdot \frac{1}{[x^2+y^2]^{\frac{1}{2}}} \quad \forall x \leq -1, \ y \geq 0.$$

Direct computations tell us that for all $x \leq -1$ and all $y \geq 0$, we have

$$\frac{sd_s}{c} \cdot \frac{1}{|(x,y)|} \leq \varphi(x,y) \ \ \leq \ \ \left(1 + \frac{sd_s}{c}\right) \cdot \frac{1}{|(x,y)|}$$

Also we will be able to get the following estimates of the auxiliary function.

$$\begin{aligned} \operatorname{div}\,[y^{1-2s}\nabla\varphi(x,y)] &= \frac{2s^2 d_s}{c} \cdot \frac{y^{1-2s}}{[x^2+y^2]^{\frac{3}{2}}} \geq 0, \\ \lim_{y\searrow 0}\ -d_s y^{1-2s}\varphi_y(x,y) &= d_s \lim_{y\searrow 0} \left[\frac{y^2 - 2sx^2}{[x^2+y^2]^{\frac{3}{2}+s}} + \frac{sd_s}{c} \cdot \frac{y^{2-2s}}{[x^2+y^2]^{\frac{3}{2}}} \right] = -\frac{2sd_s}{|x|^{1+2s}}, \\ \varphi_x(x,0) &= \frac{sd_s}{c} \cdot \frac{1}{|x|^2}. \end{aligned}$$

Since $0 < s \leq \frac{1}{2}$, we have $\frac{1}{|x|^2} \leq \frac{1}{|x|^{1+2s}}$ for all $x \leq -1$. Hence for all $x \leq -1$, we have

$$\begin{aligned}
\lim_{y\searrow 0} -d_s y^{1-2s}\varphi_y(x,y)+c\varphi_x(x,0) &= -\frac{2sd_s}{|x|^{1+2s}}+\frac{sd_s}{|x|^2}\\
&\le -\frac{2sd_s}{|x|^{1+2s}}+\frac{sd_s}{|x|^{1+2s}}\\
&= -\frac{sd_s}{|x|^{1+2s}}\\
&< 0.
\end{aligned}$$

For any $\delta>0$, let $w_\delta(x,y)=v(x,y)-\delta\varphi(x,y)$ for all $x\le -1$ and all $y\ge 0$, then w_δ satisfies

$$\begin{cases}
\operatorname{div}[y^{1-2s}\nabla w_\delta(x,y)]\le 0, \quad \forall x\le -1,\ y>0,\\
\lim_{y\searrow 0} -d_s y^{1-2s}D_y w_\delta(x,y)+cD_x w_\delta(x,0)\ge 0, \quad \forall x\le -1,\\
\lim_{|(x,y)|\to\infty} w_\delta(x,y)=0.
\end{cases} \tag{2.1.7}$$

Lemma 2.1.1. *There exists some $\delta_0>0$ such that $w_{\delta_0}(-1,y)\ge 0$ for all $y\ge 0$.*

Proof. First we see that

$$\lim_{y\to\infty}\frac{\varphi(-1,y)}{\frac{y^{2s}}{[1+y^2]^{\frac{1+2s}{2}}}}=\lim_{y\to\infty}\frac{\frac{y^{2s}}{[1+y^2]^{\frac{1+2s}{2}}}+\frac{sd_s}{c}\cdot\frac{1}{[1+y^2]^{\frac{1}{2}}}}{\frac{y^{2s}}{[1+y^2]^{\frac{1+2s}{2}}}}=1+\frac{sd_s}{c}<\infty.$$

Since $u'(x)>0$ for all $x\in\mathbb{R}$, then $u(0)>u(-1)$, which implies that there exists some constant $B_1>0$ such that

$$a_s[u(0)-u(-1)]\cdot\frac{y^{2s}}{[1+y^2]^{\frac{1+2s}{2}}}\ge B_1\varphi(-1,y), \quad \forall y\ge 1. \tag{2.1.8}$$

Since $v(x,y) = P_y * u'(x)$ for all $(x,y) \in \mathbb{R}^2_+$, by (2.1.5), for all $y \geq 1$ we have

$$
\begin{aligned}
v(-1,y) &= \int_{\mathbb{R}} \frac{a_s y^{2s}}{[(-1-x)^2+y^2]^{\frac{1+2s}{2}}} \cdot u'(x)\, dx \\
&\geq \frac{a_s y^{2s}}{[1+y^2]^{\frac{1+2s}{2}}} \int_{-1}^{0} u'(x)\, dx \\
&= a_s[u(0)-u(-1)] \cdot \frac{y^{2s}}{[1+y^2]^{\frac{1+2s}{2}}} \\
&\geq B_1 \varphi(-1,y).
\end{aligned}
$$

On the other hand, since $v(x,y) > 0$ for all $(x,y) \in \overline{\mathbb{R}^2_+}$, so there exists some $B_2 > 0$ such that

$$\inf_{0 \leq y \leq 1} v(-1,y) \geq B_2 \cdot \sup_{0 \leq y \leq 1} \varphi(-1,y).$$

Let $\delta_0 = \min\{B_1, B_2\} > 0$, we know that

$$w_{\delta_0}(-1,y) \geq 0, \quad \forall y \geq 0.$$

$\square$

Lemma 2.1.2. *For the above δ_0 in Lemma 2.1.1, there holds that*

$$w_{\delta_0}(x,y) \geq 0, \quad \forall x \leq -1,\ y \geq 0.$$

Proof. Assume $w_{\delta_0}(x_0,y_0) < 0$ for some $x_0 \leq -1$ and some $y_0 \geq 0$. Since $w_{\delta_0}(x,y) \to 0$, as $|(x,y)| \to \infty$, by Lemma 2.1.1, we know that there exists some $x_1 < -1$ and some $y_1 \geq 0$ such that

$$w_{\delta_0}(x_1,y_1) = \inf_{x \leq -1,\ y \geq 0} w_{\delta_0}(x,y) < 0.$$

By the strong maximum principle for uniformly elliptic equations, we know that $y_1 = 0$. Applying Hopf lemma as in [8], we have

$$\lim_{y \searrow 0} -d_s y^{1-2s} D_y w_{\delta_0}(x_1, y) < 0.$$

Since x_1 is an interior minimum of $w_{\delta_0}(x, 0)$ in $x < -1$, then we have

$$D_x w_{\delta_0}(x_1, 0) = 0$$

. By (2.1.7), we get

$$\lim_{y \searrow 0} -d_s y^{1-2s} D_y w_{\delta_0}(x_1, y) = \lim_{y \searrow 0} -d_s y^{1-2s} D_y w_{\delta_0}(x_1, y) + c D_x w_{\delta_0}(x_1, 0) \geq 0.$$

We get a contradiction. Therefore

$$w_{\delta_0}(x, y) \geq 0, \quad \forall x \leq -1, \; y \geq 0.$$

$\square$

Now we provide the first theorem regarding to the existence of the traveling front of (1.4.2).

Theorem 2.1.3. *Suppose that there exists some $\theta \in (0, 1)$ such that $f \in C^1(\mathbb{R})$ satisfies*

$$\int_0^1 f(u)\ du > 0, \qquad \text{and} \qquad f'(u) \geq 0, \quad \forall u \in (0, \theta]. \tag{2.1.9}$$

Then there is no solution to (1.4.2) if $0 < s \leq \frac{1}{2}$.

This theorem applies to the combustion model. For the Fisher-KPP model, i.e, $f \in C^1(\mathbb{R})$ satisfies

$$f(u) > 0 = f(0) = f(1), \quad \forall u \in (0,1), \qquad f'(0) > 0, \qquad \text{and} \qquad f'(1) < 0, \tag{2.1.10}$$

Theorem 2.1.3 implies that if $0 < s \leq 1/2$, (1.4.2) has no solution.

Proof. Assume (c, u) is a solution to (1.4.2). By Lemma 2.1.5, we know that

$$w_{\delta_0}(x, 0) \geq 0, \quad \forall x \leq -1.$$

Since $\varphi(x, 0) \geq \dfrac{sd_s}{c} \cdot \dfrac{1}{|x|}$ for all $x \leq -1$, we know that

$$u'(x) \geq \frac{\delta_0 s d_s}{c} \cdot \frac{1}{|x|}, \quad \forall x \leq -1.$$

On the other hand, we know that $\displaystyle\int_{\mathbb{R}} u'(x)\ dx = 1$. This is a contradiction which implies that there is no solution to (1.4.2).

$\square$

2.2 Generalized Fisher-KPP model

In this section, we assume there exists some $\theta \in (0,1)$, $0 < p < \infty$, $A_1 > 0$ and $A_2 > 0$ such that

$$\begin{cases} f(u) > 0 = f(0) = f(1), \quad \forall u \in (0,1), \\ A_1 u^p \le f(u) \le A_2 u^p, \quad \forall u \in [0,\theta], \\ f'(u) \ge A_1 u^{p-1}, \quad \forall u \in (0,\theta). \end{cases} \tag{2.2.1}$$

One example for (2.2.1) is the following:

$$f(u) = u^p(1-u), \quad \forall u \in \mathbb{R},$$

where $p > 0$ is the reaction power.

Our goal is to find the critical exponent $s = s(p)$ such that a solution of (1.4.2) exists if and only if $1 > s \ge s(p)$. In this section, we provide the proof of nonexistence of solutions for (1.4.2) when $s < s(p)$ by studying the asymptotics of solutions related to (1.4.2). By Theorem 2.1.3, it is readily seen that the solution to (1.4.2) does not exist when $0 < s \le 1/2$. Later, we shall discuss the existence of solutions to (1.4.2) by a similar argument as in [26].

The following lemma is important in the proof of nonexistence, and has already been proven in [26]. For completeness, we list the proof here.

Lemma 2.2.1. *Let $\frac{1}{2} < s < 1$ and $u \in C^2(\mathbb{R})$ such that $\lim_{|x|\to\infty} u'(x) = 0$ and $\lim_{x\to\pm\infty} u(x) = L^\pm$ for some $L^-, L^+ \in \mathbb{R}$, then we have*

$$\lim_{R\to\infty} \int_{-R}^{R} (-\Delta)^s u(y)\, dy = 0.$$

Proof. For any $R > 0$, we have

$$\begin{aligned}
\int_{-R}^{R} (-\Delta)^s u(y)\ dy &= C_{1,s} \left[\int_{-R}^{R} \int_{|w|\geq 1} \frac{u(y) - u(y+w)}{|w|^{1+2s}}\ dwdy \right. \\
&\qquad \left. + \int_{-R}^{R} (\text{P.V.}) \int_{|w|<1} \frac{u(y) - u(y+w)}{|w|^{1+2s}}\ dwdy \right] \\
&= C_{1,s} \left[\int_{-R}^{R} \int_{|w|\geq 1} \frac{u(y) - u(y+w)}{|w|^{1+2s}}\ dwdy \right. \\
&\qquad \left. - \int_{-R}^{R} \int_{|w|<1} \frac{u(y+w) - u(y) - u'(y)w}{|w|^{1+2s}}\ dwdy \right].
\end{aligned}$$

For $\int_{-R}^{R} \int_{|w|\geq 1} \frac{u(y) - u(y+w)}{|w|^{1+2s}}\ dwdy$, since $\frac{1}{2} < s$, by Fubini-Tonelli's theorem and the dominated convergence theorem, we know that

$$\begin{aligned}
\int_{-R}^{R} \int_{|w|\geq 1} \frac{u(y) - u(y+w)}{|w|^{1+2s}}\ dwdy &= -\int_{-R}^{R} \int_{|w|\geq 1} \int_0^1 \frac{u'(y+tw)\cdot w}{|w|^{1+2s}}\ dtdwdy \\
&= -\int_{|w|\geq 1} \frac{w}{|w|^{1+2s}} \int_0^1 \int_{-R}^{R} u'(y+tw)\ dydtdw \\
&= -\int_{|w|\geq 1} \frac{w}{|w|^{1+2s}} \int_0^1 [u(R+tw) \\
&\qquad -u(-R+tw)]\ dtdw \\
&\to \int_{|w|\geq 1} \frac{w}{|w|^{1+2s}} \cdot (L^- - L^+)\ dtdw = 0, \\
&\qquad \text{as } R \to \infty
\end{aligned}$$

For $\int_{-R}^{R} \int_{|w|<1} \frac{u(y+w) - u(y) - u'(y)w}{|w|^{1+2s}}\ dwdy$, since $s < 1$, by Fubini-Tonelli's theorem and the dominated convergence theorem, we know that

$$\begin{aligned}
&\int_{-R}^{R}\int_{|w|<1}\frac{u(y+w)-u(y)-u'(y)w}{|w|^{1+2s}}\,dwdy\\
=\;&\int_{-R}^{R}\int_{|w|<1}\int_0^1\int_0^1\frac{1}{|w|^{2s-1}}\cdot u''(y+rtw)\,drdtdwdy\\
=\;&\int_{|w|\le 1}\frac{1}{|w|^{2s-1}}\int_0^1\int_0^1\int_{-R}^{R}u''(y+rtw)\,dydrdtdw\\
=\;&\int_{|w|\le 1}\frac{1}{|w|^{2s-1}}\int_0^1\int_0^1[u'(R+rtw)-u'(-R+rtw)]\,drdtdw\\
\to\;&0,\quad \text{as } R\to\infty.
\end{aligned}$$

Therefore, we can conclude that $\displaystyle\int_{-R}^{R}(-\Delta)^s u(y)\,dy \to 0$, as $R\to\infty$.

$\square$

Remark 2.2.2. *If (c,u) is a solution to (1.4.2), since $u'\in L^1(\mathbb{R})$, by Lemma 2.2.1 and $f(u)\ge 0$ for all $u\in[0,1]$ we know that $f(u)\in L^1(\mathbb{R})$. In particular, if we know that there exists some constants $C>0$ and $r>0$ such that*

$$u'(x)\ge \frac{C}{|x|^r},\quad \forall x\le -1,$$

then we have $r>1$ by the integrability of u'. On the other hand, by (2.2.1), we know that $f(u(x))\ge A\left(\frac{C}{r-1}\cdot\frac{1}{|x|^{r-1}}\right)^p$ for all $x\le -1$. Hence it necessarily holds that $(r-1)p>1$, i.e., $r>\frac{p+1}{p}$.

In the following, we assume that (c,u) is a solution to (1.4.2) with $c>0$ and $u(-1)=\theta$. Let $\overline{u}$ be the s-harmonic extension of u on $\mathbb{R}^2_+$ and $v(x,y)=\overline{u}_x(x,y)=P_y*u'(x)$ for all $(x,y)\in\mathbb{R}^2_+$, by the same discussion as in section **??**, we know that

v satisfies

$$\begin{cases} \operatorname{div}[y^{1-2s}\nabla v(x,y)] = 0, \quad \forall (x,y) \in \mathbb{R}^2_+, \\ \lim\limits_{y \searrow 0} -d_s y^{1-2s} v_y(x,y) + c v_x(x,0) = f'(u(x))u'(x), \quad \forall x \in \mathbb{R}, \\ v(x,y) > 0, \ \ \forall (x,y) \in \overline{\mathbb{R}^2_+}, \qquad \text{and} \qquad \lim\limits_{|(x,y)| \to \infty} v(x,y) = 0. \end{cases} \tag{2.2.2}$$

For any $\alpha \in [1, 2s]$ and $\beta > 0$, we consider the axillary functions

$$\varphi_{\alpha,\beta}(x,y) = \frac{y^{2s}}{[x^2+y^2]^{\frac{1+2s}{2}}} + \frac{2\beta s d_s}{\alpha c} \cdot \frac{1}{[x^2+y^2]^{\frac{\alpha}{2}}}, \qquad \forall x \le -1, \ y \ge 0.$$

By direct computations, for all $x \le -1$ and all $y \ge 0$ we know that

$$\begin{aligned} \frac{2\beta s d_s}{2sc} \cdot \frac{1}{[x^2+y^2]^{\frac{\alpha}{2}}} &\le \varphi_{\alpha,\beta}(x,y) \le \left(1 + \frac{2\beta s d_s}{\alpha c}\right) \cdot \frac{1}{|(x,y)|}, \\ \operatorname{div}\left[y^{1-2s}\nabla \varphi_{\alpha,\beta}(x,y)\right] &= \frac{2\beta s d_s}{c} \cdot \frac{(2s-1+\alpha)y^{1-2s}}{[x^2+y^2]^{\frac{\alpha+2}{2}}} \ge 0, \\ \lim_{y \searrow 0} -d_s y^{1-2s} D_y \varphi_{\alpha,\beta}(x,y) &= d_s \lim_{y \searrow 0} \left[\frac{y^2 - 2sx^2}{[x^2+y^2]^{\frac{3}{2}+s}} + \frac{2\beta s d_s}{c} \cdot \frac{y^{2-2s}}{[x^2+y^2]^{\frac{\alpha+2}{2}}} \right] \\ &= -\frac{2sd_s}{|x|^{1+2s}}, \end{aligned}$$

Also we get

$$D_x \varphi_{\alpha,\beta}(x,0) = \frac{2\beta s d_s}{c} \cdot \frac{1}{|x|^{1+\alpha}}.$$

Hence for all $x \le -1$, we have

$$\lim_{y \searrow 0} -d_s y^{1-2s} D_y \varphi_{\alpha,\beta}(x,y) + c D_x \varphi_{\alpha,\beta}(x,0) \ = \ -\frac{2sd_s}{|x|^{1+2s}} + \frac{2\beta s d_s}{|x|^{1+\alpha}}.$$

For any $\delta \in (0,1)$, let

$$w_{\delta,\alpha,\beta}(x,y) = v(x,y) - \delta\varphi_{\alpha,\beta}(x,y), \quad \forall x \le -1,\ y \ge 0.$$

Then $w_{\delta,\alpha,\beta}$ satisfies

$$\begin{cases} \operatorname{div}[y^{1-2s}\nabla w_{\delta,\alpha,\beta}(x,y)] \le 0, \quad \forall x \le -1,\ y > 0, \\ \lim\limits_{y\searrow 0} -d_s y^{1-2s} D_y w_{\delta,\alpha,\beta}(x,y) + cD_x w_{\delta,\alpha,\beta}(x,0) = \\ \qquad f'(u(x))u'(x) - \frac{2\delta\beta s d_s}{|x|^{1+\alpha}} + \frac{2\delta s d_s}{|x|^{1+2s}}, \quad \forall x \le -1, \\ \lim\limits_{|(x,y)|\to\infty} w_{\delta,\alpha,\beta}(x,y) = 0. \end{cases} \tag{2.2.3}$$

Lemma 2.2.3. *For any fixed $\alpha \in [1,2s]$ and $\beta > 0$, for all $\delta \in (0,1]$, if we have*

$$f'(u(x))u'(x) - \frac{2\delta\beta s d_s}{|x|^{1+\alpha}} + \frac{2\delta s d_s}{|x|^{1+2s}} \ge 0, \quad \forall x \le -1,$$

then there exists some constant $C > 0$ such that

$$u'(x) \ge \frac{C}{|x|^{\alpha}}, \quad \text{and} \quad u(x) \ge \frac{C}{|x|^{\alpha-1}}, \qquad \forall x \le -1.$$

Proof. Since $\alpha \ge 1$, we know that $\dfrac{1}{[1+y^2]^{\frac{\alpha}{2}}} \le \dfrac{1}{[1+y^2]^{\frac{1}{2}}}$ for all $y \ge 0$. By taking the limit of the ratio, one can get

$$\lim_{y\to\infty} \frac{\varphi_{\alpha,\beta}(-1,y)}{\frac{y^{2s}}{[1+y^2]^{\frac{1+2s}{2}}}} \le \lim_{y\to\infty} \frac{\frac{y^{2s}}{[1+y^2]^{\frac{1+2s}{2}}} + \frac{2\alpha\beta s d_s}{c}\cdot\frac{1}{[1+y^2]^{\frac{1}{2}}}}{\frac{y^{2s}}{[1+y^2]^{\frac{1+2s}{2}}}} = 1 + \frac{2\alpha\beta s d_s}{c} > 0.$$

By the same arguments as in Lemma 2.1.1 and Lemma 2.1.2, we know that there exists some small $\delta_0 > 0$ such that

$$w_{\delta_0,\alpha,\beta}(x,y) \geq 0, \quad \forall x \leq -1,\ y \geq 0.$$

Since $\varphi_{\alpha,\beta}(x,0) \geq \frac{2\beta s d_s}{c} \cdot \frac{1}{|x|^\alpha}$ for all $x \leq -1$, we have

$$u'(x) = v(x,0) \geq \frac{2\delta_0 \beta s d_s}{c} \cdot \frac{1}{|x|^\alpha}, \quad \forall x \leq -1.$$

□

Lemma 2.2.4 (Initial Asymptotic Rate)**.** *There exists some constant $C_0 > 0$ such that*

$$u'(x) \geq \frac{C_0}{|x|^{2s}}, \quad \text{and} \quad u(x) \geq \frac{C_0}{|x|^{2s-1}}, \qquad \forall x \leq -1.$$

Proof. Let $\alpha = 2s$ and $\beta = 1$ in Lemma 2.2.3. Observe that

$$\begin{aligned} f'(u(x))u'(x) - \frac{2\delta\beta s d_s}{|x|^{1+\alpha}} + \frac{2\delta s d_s}{|x|^{1+2s}} &= f'(u(x))u'(x) - \frac{2\delta s d_s}{|x|^{1+2s}} + \frac{2\delta s d_s}{|x|^{1+2s}} \\ &= f'(u(x))u'(x) \geq 0, \quad \forall x \leq -1. \end{aligned}$$

Then Lemma 2.2.3 leads to the conclusion. □

Remark 2.2.5. *Lemma 2.2.4 provides an alternative proof of Proposition 4.2 in [26].*

As an immediate consequence of Lemma 2.2.4 and Remark 2.2.2, we have the following

Theorem 2.2.6. *Let $\frac{1}{2} < s \leq \frac{p+1}{2p}$, then there is no solution to (1.4.2). In particular, for all $0 < p \leq 1$ and $\frac{1}{2} < s < 1$, there is no solution to (1.4.2).*

Lemma 2.2.7 (Asymptotic Rate Lifting)**.** *Let* $\frac{p+1}{2p} < s < 1$ *and* $r \in (\frac{p+1}{p}, 2s]$*, we assume there exists some constant* $B_0 > 0$ *such that*

$$u'(x) \geq \frac{B_0}{|x|^r}, \quad \text{and} \quad u(x) \geq \frac{B_0}{|x|^{r-1}}, \qquad \forall x \leq -1.$$

Let $\alpha \in [1, 2s]$ *be such that* $\alpha \geq p(r-1)$*, then there exists some constant* $C > 0$ *such that*

$$u'(x) \geq \frac{C}{|x|^\alpha}, \quad \text{and} \quad u(x) \geq \frac{C}{|x|^{\alpha-1}}, \qquad \forall x \leq -1.$$

Proof. By the assumption and (2.2.1), for all $\beta > 0$, all $\delta \in (0, 1]$ and all $x \leq -1$, we know that

$$\begin{aligned} f'(u(x))u'(x) - \frac{2\delta\beta s d_s}{|x|^{1+\alpha}} + \frac{2\delta s d_s}{|x|^{1+2s}} &\geq A_1|u(x)|^{p-1}u'(x) - \frac{2\delta\beta s d_s}{|x|^{1+\alpha}} \\ &\geq A_1\left(\frac{B_0}{|x|^{r-1}}\right)^{p-1} \cdot \frac{B_0}{|x|^r} - \frac{2\delta\beta s d_s}{|x|^{1+\alpha}} \\ &= \frac{A_1 B_0^p}{|x|^{r+(p-1)(r-1)}} - \frac{2\delta\beta s d_s}{|x|^{1+\alpha}} \\ &\geq \frac{A_1 B_0^p - 2\delta\beta s d_s}{|x|^{1+\alpha}}. \end{aligned}$$

Let $\beta = \dfrac{A_1 B_0^p}{2\delta s d_s} > 0$, by Lemma 2.2.3, we complete the proof. □

Remark 2.2.8. *If* $\frac{p+1}{p} < r < \frac{p}{p-1}$*, letting* $\alpha = p(r-1)$*, we know that*

$$1 < \alpha < \frac{r}{r-1}(r-1) = r.$$

We shall show the following theorem.

Theorem 2.2.9. *Let* $p > 1$ *and* $\frac{1}{2} < s < \min\left\{1, \frac{p}{2(p-1)}\right\}$*, then (1.4.2) has no solution.*

Proof. Assume (c, u) is a solution to (1.4.2). We have Claim I: Choose $r_0 = 2s$ and let $r_{k+1} = p(r_k - 1)$ for all $k \geq 0$, then it necessarily holds that $\frac{p}{p-1} > r_k > \frac{p+1}{p}$ and there exists some constant $B_k > 0$ such that

$$u'(x) \geq \frac{B_k}{|x|^{r_k}}, \quad \text{and} \quad u(x) \geq \frac{B_k}{|x|^{r_k - 1}}, \qquad \forall x \leq -1.$$

When $k = 0$, by the assumption $2s < \frac{p}{p-1}$, Lemma 2.2.4 and Remark 2.2.2, we know that the Claim I is true. Assume the Claim I holds for $k = n$, that is, $\frac{p}{p-1} > r_n > \frac{p+1}{p}$ and there exists some constant $B_n > 0$ such that

$$u'(x) \geq \frac{B_n}{|x|^{r_n}}, \quad \text{and} \quad u(x) \geq \frac{B_n}{|x|^{r_n - 1}}, \qquad \forall x \leq -1.$$

By Remark 2.2.8 and Lemma 2.2.7, we know that $1 < r_{n+1} < r_n < \frac{p}{p-1}$ and there exists some constant $B_{n+1} > 0$ such that

$$u'(x) \geq \frac{B_{n+1}}{|x|^{r_{n+1}}}, \quad \text{and} \quad u(x) \geq \frac{B_{n+1}}{|x|^{r_{n+1} - 1}}, \qquad \forall x \leq -1.$$

By Remark 2.2.2, it necessarily holds that $r_{n+1} > \frac{p+1}{p}$. Hence we know that the Claim I is true for $k = n + 1$. By induction, we can conclude that Claim I holds for all $k \geq 0$.

Claim II: It necessarily holds that

$$2s > 1 + \frac{1}{p} + \frac{1}{p^2} + \cdots + \frac{1}{p^n}, \quad \forall n \geq 1.$$

Since $r_{k+1} = p(r_k - 1)$ for all $k \geq 0$, we know that

$$r_k = 1 + \frac{r_{k+1}}{p}, \quad \forall k \geq 0$$

Hence we obtain

$$\begin{aligned} 2s = r_0 &= 1 + \frac{r_1}{p} \\ &= 1 + \frac{1}{p}\left(1 + \frac{r_2}{p}\right) \\ &= 1 + \frac{1}{p} + \frac{1}{p^2} + \cdots + \frac{1}{p^n} + \frac{r_{n+1}}{p^{n+1}} \\ &> 1 + \frac{1}{p} + \frac{1}{p^2} + \cdots + \frac{1}{p^n}, \quad \forall n \geq 1. \end{aligned}$$

Since $p > 1$, by taking $n \to \infty$ in Claim II, we know that it necessarily holds that

$$2s \geq \frac{1}{1 - \frac{1}{p}} = \frac{p}{p-1},$$

which contradicts with our assumption. Therefore if $p > 1$ and $\frac{1}{2} < s < \min\left\{1, \frac{p}{2(p-1)}\right\}$, (1.4.2) has no solution. □

Note that $\frac{p}{2(p-1)} \geq 1$ if $1 < p \leq 2$, and $\frac{p}{2(p-1)} < 1$ if $2 < p$. Therefore, there is no solution to (1.4.2) for all $s \in (0, 1)$ if $p \leq 2$.

Now we are going to assume that f satisfies (2.2.1), $p > 2$ and $\frac{p}{2(p-1)} \leq s < 1$, we will show that a solution to (1.4.2) exists. A. Mellet, J. Roquejoffre and Y. Sire [26] have shown the existence of traveling fronts for the non local combustion model when $\frac{1}{2} < s < 1$. The proof for the generalized Fisher-KPP model follows a similar argument to that in [26]. For any $c \in \mathbb{R}$ and $b > 0$, we first consider the following truncated problem:

$$\begin{cases} (-\Delta)^s u(x) + cu'(x) = f(u(x)), \quad \forall x \in (-b, b), \\ u(x) = 0, \quad \forall x \leq -b, \\ u(x) = 1, \quad \forall x \geq b. \end{cases} \tag{2.2.4}$$

Proposition 2.2.10. *Assume* $s \geq \dfrac{p}{2(p-1)}$ *and* f *satisfies (2.2.1). Then there exists a constant* M *such that if* $b > M$ *the truncated problem 2.2.4 has a solution* (u_b, c_b)*. Furthermore, the following properties hold:*

1. *There exists* K *independent of* b *such that* $-K \leq c_b \leq K$*;*

2. u_b *is non-decreasing with respect to* x *and satisfies* $0 < u_b(x) < 1$ *for all* $x \in (-b, b)$*.*

To prove this Proposition, we need the construction of sub- and super-solutions. The construction is based on the following lemmas, same as in [26]. We would like to present the proof of the following second lemma, and especially elaborate on the sliding method mentioned in [26].

Lemma 2.2.11. *For any* $c \in \mathbb{R}$ *and* $b > 0$*, (2.2.4) has a solution* $u_{c,b}$ *such that* $0 \leq u_{c,b}(x) \leq 1$ *in* $\mathbb{R}$*,* $u_{c,b}$ *is non-decreasing in* $\mathbb{R}$ *and* $c \to u_{c,b}$ *is continuous.*

Proof. The proof is the same as the proof of Lemma 2.4 in [26]. □

Lemma 2.2.12. *There exists some constants* $M, K > 0$ *such that for all* $b > M$*, we have*

a. *If* $c > K$*, then* $u_{c,b}(0) < \theta$*;*

b. If $c<-K$, then $u_{c,b}(0)>\theta$.

Together with Lemma 2.2.11, Lemma 2.2.12 implies that there exists $c_b \in [-K,K]$ such that $u_{c,b}(0)=\theta$.

Proof. Consider the function

$$\varphi(x)=\begin{cases} \frac{1}{|x|^{2s-1}}, & \forall x\le -1,\\ 1, & \forall x>-1. \end{cases}$$

Since $2s>1$, by Lemma 2.2 in [26], we have

$$(-\Delta)^s\varphi(x)+c\varphi'(x)=-\frac{C_{1,s}}{2s|x|^{2s}}+\frac{c(2s-1)}{|x|^{2s}}+O\left(\frac{1}{|x|^{4s-1}}\right),\quad \text{as } x\to-\infty.$$

Moreover, by (2.2.1), we get

$$f(\varphi(x))\le A_2|\varphi(x)|^p\le \frac{A_2}{|x|^{(2s-1)p}},\quad \forall x\in\mathbb{R}.$$

Since $\frac{p}{p-1}\le 2s$, we have $(2s-1)p\ge 2s$, which implies that for all $c\ge \dfrac{C_{1,s}}{2s(2s-1)}+\dfrac{A_2+1}{2s-1}$, we have

$$(-\Delta)^s\varphi(x)+c\varphi'(x)-f(\varphi(x))\ge \frac{1}{|x|^{2s}}+O\left(\frac{1}{|x|^{4s-1}}\right),\quad \text{as } x\to-\infty.$$

Since $4s-1>2s$, we know that there exists some large $A>0$ which is independent of c such that for all $c\ge \dfrac{C_{1,s}}{2s(2s-1)}+\dfrac{A_2+1}{2s-1}$, we have

$$(-\Delta)^s\varphi(x)+c\varphi'(x)\ge f(\varphi(x)),\qquad x\le -A.$$

For $-A < x < -1$, we know that $(-\Delta)^s\varphi(x)$ is bounded, but $\varphi'(x) = \dfrac{2s-1}{|x|^{2s}} \geq \dfrac{2s-1}{A^{2s}}$, so there exists some $K > 0$ such that for all $c \geq K$, we have

$$(-\Delta)^s\varphi(x) + c\varphi'(x) \geq \sup_{x\in[-A,-1]} f(\varphi(x)).$$

Hence for all $c \geq K$, we have

$$(-\Delta)^s\varphi(x) + c\varphi'(x) \geq f(\varphi(x)), \quad \forall x \leq -1.$$

On the other hand, by the definition of $\varphi(x)$ and (2.1.1), we know that for all $x \geq -1$, $(-\Delta)^s\varphi(x) > 0$, $\varphi'(x) = 0$ and $f(\varphi(x)) = 0$. In summary, for all $c \geq K$, we have $\varphi(x)$ is a super-solution for (2.2.4). Now fix some large $M > 0$ such that $\varphi(-M) = \frac{1}{M^{2s-1}} < \theta$.

Claim: For all $c \geq K$ and all $b \geq M$, we have $u_{c,b}(x) \leq \varphi(x-M)$ for all $x \in \mathbb{R}$, in particular, $u_{c,b}(0) < \theta$.

Let $\phi(x) = \varphi(x-M)$, define

$$\Psi_t(x) = \phi(x+t) - u_c(x), \qquad x \in \mathbb{R}.$$

Let

$$\mathcal{O} = \{t \geq 0 : \Psi_t(x) = \phi(x+t) - u_c(x) \geq 0, \qquad x \in \mathbb{R}\},$$

then $\mathcal{O}$ is nonempty since $\{t \geq 2b\} \subset \mathcal{O}$. $\mathcal{O}$ is clearly closed. Take a convergent sequence $\{t_n\} \subset \mathcal{O}$, $t_n \to t$ as $n \to \infty$ then

$$\lim_{n\to\infty} \Psi_{t_n}(x) = \lim_{n\to\infty} \phi(x+t_n) - u_c(x) \geq 0, \qquad x \in \mathbb{R}$$

so $t \in \mathcal{O}$.

Next we show that for any $t \in \mathcal{O}$,

$$\Psi_t(x) = \phi(x+t) - u_c(x) > 0 \text{ for all } x \in (-b, b).$$

In fact, if there exists $x_0 \in (-b, b)$ such that $\Psi_t(x_0) = \phi(x_0+t) - u_c(x_0) = 0$, then

$$0 > (-\Delta)^s \Psi_t(x_0) + c\Psi_t'(x_0) \geq f(\phi(x_0+t)) - f(u_c(x_0)) = 0.$$

This is a contradiction.

It follows that $\mathcal{O}$ is open. Together with the fact that $\mathcal{O}$ is closed, we get $\mathcal{O} = [0, \infty)$. By the above sliding argument we know

$$u_c(0) \leq \varphi(-M) < \theta.$$

Similarly, for a lower bound we define $\varphi_1(x) = 1 - \varphi(-x)$. Then if $c \leq -K$, $x > 1$,

$$(-\Delta)^s \varphi_1(x) + c\varphi_1'(x) = -[(-\Delta)^s \varphi(-x) - c\varphi'(x)] \leq 0 \leq f(\varphi_1).$$

Moreover $\varphi_1(x) = 0$ for $x \leq 1$. Take M so that $\varphi_1(-M) = 1 - t_0$, then $\varphi_1(x) > \theta$ for $x \geq M$. Define $\varphi_{1,M}(x) = \varphi_1(x+M)$, then $\varphi_{1,M}$ is a sub-solution to 2.2.4. Therefore by the same argument as above $u_c(0) \geq \varphi_{1,M}(0) > \theta$ for $c < -K$. $\square$

Theorem 2.2.13. *Under the conditions of Proposition 2.2.10, there exists a subsequence $b_n \to \infty$ such that $u_{b_n} \to u_0$ and $c_{b_n} \to c_0$. Furthermore, $c_0 \in (0, K]$ and u_0 is a monotone increasing solution of (1.4.2).*

Proof. By Lemma 2.2.12, $c_b \in [-K, K]$ we have the elliptic estimate for u_b:

$$\|u_b\|_{C^{2,\alpha}} \le C$$

for some $\alpha \in (0,1)$. Thus there exists a subsequence $b_n \to \infty$ such that

$$\begin{aligned} c_n &:= c_{b_n} \to c_0 \in [-K, K] \\ u_n &:= u_{b_n} \to u_0, \quad \text{as } n \to \infty. \end{aligned}$$

Thus u_0 satisfies $(-\Delta)^s u_0 + c_0 u_0' = f(u_0)$. Also we know u_0 is monotone increasing, $u_0(0) = \theta$ and u_0 is bounded. By a compactness argument, there exist γ_0, γ_1 such that $\lim_{x\to-\infty} u_0(x) = \gamma_0$ and $\lim_{x\to\infty} u_0(x) = \gamma_1$ with

$$0 \le \gamma_0 \le \theta \le \gamma_1 \le 1.$$

We know both γ_0 and γ_1 satisfy $f(\gamma_0) = 0$ and $f(\gamma_1) = 0$ which implies $\gamma_0 = 0,\ \gamma_1 = 1$. Moreover, by integrating $(-\Delta)^s u_0 + c_0 u_0' = f(u_0)$ over $\mathbb{R}$, together with Lemma 2.2.1, we know

$$c_0 = \int_{\mathbb{R}} f(u_0(x))dx > 0.$$

$\square$

Chapter 3

Asymptotic Rates and Stability

3.1 Assymptotic rates at Infinity

In this chapter, we will study asymptotic behaviors of solutions to (1.4.2) when $x \to \pm\infty$. Let $f \in C^1(\mathbb{R})$ satisfy (2.2.1) and (c, u) be a solution to (1.4.2). First we investigate the asymptotic behavior of u when $x \to \infty$. Let $M = \|f\|_{C^1([0,1])} > 0$, by (2.2.2), we know that

$$\begin{cases} \operatorname{div}[y^{1-2s}\nabla v(x,y)] = 0, \quad \forall (x,y) \in \mathbb{R}^2_+, \\ \lim\limits_{y \searrow 0} -d_s y^{1-2s} v_y(x,y) + cv_x(x,0) + Mv(x,0) = [M + f'(u(x))]u'(x) \geq 0, \quad \forall x \in \mathbb{R}, \\ v(x,y) > 0, \ \ \forall (x,y) \in \overline{\mathbb{R}^2_+}, \qquad \text{and} \qquad \lim\limits_{|(x,y)| \to \infty} v(x,y) = 0. \end{cases} \tag{3.1.1}$$

We consider the axillary function

$$\varphi(x,y) = \frac{y^{2s}}{[x^2+y^2]^{\frac{1+2s}{2}}} + \frac{2sd_s}{M} \cdot \frac{1}{[x^2+y^2]^{\frac{1+2s}{2}}}, \qquad \forall x \geq 1, \ y \geq 0.$$

For all $x \geq 1$ and all $y \geq 0$, we can get the following estimates.

$$
\begin{aligned}
\frac{2sd_s}{M} \cdot \frac{1}{[x^2+y^2]^{\frac{1+2s}{2}}} &\leq \varphi(x,y) \leq \left(1 + \frac{2sd_s}{M}\right) \cdot \frac{1}{|(x,y)|}, \\
\text{div } [y^{1-2s}\nabla\varphi(x,y)] &= \frac{2sd_s}{M} \cdot \frac{(4s)(1+2s)y^{1-2s}}{[x^2+y^2]^{\frac{2s+3}{2}}} \geq 0, \\
\lim_{y\searrow 0} \; -d_s y^{1-2s}\varphi_y(x,y) &= d_s \lim_{y\searrow 0} \left[\frac{y^2 - 2sx^2}{[x^2+y^2]^{\frac{3}{2}+s}} + \frac{2sd_s}{M} \cdot \frac{y^{2-2s}}{[x^2+y^2]^{\frac{\alpha+2}{2}}}\right] \\
&= -\frac{2sd_s}{|x|^{1+2s}}, \\
D_x\varphi(x,0) &= -\frac{2sd_s}{M} \cdot \frac{2s}{|x|^{2+2s}}.
\end{aligned}
$$

Hence for all $x \geq 1$, we have

$$
\begin{aligned}
& \lim_{y\searrow 0} \; -d_s y^{1-2s}\varphi_y(x,y) + c\varphi_x(x,0) + M\varphi(x,0) \\
= \; & -\frac{2sd_s}{|x|^{1+2s}} - \frac{2csd_s}{M} \cdot \frac{2s}{|x|^{2+2s}} + M \cdot \frac{2sd_s}{M} \cdot \frac{1}{|x|^{1+2s}}, \\
= \; & -\frac{2csd_s}{M} \cdot \frac{2s}{|x|^{2+2s}} \leq 0.
\end{aligned}
$$

For any $\delta > 0$, let

$$
w_\delta(x,y) = v(x,y) - \delta\varphi(x,y), \quad \forall x \geq 1, \; y \geq 0.
$$

Then w_δ satisfies

$$
\begin{cases}
\text{div}[y^{1-2s}\nabla w_\delta(x,y)] \leq 0, \quad \forall x \geq 1, \; y > 0, \\
\lim\limits_{y\searrow 0} -d_s y^{1-2s} D_y w_\delta(x,y) + cD_x w_\delta(x,0) + M w_\delta(x,0) \geq 0, \quad \forall x \geq 1, \\
\lim\limits_{|(x,y)|\to\infty} \; w_\delta(x,y) = 0.
\end{cases} \tag{3.1.2}
$$

We have the following

Proposition 3.1.1. *There exists some constant $C > 0$ such that*

$$u'(x) \geq \frac{C}{|x|^{1+2s}}, \quad \forall x \geq 1.$$

Proof. By the same argument as in Lemma 2.1.2, we know that there is a positive constant δ_0 such that

$$v(x,y) \geq \delta_0 \varphi(x,y), \quad \forall x \geq 1, \ y \geq 0.$$

In particular, we know that

$$u'(x) = v(x,0) \geq \delta_0 \varphi(x,0) = \frac{2\delta_0 s d_s}{|x|^{1+2s}}, \quad \forall x \geq 0.$$

$\square$

Lemma 3.1.2. *Let $\beta > 0$, we consider the function*

$$\psi_\beta(x) = \begin{cases} \dfrac{1}{|x|^\beta}, & \forall x < -1, \\ 0, & \forall x \geq -1. \end{cases}$$

Then

a. If $0 < \beta < 1$, we have

$$(-\Delta)^s \psi_\beta(x) = -\frac{C_{1,s} \cdot B(2s+\beta, 1-\beta)}{x^{2s+\beta}} + o\left(\frac{1}{x^{2s+\beta}}\right), \quad \text{as } x \to \infty.$$

b. If $\beta > 1$, we have

$$(-\Delta)^s\psi_\beta(x) = -\frac{C_{1,s}}{\beta-1}\cdot\frac{1}{x^{1+2s}} + o\left(\frac{1}{x^{1+2s}}\right), \quad \text{as } x\to\infty.$$

c. If $\beta = 1$, we have

$$(-\Delta)^s\psi_1(x) = -\frac{C_{1,s}\ln x}{x^{2s+1}} + o\left(\frac{\ln x}{x^{2s+1}}\right), \quad \text{as } x\to\infty.$$

Proof. In fact, for all $x\geq 2$, by changing of variables, we know that

$$\begin{aligned}(-\Delta)^s\psi_\beta(x) &= C_{1,s}\left[\int_{-\infty}^{-x-1}\frac{\psi_\beta(x)-\psi_\beta(x+y)}{|y|^{1+2s}}\,dy + (\text{P.V.})\int_{-x-1}^{\infty}\frac{\psi_\beta(x)-\psi_\beta(x+y)}{|y|^{1+2s}}\,dy\right]\\ &= C_{1,s}\int_{-\infty}^{-x-1}\frac{-1}{|x+y|^\beta|y|^{1+2s}}\,dy = -\frac{C_{1,s}}{x^{2s+\beta}}\int_{-\infty}^{-1-\frac{1}{x}}\frac{1}{|z+1|^\beta|z|^{1+2s}}\,dz.\end{aligned}$$

a. When $0<\beta<1$, we have

$$\int_{-2}^{-1}\frac{1}{|z+1|^\beta}\,dz < \infty.$$

By the dominated convergence theorem, we know that

$$\int_{-\infty}^{-1-\frac{1}{x}}\frac{1}{|z+1|^\beta|z|^{1+2s}}\,dz \to \int_{-\infty}^{-1}\frac{1}{|z+1|^\beta|z|^{1+2s}}\,dz, \quad \text{as } x\to\infty.$$

On the other hand, we know that

$$\begin{aligned}\int_{-\infty}^{-1}\frac{1}{|z+1|^\beta|z|^{1+2s}}\,dz &= \int_0^1\frac{y^{1+2s}}{\left|-\frac{1}{y}+1\right|^\beta}\cdot\frac{1}{y^2}\,dy \quad (\text{by letting } z=-\frac{1}{y})\\ &= \int_0^1 y^{2s+\beta-1}(1-y)^{-\beta}\,dy = B(2s+\beta,1-\beta)>0.\end{aligned}$$

So we know that

$$(-\Delta)^s \psi_\beta(x) = -\frac{C_{1,s} \cdot B(2s+\beta, 1-\beta)}{x^{2s+\beta}} + o\left(\frac{1}{x^{2s+\beta}}\right), \quad \text{as } x \to \infty.$$

b. When $\beta > 1$, we know that $\int_{-2}^{-1} \frac{1}{|z+1|^\beta}\, dz = \infty$, which implies that

$$\int_{-\infty}^{-1-\frac{1}{x}} \frac{1}{|z+1|^\beta |z|^{1+2s}}\, dz \to \infty, \quad \text{as } x \to \infty.$$

By L'Hospital rule, we have

$$\lim_{x\to\infty} \frac{\int_{-\infty}^{-1-\frac{1}{x}} \frac{1}{|z+1|^\beta |z|^{1+2s}}\, dz}{x^{\beta-1}} = \lim_{x\to\infty} \frac{x^\beta \cdot \left|1+\frac{1}{x}\right|^{-1-2s} \cdot \frac{1}{x^2}}{(\beta-1)x^{\beta-2}} = \frac{1}{\beta-1}.$$

So we derive

$$(-\Delta)^s \psi_\beta(x) = -\frac{C_{1,s}}{\beta-1} \cdot \frac{1}{x^{1+2s}} + o\left(\frac{1}{x^{1+2s}}\right), \quad \text{as } x \to \infty.$$

c. When $\beta = 1$, we know that $\int_{-2}^{-1} \frac{1}{|z+1|}\, dz = \infty$, which implies that

$$\int_{-\infty}^{-1-\frac{1}{x}} \frac{1}{|z+1||z|^{1+2s}}\, dz \to \infty, \quad \text{as} x \to \infty.$$

By L'Hospital rule, we know that

$$\lim_{x\to\infty} \frac{\int_{-\infty}^{-1-\frac{1}{x}} \frac{1}{|z+1||z|^{1+2s}}\, dz}{\ln x} = \lim_{x\to\infty} \frac{|x| \cdot \left|1+\frac{1}{x}\right|^{-1-2s} \cdot \frac{1}{x^2}}{\frac{1}{x}} = 1.$$

Therefore we have

$$(-\Delta)^s \psi_1(x) = -\frac{C_{1,s} \ln x}{x^{2s+1}} + o\left(\frac{\ln x}{x^{2s+1}}\right), \quad \text{as } x \to \infty.$$

$\square$

Lemma 3.1.3. *Let $\beta > 0$, $\psi_\beta(x)$ be defined as in Lemma 3.1.2, then we have the following estimates:*

a. If $0 < \beta < 1$, there holds that

$$(-\Delta)^s \psi_\beta(x) = -\frac{C_{1,s} \cdot A(s,\beta)}{|x|^{2s+\beta}} + o\left(\frac{1}{|x|^{2s+\beta}}\right), \quad \text{as } x \to -\infty;$$

where

$$A(s,\beta) = \int_1^\infty \frac{1}{|z|^{1+2s}|z+1|^\beta}\, dz - \frac{1}{s} + \int_0^1 \frac{\frac{1}{|z-1|^\beta} + \frac{1}{|z+1|^\beta} - 2}{|z|^{1+2s}}\, dz;$$

b. If $\beta > 1$, we have

$$(-\Delta)^s \psi_\beta(x) = -\frac{C_{1,s}}{\beta - 1} \cdot \frac{1}{|x|^{1+2s}} + o\left(\frac{1}{|x|^{2s+1}}\right), \quad \text{as } x \to -\infty;$$

c. If $\beta = 1$, we have

$$(-\Delta)^s \psi_1(x) = -\frac{C_{1,s} \ln |x|}{|x|^{2s+1}} + o\left(\frac{\ln |x|}{|x|^{2s+1}}\right), \quad \text{as } x \to -\infty.$$

Proof. For all $x<-2$, we know that $x+1<-x-1$ and

$$\begin{aligned}
(-\Delta)^s\psi_\beta(x) &= -\frac{C_{1,s}}{2}\int_{\mathbb{R}}\frac{\psi_\beta(x+y)+\psi_\beta(x-y)-2\psi_\beta(x)}{|y|^{1+2s}}\,dy\\
&= -\frac{C_{1,s}}{2}\left[\int_{-\infty}^{x+1}\frac{\frac{1}{|x+y|^\beta}-\frac{2}{|x|^\beta}}{|y|^{1+2s}}\,dy+\int_{x+1}^{-x-1}\frac{\frac{1}{|x+y|^\beta}+\frac{1}{|x-y|^\beta}-\frac{2}{|x|^\beta}}{|y|^{1+2s}}\,dy\right.\\
&\qquad\left.+\int_{-x-1}^{\infty}\frac{\frac{1}{|x-y|^\beta}-\frac{2}{|x|^\beta}}{|y|^{1+2s}}\,dy\right]\\
&= -\frac{C_{1,s}}{2|x|^{2s+\beta}}\left[\int_{-\infty}^{-1-\frac{1}{x}}\frac{\frac{1}{|z-1|^\beta}-2}{|z|^{1+2s}}\,dz+\int_{-1-\frac{1}{x}}^{1+\frac{1}{x}}\frac{\frac{1}{|z-1|^\beta}+\frac{1}{|z+1|^\beta}-2}{|z|^{1+2s}}\,dz\right.\\
&\qquad\left.+\int_{1+\frac{1}{x}}^{\infty}\frac{\frac{1}{|z+1|^\beta}-2}{|z|^{1+2s}}\,dz\right]\quad \text{Let } y=-xz\\
&= -\frac{C_{1,s}}{|x|^{2s+\beta}}\left[\int_{1+\frac{1}{x}}^{\infty}\frac{\frac{1}{|z+1|^\beta}-2}{|z|^{1+2s}}\,dz+\int_{0}^{1+\frac{1}{x}}\frac{\frac{1}{|z-1|^\beta}+\frac{1}{|z+1|^\beta}-2}{|z|^{1+2s}}\,dz\right]
\end{aligned}$$

For the first term inside the bracket, we know that

$$\lim_{x\to-\infty}\int_{1+\frac{1}{x}}^{\infty}\frac{\frac{1}{|z+1|^\beta}-2}{|z|^{1+2s}}\,dz = \int_1^\infty\frac{1}{|z|^{1+2s}|z+1|^\beta}\,dz-\frac{1}{s}.$$

a. Since $\beta\in(0,1)$, we know that $\int_0^1\frac{1}{|z-1|^\beta}\,dz<\infty$, which implies that

$$\lim_{x\to-\infty}\int_0^{1+\frac{1}{x}}\frac{\frac{1}{|z-1|^\beta}+\frac{1}{|z+1|^\beta}-2}{|z|^{1+2s}}\,dz=\int_0^1\frac{\frac{1}{|z-1|^\beta}+\frac{1}{|z+1|^\beta}-2}{|z|^{1+2s}}\,dz.$$

Let

$$A(s,\beta)=\int_1^\infty\frac{1}{|z|^{1+2s}|z+1|^\beta}\,dz-\frac{1}{s}+\int_0^1\frac{\frac{1}{|z-1|^\beta}+\frac{1}{|z+1|^\beta}-2}{|z|^{1+2s}}\,dz,$$

then we have

$$(-\Delta)^s\psi_\beta(x) = -\frac{C_{1,s}\cdot A(s,\beta)}{|x|^{2s+\beta}} + o\left(\frac{1}{|x|^{2s+\beta}}\right), \quad \text{as } x \to -\infty.$$

b. Since $\beta > 1$, we know that $\int_0^1 \frac{1}{|z-1|^\beta}\ dz = \infty$, which implies that

$$\int_0^{1+\frac{1}{x}} \frac{\frac{1}{|z-1|^\beta} + \frac{1}{|z+1|^\beta} - 2}{|z|^{1+2s}}\ dz \to \infty, \quad \text{as } x \to -\infty.$$

By L'Hospital rule, we know that

$$\lim_{x\to-\infty} \frac{\int_0^{1+\frac{1}{x}} \frac{\frac{1}{|z-1|^\beta} + \frac{1}{|z+1|^\beta} - 2}{|z|^{1+2s}}\ dz}{(-x)^{\beta-1}} = \lim_{x\to-\infty} \frac{\left[|x|^\beta + \frac{1}{2^\beta} - 2\right]\cdot\left(-\frac{1}{x^2}\right)}{-(\beta-1)(-x)^{\beta-2}} = \frac{1}{\beta-1}.$$

Hence we have

$$(-\Delta)^s\psi_\beta(x) = -\frac{C_{1,s}}{\beta-1}\cdot\frac{1}{|x|^{2s+1}} + o\left(\frac{1}{|x|^{1+2s}}\right), \quad \text{as } x \to -\infty.$$

c. Since $\beta = 1$, we know that $\int_0^1 \frac{1}{|z-1|}\ dz = \infty$, which implies that

$$\int_0^{1+\frac{1}{x}} \frac{\frac{1}{|z-1|^\beta} + \frac{1}{|z+1|^\beta} - 2}{|z|^{1+2s}}\ dz \to \infty, \quad \text{as } x \to -\infty.$$

By L'Hospital rule, we know that

$$\lim_{x\to-\infty} \frac{\int_0^{1+\frac{1}{x}} \frac{\frac{1}{|z-1|} + \frac{1}{|z+1|} - 2}{|z|^{1+2s}}\ dz}{\ln(-x)} = \lim_{x\to-\infty} \frac{\left[|x| + \frac{1}{2} - 2\right]\cdot\left(-\frac{1}{x^2}\right)}{\frac{1}{x}} = 1.$$

Hence we have

$$(-\Delta)^s \psi_1(x) = -\frac{C_{1,s} \ln |x|}{|x|^{2s+1}} + o\left(\frac{\ln |x|}{|x|^{2s+1}}\right), \quad \text{as } x \to -\infty.$$

□

Lemma 3.1.4. *Consider the function*

$$\phi(x) = \begin{cases} 1, & \forall x \leq -1, \\ 0, & \forall x > -1. \end{cases}$$

Then

$$(-\Delta)^s \phi(x) = -\frac{C_{1,s}}{2s} \cdot \frac{1}{|x|^{2s}} + o\left(\frac{1}{|x|^{2s}}\right), \quad \text{as } x \to \infty, \quad \textit{and}$$

$$(-\Delta)^s \phi(x) = \frac{C_{1,s}}{2s} \cdot \frac{1}{|x|^{2s}} + o\left(\frac{1}{|x|^{2s}}\right), \quad \text{as } x \to -\infty.$$

Proof. a. In fact, for all $x \geq 2$, we have

$$\begin{aligned}
(-\Delta)^s \phi(x) &= C_{1,s} \left[\int_{-\infty}^{-x-1} \frac{\phi(x) - \phi(x+y)}{|y|^{1+2s}}\, dy + (\text{P.V.}) \int_{-x-1}^{\infty} \frac{\phi(x) - \phi(x+y)}{|y|^{1+2s}}\, dy \right] \\
&= -C_{1,s} \int_{-\infty}^{-x-1} \frac{1}{|y|^{1+2s}}\, dy \\
&= -\frac{C_{1,s}}{2s} \cdot \frac{1}{|x+1|^{2s}} \\
&= -\frac{C_{1,s}}{2s} \cdot \frac{1}{|x|^{2s}} + o\left(\frac{1}{|x|^{2s}}\right), \quad \text{as } x \to \infty.
\end{aligned}$$

b. If $x \leq -2$, we have

$$
\begin{aligned}
(-\Delta)^s \phi(x) &= C_{1,s} \left[(\text{P.V.}) \int_{-\infty}^{-x-1} \frac{\phi(x) - \phi(x+y)}{|y|^{1+2s}}\, dy + \int_{-x-1}^{\infty} \frac{\phi(x) - \phi(x+y)}{|y|^{1+2s}}\, dy \right] \\
&= C_{1,s} \int_{-x-1}^{\infty} \frac{1}{|y|^{1+2s}}\, dy \\
&= \frac{C_{1,s}}{2s} \cdot \frac{1}{|x+1|^{2s}} \\
&= \frac{C_{1,s}}{2s} \cdot \frac{1}{|x|^{2s}} + o\left(\frac{1}{|x|^{2s}} \right), \quad \text{as } x \to -\infty.
\end{aligned}
$$

□

Below we show a form of the maximal principle which is a slight variation of those in [**?**] and [20].

Lemma 3.1.5 (The Maximum Principle)**.** *Let H be a nonempty open subset of $\mathbb{R}$, assume $d(x) \geq 0$ for all $x \in H$ and $w \in C^1(\overline{H})$ satisfies*

$$
\begin{cases}
(-\Delta)^s w(x) + c w'(x) + d(x) w(x) \geq 0, \quad \forall x \in H, \\
\lim\limits_{|x| \to \infty} w(x) = 0, \\
w(x) \geq 0, \quad \forall x \notin H.
\end{cases}
$$

Then $w(x) \geq 0$ for all x in $\mathbb{R}$.

Proof. Assume $w(x_0) < 0$ for some $x_0 \in \mathbb{R}$, since $w(x) \geq 0$ for all $x \notin H$, $\lim\limits_{|x| \to \infty} w(x) = 0$, and $w \in C^1(\overline{H})$, then there exists some $x_1 \in H$ such that

$$
w(x_1) = \inf_{x \in \mathbb{R}} w(x) < 0.
$$

Since x_1 is a global minimum of w in $\mathbb{R}$, $x_1 \in H$ and $w \in C^1(H)$, then

$$(-\Delta)^s w(x_1) < 0, \quad \text{and} \quad w'(x_1) = 0.$$

Since $d(x) \geq 0$ for all $x \in H$, and $x_1 \in H$, so we have

$$(-\Delta)^s w(x_1) + cw'(x_1) + d(x_1)w(x_1) < 0,$$

which contradicts with the assumption.

□

The following two propositions give suitable lower and upper bounds of the asymptotic decay rates of u' and $1-u$ at ∞, which are expected to be a power of $1+2s$ and $2s$, respectively.

Proposition 3.1.6. *Let $\frac{1}{2} < s < 1$ and (c, u) be a solution to (1.4.2) with $c > 0$. Assume that $f'(1) < 0$, then there exists some constant $C > 0$ such that*

$$u'(x) \leq \frac{C}{|x|^{2s}}, \quad \text{and} \quad 1 - u(x) \leq \frac{C}{|x|^{2s}}, \qquad x \geq 1.$$

Proof. Since $f'(1) < 0$, there exists some $m > 0$ and $\theta_0 \in (0,1)$ such that $f'(u) \leq -m$ for all $u \in [\theta_0, 1]$. Let $\epsilon > 0$ be such that $-\frac{C_{1,s}}{2s} + m\epsilon^{-2s} = \frac{m}{2}\epsilon^{-2s}$, that is, $\epsilon = \left(\frac{sm}{C_{1,s}}\right)^{\frac{1}{2s}}$. Consider

$$\Psi(x) = \phi\left(x - \epsilon^{-1} - 1\right) + \psi_{2s}(-\epsilon x) \quad \forall x \in \mathbb{R}.$$

we know that

$$\Psi(x) = \epsilon^{-2s} \cdot \frac{1}{|x|^{2s}}, \quad \text{and} \quad \Psi'(x) = -2s\epsilon^{-2s} \cdot \frac{1}{|x|^{1+2s}}, \qquad \forall x > \frac{1}{\epsilon}.$$

By Lemma 3.1.4 and Lemma 3.1.3, we know that

$$(-\Delta)^s \Psi(x) = -\frac{C_{1,s}}{2s} \cdot \frac{1}{|x|^{2s}} + o\left(\frac{1}{|x|^{2s}}\right), \quad \text{as } x \to \infty.$$

Hence we have

$$\begin{aligned} (-\Delta)^s \Psi(x) + c\Psi'(x) + m\Psi(x) &= \left[-\frac{C_{1,s}}{2s} + m\epsilon^{-2s}\right] \cdot \frac{1}{|x|^{2s}} + o\left(\frac{1}{|x|^{2s}}\right) \\ &= \frac{m}{2}\epsilon^{-2s} \cdot \frac{1}{|x|^{2s}} + o\left(\frac{1}{|x|^{2s}}\right), \quad \text{as } x \to \infty. \end{aligned}$$

So there exists some large $R > 0$ such that

$$(-\Delta)^s \Psi(x) + c\Psi'(x) + m\Psi(x) \geq 0, \quad \forall x \geq R.$$

Up to a translation, without the loss of generality, we assume $u(0) = \theta_0$. Notice that $v(x) = u'(x) > 0$ in $\mathbb{R}$ satisfies

$$(-\Delta)^s v(x) + cv'(x) + mv(x) = [m + f'(u(x))]v(x) \leq 0, \quad \forall x \geq R.$$

Since $\Psi(x) > 0$ for all $x \in \mathbb{R}$, there exists some $C > 0$ such that

$$C > \|v\|_{C(\mathbb{R})} \quad \text{and} \quad C \inf_{x \in [\epsilon^{-1}, R]} \Psi(x) \geq \|v\|_{C(\mathbb{R})}.$$

Since $\Psi(x) = \phi\left(x - \epsilon^{-1} - 1\right) = 1$ for all $x \leq \epsilon^{-1}$, we get $C\Psi(x) = C \geq \|v\|_{C(\mathbb{R})}$ for all $x \leq \epsilon^{-1}$. In summary, we know that

$$C\Psi(x) \geq v(x), \quad \forall x \leq R.$$

Let $w(x) = C\Psi(x) - v(x)$ for all $x \in \mathbb{R}$, we have

$$\begin{cases} (-\Delta)^s w(x) + cw'(x) + mw(x) \geq 0, \quad \forall x \geq R, \\ \lim\limits_{x\to\infty} w(x) = 0, \\ w(x) \geq 0, \quad \forall x \leq R. \end{cases}$$

By Lemma 3.1.5, we have $w(x) \geq 0$ in $\mathbb{R}$, which implies that

$$\frac{C}{|x|^{2s}} \geq v(x) = u'(x), \quad \forall x \geq 1.$$

□

Proposition 3.1.7. *Let $\frac{1}{2} < s < 1$, assume that $f'(1) < 0$, let (c, u) be a solution to (1.4.2) with $c > 0$. Then there exists some constant $C > 0$ such that*

$$u'(x) \leq \frac{C}{|x|^{1+2s}}, \quad \text{and} \quad 1 - u(x) \leq \frac{C}{|x|^{1+2s}}, \qquad x \geq 1.$$

Proof. Since $f'(1) < 0$, there exists some $m > 0$ and $\theta_0 \in (0,1)$ such that $f'(u) \leq -m$ for all $u \in [\theta_0, 1]$. Let $\epsilon > 0$ be such that

$$-\epsilon^{-1} \cdot \frac{C_{1,s}}{2s-1} - \epsilon^{-1} \cdot \frac{C_{1,s}}{2s} + m\epsilon^{-1-2s} = \frac{m}{2} \cdot \epsilon^{-1-2s}.$$

That is, we have

$$\frac{\epsilon^{2s}}{2s} + \frac{\epsilon^{2s}}{2s-1} = \frac{m}{2C_{1,s}}.$$

Look at the function $\Psi(x) = \psi_{2s}(\epsilon x - 2) + \psi_{1+2s}(-\epsilon x)$ for all $x \in \mathbb{R}$, we know that

$$\Psi(x) = \epsilon^{-1-2s} \cdot \frac{1}{|x|^{1+2s}}, \quad \text{and} \quad \Psi'(x) = -\epsilon^{-1-2s} \cdot \frac{1+2s}{|x|^{2+2s}}, \quad \forall x > \epsilon^{-1}.$$

By Lemma 3.1.2 and Lemma 3.1.3, we know that

$$(-\Delta)^s \Psi(x) = -\epsilon^{-1} \cdot \frac{C_{1,s}}{2s-1} \cdot \frac{1}{|x|^{1+2s}} - \epsilon^{-1} \cdot \frac{C_{1,s}}{2s} \cdot \frac{1}{|x|^{1+2s}} + o\left(\frac{1}{|x|^{1+2s}}\right), \quad \text{as } x \to \infty.$$

So we get

$$\begin{aligned} (-\Delta)^s \Psi(x) + c\Psi'(x) + m\Psi(x) &= \left[-\epsilon^{-1} \cdot \frac{C_{1,s}}{2s-1} - \epsilon^{-1} \cdot \frac{C_{1,s}}{2s} + m\epsilon^{-1-2s}\right] \cdot \frac{1}{|x|^{1+2s}} + o\left(\frac{1}{|x|^{1+2s}}\right) \\ &= \frac{m}{2} \cdot \epsilon^{-1-2s} \cdot \frac{1}{|x|^{1+2s}} + o\left(\frac{1}{|x|^{1+2s}}\right), \quad \text{as } x \to \infty. \end{aligned}$$

Hence there exists some large $R > 0$ such that

$$(-\Delta)^s \Psi(x) + c\Psi'(x) + m\Psi(x) \geq 0, \quad \forall x \geq R.$$

Without the loss of generality, we assume $u\left(\epsilon^{-1}\right) = \theta_0$, we know that $v = u'$ satisfies

$$(-\Delta)^s v(x) + cv'(x) + mv(x) = [m + f'(u(x))]v(x) \leq 0, \quad \forall x \geq \epsilon^{-1}.$$

For all $x \leq \epsilon^{-1}$, we have $\epsilon x - 2 \leq -1$ and $-\epsilon x \geq -1$, which implies that

$$\Psi(x) = \psi_{2s}(\epsilon x - 2) = \frac{1}{|\epsilon x - 2|^{2s}}.$$

By Proposition 3.1.6), we know that there exists some constant $C_1 > 0$ such that

$$u'(x) = v(x) \leq C_1 \Psi(x), \quad \forall x \leq \epsilon^{-1}.$$

Notice that for all $x \geq \epsilon^{-1}$, $\Psi(x) \geq \psi_{1+2s}(-\epsilon x) > 0$, which implies that there exists some $C_2 > 0$ such that

$$C_2 \inf_{x \in [\frac{1}{\epsilon}, R]} \Psi(x) \geq \sup_{x \in [\epsilon^{-1}, R]} v(x).$$

Let $C = \max\{C_1, C_2\} > 0$ and $w(x) = C\Psi(x) - v(x)$ for all $x \in \mathbb{R}$, then

$$\begin{cases} (-\Delta)^s w(x) + cw'(x) \geq 0, \quad \forall x \geq R, \\ \lim\limits_{x \to \infty} w(x) = 0, \\ w(x) \geq 0, \quad \forall x \leq R. \end{cases}$$

By Lemma 3.1.5, we know that $w(x) \geq 0$ in $\mathbb{R}$, which implies

$$\frac{C}{|x|^{1+2s}} \geq v(x) = u'(x), \quad \forall x \geq 1.$$

□

Proposition 3.1.8. *Let* $\frac{1}{2} < s < 1$,*let* (c, u) *be a solution to (1.4.2) with* $c > 0$ *in*

Theorem 2.2.13. Then there exists some constant $C > 0$ such that

$$\frac{1}{C|x|^{2s}} \leq u'(x), \quad \text{and} \quad \frac{1}{C|x|^{2s-1}} \leq u(x) \leq \frac{C}{|x|^{2s-1}}, \qquad \forall x \leq -1.$$

Proof. We have shown in the proof of Theorem 2.2.13 that there exists some constant $C > 0$ such that

$$u(x) \leq \frac{C}{|x|^{2s-1}}, \qquad \forall x \leq -1.$$

Now it suffices to show that there exists some constant $C > 0$ such that

$$\frac{1}{C|x|^{2s}} \leq u'(x), \qquad \forall x \leq -1.$$

Let $\epsilon > 0$ be such that $-\dfrac{C_{1,s}}{2s-1} + 2sc\epsilon^{1-2s} = -\dfrac{C_{1,s}}{2(2s-1)}$, that is,

$$\epsilon^{1-2s} = \frac{C_{1,s}}{4sc(2s-1)}.$$

Let $\Phi(x) = \psi_{2s}(\epsilon x)$ in $\mathbb{R}$, then

$$\Phi(x) = \epsilon^{-2s} \cdot \frac{1}{|x|^{2s}}, \quad \text{and} \quad \Phi'(x) = \epsilon^{-2s} \cdot \frac{2s}{|x|^{1+2s}}, \qquad \forall x \leq -\epsilon^{-1}.$$

By Lemma 3.1.3, we have

$$(-\Delta)^s \Phi(x) = -\frac{C_{1,s}}{2s-1} \cdot \frac{\epsilon^{-1}}{|x|^{1+2s}} + o\left(\frac{1}{|x|^{1+2s}}\right), \quad \text{as } x \to -\infty.$$

So we get

$$
\begin{aligned}
(-\Delta)^s\Phi(x) + c\Phi'(x) &= -\frac{C_{1,s}}{2s-1}\cdot\frac{\epsilon^{-1}}{|x|^{1+2s}} + c\epsilon^{-2s}\cdot\frac{2s}{|x|^{1+2s}} + o\left(\frac{1}{|x|^{1+2s}}\right) \\
&= \left[-\frac{C_{1,s}}{2s-1} + 2sc\epsilon^{1-2s}\right]\frac{\epsilon^{-1}}{|x|^{2s}} + o\left(\frac{1}{|x|^{1+2s}}\right) \\
&= -\frac{C_{1,s}}{2(2s-1)}\cdot\frac{\epsilon^{-1}}{|x|^{1+2s}} + o\left(\frac{1}{|x|^{1+2s}}\right), \quad \text{as } x \to -\infty
\end{aligned}
$$

So there exists some large $R > 0$ such that

$$(-\Delta)^s\Phi(x) + c\Phi'(x) \le 0, \quad \forall x \le -R.$$

Since $f'(t) \ge 0$ for all $t \in [0, \theta_0]$, without the loss of generality, we assume $u(-\epsilon^{-1}) = \theta_0$. Notice that $v(x) = u'(x) > 0$ in $\mathbb{R}$ satisfies

$$(-\Delta)^s v(x) + cv'(x) = f'(u(x))(x) \ge 0, \quad \forall x \le -\epsilon^{-1}.$$

Since $\Phi(x) = 0$ for all $x \ge -\epsilon^{-1}$, so we get $\Phi(x) \le v(x)$ for all $x \ge -\epsilon^{-1}$. Since $v(x) > 0$ in $\mathbb{R}$, there exists some $C > 1$ such that

$$C \inf_{x\in[-R,-\epsilon^{-1}]} v(x) \ge \sup_{x\in[-R,-\epsilon^{-1}]} \Phi(x).$$

Let $w(x) = Cv(x) - \Phi(x)$ for all $x \in \mathbb{R}$, we have

$$
\begin{cases}
(-\Delta)^s w(x) + cw'(x) \ge 0, \quad \forall x \le -R, \\
\lim\limits_{x\to-\infty} w(x) = 0, \\
w(x) \ge 0, \quad \forall x \ge -R.
\end{cases}
$$

By Lemma 3.1.5, we have $w(x) \geq 0$ in $\mathbb{R}$, which implies that

$$\frac{C}{|x|^{2s}} \leq v(x) = u'(x), \quad \forall x \leq -1.$$

$\square$

3.2 Stability results

Another question of interests is the the approach of solution to a traveling wave solution for the initial value problem corresponding to (1.1.2) with fractional laplacian. That is, we would like to focus on the fate of solutions whose initial conditions are small perturbations of the traveling wave under consideration. If any such solution stays close to the set of all translates of the traveling wave $u(\cdot)$ for all positive times, then we say that the traveling wave $u(\cdot)$ is stable. If there are initial conditions arbitrarily close to the wave such that the associated solutions leave a small neighborhood of the wave and its translates, then the wave is said to be unstable.

Fife and McLeod [17] considered the pure initial value problem for the nonlinear equation

$$u_t - u_{xx} - f(u) = 0, \quad x \in (-\infty, \infty),\ t > 0, \tag{3.2.1}$$

in the case

$$f(0) = f(1) = 0, \quad f'(x) < 0, \quad f'(1) < 0.$$

The initial value being, say

$$u(x,0) = \phi_0(x), \quad -\infty < x < \infty. \tag{3.2.2}$$

One of the central questions for this problem is the behavior as $t \to \infty$ of the solution $u(x,t)$; in particular one would like to determine under what circumstances it does (or does not) tend to a traveling front solution.

If we assume $f \in C^1$ with $f(0) = f(1) = 0$, so that $u \equiv 0$ and $u \equiv 1$ are particular solutions of (3.2.1), it is a standard result that if u_0 is piecewise continuous and $0 \le u_0 \le 1$, then there exists one and only one bounded classical solution $u(x,t)$ of (3.2.1)-(3.2.2), and $0 \le u(x,t) \le 1$ for all x, t. Indeed, they have shown the following theorem:

Theorem 3.2.1. *Consider*

$$u_t - u_{xx} - f(u) = 0, \;\; x \in (-\infty, \infty),$$

in the case

$$f(0) = f(1) = 0, \;\; f'(0) < 0, \;\; f'(1) < 0$$

Let $u(x,0) = u_0(x)$ *satisfy* $0 \le u_0 \le 1$. *Let*

$$a_- = \limsup_{x \to -\infty} u_0(x), \;\; a_+ = \liminf_{x \to \infty} u_0(x).$$

Then u *approaches a translate of* U *uniformly in* x *and exponentially in time, if* a_- *is not too far from* 0, *and* a_+ *not too far from* 1. *Here* $u = U(x - ct)$, $U(-\infty) = 0$, $U(\infty) = 1$ *is the traveling front solution. More precisely, there are constants* z_0,

$K > 0$ and $w > 0$, and

$$|u(x,t) - U(x - ct - z_0)| < Ke^{-wt}.$$

Their approach relies on a priori estimates and the standard comparison theorems for parabolic equations, i.e., let N be the nonlinear differential operator, acting on functions of x and t, defined by,

$$Nu \equiv u_t - u_{xx} - cu_x - f(u).$$

Consider the initial value problem

$$\begin{aligned} Nu &= 0 \quad \text{for } (x,t) \in (-\infty, \infty) \times (0, \infty), & (3.2.3) \\ u(x,0) &= u_0(x). & (3.2.4) \end{aligned}$$

Comparison Theorem. Let $\underline{u}$ be a sub solution, and $\bar{u}$ a super solution, of (3.2.3). Then $\underline{u}(x,t) \leq \bar{u}(x,t)$ in $(-\infty, \infty) \times [0, T)$.

It turns out that when we consider the same problem in the fractional Laplacian setting, we could obtain a similar stability result by establishing the corresponding comparison principle.

In what follows, we establish the preliminary results related to the stability of traveling fronts of the reaction diffusion equations involving the fractional Laplacian in the bistable case. We will consider the following initial value problem:

$$u_t + (-\Delta)^s u = f(u) \quad \text{in } (0, \infty) \times \mathbb{R} \tag{3.2.5}$$

with the following initial value,

$$u(x,0) = u_0(x), \quad -\infty < x < \infty. \tag{3.2.6}$$

We will also assume the nonlinearity $f \in C^1[0,1]$ satisfy

$$\begin{aligned} f(0) &= f(1) = 0,\ f'(0) < 0, f'(1) < 0, && (3.2.7)\\ f(u) &< 0 \quad \text{for } 0 < u < \alpha, && (3.2.8)\\ f(u) &> 0 \quad \text{for } \alpha < u < 1, && (3.2.9)\end{aligned}$$

where $0 < \alpha < 1$.

The existence of traveling fronts has been shown by X. Cabre and Y. Sire in [?] when the bistable nonlinearity is balanced and by C. Gui and M. Zhao in [20] when the bistable nonlinearity is unbalanced. Assume the existence and uniqueness of the solution to the initial value problem (3.2.5)-(3.2.7), and we are expecting a similar stability result of the traveling fronts.

The main tool needed here is the comparison principle for (1.1.2) where L is the generator of of a strong continuous semigroup in a Banach space X. It is applicable to This comparison principle can be found in [12]. For completion, we will state the comparison principle here.

We first consider the non homogeneous linear problem

$$\begin{cases} u_t + Lu = f(u) \text{ in } (0,\infty)\times\mathbb{R}^n, \\ u(0,\cdot) = u_0 \text{ in } \mathbb{R}^n, \end{cases} \tag{3.2.10}$$

for u_0 in any of both Banach spaces.

Assume that $f_1,\ f_2 \in C^1(\mathbb{R})$ is globally Lipschitz and f' is uniformly continuous in $\mathbb{R}$, and $f_1 \le f_2$ in $\mathbb{R}$. We then have

$$\text{if } u_1(0,\cdot) \le u_2(0,\cdot) \text{ belong to } C_{u,b}(\mathbb{R}^n),\ \text{ then } u_1(t,\cdot) \le u_2(t,\cdot)$$

for all $t \in [0,\infty)$, where u_1 and u_2 are the respective solutions of the nonlinear problem (3.2.10) with f and u_0 replaced by f_i and $u_i(0,\cdot)$. $C_{u,b}(\mathbb{R}^n) = \{u : \mathbb{R}^n \to \mathbb{R} :$ u is bounded and uniformly continuous in $\mathbb{R}^n\}$.

We set $z = x - ct$, and write the solution of (3.2.5)-(3.2.6) as

$$v(z,t) = u(x,t) = u(z+ct,t)$$

Followed by a comparison argument, we get the stability result when the traveling wave solution exists. More precisely, we have obtained the following theorem:

Theorem 3.2.2. *Let $0 < s < 1$, consider equations (3.2.5)-(3.2.7). Assume the traveling front solution is U with speed c, let u_0 satisfies $0 \le u_0 \le 1$, and suppose*

$$\limsup_{x\to-\infty} u_0(x) < \alpha, \quad \liminf_{x\to\infty} u_0(x) > \alpha,$$

then there exists constants z_1, z_2, q_0, and μ (the last two positive), such that

$$U(z-z_1) - q_0 e^{-\mu t} \le v(z,t) \le U(z-z_2) + q_0 e^{-\mu t}. \tag{3.2.11}$$

The proof of the theorem relies on the comparison principle for fractional Laplacian. The comparison principle can be found in [12], in which the authors consider the Cauchy problem

$$\begin{cases} u_t + Au = f(u) \quad \text{in } (0, +\infty) \times \mathbb{R}^n, \\ u(0, \cdot) = u_0 \quad \text{in } \mathbb{R}^n, \quad 0 \le u_0 \le 1 \end{cases}$$

where A is the infinitesimal generator of a Feller semigroup. Examples includes Δ (the classical Laplacian) and $(-\Delta)^s$ with $s \in (0, 1)$ (the fractional Laplacian). By adding a drift term to the classical Laplacian or the fractional Laplacian, the resulting operator will still be an infinitesimal generator of a Feller semigroup. According to [12], we have the following comparison principle.

Lemma 3.2.3. *Assume that $f_1,\ f_2 \in C^1(\mathbb{R})$ are globally Lipschitz and $f_i',\ i = 1,\ 2$ is uniformly continuous in $\mathbb{R}$. If $f_1 \le f_2$ in $\mathbb{R}$, we then have:*

$$\text{if } u_1(0, \cdot) \le u_2(0, \cdot) \text{ are both bounded and uniformly continuous in} \mathbb{R}^n, \text{ then } u_1(t, \cdot) \le u_2(t, \cdot).$$

The proof of the theorem then follows directly from [17]. It then establishes the stability of traveling fronts in the C^0 norm. The C^1 norm stability for the traveling fronts still remains as an open question.

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