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# Stirling's Formula in Number Fields

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# Stirling's Formula in Number Fields

Matthew Lamoureux, Ph.D.

University of Connecticut, 2014

## ABSTRACT

In 1997 Bhargava generalized the factorial sequence to factorials in any Dedekind domain. He asked if there is an analogue of Stirling's formula for generalized factorials. Using techniques of analytic number theory, my thesis presents such an analogue when the Dedekind domain is the ring of integers in a number field. Unlike the classical case of Stirling's formula, which corresponds to the number field  $\mathbf{Q}$ , its generalization to the factorials in a number field other than  $\mathbf{Q}$  has a surprising ingredient: the formula involves a sum over nontrivial zeros of the zeta-function of the number field.

# Stirling's Formula in Number Fields

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M.S. University of Connecticut, 2010

B.A. Quinnipiac University, 2009

A Dissertation

Submitted in Partial Fulfillment of the

Requirements for the Degree of

Doctor of Philosophy

at the

University of Connecticut

2014

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Matthew Lamoureux

2014

# APPROVAL PAGE

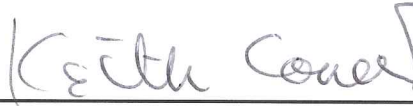
Doctor of Philosophy Dissertation

## Stirling's Formula in Number Fields

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# Chapter 1

## Preliminaries

### 1.1 Introduction

The factorial function, defined on nonnegative integers, can be described by prime factorization:

$$n! = \prod_p p^{\sum_{k \geq 1} [n/p^k]}, \quad (1.1.1)$$

where  $[\cdot]$  denotes the greatest integer function. This is due to Legendre [8]. Formula (1.1.1) generalizes to a factorial function on ideals of a number field  $K$ , which is a special case of Bhargava's definition in [2]. For a nonzero ideal  $\mathfrak{a} \subset \mathcal{O}_K$ , define

$$\mathfrak{a}! = \prod_{\mathfrak{p}} \mathfrak{p}^{\sum_{k \geq 1} [\mathrm{N}\mathfrak{a}/\mathrm{N}\mathfrak{p}^k]}.$$



This ideal only depends on  $\mathfrak{a}$  through its norm  $N\mathfrak{a}$ , and the norm of  $\mathfrak{a}!$  is

$$N(\mathfrak{a}!) = \prod_{\mathfrak{p}} N\mathfrak{p}^{\sum_{k \geq 1} [N\mathfrak{a}/N\mathfrak{p}^k]}.$$

We define the integers  $n!_K$  for  $n \geq 0$  by

$$n!_K = \prod_{\mathfrak{p}} N\mathfrak{p}^{\sum_{k \geq 1} [n/N\mathfrak{p}^k]}, \quad (1.1.2)$$

so  $N(\mathfrak{a}!) = (N\mathfrak{a})!_K$ . As a comparison to  $0! = 1, 1! = 1, 2! = 2, 3! = 6, 4! = 24$ , and  $5! = 120$ , we have for  $K = \mathbb{Q}(i)$  that  $0!_K = 1, 1!_K = 1, 2!_K = 2, 3!_K = 2, 4!_K = 8$ , and  $5!_K = 200$ .

One example of the similarities between  $n!_K$  and  $n!$  is their common role in the description of integral-valued polynomials on  $\mathbb{Z}$  and  $\mathcal{O}_K$ . Define  $\text{Int}(\mathcal{O}_K)$  to be the subring of  $K[x]$  consisting of polynomials  $f(x)$  such that  $f(\mathcal{O}_K) \subseteq \mathcal{O}_K$ . The ring  $\text{Int}(\mathbb{Z})$  is a free  $\mathbb{Z}$ -module spanned by the binomial coefficient polynomials  $\binom{x}{n}$ , so the leading coefficients of degree  $n$  polynomials in  $\text{Int}(\mathbb{Z})$ , together with 0, form the fractional ideal  $\frac{1}{n!}\mathbb{Z}$ . Although  $\text{Int}(\mathcal{O}_K)$  need not be a free  $\mathcal{O}_K$ -module, Bhargava [2, Theorem 12] showed that the set of leading coefficients of degree  $n$  polynomials in  $\text{Int}(\mathcal{O}_K)$ , together with 0, is the ideal

$$\underline{n!}_K = \prod_{\mathfrak{p}} \mathfrak{p}^{-\sum_{k \geq 1} [n/N\mathfrak{p}^k]},$$

and the norm of this ideal is  $n!_K^{-1}$ . For another application of  $\underline{n!}_K$ , if  $f(x) \in \mathcal{O}_K[x]$  is primitive of degree  $n$ , the ideal generated by  $f(\mathcal{O}_K)$  divides  $\underline{n!}_K$  [3, Theorem 2], and there exists an  $f(x)$  such that this ideal is exactly  $\underline{n!}_K$  [3, Theorem 4].

Stirling's formula for  $n!$  and  $\log(n!)$  says

$$n! \sim \frac{n^n}{e^n} \sqrt{2\pi n}, \quad \log(n!) = n \log n - n + \frac{1}{2} \log n + \frac{1}{2} \log(2\pi) + o(1) \quad (1.1.3)$$

as  $n \rightarrow \infty$ , and the goal of this thesis is to develop an analogue of these with  $n!_K$  in place of  $n!$ . We cannot start from any of the standard proofs of Stirling's formula, since those are based on the recursion  $(n+1)! = (n+1)n!$  or Euler's integral formula  $n! = \int_0^\infty t^n e^{-t} dt$ , and neither of these seem to carry over nicely to  $n!_K$ . For instance, if  $[K : \mathbb{Q}] = 2$  then

$$\frac{(n+1)!_K}{n!_K} = (n+1) \prod_{\substack{p|(n+1) \\ p \text{ splits in } \mathcal{O}_K}} p^{\text{ord}_p(n+1)} \prod_{\substack{p|(n+1) \\ p \text{ inert in } \mathcal{O}_K}} \frac{1}{p^{a_p(n)}}, \quad (1.1.4)$$

where  $a_p(n) = 1$  if  $\text{ord}_p(n+1) \equiv 1 \pmod{2}$  and  $a_p(n) = 0$  if  $\text{ord}_p(n+1) \equiv 0 \pmod{2}$  (see Section 5.1 for a proof). In a precise sense that we'll explain in Section 5.2, there is a way to test for each  $K$  whether it is possible to write  $n!_K = \int_0^\infty t^n d\mu(t)$  for all  $n \geq 0$  and some measure  $\mu$  on  $(0, \infty)$ , and for many low-degree number fields other than  $\mathbb{Q}$  the test has a negative answer. Developing a Stirling-type formula for  $n!_K$  or  $\log(n!_K)$  must start from the definition (1.1.2), which would be analogous to a proof of Stirling's formula starting from the prime factorization in (1.1.1).

In Chapter 1, we discuss the dominant terms in the formula for  $\log(n!_K)$ , analogous to  $\log(n!) = n \log n - n + o(n)$ . For several low-degree number fields  $K$  other than  $\mathbb{Q}$ , the difference between  $\log(n!_K)$  and its two dominant terms is completely different from the classical case where the difference is approximately  $\frac{1}{2} \log(2\pi n)$ . The difference has magnitude far larger than  $\log(2\pi n)$  and oscillates between positive and negative values. In Chapter 2, we develop an integral representation for  $\log(n!_K)$ ,

and the explanation of these oscillations will involve a sum over nontrivial zeros of  $\zeta_K(s)$ . In Chapter 3, we use GRH to develop estimates on the sum over nontrivial zeros, achieving the following truncated explicit formula for  $\log(n!_K)$ :

**Theorem.** *Assume GRH for  $\zeta(s)$  and  $\zeta_K(s)$ . For  $0 < \delta < \frac{5}{18}$ ,*

$$\log(n!_K) = n \log n - (1 + \gamma_K - \gamma)n + O_{\delta,K}(n^{2/3+\delta})$$

as  $n \rightarrow \infty$  in  $\mathbb{Z}^+$ , where  $\gamma_K$  is the constant term in the Laurent expansion of  $\zeta'_K(s)/\zeta_K(s)$  at  $s = 1$ .

Some of the technical proofs from Chapter 2 are postponed for the sake of readability, and they are presented in Chapter 4. Chapter 5 discusses some non-analogues between  $n!$  and  $n!_K$  when  $K \neq \mathbb{Q}$ .

## 1.2 Dominant Terms of $\log(n!_K)$

To determine the behavior of  $\log(n!_K)$ , we start with (1.1.2) and take logarithms of both sides of the equation:

$$\log(n!_K) = \sum_{\mathfrak{p}} \sum_{k \geq 1} \log N\mathfrak{p} \left[ \frac{n}{N\mathfrak{p}^k} \right].$$

Writing this as a single sum over integers, we have

$$\log(n!_K) = \sum_{m \leq n} b_K(m) \left[ \frac{n}{m} \right], \tag{1.2.1}$$

where

$$b_K(m) = \sum_{N\mathfrak{p}^k=m} \log N\mathfrak{p},$$

which is the  $m$ -th Dirichlet coefficient of  $-\zeta'_K(s)/\zeta_K(s)$ . The sum in (1.2.1) is over  $m \leq n$  since  $[n/m] = 0$  when  $m > n$ .

We can determine the first two dominant terms in the analogue of (1.1.3) for  $\log(n!_K)$  by using a Tauberian theorem for sums of nonnegative Dirichlet coefficients.

**Theorem 1.2.1.** *Suppose  $a_m \geq 0$  for each  $m$  and*

$$f(s) = \sum_{m \geq 1} \frac{a_m}{m^s}$$

*converges on the half-plane  $\operatorname{Re}(s) > 1$  and has an analytic continuation to  $\operatorname{Re}(s) = 1$  except for a simple pole at  $s = 1$  with residue  $r$ . Then*

$$\begin{aligned} \text{(a)} \quad & \sum_{m \leq n} a_m = rn + o(n), \\ \text{(b)} \quad & \sum_{m \leq n} \frac{a_m}{m} = r \log n + c + o(1), \end{aligned}$$

*where  $c$  is the constant term in the Laurent expansion of  $f(s)$  at  $s = 1$ .*

*Proof.* See [7, Theorem 3.2.4]. ■

**Lemma 1.2.2.** *If  $c_m \in \mathbb{R}$ ,  $\sum_{m \geq 1} c_m/m$  converges, and  $\sum_{m \leq n} c_m = O(n)$ , then*

$$\frac{1}{n} \sum_{m \leq n} c_m \left\{ \frac{n}{m} \right\} \rightarrow 0$$

*as  $n \rightarrow \infty$ , where  $\{\cdot\}$  denotes the fractional part function.*

*Proof.* See [10, p. 179-180]. ■

**Theorem 1.2.3.** *If  $a_m \in \mathbb{R}$  satisfies  $\sum_{m \leq x} a_m = O(x)$  and  $\sum_{m \leq x} a_m/m = r \log x + c + o(1)$  for some  $r$  and  $c$ , then*

$$\frac{1}{n} \sum_{m \leq n} a_m \left\{ \frac{n}{m} \right\} \rightarrow r(1 - \gamma)$$

as  $n \rightarrow \infty$ , where  $\gamma$  is Euler's constant.

*Proof.* Let  $c_m = a_m - r$ . Then

$$\begin{aligned} \sum_{m \leq x} \frac{c_m}{m} &= \sum_{m \leq x} \frac{a_m}{m} - r \sum_{m \leq x} \frac{1}{m} \\ &= (r \log x + c + o(1)) - r(\log x + \gamma + o(1)) \\ &= c - r\gamma + o(1), \end{aligned}$$

so  $\sum_{m \geq 1} c_m/m$  converges. Moreover,

$$\sum_{m \leq x} c_m = \sum_{m \leq x} a_m - r[x] = O(x),$$

so by Lemma 1.2.2,

$$\frac{1}{n} \sum_{m \leq n} c_m \left\{ \frac{n}{m} \right\} = o(1)$$

as  $n \rightarrow \infty$ . At the same time,

$$\frac{1}{n} \sum_{m \leq n} c_m \left\{ \frac{n}{m} \right\} = \frac{1}{n} \sum_{m \leq n} (a_m - r) \left\{ \frac{n}{m} \right\} = \frac{1}{n} \sum_{m \leq n} a_m \left\{ \frac{n}{m} \right\} - \frac{r}{n} \sum_{m \leq n} \left\{ \frac{n}{m} \right\},$$

so

$$\begin{aligned}
\frac{1}{n} \sum_{m \leq n} a_m \left\{ \frac{n}{m} \right\} &= \frac{r}{n} \sum_{m \leq n} \left\{ \frac{n}{m} \right\} + o(1) \\
&= \frac{r}{n} \sum_{m \leq n} \left( \frac{n}{m} - \left[ \frac{n}{m} \right] \right) + o(1) \\
&= r \sum_{m \leq n} \frac{1}{m} - \frac{r}{n} \sum_{m \leq n} \left[ \frac{n}{m} \right] + o(1) \\
&= r(\log n + \gamma) - \frac{r}{n} \sum_{m \leq n} \left[ \frac{n}{m} \right] + o(1). \tag{1.2.2}
\end{aligned}$$

It is well-known [1, p. 57] that

$$\sum_{m \leq n} \left[ \frac{n}{m} \right] = n \log n + (2\gamma - 1)n + o(n),$$

and we divide this through by  $n$  and apply the result to (1.2.2) to get

$$\begin{aligned}
\frac{1}{n} \sum_{m \leq n} a_m \left\{ \frac{n}{m} \right\} &= r(\log n + \gamma) - r(\log n + (2\gamma - 1) + o(1)) \\
&= r(\gamma - 2\gamma + 1) + o(1) \\
&= r(1 - \gamma) + o(1).
\end{aligned}$$

■

Now we give the first two dominant terms for  $\log(n!_K)$ .

**Theorem 1.2.4.** *As  $n \rightarrow \infty$  in  $\mathbb{Z}^+$ ,*

$$\log(n!_K) = n \log n - (\gamma_K - \gamma + 1)n + o(n),$$

where  $\gamma_K$  is the constant term of the Laurent expansion of  $\zeta'_K(s)/\zeta_K(s)$  at  $s = 1$ .

Before proving Theorem 1.2.4, we discuss  $\gamma_K$ , the Euler–Kronecker constant for  $K$ , which was introduced by Ihara [5]. There are two related ways that  $\gamma$  appears in the theory of the Riemann zeta-function:

$$\zeta(s) = \frac{1}{s-1} + \gamma + O(s-1), \quad \frac{\zeta'(s)}{\zeta(s)} = -\frac{1}{s-1} + \gamma + O(s-1). \quad (1.2.3)$$

Ihara took the second formula as more basic:  $\gamma_K$  is defined to be the constant term in the Laurent expansion of  $\zeta'_K(s)/\zeta_K(s)$  at  $s = 1$ :

$$\frac{\zeta'_K(s)}{\zeta_K(s)} = -\frac{1}{s-1} + \gamma_K + O(s-1). \quad (1.2.4)$$

When  $K = \mathbb{Q}$ ,  $\gamma_K = \gamma$  and Theorem 1.2.4 becomes

$$\log(n!) = n \log n - n + o(n),$$

a weak form of Stirling's formula.

Writing  $\zeta_K(s) = \rho_K/(s-1) + a_K + O(s-1)$ , where  $\rho_K$  is the residue of  $\zeta_K(s)$  at  $s = 1$ , we get  $\zeta'_K(s)/\zeta_K(s) = -1/(s-1) + a_K/\rho_K + O(s-1)$  since for any meromorphic function  $f(s)$  with a simple pole at  $s = a$ ,

$$f(s) = \frac{r}{s-a} + c + O(s-a) \implies \frac{f'(s)}{f(s)} = -\frac{1}{s-a} + \frac{c}{r} + O(s-a). \quad (1.2.5)$$

Thus  $a_K = \rho_K \gamma_K$ , so near  $s = 1$ ,

$$\zeta_K(s) = \frac{\rho_K}{s-1} + \rho_K \gamma_K + O(s-1). \quad (1.2.6)$$

Comparing (1.2.4) and (1.2.6) to (1.2.3), the appearance of  $\gamma$  in both constant terms of (1.2.3) is due to  $\zeta(s)$  having residue 1 at  $s = 1$ , which is likely untrue of  $\zeta_K(s)$  for  $K \neq \mathbb{Q}$ .

When  $[K : \mathbb{Q}] = 2$ , we can compute  $\gamma_K$  in terms of a Dirichlet  $L$ -function at  $s = 1$ . Write  $\zeta_K(s) = \zeta(s)L(s, \chi)$  for a certain quadratic Dirichlet character  $\chi$ , so

$$\frac{\zeta'_K(s)}{\zeta_K(s)} = \frac{\zeta'(s)}{\zeta(s)} + \frac{L'(s, \chi)}{L(s, \chi)}.$$

Thus  $\gamma_K = \gamma + L'(1, \chi)/L(1, \chi)$  in this case. Using this formula, we have  $\gamma_{\mathbb{Q}(i)} \approx .823$  and  $\gamma_{\mathbb{Q}(\sqrt{-47})} \approx -.063$ , so  $\gamma_K$  could be positive or negative.

*Proof of Theorem 1.2.4.* Substituting  $[n/m] = n/m - \{n/m\}$  into (1.2.1) we get

$$\log(n!_K) = \sum_{m \leq n} b_K(m) \left[ \frac{n}{m} \right] = n \sum_{m \leq n} \frac{b_K(m)}{m} - \sum_{m \leq n} b_K(m) \left\{ \frac{n}{m} \right\}. \quad (1.2.7)$$

For the first sum on the right of (1.2.7), we use Theorem 1.2.1 (b). The residue of  $-\zeta'_K(s)/\zeta_K(s)$  at  $s = 1$  is 1, and the constant term in its Laurent expansion at  $s = 1$  is  $-\gamma_K$ . Thus by Theorem 1.2.1,

$$\sum_{m \leq n} b_K(m) = n + o(n), \quad \sum_{m \leq n} \frac{b_K(m)}{m} = \log n - \gamma_K + o(1) \quad (1.2.8)$$

as  $n \rightarrow \infty$ . By (1.2.8) and Theorem 1.2.3, the second sum on the right of (1.2.7) is

$$\sum_{m \leq n} b_K(m) \left\{ \frac{n}{m} \right\} = (1 - \gamma)n + o(n). \quad (1.2.9)$$



Substituting (1.2.8) and (1.2.9) into (1.2.7) gives us

$$\log(n!_K) = n \log n - (\gamma_K - \gamma + 1)n + o(n)$$

as  $n \rightarrow \infty$ . □

We can write (1.2.1) in terms of  $\psi_K(x) = \sum_{m \leq x} b_K(m)$ , which is used to prove the Prime Ideal Theorem:

$$\sum_{m \leq n} b_K(m) \left[ \frac{n}{m} \right] = \sum_{m \leq n} b_K(m) \left( \sum_{k \leq n/m} 1 \right) = \sum_{k \leq n} \sum_{m \leq n/k} b_K(m) = \sum_{k \leq n} \psi_K \left( \frac{n}{k} \right). \quad (1.2.10)$$

Using (1.2.10), can estimates on  $\psi_K(x)$  help us estimate  $\log(n!_K)$  better than in Theorem 1.2.4? The elementary estimate  $\psi_K(x) = O(x)$  implies

$$\log(n!_K) = \sum_{k \leq n} O \left( \frac{n}{k} \right) = O \left( n \sum_{k \leq n} \frac{1}{k} \right) = O(n \log n). \quad (1.2.11)$$

If we assume GRH for  $\zeta_K(s)$  so that

$$\psi_K(x) = x + O_K(\sqrt{x} \log^2 x) = x + O_{\varepsilon, K}(x^{1/2+\varepsilon})$$

for any  $\varepsilon > 0$ , then we can improve (1.2.11):

$$\begin{aligned} \log(n!_K) = \sum_{k \leq n} \psi_K \left( \frac{n}{k} \right) &= \sum_{k \leq n} \left( \frac{n}{k} + O_{\varepsilon, K} \left( \left( \frac{n}{k} \right)^{1/2+\varepsilon} \right) \right) \\ &= n(\log n + \gamma + o(1)) + O_{\varepsilon, K} \left( n^{1/2+\varepsilon} \sum_{k \leq n} \frac{1}{k^{1/2+\varepsilon}} \right). \end{aligned}$$

Since

$$\sum_{k \leq n} \frac{1}{k^{1/2+\varepsilon}} \leq 1 + \int_1^n \frac{1}{t^{1/2+\varepsilon}} dt = O_\varepsilon(n^{1/2-\varepsilon}),$$

we have

$$\begin{aligned} \sum_{k \leq n} \psi_K \left( \frac{n}{k} \right) &= n \log n + \gamma n + o(n) + O_{\varepsilon, K}(n^{1/2+\varepsilon} n^{1/2-\varepsilon}) \\ &= n \log n + O_{\varepsilon, K}(n). \end{aligned}$$

Thus, even assuming GRH for  $\zeta_K(s)$ , estimates on  $\psi_K(x)$  and (1.2.10) only lead to the first term in Theorem 1.2.4. In Chapter 3 we will see how we can use GRH on  $\zeta_K(s)$  and on  $\zeta(s)$  to sharpen the  $o(n)$ -estimate in Theorem 1.2.4.

### 1.3 Numerical Data

Theorem 1.2.4 tells us that  $\log(n!_K) = n \log n - A_K n + o(n)$  as  $n \rightarrow \infty$ , where  $A_K = \gamma_K - \gamma + 1$ . By the classical Stirling formula, we can replace  $o(n)$  with  $\frac{1}{2} \log n + \frac{1}{2} \log(2\pi) + o(1)$  when  $K = \mathbb{Q}$ . To see what lower-order terms in a Stirling-type formula for  $n!_K$  should be when  $K \neq \mathbb{Q}$ , we compute  $\log(n!_K) - (n \log n - A_K n)$  using  $A_K \approx -1.24560959$  when  $K = \mathbb{Q}(i)$  and compare to  $\log(n!) - (n \log n - n)$  in Table 1.1.

$n$	$\log n! - (n \log n - n)$	$\log(n!_{\mathbb{Q}(i)}) - (n \log n - A_{\mathbb{Q}(i)}n)$
100000	6.67540210	109.90885
200000	7.02197527	373.48412
300000	7.22470769	249.25102
400000	7.36854865	356.14378
500000	7.48012039	-381.59259
600000	7.57128114	776.08396
700000	7.64835646	-41.32520

TABLE 1.1: Subtracting  $n \log n - A_K n$  from  $\log(n!_K)$  if  $K = \mathbb{Q}$  and  $\mathbb{Q}(i)$ .

Since  $\frac{1}{2} \log n + \frac{1}{2} \log(2\pi)$  is approximately 6.675 for  $n = 100000$  and 7.648 for  $n = 700000$ , there are no surprises in the left column. However, the column for  $n!_{\mathbb{Q}(i)}$  is not behaving like  $B \log n + C$  for any  $B$  or  $C$ . Similar data occur for other number fields  $K \neq \mathbb{Q}$ : the difference  $\log(n!) - (n \log n - A_K n)$  seems to oscillate as  $n$  grows. Letting  $D_K(n)$  denote the difference  $\log(n!_K) - (n \log n - A_K n)$ , Table 1.2 shows oscillatory data similar to the  $\mathbb{Q}(i)$ -case for these other choices of  $K$  and uses  $A_{\mathbb{Q}(\sqrt{2})} \approx 1.632115$ ,  $A_{\mathbb{Q}(\sqrt{10})} \approx 1.058425$ , and  $A_{\mathbb{Q}(\sqrt[3]{2})} \approx 1.431688$ .

$n$	$D_{\mathbb{Q}(\sqrt{2})}(n)$	$D_{\mathbb{Q}(\sqrt{10})}(n)$	$D_{\mathbb{Q}(\sqrt[3]{2})}(n)$
100000	-281.66912	-118.63625	-640.54742
200000	325.78290	103.46164	-57.32647
300000	-302.49577	-211.21162	-1062.68283
400000	251.86486	-753.16007	623.15870
500000	-529.83456	-1352.73724	164.07090
600000	-668.63362	-126.68622	473.32765
700000	116.28154	392.86425	-498.49669

TABLE 1.2: Differences  $D_K(n) = \log(n!_K) - (n \log n - A_K n)$  for other number fields  $K$ .

What could cause this? Oscillations in asymptotic formulas in number theory are often related to zeros of zeta-functions or  $L$ -functions. If this is the explanation of the data for  $\log(n!_K)$ , why don't we see this phenomenon when  $K = \mathbb{Q}$ ? There are no zeros of  $\zeta(s)$  in Stirling's formula. We will see the answer in Chapter 2.

**Remark 1.3.1.** The fields  $\mathbb{Q}(i)$ ,  $\mathbb{Q}(\sqrt{2})$ ,  $\mathbb{Q}(\sqrt{10})$ , and  $\mathbb{Q}(\sqrt[3]{2})$  all have  $A_K > 0$ . There are examples where  $A_K < 0$ , or equivalently where  $\gamma_K < \gamma - 1 \approx -0.42278$ : using tables from [5, p. 40], when  $K$  is the degree 11 cyclic extension of  $\mathbb{Q}$  inside  $\mathbb{Q}(\zeta_{121})$ , we have  $\gamma_K \approx -1.4330$ , so  $\gamma_K - \gamma + 1 \approx -1.0102$ .

## Chapter 2

# An Integral Formula for $\log(n!_K)$

### 2.1 Perron's Formula and Laurent Series

To determine a sharper analogue of Stirling's formula for  $\log(n!_K)$  than Theorem 1.2.4, we will apply Perron's formula in the same way that it is used to determine von Mangoldt's explicit formula for  $\psi(x) = \sum_{p^k \leq x} \log p$ .

**Theorem 2.1.1** (Perron's formula). *If  $f(s) = \sum_{k \geq 1} \frac{a_k}{k^s}$  is absolutely convergent for  $\operatorname{Re}(s) > 1$ , then for  $x > 0$  and  $c > 1$  we have*

$$\sum_{k \leq x}^* a_k = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} f(s) \frac{x^s}{s} ds, \quad (2.1.1)$$

where  $\sum^*$  means that the final term in the sum is multiplied by  $\frac{1}{2}$  if  $x \in \mathbb{Z}$ .

*Proof.* See [1, p. 245]. ■

Here and in all future occurrences,  $\int_{c-i\infty}^{c+i\infty}$  means  $\lim_{T \rightarrow \infty} \int_{c-iT}^{c+iT}$ . The basis for

Perron's formula is the discontinuous integral

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{x^s}{s} ds = \begin{cases} 1, & \text{if } x > 1, \\ \frac{1}{2}, & \text{if } x = 1, \\ 0, & \text{if } 0 < x < 1 \end{cases} \quad (2.1.2)$$

when  $c > 0$ . For instance, when  $f(s) = \zeta(s)$ ,  $x > 0$ , and  $c > 1$ , applying Perron's formula gives us

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \zeta(s) \frac{x^s}{s} ds = \sum_{k \leq x}^* 1 = \begin{cases} [x], & \text{if } x \notin \mathbb{Z}, \\ x - \frac{1}{2}, & \text{if } x \in \mathbb{Z}. \end{cases}$$

We can rewrite this as an integral representation for  $[x]$  when  $x > 0$ :

$$[x] = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \zeta(s) \frac{x^s}{s} ds + \begin{cases} 0, & \text{if } x \notin \mathbb{Z}, \\ \frac{1}{2}, & \text{if } x \in \mathbb{Z}. \end{cases} \quad (2.1.3)$$

This will be useful for us since

$$\log(n!_K) = \sum_{\mathfrak{p}} \sum_{k \geq 1} \log N\mathfrak{p} \left[ \frac{n}{N\mathfrak{p}^k} \right] = \sum_{m \leq n} b_K(m) \left[ \frac{n}{m} \right], \quad (2.1.4)$$

where  $b_K(m)$  is the  $m$ -th coefficient of  $-\zeta'_K(s)/\zeta_K(s)$ , defined in (1.2.1). We will use (2.1.3) and (2.1.4) to produce a version of Perron's formula that is tailored to  $\log(n!_K)$ . First we will need a lemma.

**Lemma 2.1.2.** For  $c > 1$ ,  $T > 0$ , and  $0 < r < 1$ , we have

$$\left| \int_{c-iT}^{c+iT} \zeta(s) \frac{r^s}{s} ds \right| \leq \frac{2\zeta(c)r^c}{T \log(1/r)}.$$

*Proof.* The proof is similar to [1, p. 243]. Since the integrand has no poles on or to the right of  $\text{Re}(s) = c$ , we can use the residue theorem to get for any  $a > c$  that

$$\int_{c-iT}^{c+iT} \zeta(s) \frac{r^s}{s} ds = \int_{c-iT}^{a-iT} \zeta(s) \frac{r^s}{s} ds + \int_{a-iT}^{a+iT} \zeta(s) \frac{r^s}{s} ds + \int_{a+iT}^{c+iT} \zeta(s) \frac{r^s}{s} ds. \quad (2.1.5)$$

See Figure 2.1.

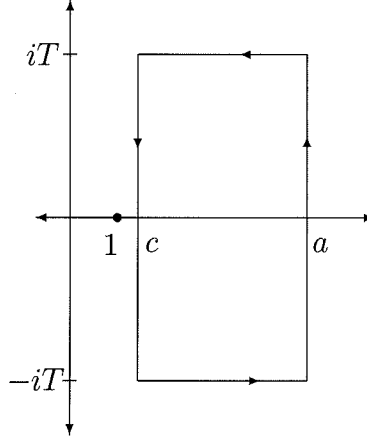


FIGURE 2.1: Using the residue theorem on  $\zeta(s)r^s/s$ .

We will estimate each of the three integrals on the right side of (2.1.5). For the first integral, we have

$$\left| \int_{c-iT}^{a-iT} \zeta(s) \frac{r^s}{s} ds \right| \leq \int_c^a |\zeta(x - iT)| \frac{r^x}{\sqrt{x^2 + T^2}} dx.$$

We know  $|\zeta(x - iT)| \leq |\zeta(x)| \leq |\zeta(c)|$  and  $\sqrt{x^2 + T^2} > T$  for  $T > 0$  and  $c \leq x \leq a$ ,

so we have

$$\begin{aligned} \int_c^a |\zeta(x - iT)| \frac{r^x}{\sqrt{x^2 + T^2}} dt &\leq \frac{\zeta(c)}{T} \int_c^a r^x dx \\ &= \frac{\zeta(c)}{T} \left( \frac{r^a}{\log r} - \frac{r^c}{\log r} \right). \end{aligned} \quad (2.1.6)$$

For the third integral in (2.1.5), taking complex conjugates gives us the same bound as in (2.1.6). To estimate the second integral on the right side of (2.1.5), we have

$$\begin{aligned} \left| \int_{a-iT}^{a+iT} \zeta(s) \frac{r^s}{s} ds \right| &\leq \int_{-T}^T |\zeta(a + iy)| \frac{r^a}{\sqrt{a^2 + y^2}} dy \\ &\leq 2T \zeta(a) \frac{r^a}{a}. \end{aligned} \quad (2.1.7)$$

Feeding (2.1.6) and (2.1.7) into (2.1.5), we have

$$\left| \int_{c-iT}^{c+iT} \zeta(s) \frac{r^s}{s} ds \right| \leq 2 \frac{\zeta(c)}{T} \left( \frac{r^a}{\log r} - \frac{r^c}{\log r} \right) + 2T \zeta(a) \frac{r^a}{a},$$

and letting  $a \rightarrow \infty$  yields

$$\left| \int_{c-iT}^{c+iT} \zeta(s) \frac{r^s}{s} ds \right| \leq \frac{2r^c \zeta(c)}{T \log(1/r)}.$$

■

**Theorem 2.1.3.** *If  $f(s) = \sum_{k \geq 1} \frac{a_k}{k^s}$  is absolutely convergent for  $\operatorname{Re}(s) > 1$ , then for  $x > 0$  and  $c > 1$ ,*

$$\sum_{k \leq x} a_k \left[ \frac{x}{k} \right] = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} f(s) \zeta(s) \frac{x^s}{s} ds + \frac{1}{2} \sum_{d|x} a_d, \quad (2.1.8)$$



where the term  $\frac{1}{2} \sum_{d|x} a_d$  is only present if  $x \in \mathbb{Z}$ .

*Proof.* Since  $\sum_{k \geq 1} a_k/k^s$  converges for  $\operatorname{Re}(s) > 1$ , it converges uniformly on the compact set  $[c - iT, c + iT]$  by [1, p. 235], so

$$\begin{aligned} \frac{1}{2\pi i} \int_{c-iT}^{c+iT} f(s) \zeta(s) \frac{x^s}{s} ds &= \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \left( \sum_{k \geq 1} \frac{a_k}{k^s} \right) \zeta(s) \frac{x^s}{s} ds \\ &= \sum_{k \geq 1} a_k \left( \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \zeta(s) \frac{(x/k)^s}{s} ds \right). \end{aligned}$$

We split this sum into two parts: the first over  $k \leq x$  and the second over  $k > x$ :

$$\sum_{k \leq x} a_k \left( \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \zeta(s) \frac{(x/k)^s}{s} ds \right) + \sum_{k > x} a_k \left( \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \zeta(s) \frac{(x/k)^s}{s} ds \right). \quad (2.1.9)$$

Since the first sum in (2.1.9) is finite, by (2.1.3) we can let  $T \rightarrow \infty$  and it becomes

$$\sum_{k \leq x} a_k \left( \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \zeta(s) \frac{(x/k)^s}{s} ds \right) = \sum_{k \leq x} a_k \left( \left[ \frac{x}{k} \right] - \frac{1}{2} \mu_k(x) \right),$$

where  $\mu_k(x) = 1$  if  $\frac{x}{k} \in \mathbb{Z}$  and  $\mu_k(x) = 0$  otherwise. When  $x$  is an integer, this becomes

$$\sum_{k \leq x} a_k \left( \left[ \frac{x}{k} \right] - \frac{1}{2} \mu_k(x) \right) = \sum_{k \leq x} a_k \left[ \frac{x}{k} \right] - \frac{1}{2} \sum_{d|x} a_d.$$

It remains to show that the sum over  $k > x$  in (2.1.9) tends to 0 as  $T \rightarrow \infty$ . We use

Lemma 2.1.2 with  $r = x/k$ :

$$\begin{aligned}
\left| \sum_{k>x} a_k \left( \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \zeta(s) \frac{(x/k)^s}{s} ds \right) \right| &\leq \sum_{k>x} \frac{|a_k|}{2\pi} \left| \int_{c-iT}^{c+iT} \zeta(s) \frac{(x/k)^s}{s} ds \right| \\
&\leq \sum_{k>x} |a_k| \left( \frac{\zeta(c)(x/k)^c}{\pi T \log(k/x)} \right) \\
&\leq \frac{\zeta(c)x^c}{\pi T \log([x]+1)/x)} \sum_{k>x} \frac{|a_k|}{k^c} \\
&\leq \frac{\zeta(c)x^c}{\pi T \log([x]+1)/x)} \sum_{k\geq 1} \frac{|a_k|}{k^c}, \quad (2.1.10)
\end{aligned}$$

where we have used the absolute convergence of the Dirichlet series on  $\operatorname{Re}(s) = c$ . As  $T \rightarrow \infty$ , (2.1.10) tends to 0.  $\blacksquare$

To apply Theorem 2.1.3 to  $\log(n!_K)$ , compare (2.1.4) and (2.1.8). Using the notation from (1.2.1), set  $a_k = b_K(k)$ , so  $\sum_{k\geq 1} a_k/k^s = -\zeta'_K(s)/\zeta_K(s)$ . The sum  $\sum_{d|n} a_d$  is the  $n$ th Dirichlet coefficient of  $-(\zeta'_K(s)/\zeta_K(s))\zeta(s)$ , which we will denote by  $B_K(n)$ . Tying this all together, we have

$$\log(n!_K) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} -\frac{\zeta'_K(s)}{\zeta_K(s)} \zeta(s) \frac{n^s}{s} ds + \frac{1}{2} B_K(n) \quad (2.1.11)$$

for all  $n \geq 1$  and  $c > 1$ . When  $K = \mathbb{Q}$  we have  $B_K(n) = \log n$ , and since  $-\zeta'_K(s)/\zeta_K(s)$  and  $\zeta(s)$  have nonnegative coefficients,  $B_K(n) \geq 0$ . The following theorem gives an upper bound on  $B_K(n)$  in general.

**Theorem 2.1.4.** *Let  $K$  be a number field. The  $n$ th Dirichlet coefficient  $B_K(n)$  of  $-(\zeta'_K(s)/\zeta_K(s))\zeta(s)$  satisfies*

$$B_K(n) \leq [K : \mathbb{Q}] \log n,$$

with equality exactly when every prime dividing  $n$  splits completely in  $K$ .

*Proof.* Write  $B_K(n) = \sum_{\mathfrak{N}\mathfrak{a}|n} \Lambda_K(\mathfrak{a})$ , where  $\Lambda_K(\mathfrak{a}) = \log \mathfrak{N}\mathfrak{p}$  if  $\mathfrak{a} = \mathfrak{p}^k$  with  $k \geq 1$  and  $\Lambda_K(\mathfrak{a}) = 0$  otherwise. Then

$$B_K(n) = \sum_{p|n} \sum_{\mathfrak{N}\mathfrak{a}|p^{e_p}} \Lambda_K(\mathfrak{a}),$$

where  $e_p$  is the maximum power of  $p$  dividing  $n$ . For each prime  $p$ , let  $\mathfrak{p}_1, \dots, \mathfrak{p}_r$  be all of the prime ideals in  $\mathcal{O}_K$  dividing  $p$ . Then each nonzero  $\Lambda_K(\mathfrak{a})$ , where  $\mathfrak{N}\mathfrak{a} | p^{e_p}$ , is some  $\log \mathfrak{N}\mathfrak{p}_i = \log p^{f_i}$ , where  $f_i$  is the residue field degree of  $\mathfrak{p}_i$ . Since  $\Lambda_K(\mathfrak{a}) \neq 0$  and  $\mathfrak{N}\mathfrak{a} | p^{e_p}$  if and only if  $\mathfrak{a} = \mathfrak{p}_i^k$ , where  $1 \leq k \leq e_p/f_i$ , we have

$$\sum_{\mathfrak{N}\mathfrak{a}|p^{e_p}} \Lambda_K(\mathfrak{a}) = \sum_{i=1}^r \sum_{\mathfrak{N}\mathfrak{p}_i^k|p^{e_p}} \Lambda_K(\mathfrak{p}_i^k).$$

There are at most  $e_p/f_i$  occurrences of  $\log \mathfrak{N}\mathfrak{p}_i$  in the sum when  $\mathfrak{a}$  is a power of  $\mathfrak{p}_i$ , so

$$\sum_{i=1}^r \sum_{\mathfrak{N}\mathfrak{p}_i^k|p^{e_p}} \Lambda_K(\mathfrak{p}_i^k) \leq \sum_{i=1}^r \frac{e_p}{f_i} \log p^{f_i} = \#\{\mathfrak{p} | p\} \log p^{e_p} \leq [K : \mathbb{Q}] \log p^{e_p}. \quad (2.1.12)$$

Summing this over all  $p$  dividing  $n$ , we have

$$\sum_{\mathfrak{N}\mathfrak{a}|n} \Lambda_K(\mathfrak{a}) \leq [K : \mathbb{Q}] \log n.$$

Equality in (2.1.12) occurs exactly when  $e_p/f_i \in \mathbb{Z}$  for each  $f_i$  and  $\#\{\mathfrak{p} | p\} = [K : \mathbb{Q}]$  for all  $p$  dividing  $n$ , which occur exactly when every prime dividing  $n$  splits completely in  $K$ . ■

We want to evaluate the integral in (2.1.11) by shifting the contour to the left.

To apply the residue theorem, we first truncate the integral.

**Lemma 2.1.5.** *Suppose  $f(s)$  is a Dirichlet series that converges absolutely for  $\operatorname{Re}(s) > 1$ .*

1. *For  $c > 1$ ,  $T > 1$ , and  $n \in \mathbb{Z}^+$ ,*

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} f(s) \frac{n^s}{s} ds = \frac{1}{2\pi i} \int_{c-iT}^{c+iT} f(s) \frac{n^s}{s} ds + O_{c,f} \left( \frac{n^{1+c}}{T} \right). \quad (2.1.13)$$

*Proof.* Set  $f(s) = \sum_{k \geq 1} a_k/k^s$ . Since  $\sum_{k \geq 1} (a_k/k^s)(n^s/s)$  converges uniformly on compact subsets of  $\operatorname{Re}(s) > 1$ , it follows that

$$\frac{1}{2\pi i} \int_{c-iT}^{c+iT} f(s) \frac{n^s}{s} ds = \sum_{k \geq 1} \frac{a_k}{2\pi i} \int_{c-iT}^{c+iT} \frac{1}{k^s} \frac{n^s}{s} ds. \quad (2.1.14)$$

Using Perron's formula and (2.1.14),

$$\begin{aligned} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} f(s) \frac{n^s}{s} ds - \frac{1}{2\pi i} \int_{c-iT}^{c+iT} f(s) \frac{n^s}{s} ds \\ = \sum_{k < n} a_k + \frac{1}{2} a_n - \sum_{k \geq 1} a_k \left( \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \frac{(n/k)^s}{s} ds \right), \end{aligned}$$

and the right side is equal to

$$\begin{aligned} \sum_{k < n} a_k \left( 1 - \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \frac{(n/k)^s}{s} ds \right) + a_n \left( \frac{1}{2} - \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \frac{1}{s} ds \right) \\ - \sum_{k > n} a_k \left( \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \frac{(n/k)^s}{s} ds \right). \end{aligned}$$

From [1, p. 243] we have for  $T > 0$  the bounds

$$\left| 1 - \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \frac{x^s}{s} ds \right| \leq \frac{x^c}{\pi T \log x} \quad \text{for } x > 1, \quad (2.1.15)$$

$$\left| \frac{1}{2} - \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \frac{1}{s} ds \right| \leq \frac{c}{\pi T}, \quad \text{and} \quad (2.1.16)$$

$$\left| \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \frac{x^s}{s} ds \right| \leq \frac{x^c}{\pi T \log(1/x)} \quad \text{for } 0 < x < 1. \quad (2.1.17)$$

Using (2.1.15) and absolute convergence of  $f(s)$  on  $\text{Re}(s) = c$ , we see for  $n \geq 2$  that

$$\begin{aligned} \left| \sum_{k < n} a_k \left( 1 - \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \frac{(n/k)^s}{s} ds \right) \right| &\leq \sum_{k < n} |a_k| \frac{(n/k)^c}{\pi T \log(n/k)} \\ &\leq \frac{n^c}{\pi T \log(\frac{n}{n-1})} \sum_{k < n} \frac{|a_k|}{k^c} \\ &\leq \frac{n^c}{\pi T \log(\frac{n}{n-1})} \sum_{k \geq 1} \frac{|a_k|}{k^c} \\ &= O_{c,f} \left( \frac{n^c}{T \log(\frac{n}{n-1})} \right). \end{aligned}$$

From (2.1.16) we get

$$\begin{aligned} \left| a_n \left( \frac{1}{2} - \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \frac{1}{s} ds \right) \right| &\leq |a_n| \frac{c}{\pi T} \\ &= \frac{c}{\pi} \frac{|a_n|}{n^c} \frac{n^c}{T} \\ &\leq \frac{c}{\pi} \left( \sum_{k \geq 1} \frac{|a_k|}{k^c} \right) \frac{n^c}{T} \\ &= O_{c,f} \left( \frac{n^c}{T} \right), \end{aligned}$$

and using (2.1.17) we get

$$\begin{aligned}
\sum_{k>n} \left| a_k \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \frac{(n/k)^s}{s} ds \right| &\leq \sum_{k>n} |a_k| \left( \frac{(n/k)^c}{\pi T \log(k/n)} \right) \\
&\leq \frac{n^c}{\pi T \log(\frac{n+1}{n})} \sum_{k>n} \frac{|a_k|}{k^c} \\
&\leq \frac{n^c}{\pi T \log(\frac{n+1}{n})} \sum_{k\geq 1} \frac{|a_k|}{k^c} \\
&= O_{c,f} \left( \frac{n^c}{T \log(\frac{n+1}{n})} \right).
\end{aligned}$$

Since  $\log(1+x) \geq \frac{1}{2}x$  for  $0 \leq x \leq 1$ , it follows that  $1/\log(1+1/n) \leq 2n$  for  $n \geq 1$ , so  $1/\log((n+1)/n) = O(n)$  as  $n \rightarrow \infty$ . This implies that  $1/\log(n/(n-1)) = O(n-1) = O(n)$  as  $n \rightarrow \infty$ . So we conclude that

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} f(s) \frac{n^s}{s} ds - \frac{1}{2\pi i} \int_{c-iT}^{c+iT} f(s) \frac{n^s}{s} ds = O_{c,f} \left( \frac{n^{1+c}}{T} \right).$$

■

In Section 2.2 we will need several Laurent expansions in order to compute the residues of the integrand  $-(\zeta'_K(s)/\zeta_K(s))\zeta(s)n^s/s$  in (2.1.11). It has poles at  $s = 0$ ,  $s = 1$ , and the zeros of  $\zeta_K(s)$ .

We will list the Laurent series that will be needed in the calculations that follow. Near  $s = 1$  we have (see (1.2.3) and (1.2.4))

$$\zeta(s) = \frac{1}{s-1} + \gamma + O(s-1) \tag{2.1.18}$$

$$\frac{\zeta'_K(s)}{\zeta_K(s)} = -\frac{1}{s-1} + \gamma_K + O(s-1) \tag{2.1.19}$$

$$\frac{n^s}{s} = n + (n \log n - n)(s-1) + O((s-1)^2), \tag{2.1.20}$$

and near  $s = 0$  we have

$$\zeta(s) = -\frac{1}{2} - \frac{1}{2}\log(2\pi)s + O(s^2) \quad (2.1.21)$$

$$\frac{\zeta'_K(s)}{\zeta_K(s)} = \frac{r_1 + r_2 - 1}{s} + C_K + O(s) \quad (2.1.22)$$

$$\frac{n^s}{s} = \frac{1}{s} + \log n + O(s), \quad (2.1.23)$$

where  $C_K$  is a constant. When  $K$  is  $\mathbb{Q}$  or imaginary quadratic,  $C_K = \zeta'_K(0)/\zeta_K(0)$ . In particular,  $C_{\mathbb{Q}} = \zeta'(0)/\zeta(0) = \log(2\pi)$ . Here is a general formula for  $C_K$ .

**Theorem 2.1.6.** *For any number field  $K$ , we have  $C_K = [K : \mathbb{Q}](\gamma + \log(2\pi)) - \log|d_K| - \gamma_K$ , where  $d_K$  is the discriminant of  $K$ .*

*Proof.* See Section 4.1. ■

When  $K = \mathbb{Q}$ , Theorem 2.1.6 recovers the formula we gave above for  $C_{\mathbb{Q}}$ :

$$C_{\mathbb{Q}} = (\gamma + \log(2\pi)) - \log(1) - \gamma_{\mathbb{Q}} = \log(2\pi).$$

## 2.2 Contours, Residues, and Zeros of $\zeta_K(s)$

Now we return to calculating the asymptotic formula for  $\log(n!_K)$ . Let  $c > 1$ . Applying Lemma 2.1.5 to (2.1.11), for each positive integer  $n$  we have

$$\begin{aligned} \log(n!_K) &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} -\frac{\zeta'_K(s)}{\zeta_K(s)} \zeta(s) \frac{n^s}{s} ds + \frac{1}{2} B_K(n) \\ &= \frac{1}{2\pi i} \int_{c-iT}^{c+iT} -\frac{\zeta'_K(s)}{\zeta_K(s)} \zeta(s) \frac{n^s}{s} ds + \frac{1}{2} B_K(n) + O_{c,K} \left( \frac{n^{1+c}}{T} \right) \end{aligned} \quad (2.2.1)$$

for all  $T > 1$ . Let  $T$  *not* be the height of a zero of  $\zeta_K(s)$ , i.e., a pole of  $\zeta'_K(s)/\zeta_K(s)$  (we will see how to avoid such heights in Theorem 2.2.1). Treat the vertical line from  $c - iT$  to  $c + iT$  as the right side of a rectangle oriented clockwise with corners  $c - iT$ ,  $c + iT$ ,  $-b + iT$ , and  $-b - iT$ , where  $b > 0$ . See Figure 2.2. When  $K$  is not totally real, the integrand has poles at negative odd integers, so it is imperative that  $b \notin \mathbb{Z}^+$ . The integral along the right side can be written in terms of the integrals along the top, left, and bottom plus the sum of the residues inside the boundary. We will use  $0 < b < 1$ .

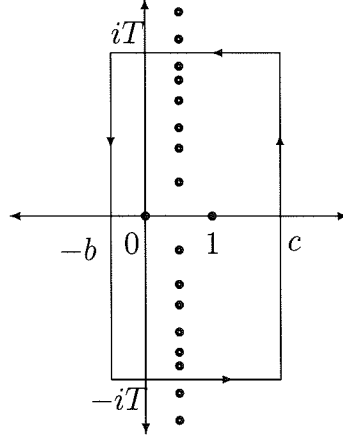


FIGURE 2.2: A contour to calculate the integral in (2.2.1). Dots are poles of the integrand.

Now we will compute the residues for each pole inside the rectangle in Figure 2.2.

- Near  $s = 1$  we multiply the Laurent series for each factor of the integrand, given in (2.1.18), (2.1.19), and (2.1.20), to get

$$-\frac{\zeta'_K(s)}{\zeta_K(s)}\zeta(s)\frac{n^s}{s} = \frac{n}{(s-1)^2} + \frac{n \log n - n + n(\gamma - \gamma_K)}{s-1} + \dots,$$

and thus

$$\text{Res}_{s=1} -\frac{\zeta'_K(s)}{\zeta_K(s)}\zeta(s)\frac{n^s}{s} = n \log n - (\gamma_K - \gamma + 1)n. \quad (2.2.2)$$



The residue at the pole with the largest real part aligns perfectly with the dominant terms we discovered in Chapter 1.

- Near  $s = 0$ , we multiply the Laurent series given in (2.1.21), (2.1.22), and (2.1.23) to get

$$-\frac{\zeta'_K(s)}{\zeta_K(s)}\zeta(s)\frac{n^s}{s} = \frac{(r_1 + r_2 - 1)/2}{s^2} + \frac{(r_1 + r_2 - 1)(\log n + \log(2\pi))/2 + C_K/2}{s} + \dots,$$

so

$$\text{Res}_{s=0} -\frac{\zeta'_K(s)}{\zeta_K(s)}\zeta(s)\frac{n^s}{s} = \frac{r_1 + r_2 - 1}{2}(\log n + \log(2\pi)) + \frac{1}{2}C_K. \quad (2.2.3)$$

Recall that  $C_K = [K : \mathbb{Q}](\gamma + \log(2\pi)) - \log |d_K| - \gamma_K$  by Theorem 2.1.6.

- At any nontrivial zero  $\rho$  of  $\zeta_K(s)$ , there is at worst a simple pole of the integrand, with

$$\text{Res}_{s=\rho} -\frac{\zeta'_K(s)}{\zeta_K(s)}\zeta(s)\frac{n^s}{s} = -(\text{ord}_{s=\rho} \zeta_K(s))\zeta(\rho)\frac{n^\rho}{\rho}. \quad (2.2.4)$$

If  $\rho$  is also a zero of  $\zeta(s)$ , the residue in (2.2.4) is 0, and thus we only have a nonzero residue at each nontrivial zero of  $\zeta_K(s)$  that is *not* a zero of  $\zeta(s)$ . When  $K = \mathbb{Q}$  there is no contribution from any zeta-zeros, which we now understand as accounting for “why” the classical Stirling formula for  $\log(n!)$  is accessible to methods that don’t involve number theory (no nontrivial zeros of  $\zeta(s)$  appear in the asymptotic estimate for  $\log(n!)$ ). It is widely believed that  $\zeta_K(s)/\zeta(s)$  is holomorphic (it is known if  $K/\mathbb{Q}$  is abelian, but not in general), so every zero of  $\zeta(s)$  should occur among the zeros of  $\zeta_K(s)$ .

Adding up the residues of  $-(\zeta'_K(s)/\zeta_K(s))\zeta(s)n^s/s$  in (2.2.2), (2.2.3), and (2.2.4), we get from (2.2.1) and the residue theorem that, for  $n \geq 1$  and for  $T$  not the height of a zero of  $\zeta_K(s)$ ,

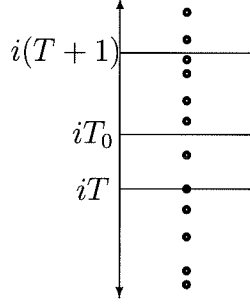
$$\begin{aligned} \log(n!_K) &= n \log n - (\gamma_K - \gamma + 1)n + \frac{r_1 + r_2 - 1}{2}(\log(2\pi n)) \\ &\quad - \sum_{|\operatorname{Im} \rho| < T} \zeta(\rho) \frac{n^\rho}{\rho} + \frac{1}{2}B_K(n) + \frac{1}{2}C_K \\ &\quad + \frac{1}{2\pi i} \int_{\omega(b,c,T)} \frac{\zeta'_K(s)}{\zeta_K(s)} \zeta(s) \frac{n^s}{s} ds + O_{c,K} \left( \frac{n^{1+c}}{T} \right), \end{aligned} \quad (2.2.5)$$

where the sum is over nontrivial zeros  $\rho$  of  $\zeta_K(s)$  counted with multiplicities and  $\omega(b, c, T)$  is the contour traversing three edges of the boundary in Figure 2.2: from  $c + iT$  to  $-b + iT$ ,  $-b + iT$  to  $-b - iT$ , and  $-b - iT$  to  $c - iT$ .

The formula in (2.2.5) needs care with the choice of  $T$ , the height of the box above and below the  $x$ -axis. The integrand of (2.2.1) has poles at the zeros of  $\zeta_K(s)$  that are not zeros of  $\zeta(s)$ . If the lines  $\operatorname{Im} s = \pm T$  intersect such poles, then (2.2.5) is invalid. To allow such  $T$ , we use information about the density of zeros in the critical strip [6, p. 102]:

**Theorem 2.2.1.** *For  $T > 1$ , the square defined by  $0 \leq \operatorname{Re}(s) \leq 1$  and  $T \leq \operatorname{Im}(s) \leq T + 1$  contains  $O_K(\log T)$  zeros of  $\zeta_K(s)$ , counted with multiplicity.*

Given  $T > 1$ , in order to develop a version of (2.2.5) that can use  $T$  even if it is the height of a zero of  $\zeta_K(s)$ , it may be necessary to adjust the height of the box so that the contour of the integral in (2.2.5) is not too close to a pole of  $-(\zeta'_K(s)/\zeta_K(s))\zeta(s)n^s/s$ . See Figure 2.3.

FIGURE 2.3: Determining an appropriate height  $T_0$ .

**Corollary 2.2.2.** *There is an  $\alpha_K > 0$  such that for each  $T > 1$  there exists  $T_0$  with  $T < T_0 < T + 1$  such that*

$$|s - \rho| \geq \frac{\alpha_K}{\log T}$$

*for all  $s$  with  $\text{Im}(s) = T_0$  and all nontrivial zeros  $\rho$  of  $\zeta_K(s)$ .*

*Proof.* By Theorem 2.2.1 there exists a positive constant  $\alpha_K$  such that if we divide the square defined by  $0 < \text{Re}(s) < 1$  and  $T \leq \text{Im}(s) \leq T+1$  into successive horizontal strips that each have vertical width  $2\alpha_K/(\log T)$ , at least one of these strips does not contain any zero of  $\zeta_K(s)$ . By choosing  $T_0$  in the center of this strip, we can guarantee that the distance between the line  $\text{Im}(s) = T_0$  and the nearest zero of  $\zeta_K(s)$  will be at least  $\alpha_K/(\log T)$ . ■

For any  $T > 1$ , choose  $T_0$  based on Corollary 2.2.2, satisfying  $T < T_0 < T + 1$  such that the interior of the rectangle in the critical strip with top and bottom edges along  $\text{Im}(s) = T_0 \pm \alpha_K/(\log T)$  has no zeros of  $\zeta_K(s)$ . The conjugate box in the lower half-plane also has no zeros of  $\zeta_K(s)$ . We can build a rectangle similar to that in Figure 2.2 using height  $T_0$  instead of  $T$ , and then we can apply the residue theorem

as in (2.2.5), getting

$$\begin{aligned}
\log(n!_K) &= n \log n - (\gamma_K - \gamma + 1)n + \frac{r_1 + r_2 - 1}{2}(\log(2\pi n)) \\
&\quad - \sum_{|\operatorname{Im} \rho| < T_0} \zeta(\rho) \frac{n^\rho}{\rho} + \frac{1}{2}B_K(n) + \frac{1}{2}C_K \\
&\quad + \frac{1}{2\pi i} \int_{\omega(b, c, T_0)} \frac{\zeta'_K(s)}{\zeta_K(s)} \zeta(s) \frac{n^s}{s} ds + O_{c, K} \left( \frac{n^{1+c}}{T_0} \right) \quad (2.2.6)
\end{aligned}$$

for all  $n \in \mathbb{Z}^+$ . To replace  $T_0$  by  $T$  throughout (2.2.6), we will absorb the horizontal portions from the contour  $\omega(b, c, T_0)$  into an error term. Up to now we took  $0 < b < 1$ .

We will now restrict  $b$  further.

**Lemma 2.2.3.** *For  $0 < b < \frac{1}{2}$ ,  $c > 1$ , and  $\varepsilon > 0$ , and  $T_0$  chosen as in Corollary 2.2.2,*

$$\left| \frac{1}{2\pi i} \int_{c+iT_0}^{-b+iT_0} \frac{\zeta'_K(s)}{\zeta_K(s)} \zeta(s) \frac{n^s}{s} ds \right| = \left| \frac{1}{2\pi i} \int_{-b-iT_0}^{c-iT_0} \frac{\zeta'_K(s)}{\zeta_K(s)} \zeta(s) \frac{n^s}{s} ds \right| = O_{b, c, \varepsilon, K} \left( \frac{n^c}{T_0^{1/2-b-\varepsilon}} \right).$$

*Proof.* Making the substitution  $s = x + iT_0$  and using the triangle inequality,

$$\begin{aligned}
\left| \frac{1}{2\pi i} \int_{c+iT_0}^{-b+iT_0} \frac{\zeta'_K(s)}{\zeta_K(s)} \zeta(s) \frac{n^s}{s} ds \right| &= \left| \frac{1}{2\pi i} \int_{-b}^c \frac{\zeta'_K(x + iT_0)}{\zeta_K(x + iT_0)} \zeta(x + iT_0) \frac{n^{x+iT_0}}{x + iT_0} dx \right| \\
&\leq \frac{n^c}{2\pi T_0} \int_{-b}^c \left| \frac{\zeta'_K(x + iT_0)}{\zeta_K(x + iT_0)} \right| |\zeta(x + iT_0)| dx.
\end{aligned}$$

For  $0 < b < \frac{1}{2}$ ,  $c > 1$ , and  $T_0 > 1$ , we will show that

$$\max_{-b \leq x \leq c} \left| \frac{\zeta'_K(x + iT_0)}{\zeta_K(x + iT_0)} \right| = O_K(\log^2 T_0). \quad (2.2.7)$$

By [6, p. 103], for  $-\frac{1}{2} < \operatorname{Re}(s) < c$  and  $|\operatorname{Im}(s)| > 1$  with  $\zeta_K(s) \neq 0$ ,

$$\frac{\zeta'_K(s)}{\zeta_K(s)} + \frac{\rho_K}{s} + \frac{\rho_K}{s-1} - \sum_{|\rho-s|<1} \frac{1}{\rho-s} = O(\log |d_K| + [K : \mathbb{Q}] \log |s|) = O_K(\log |s|),$$

where  $\rho_K$  is the residue of  $\zeta_K(s)$  at  $s = 1$  and the sum is over nontrivial zeros  $\rho$  of  $\zeta_K(s)$ , counted with multiplicity. The terms  $\rho_K/s$  and  $\rho_K/(s-1)$  are  $O_K(1)$  for  $|\operatorname{Im}(s)| > 1$ . By Theorem 2.2.1, the number of zeros of  $\zeta_K(s)$  (counted with multiplicity) in the rectangle within the critical strip between  $\operatorname{Im} s = T_0$  and  $\operatorname{Im} s = T_0 + 1$  is  $O_K(\log T_0)$ . Between  $\operatorname{Im} s = T_0$  and  $\operatorname{Im} s = T_0 - 1$ , the number of zeros is  $O_K(\log(T_0 - 1)) = O_K(\log T_0)$ . By Corollary 2.2.2 we have  $1/|\rho - s| = O_K(\log T_0)$  for  $\operatorname{Im}(s) = T_0$ , so

$$\sum_{|\rho-s|<1} \frac{1}{|\rho-s|} = O_K(\log^2 T_0), \quad (2.2.8)$$

and (2.2.7) follows. Furthermore, for  $\varepsilon > 0$ ,  $b > 0$ ,  $c > 1$ , and  $T_0 > 1$ , we have by [9, p. 23]<sup>1</sup> that

$$\max_{-b \leq x \leq c} |\zeta(x + iT_0)| = O_{b,c,\varepsilon}(T_0^{1/2+b+\varepsilon/2}). \quad (2.2.9)$$

Combining (2.2.7) and (2.2.9) gives us

$$\begin{aligned} \left| \frac{1}{2\pi i} \int_{c+iT_0}^{-b+iT_0} -\frac{\zeta'_K(s)}{\zeta_K(s)} \zeta(s) \frac{n^s}{s} ds \right| &= O_{b,c,\varepsilon,K} \left( \frac{n^c}{T_0} \cdot \log^2 T_0 \cdot T_0^{1/2+b+\varepsilon/2} \right) \\ &= O_{b,c,\varepsilon,K} \left( \frac{n^c \log^2 T_0}{T_0^{1/2-b-\varepsilon/2}} \right) \\ &= O_{b,c,\varepsilon,K} \left( \frac{n^c}{T_0^{1/2-b-\varepsilon}} \right) \end{aligned}$$

since  $\log^2 t = O_\varepsilon(t^{\varepsilon/2})$  for  $t \geq 1$ . Applying complex conjugation throughout this

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<sup>1</sup>The proof of this estimate uses the classical Stirling's formula.

process yields the same result for the horizontal integral along  $\text{Im}(s) = -T_0$ .  $\blacksquare$

Applying Lemma 2.2.3 to (2.2.6) yields the formula

$$\begin{aligned} \log(n!_K) &= n \log n - (\gamma_K - \gamma + 1)n + \frac{r_1 + r_2 - 1}{2} (\log n + \log 2\pi) \\ &\quad - \sum_{|\text{Im } \rho| < T_0} \zeta(\rho) \frac{n^\rho}{\rho} + \frac{1}{2} B_K(n) + \frac{1}{2} C_K - \frac{1}{2\pi i} \int_{-b-iT_0}^{-b+iT_0} \frac{\zeta'_K(s)}{\zeta_K(s)} \zeta(s) \frac{n^s}{s} ds \\ &\quad + O_{c,K} \left( \frac{n^{1+c}}{T_0} \right) + O_{b,c,\varepsilon,K} \left( \frac{n^c}{T_0^{1/2-b-\varepsilon}} \right), \end{aligned} \quad (2.2.10)$$

and in the two error terms we can replace  $T_0$  with  $T$  since  $T < T_0 < T + 1$ . How much error is created if we replace  $T_0$  with  $T$  in the sum over  $\rho$  and in the integral along  $\text{Re}(s) = -b$ ?

**Lemma 2.2.4.** *For  $n \in \mathbb{Z}^+$ ,  $T > 1$ , and  $T < T_0 < T + 1$ ,*

$$\sum_{|\text{Im } \rho| < T_0} \zeta(\rho) \frac{n^\rho}{\rho} = \sum_{|\text{Im } \rho| < T} \zeta(\rho) \frac{n^\rho}{\rho} + O_K \left( \frac{n^{\beta_K(T)}}{T^{1/4}} \right),$$

where  $\beta_K(T) = 1 - \lambda_K / \log(T + 3)$  for some positive constant  $\lambda_K$  depending only on  $K$  and  $\frac{1}{2} \leq \beta_K(T) < 1$  (see Theorem 3.2.1). Under GRH for  $\zeta_K(s)$ ,  $\beta_K(T) = \frac{1}{2}$  for all  $T$ .

**Lemma 2.2.5.** *For  $0 < b < \frac{1}{2}$ ,  $n \in \mathbb{Z}^+$ ,  $T > 1$ ,  $T < T_0 < T + 1$ , and all  $\varepsilon > 0$ , we have*

$$\int_{-b-iT_0}^{-b+iT_0} \frac{\zeta'_K(s)}{\zeta_K(s)} \zeta(s) \frac{n^s}{s} ds = \int_{-b-iT}^{-b+iT} \frac{\zeta'_K(s)}{\zeta_K(s)} \zeta(s) \frac{n^s}{s} ds + O_{b,\varepsilon,K} \left( \frac{1}{T^{1/2-b-\varepsilon}} \right).$$

See Section 4.2 for the proofs of Lemmas 2.2.4 and 2.2.5. Applying these lemmas to (2.2.10), we now have a formula for all  $T > 1$ :

**Theorem 2.2.6.** *Let  $n \in \mathbb{Z}^+$  and  $T > 1$ . Then for  $0 < b < \frac{1}{2}$ ,  $c > 1$ , and  $\varepsilon > 0$ ,*

$$\begin{aligned} \log(n!_K) &= n \log n - (\gamma_K - \gamma + 1)n + \frac{r_1 + r_2 - 1}{2}(\log(2\pi n)) \\ &\quad - \sum_{|\operatorname{Im} \rho| < T} \zeta(\rho) \frac{n^\rho}{\rho} + \frac{1}{2}B_K(n) + \frac{1}{2}C_K - \frac{1}{2\pi i} \int_{-b-iT}^{-b+iT} \frac{\zeta'_K(s)}{\zeta_K(s)} \zeta(s) \frac{n^s}{s} ds \\ &\quad + O_{c,K} \left( \frac{n^{1+c}}{T} \right) + O_K \left( \frac{n^{\beta_K(T)}}{T^{1/4}} \right) + O_{b,c,\varepsilon,K} \left( \frac{n^c}{T^{1/2-b-\varepsilon}} \right), \end{aligned}$$

where  $\frac{1}{2} \leq \beta_K(T) < 1$ .

All three error terms tend to 0 as  $T \rightarrow \infty$  with  $n$  fixed, so the convergence of the sum over  $\rho$  in Theorem 2.2.6 as  $T \rightarrow \infty$  is equivalent to convergence of the integral over  $[-b-iT, -b+iT]$  as  $T \rightarrow \infty$ . The second  $O$ -term can be replaced with  $O(\sqrt{n}/T^{1/4})$  under GRH for  $\zeta_K(s)$ .

For the case  $K = \mathbb{Q}$ , we have  $C_{\mathbb{Q}} = \log(2\pi)$  and  $B_{\mathbb{Q}}(n) = \log n$ , so letting  $T \rightarrow \infty$  in Theorem 2.2.6 gives us

$$\log(n!) = n \log n - n + \frac{1}{2} \log n + \frac{1}{2} \log(2\pi) - \frac{1}{2\pi i} \int_{-b-i\infty}^{-b+i\infty} \zeta'(s) \frac{n^s}{s} ds$$

for  $n \geq 1$  and  $0 < b < \frac{1}{2}$ . Stirling's formula (classically) is equivalent to

$$\int_{-b-i\infty}^{-b+i\infty} \zeta'(s) \frac{n^s}{s} ds \rightarrow 0 \tag{2.2.11}$$

as  $n \rightarrow \infty$ , where  $0 < b < \frac{1}{2}$ .

What about when  $K \neq \mathbb{Q}$ ? Let's compare the formula in Theorem 2.2.6 up through the term  $\frac{1}{2}C_K$  with the oscillatory data in Table 1.1. In Theorem 2.2.6, the coefficient  $\gamma_K - \gamma + 1$  was denoted  $A_K$  in Chapter 1. Let  $S_{K,T}(n)$  be the sum over

zeros  $\rho$  up to height  $T$ . To test the convergence of this sum, we use Sage to compute  $S_{\mathbb{Q}(i),T}(2)$ ,  $S_{\mathbb{Q}(i),T}(10)$ , and  $S_{\mathbb{Q}(i),T}(50)$  with increasing values of  $T$  in each case. See Table 2.1.

$T$	$S_{\mathbb{Q}(i),T}(2)$	$S_{\mathbb{Q}(i),T}(10)$	$S_{\mathbb{Q}(i),T}(50)$
20000	-2.20361	4.92109	-8.13464
40000	-2.20509	4.92394	-8.13932
60000	-2.20488	4.92055	-8.14834
80000	-2.20630	4.92159	-8.13969
100000	-2.20642	4.92258	-8.14054

TABLE 2.1: Testing the convergence for  $S_{\mathbb{Q}(i),T}(n)$  using fixed integers  $n$ .

As  $T$  increases, the sum  $S_{\mathbb{Q}(i),T}(n)$  appears to remain stable. Since  $r_1 + r_2 - 1 = 0$  for  $\mathbb{Q}(i)$ , we expect

$$\log(n!_{\mathbb{Q}(i)}) - n \log n + A_{\mathbb{Q}(i)}n \approx -S_{\mathbb{Q}(i),T}(n) + \frac{1}{2}B_{\mathbb{Q}(i)}(n) + \frac{1}{2}C_{\mathbb{Q}(i)} \quad (2.2.12)$$

for large  $T$  and  $n$ . Using Sage, we use values from Table 1.1 (with  $A_{\mathbb{Q}(i)} = \gamma_{\mathbb{Q}(i)} - \gamma + 1 \approx -1.24560959$ ) and compare these in Table 2.2 to estimates of  $-S_{\mathbb{Q}(i),200000}(n) + \frac{1}{2}B_{\mathbb{Q}(i)}(n) + \frac{1}{2}C_{\mathbb{Q}(i)}$ , where  $C_{\mathbb{Q}(i)} = \zeta'_{\mathbb{Q}(i)}(0)/\zeta_{\mathbb{Q}(i)}(0) = 2\gamma + 2\log \pi - \gamma_{\mathbb{Q}(i)}$  by Theorem 2.1.6.



$n$	$\log(n!_{\mathbb{Q}(i)})$ $-n \log n + A_{\mathbb{Q}(i)}n$	$-S_{\mathbb{Q}(i),200000}(n) + \frac{1}{2}B_{\mathbb{Q}(i)}(n) + \frac{1}{2}C_{\mathbb{Q}(i)}$
50000	10.05559	$-1.29390 + 9.43348 + 1.31055 = 9.45013$
75000	49.16775	$39.22758 + 9.08691 + 1.31055 = 49.62504$
100000	109.90885	$97.98759 + 9.78006 + 1.31055 = 109.07820$
125000	-369.47232	$-383.77638 + 10.69635 + 1.31055 = -371.76948$
150000	-70.18000	$-79.25356 + 9.43348 + 1.31055 = -68.50953$
175000	256.94663	$247.70379 + 9.08691 + 1.31055 = 258.10125$
200000	373.48412	$364.27975 + 10.12663 + 1.31055 = 375.71693$

TABLE 2.2: Testing the formula for  $\log(n!_K)$  in Theorem 2.2.6 when  $K = \mathbb{Q}(i)$ .

These numerical estimates suggest that the integral over  $[-b - iT, -b + iT]$  in Theorem 2.2.6 is small when  $T$  is large. This same phenomenon occurs in data calculated for  $\mathbb{Q}(\sqrt{2})$ . The analogue of (2.2.12) for  $\mathbb{Q}(\sqrt{2})$ , where  $A_{\mathbb{Q}(\sqrt{2})} \approx 1.632115$  and  $r_1 + r_2 - 1 = 1$ , is

$$\begin{aligned} \log(n!_{\mathbb{Q}(\sqrt{2})}) - n \log n + A_{\mathbb{Q}(\sqrt{2})}n \\ \approx -S_{\mathbb{Q}(\sqrt{2}),T}(n) + \frac{1}{2} \log(2\pi n) + \frac{1}{2}B_{\mathbb{Q}(\sqrt{2})}(n) + \frac{1}{2}C_{\mathbb{Q}(\sqrt{2})}. \end{aligned}$$

See Table 2.3.

$n$	$\log(n!_{\mathbb{Q}(\sqrt{2})})$ $-n \log n + A_{\mathbb{Q}(\sqrt{2})}n$	$-S_{\mathbb{Q}(\sqrt{2}), 200000}(n) + \frac{1}{2} \log(2\pi n) + \frac{1}{2}B_{\mathbb{Q}(\sqrt{2})}(n) + \frac{1}{2}C_{\mathbb{Q}(\sqrt{2})}$
50000	-142.82749	$-152.75858 + 6.32883 + 4.60517 + 1.05932 = -140.76526$
75000	-251.78682	$-263.32389 + 6.53156 + 4.25860 + 1.05932 = -251.47441$
100000	-281.66912	$-292.58397 + 6.67540 + 4.95174 + 1.05932 = -279.89751$
125000	-257.53372	$-272.69135 + 6.78697 + 5.86803 + 1.05932 = -258.97703$
150000	-480.37984	$-487.71736 + 6.87813 + 4.60517 + 1.05932 = -475.17474$
175000	217.51523	$202.77117 + 6.95521 + 6.20451 + 1.05932 = 216.99021$
200000	325.78290	$312.17810 + 7.02197 + 5.29832 + 1.05932 = 325.55771$

TABLE 2.3: Testing the formula for  $\log(n!_K)$  in Theorem 2.2.6 when  $K = \mathbb{Q}(\sqrt{2})$ .

## 2.3 An Explicit Formula for $\log(n!_K)$

Now we wish to estimate the value of the integral along the left side of the rectangle in Figure 2.2:

$$\frac{1}{2\pi i} \int_{-b-iT}^{-b+iT} \frac{\zeta'_K(s)}{\zeta_K(s)} \zeta(s) \frac{n^s}{s} ds = \frac{1}{2\pi} \int_{-T}^T \frac{\zeta'_K(-b+iy)}{\zeta_K(-b+iy)} \zeta(-b+iy) \frac{n^{-b+iy}}{-b+iy} dy. \quad (2.3.1)$$

To get an asymptotic formula for  $\log(n!_K)$  from Theorem 2.2.6, we want to let  $T \rightarrow \infty$ , and the integral in (2.3.1) would become

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\zeta'_K(-b+iy)}{\zeta_K(-b+iy)} \zeta(-b+iy) \frac{n^{-b+iy}}{-b+iy} dy = \frac{1}{2\pi n^b} \int_{-\infty}^{\infty} f_{b,K}(y) e^{iy \log n} dy,$$

where

$$f_{b,K}(y) = \frac{\zeta'_K(-b+iy)}{\zeta_K(-b+iy)} \frac{\zeta(-b+iy)}{-b+iy}.$$

By (2.2.7) and (2.2.9), for each  $\varepsilon > 0$  we have

$$f_{b,K}(y) = O_{b,\varepsilon,K} \left( \frac{\log^2 |y|}{|y|^{1/2-b-\varepsilon/2}} \right),$$

so by choosing  $\varepsilon < 2(\frac{1}{2} - b)$ , we see that  $f_{b,K}(y) \rightarrow 0$  as  $|y| \rightarrow \infty$ .

**Conjecture 2.3.1.** *Let  $0 < b < \frac{1}{2}$ . The improper integral*

$$\int_{-\infty}^{\infty} f_{b,K}(y) e^{ixy} dy = \lim_{T \rightarrow \infty} \int_{-T}^T f_{b,K}(y) e^{ixy} dy \quad (2.3.2)$$

*exists for  $x > 0$  and*

$$\lim_{x \rightarrow \infty} \int_{-\infty}^{\infty} f_{b,K}(y) e^{ixy} dy = 0.$$

If  $f_{b,K} \in L^1(\mathbb{R})$ , then Conjecture 2.3.1 is true by the Riemann-Lebesgue lemma, but we have no reason to believe that  $f_{b,K} \in L^1(\mathbb{R})$ . Since  $f_{b,K}$  is differentiable, we can apply integration by parts to the right side of (2.3.2), and we get

$$\begin{aligned} \int_{-T}^T f_{b,K}(y) e^{ixy} dy &= f_{b,K}(y) \frac{e^{ixy}}{ix} \Big|_{-T}^T - \int_{-T}^T f'_{b,K}(y) \frac{e^{ixy}}{ix} dy \\ &= f_{b,K}(T) \frac{e^{ixT}}{ix} - f_{b,K}(-T) \frac{e^{-ixT}}{ix} - \int_{-T}^T f'_{b,K}(y) \frac{e^{ixy}}{ix} dy. \end{aligned}$$

Letting  $T \rightarrow \infty$ , the first two terms on the right tend to 0, and we see

$$\int_{-\infty}^{\infty} f_{b,K}(y) e^{ixy} dy = -\frac{1}{ix} \int_{-\infty}^{\infty} f'_{b,K}(y) e^{ixy} dy$$

if either integral exists. Thus, to prove Conjecture 2.3.1, it suffices to show that  $f'_{b,K} \in L^1(\mathbb{R})$ , but this is not a necessary condition. It would be sufficient to prove that

1.  $\widehat{f}_{b,K}(x) = \lim_{T \rightarrow \infty} \int_{-T}^T f_{b,K}(y) e^{ixy} dy$  exists for each  $x$ , and
2.  $\widehat{f}_{b,K}(x) = O(x^a)$  as  $|x| \rightarrow \infty$  for some  $a > 0$ .

Nonetheless, if the conjecture is proven, it will lead to the following theorem.

**Theorem 2.3.2.** *Let  $K$  be a number field and assume Conjecture 2.3.1. As  $n \rightarrow \infty$  in  $\mathbb{Z}^+$ , we have*

$$\begin{aligned} \log(n!_K) &= n \log n - (\gamma_K - \gamma + 1)n - \sum_{\substack{\zeta_K(\rho)=0 \\ 0 < \operatorname{Re}(\rho) < 1}} \zeta(\rho) \frac{n^\rho}{\rho} \\ &\quad + \frac{r_1 + r_2 - 1}{2} \log(2\pi n) + \frac{1}{2} B_K(n) + \frac{1}{2} C_K + o(1), \end{aligned}$$

where each term in the sum over  $\rho$  is taken with its multiplicity,  $B_K(n)$  is the  $n$ th Dirichlet coefficient of  $-(\zeta'_K(s)/\zeta_K(s))\zeta(s)$ , and  $C_K$  is the constant term in the Laurent expansion of  $\zeta'_K(s)/\zeta_K(s)$  at  $s = 0$ . Equivalently,

$$n!_K \sim \frac{n^n}{e^{(\gamma_K - \gamma + 1)n} e^{S_K(n)}} \sqrt{2\pi n}^{r_1 + r_2 - 1} \sqrt{e^{B_K(n)} e^{C_K}},$$

where  $S_K(n) = \sum_{\substack{\zeta_K(\rho)=0 \\ 0 < \operatorname{Re}(\rho) < 1}} \zeta(\rho) \frac{n^\rho}{\rho}.$

By (2.2.11) we know Conjecture 2.3.1 is true for  $K = \mathbb{Q}$ .

# Chapter 3

## A Truncated Formula for $\log(n!_K)$

When using the explicit formula for  $\psi(x) = \sum_{p^k \leq x} \log p$  to prove the Prime Number Theorem, one uses a truncated version to estimate the sum over nontrivial zeros. In this chapter we will present an analogue of this for  $\log(n!_K)$ .

### 3.1 An Estimate on the Vertical Integral

To present a truncated version of the formula in Theorem 2.2.6, we will need an estimate on the vertical integral and the sum over zeros, both in terms of  $n$  and  $T$ . We start with the integral.

**Theorem 3.1.1.** *For  $T > 1$ ,  $n > 1$ ,  $0 < b < \frac{1}{2}$ , and  $\varepsilon > 0$ ,*

$$\int_{-b-iT}^{-b+iT} \frac{\zeta'_K(s)}{\zeta_K(s)} \zeta(s) \frac{n^s}{s} ds = O_{b,\varepsilon,K} \left( \frac{T^{1/2+b+\varepsilon}}{n^b} \right).$$

*Proof.* By (2.2.7),

$$\left| \frac{\zeta'_K(-b+iy)}{\zeta_K(-b+iy)} \right| = O_{b,K}(\log^2 T)$$

for  $1 \leq y \leq T$ . By (2.2.9), we have  $|\zeta(-b+iy)| = O_{b,\varepsilon}(T^{1/2+b+\varepsilon/2})$  for  $1 \leq y \leq T$ .

This implies that

$$\begin{aligned} \left| \int_{-b+i}^{-b+iT} \frac{\zeta'_K(s)}{\zeta_K(s)} \zeta(s) \frac{n^s}{s} ds \right| &\leq \int_1^T \left| \frac{\zeta'_K(-b+iy)}{\zeta_K(-b+iy)} \zeta(-b+iy) \right| \frac{n^{-b}}{y} dy \\ &= O_{b,\varepsilon,K} \left( \frac{(\log^2 T) T^{1/2+b+\varepsilon/2}}{n^b} \int_1^T \frac{dy}{y} \right) \\ &= O_{b,\varepsilon,K} \left( \frac{T^{1/2+b+\varepsilon/2} \log^3 T}{n^b} \right). \end{aligned}$$

By taking conjugates, we have

$$\left| \int_{-b-iT}^{-b-i} \frac{\zeta'_K(s)}{\zeta_K(s)} \zeta(s) \frac{n^s}{s} ds \right| = O_{b,\varepsilon,K} \left( \frac{T^{1/2+b+\varepsilon/2} \log^3 T}{n^b} \right),$$

and it is clear that

$$\left| \int_{-b-i}^{-b+i} \frac{\zeta'_K(s)}{\zeta_K(s)} \zeta(s) \frac{n^s}{s} ds \right| = O_{b,K} \left( \frac{1}{n^b} \right).$$

We use  $\log^3 T = O_\varepsilon(T^{\varepsilon/2})$  to get the desired result. ■

### 3.2 Estimates on the Sum over Zeros

For a number field  $K$ , define  $Z_K$  to be the set of nontrivial zeros of  $\zeta_K(s)$ . Now we wish to estimate the size of the finite sum in Theorem 2.2.6:

$$S_{K,T}(n) = \sum_{\substack{\rho \in Z_K \\ |\operatorname{Im}(\rho)| < T}} \zeta(\rho) \frac{n^\rho}{\rho},$$

where  $n \in \mathbb{Z}^+$  and the term in the sum associated to  $\rho$  is counted with the multiplicity of  $\rho$ . To find an upper bound on  $|S_{K,T}(n)|$  we will need to use a density estimate on the zeros of  $\zeta_K(s)$ .

**Theorem 3.2.1.** *There is a  $\lambda_K > 0$  such that, for all nontrivial zeros  $\rho$  of  $\zeta_K(s)$ ,*

$$\operatorname{Re}(\rho) \leq \beta_K(y) = 1 - \frac{\lambda_K}{\log(|y| + 3)},$$

where  $y = \operatorname{Im}(\rho)$ . Under GRH for  $\zeta_K(s)$ ,  $\operatorname{Re}(\rho) = \beta_K(y) = \frac{1}{2}$  for all  $y$ .

*Proof.* See [6, p. 128]. ■

For each  $K$ , there are only finitely many nontrivial zeros of  $\zeta_K(s)$  in the rectangle  $0 \leq \operatorname{Re}(s) \leq 1$  and  $|\operatorname{Im}(s)| < 1$ , and they do not include  $s = 0$ .

### 3.2.1 Assuming GRH

Let's assume the nontrivial zeros of  $\zeta_K(s)$  lie on the line  $\text{Re}(s) = \frac{1}{2}$ . Using Theorem 3.2.1, we have

$$\begin{aligned} |S_{K,T}(n)| &\leq \left| \sum_{\substack{\rho \in Z_K \\ 1 \leq |\text{Im}(\rho)| < T}} \zeta(\rho) \frac{n^\rho}{\rho} \right| + \left| \sum_{\substack{\rho \in Z_K \\ |\text{Im}(\rho)| < 1}} \zeta(\rho) \frac{n^\rho}{\rho} \right| \\ &= \left| \sum_{\substack{\rho \in Z_K \\ 1 \leq |\text{Im}(\rho)| < T}} \zeta(\rho) \frac{n^\rho}{\rho} \right| + O_K(\sqrt{n}). \end{aligned}$$

Now we need an upper bound on the growth of  $\zeta(s)$  in the critical strip. Under the Lindelöf Hypothesis (which is implied by RH for  $\zeta(s)$ , which in turn is implied by GRH for  $\zeta_K(s)$  if  $\zeta_K(s)/\zeta(s)$  is holomorphic, and the latter implication is known if  $K/\mathbb{Q}$  is abelian), we have

$$|\zeta(s)| = O_\varepsilon(|\text{Im}(s)|^{\varepsilon/2}) \quad (3.2.1)$$

for all  $\varepsilon > 0$ . Applying (3.2.1) to  $S_{K,T}(n)$ , we have under GRH for  $\zeta_K(s)$  and RH for  $\zeta(s)$  that

$$\begin{aligned} \left| \sum_{\substack{\rho \in Z_K \\ 1 \leq |\text{Im}(\rho)| < T}} \zeta(\rho) \frac{n^\rho}{\rho} \right| &\leq \sum_{\substack{\rho \in Z_K \\ 1 \leq |\text{Im}(\rho)| < T}} |\zeta(\rho)| \frac{n^{\text{Re}(\rho)}}{|\text{Im}(\rho)|} \\ &= O_\varepsilon \left( \sqrt{n} \sum_{\substack{\rho \in Z_K \\ 1 \leq |\text{Im}(\rho)| < T}} \frac{1}{|\text{Im}(\rho)|^{1-\varepsilon/2}} \right). \end{aligned} \quad (3.2.2)$$



By Theorem 2.2.1, for each  $H > 1$  the number of zeros of  $\zeta_K(s)$  counted with multiplicity in the critical strip such that  $H-1 \leq |\operatorname{Im}(s)| \leq H$  is  $O_K(\log(H-1)) = O_K(\log H)$ ,

so

$$\sum_{\substack{\rho \in Z_K \\ 1 \leq |\operatorname{Im}(\rho)| < T}} \frac{1}{|\operatorname{Im}(\rho)|^{1-\varepsilon/2}} = \sum_{1 \leq m < T} \frac{O_K(\log m)}{m^{1-\varepsilon/2}} = O_K(\log T) \sum_{1 \leq m < T} \frac{1}{m^{1-\varepsilon/2}}.$$

Approximating this sum with an integral gives us, for each  $\varepsilon < 1$  and  $N = [T]$ ,

$$\begin{aligned} \sum_{1 \leq m < T} \frac{1}{m^{1-\varepsilon/2}} &\leq \sum_{1 \leq m \leq N} \frac{1}{m^{1-\varepsilon/2}} \leq 1 + \int_1^N \frac{1}{t^{1-\varepsilon/2}} dt \\ &= 1 + \frac{N^{\varepsilon/2}}{\varepsilon/2} - \frac{1}{\varepsilon/2} \\ &= O_\varepsilon(T^{\varepsilon/2}). \end{aligned} \tag{3.2.3}$$

Combining (3.2.2) and (3.2.3),

$$\begin{aligned} |S_{K,T}(n)| &= O_{\varepsilon,K}(\sqrt{n}T^{\varepsilon/2} \log T) + O_K(\sqrt{n}) \\ &= O_{\varepsilon,K}(\sqrt{n}T^\varepsilon) + O_K(\sqrt{n}). \end{aligned} \tag{3.2.4}$$

We will apply Theorem 3.1.1 and (3.2.4) to the formula in Theorem 2.2.6.

**Theorem 3.2.2.** *Assume GRH for  $\zeta(s)$  and  $\zeta_K(s)$ . For  $0 < \delta < \frac{5}{18}$ ,*

$$\log(n!_K) = n \log n - (1 + \gamma_K - \gamma)n + O_{\delta,K}(n^{2/3+\delta})$$

as  $n \rightarrow \infty$  in  $\mathbb{Z}^+$ .

*Proof.* Apply the estimates in Theorem 3.1.1 and (3.2.4) to Theorem 2.2.6 to get an

estimate for  $\log(n!_K)$  for all  $n \in \mathbb{Z}^+$  and  $T > 1$ :

$$\begin{aligned} \log(n!_K) &= n \log n - (\gamma_K - \gamma + 1)n + \frac{r_1 + r_2 - 1}{2} \log(2\pi n) + \frac{1}{2}B_K(n) + \frac{1}{2}C_K \\ &\quad + O_{\varepsilon,K}(\sqrt{n} T^\varepsilon) + O_K(\sqrt{n}) + O_{b,\varepsilon,K}\left(\frac{T^{1/2+b+\varepsilon}}{n^b}\right) \\ &\quad + O_{c,K}\left(\frac{n^{1+c}}{T}\right) + O_K\left(\frac{\sqrt{n}}{T^{1/4}}\right) + O_{b,c,\varepsilon,K}\left(\frac{n^c}{T^{1/2-b-\varepsilon}}\right). \end{aligned} \quad (3.2.5)$$

We will choose  $T = n^a$  for some  $a > 1$  to be determined. To choose an appropriate  $a$ , heuristically we set  $b = 0$ ,  $c = 1$ , and  $\varepsilon = 0$ . The six  $O$ -terms become

$$O_K(\sqrt{n}) + O_K(\sqrt{n}) + O_K(n^{a/2}) + O_K(n^{2-a}) + O_K(n^{1/2-a/4}) + O_K(n^{1-a/2}).$$

By Figure 3.1, the  $a$  where the maximum exponent is minimal is  $a = \frac{4}{3}$ .

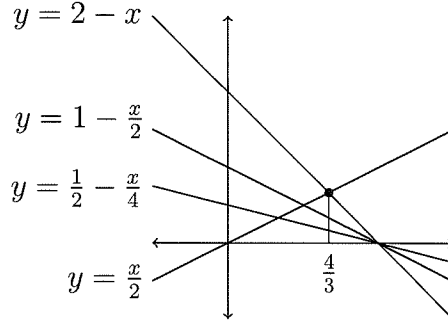


FIGURE 3.1: Choosing an optimal  $a$  for  $T = n^a$  in (3.2.5).

Returning to (3.2.5), we have  $(r_1 + r_2 - 1)(\frac{1}{2} \log(2\pi n)) = O_K(\log n)$ ,  $\frac{1}{2}B_K(n) = O_K(\log n)$  by Theorem 2.1.4, and  $\frac{1}{2}C_K = O_K(1)$ , and now we choose  $T = n^{4/3}$ ,  $b = \varepsilon$ ,

and  $c = 1 + \varepsilon$ . This gives us

$$\begin{aligned} \log(n!_K) &= n \log n - (1 + \gamma_K - \gamma)n + O_K(\log n) + O_K(\log n) + O_K(1) \\ &\quad + O_{\varepsilon,K}(n^{1/2+4\varepsilon/3}) + O_K(n^{1/2}) + O_{\varepsilon,K}(n^{2/3+5\varepsilon/3}) \\ &\quad + O_K(n^{2/3+\varepsilon}) + O_K(n^{1/6}) + O_{\varepsilon,K}(n^{1/3+11\varepsilon/3}). \end{aligned}$$

As  $\varepsilon \rightarrow 0$ , the largest exponent is  $\frac{2}{3} + \frac{5}{3}\varepsilon$ . It is the largest for  $0 < \varepsilon \leq \frac{1}{6}$ , so we set  $\delta = \frac{5}{3}\varepsilon$  and we have

$$\log(n!_K) = n \log n - (1 + \gamma_K - \gamma)n + O_{\delta,K}(n^{2/3+\delta})$$

as  $n \rightarrow \infty$ . ■

### 3.2.2 Assuming Weak Form of GRH

Rather than assuming all nontrivial zeros  $\rho$  of  $\zeta_K(s)$  lie on  $\text{Re}(s) = \frac{1}{2}$ , assume  $\text{Re}(\rho) \leq \frac{1}{2} + d$ , for some positive  $d < \frac{1}{2}$ . Then for each  $\varepsilon > 0$  we can replace the bound in (3.2.1) with the weaker bound [9, p. 23]

$$|\zeta(\rho)| = O_\varepsilon(|\text{Im}(\rho)|^{1/4+d/2+\varepsilon/2}),$$

and (3.2.2) becomes

$$\left| \sum_{\substack{\rho \in Z_K \\ 1 \leq |\text{Im}(\rho)| < T}} \zeta(\rho) \frac{n^\rho}{\rho} \right| = O_\varepsilon \left( n^{1/2+d} \sum_{\substack{\rho \in Z_K \\ 1 \leq |\text{Im}(\rho)| < T}} \frac{1}{|\text{Im}(\rho)|^{3/4-d/2-\varepsilon/2}} \right). \quad (3.2.6)$$

Since

$$\begin{aligned}
\sum_{\substack{\rho \in Z_K \\ 1 \leq |\operatorname{Im}(\rho)| < T}} \frac{1}{|\operatorname{Im}(\rho)|^{3/4-d/2-\varepsilon/2}} &= O_K \left( \log T \sum_{1 \leq m < T} \frac{1}{m^{3/4-d/2-\varepsilon/2}} \right) \\
&= O_{\varepsilon, K} \left( (\log T) T^{1/4+d/2+\varepsilon/2} \right) \\
&= O_{\varepsilon, K} (T^{1/4+d/2+\varepsilon}), \tag{3.2.7}
\end{aligned}$$

combining (3.2.6) and (3.2.7) gives an analogue of (3.2.4):

$$|S_{K,T}(n)| = O_{\varepsilon, K} (n^{1/2+d} T^{1/4+d/2+\varepsilon}) + O_K(n^{1/2+d}). \tag{3.2.8}$$

To get an analogue of Theorem 3.2.2, we start with the analogue of (3.2.5). For all  $n \in \mathbb{Z}^+$  and  $T > 1$ ,

$$\begin{aligned}
\log(n!_K) &= n \log n - (\gamma_K - \gamma + 1)n + \frac{r_1 + r_2 - 1}{2} \log(2\pi n) + \frac{1}{2}B_K(n) + \frac{1}{2}C_K \\
&\quad + O_{\varepsilon, K} (n^{1/2+d} T^{1/4+d/2+\varepsilon}) + O_K(n^{1/2+d}) + O_{b, \varepsilon, K} \left( \frac{T^{1/2+b+\varepsilon}}{n^b} \right) \\
&\quad + O_{c, K} \left( \frac{n^{1+c}}{T} \right) + O_K \left( \frac{n^{1/2+d}}{T^{1/4}} \right) + O_{b, c, \varepsilon, K} \left( \frac{n^c}{T^{1/2-b-\varepsilon}} \right). \tag{3.2.9}
\end{aligned}$$

Letting  $T = n^a$  for some  $a > 1$  and heuristically setting  $b = 0$ ,  $c = 1$ ,  $d = 0$ , and  $\varepsilon = 0$ , the last six terms are

$$O_K(n^{1/2+a/4}) + O_K(n^{1/2}) + O_K(n^{a/2}) + O_K(n^{2-a}) + O_K(n^{1/2-a/4}) + O_K(n^{1-a/2}).$$

What is the optimal choice of  $a$  in this case? As  $a \rightarrow 1^+$ , the largest exponent is

$2 - a$ . It is the largest for  $1 < a \leq \frac{6}{5}$ . See Figure 3.2.

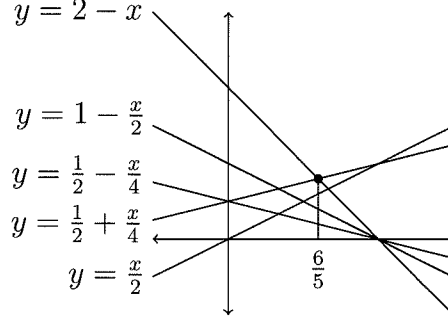


FIGURE 3.2: Choosing an optimal  $a$  for  $T = n^a$  in (3.2.9).

Choosing  $T = n^{6/5}$ ,  $b = \varepsilon$ , and  $c = 1 + \varepsilon$ , (3.2.9) becomes

$$\begin{aligned}
 \log(n!_K) &= n \log n - (\gamma_K - \gamma + 1)n + \frac{r_1 + r_2 - 1}{2} \log(2\pi n) + \frac{1}{2}B_K(n) + \frac{1}{2}C_K \\
 &\quad + O_{\varepsilon,K}(n^{4/5+8d/5+6\varepsilon/5}) + O_K(n^{1/2+d}) + O_{\varepsilon,K}(T^{3/5+7\varepsilon/5}) \\
 &\quad + O_K(n^{4/5+\varepsilon}) + O_K(n^{1/5+d}) + O_{\varepsilon,K}(n^{2/5+17\varepsilon/5}). \tag{3.2.10}
 \end{aligned}$$

As  $\varepsilon \rightarrow 0$  and  $d \rightarrow 0$ , the first  $O$ -term is dominant. It remains dominant for  $\varepsilon \leq (2 + 8d)/11$ .

**Theorem 3.2.3.** *Assume the real parts of the nontrivial zeros of  $\zeta_K(s)$  are at most  $\frac{1}{2} + d$  for some positive  $d < \frac{1}{2}$ . For  $0 < \varepsilon \leq (2 + 8d)/11$ ,*

$$\log(n!_K) = n \log n - (1 + \gamma_K - \gamma)n + O_{\varepsilon,K}(n^{4/5+8d/5+6\varepsilon/5})$$

as  $n \rightarrow \infty$  in  $\mathbb{Z}^+$ .

For the error term to be  $o(n)$ , the exponent needs to be less than 1, which corre-

sponds to  $\varepsilon < (1 - 8d)/6$ , so we need  $d < \frac{1}{8}$ . Thus  $\frac{1}{2} + d < \frac{5}{8}$ .

# Chapter 4

## Proofs from Chapter 2

### 4.1 Constant Term of the Laurent Expansion of $\zeta'_K(s)/\zeta_K(s)$ at $s = 0$

The proof of Theorem 2.1.6 will rely on two lemmas about the logarithmic derivative of the  $\Gamma$ -function.

**Lemma 4.1.1.** *Near  $s = 0$ ,*

$$\frac{\Gamma'(s)}{\Gamma(s)} = -\frac{1}{s} - \gamma + O(s), \quad \frac{1}{2} \frac{\Gamma'(s/2)}{\Gamma(s/2)} = -\frac{1}{s} - \frac{\gamma}{2} + O(s).$$

*Proof.* The  $\Gamma$ -function has simple poles at nonpositive integers. Writing  $\Gamma(s) = a/s + b + O(s)$  near  $s = 0$ , multiplying by  $s$  gives us  $\Gamma(s+1) = a + bs + O(s^2)$ , so setting  $s = 0$  implies  $a = \Gamma(1) = 1$ . Then differentiating  $\Gamma(s+1) = 1 + bs + \dots$  we get  $\Gamma'(s+1) = b + O(s)$ , so  $b = \Gamma'(1) = -\gamma$ . Thus  $\Gamma(s) = 1/s - \gamma + O(s)$ , from which the first two terms in the Laurent expansion of  $\Gamma'(s)/\Gamma(s)$  at  $s = 0$  follow by (1.2.5).

Replacing  $s$  with  $s/2$  in  $\Gamma'(s)/\Gamma(s) = -1/s - \gamma + O(s)$  gives the Laurent expansion of  $(1/2)\Gamma'(s/2)/\Gamma(s/2)$  at  $s = 0$ . ■

**Lemma 4.1.2.**  $\Gamma'(1/2)/\Gamma(1/2) = -(\gamma + \log 4)$ .

*Proof.* Take the logarithmic derivative of the duplication formula  $\Gamma(s)\Gamma(s + 1/2) = 2^{1-2s}\sqrt{\pi}\Gamma(2s)$ :

$$\frac{\Gamma'(s)}{\Gamma(s)} + \frac{\Gamma'(s + 1/2)}{\Gamma(s + 1/2)} = -\log 4 + 2\frac{\Gamma'(2s)}{\Gamma(2s)}.$$

Set  $s = 1/2$  and rearrange terms to get  $\Gamma'(1/2)/\Gamma(1/2) = -\log 4 + \Gamma'(1)/\Gamma(1) = -\log 4 - \gamma$ . ■

Now we prove Theorem 2.1.6.

*Proof.* We will use the completed zeta-function of  $K$ ,

$$Z_K(s) = |d_K|^{s/2} \Gamma_{\mathbb{R}}(s)^{r_1} \Gamma_{\mathbb{C}}(s)^{r_2} \zeta_K(s), \quad (4.1.1)$$

where  $\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2)$  and  $\Gamma_{\mathbb{C}}(s) = (2\pi)^{-s}\Gamma(s)$ . Taking logarithmic derivatives of the functional equation  $Z_K(s) = Z_K(1-s)$ ,

$$\frac{Z'_K(s)}{Z_K(s)} = -\frac{Z'_K(1-s)}{Z_K(1-s)}. \quad (4.1.2)$$

Substituting (4.1.1) into (4.1.2),

$$\begin{aligned} \frac{1}{2} \log |d_K| + r_1 \frac{\Gamma'_{\mathbb{R}}(s)}{\Gamma_{\mathbb{R}}(s)} + r_2 \frac{\Gamma'_{\mathbb{C}}(s)}{\Gamma_{\mathbb{C}}(s)} + \frac{\zeta'_K(s)}{\zeta_K(s)} \\ = -\frac{1}{2} \log |d_K| - r_1 \frac{\Gamma'_{\mathbb{R}}(1-s)}{\Gamma_{\mathbb{R}}(1-s)} - r_2 \frac{\Gamma'_{\mathbb{C}}(1-s)}{\Gamma_{\mathbb{C}}(1-s)} - \frac{\zeta'_K(1-s)}{\zeta_K(1-s)}, \end{aligned}$$



so

$$\begin{aligned} \frac{\zeta'_K(s)}{\zeta_K(s)} &= -\log |d_K| - r_1 \left( \frac{\Gamma'_\mathbb{R}(s)}{\Gamma_\mathbb{R}(s)} + \frac{\Gamma'_\mathbb{R}(1-s)}{\Gamma_\mathbb{R}(1-s)} \right) \\ &\quad - r_2 \left( \frac{\Gamma'_\mathbb{C}(s)}{\Gamma_\mathbb{C}(s)} + \frac{\Gamma'_\mathbb{C}(1-s)}{\Gamma_\mathbb{C}(1-s)} \right) - \frac{\zeta'_K(1-s)}{\zeta_K(1-s)}. \end{aligned} \quad (4.1.3)$$

We will replace each term on the right side of (4.1.3) with its Laurent expansion at  $s = 0$  up through the constant term. Since

$$\Gamma_\mathbb{R}(s) = \pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \implies \frac{\Gamma'_\mathbb{R}(s)}{\Gamma_\mathbb{R}(s)} = -\frac{1}{2} \log \pi + \frac{1}{2} \frac{\Gamma'(s/2)}{\Gamma(s/2)}, \quad (4.1.4)$$

$$\Gamma_\mathbb{C}(s) = (2\pi)^{-s} \Gamma(s) \implies \frac{\Gamma'_\mathbb{C}(s)}{\Gamma_\mathbb{C}(s)} = -\log(2\pi) + \frac{\Gamma'(s)}{\Gamma(s)}, \quad (4.1.5)$$

by Lemma 4.1.1 and (4.1.4)

$$\frac{\Gamma'_\mathbb{R}(s)}{\Gamma_\mathbb{R}(s)} = -\frac{1}{s} - \frac{1}{2}(\gamma + \log \pi) + O(s)$$

near  $s = 0$ , while Lemma 4.1.2 and (4.1.4) give us

$$\left. \frac{\Gamma'_\mathbb{R}(1-s)}{\Gamma_\mathbb{R}(1-s)} \right|_{s=0} = \frac{\Gamma'_\mathbb{R}(1)}{\Gamma_\mathbb{R}(1)} = -\frac{1}{2} \log \pi + \frac{1}{2} \frac{\Gamma'(1/2)}{\Gamma(1/2)} = -\frac{1}{2}(\log \pi + \gamma + \log 4),$$

so near  $s = 0$

$$\begin{aligned} \frac{\Gamma'_\mathbb{R}(s)}{\Gamma_\mathbb{R}(s)} + \frac{\Gamma'_\mathbb{R}(1-s)}{\Gamma_\mathbb{R}(1-s)} &= -\left( \frac{1}{s} + \frac{1}{2}(\gamma + \log \pi) + \frac{1}{2}(\log \pi + \gamma + \log 4) \right) + O(s) \\ &= -\left( \frac{1}{s} + (\gamma + \log(2\pi)) \right) + O(s). \end{aligned} \quad (4.1.6)$$

Feeding Lemma 4.1.1 into (4.1.5), near  $s = 0$

$$\frac{\Gamma'_{\mathbb{C}}(s)}{\Gamma_{\mathbb{C}}(s)} = -\frac{1}{s} - (\gamma + \log(2\pi)) + O(s)$$

and

$$\left. \frac{\Gamma'_{\mathbb{C}}(1-s)}{\Gamma_{\mathbb{C}}(1-s)} \right|_{s=0} = \frac{\Gamma'_{\mathbb{C}}(1)}{\Gamma_{\mathbb{C}}(1)} = -\log(2\pi) + \Gamma'(1) = -\log(2\pi) - \gamma,$$

so near  $s = 0$

$$\begin{aligned} \frac{\Gamma'_{\mathbb{C}}(s)}{\Gamma_{\mathbb{C}}(s)} + \frac{\Gamma'_{\mathbb{C}}(1-s)}{\Gamma_{\mathbb{C}}(1-s)} &= -\left(\frac{1}{s} + (\gamma + \log(2\pi)) + \log(2\pi) + \gamma\right) + O(s) \\ &= -\left(\frac{1}{s} + 2(\gamma + \log(2\pi))\right) + O(s). \end{aligned} \quad (4.1.7)$$

By definition of  $\gamma_K$ ,

$$\begin{aligned} \frac{\zeta'_K(s)}{\zeta_K(s)} &= -\frac{1}{s-1} + \gamma_K + O(s-1) \text{ near } s = 1 \implies \\ \frac{\zeta'_K(1-s)}{\zeta_K(1-s)} &= \frac{1}{s} + \gamma_K + O(s) \text{ near } s = 0. \end{aligned} \quad (4.1.8)$$

Feeding (4.1.6), (4.1.7), and (4.1.8) into (4.1.3),

$$\begin{aligned} \frac{\zeta'_K(s)}{\zeta_K(s)} &= -\log|d_K| + r_1 \left(\frac{1}{s} + (\gamma + \log(2\pi))\right) + r_2 \left(\frac{1}{s} + 2(\gamma + \log(2\pi))\right) \\ &\quad - \left(\frac{1}{s} + \gamma_K\right) + O(s) \\ &= \frac{r_1 + r_2 - 1}{s} - \log|d_K| + (r_1 + 2r_2)(\gamma + \log(2\pi)) - \gamma_K + O(s) \\ &= \frac{r_1 + r_2 - 1}{s} + [K : \mathbb{Q}](\gamma + \log(2\pi)) - \log|d_K| - \gamma_K + O(s), \end{aligned}$$

so  $C_K = [K : \mathbb{Q}](\gamma + \log(2\pi)) - \log|d_K| - \gamma_K$ . ■

## 4.2 Height of the Rectangle

In this section we prove Lemmas 2.2.4 and 2.2.5. We start with equation (2.2.10), which we restate here:

$$\begin{aligned} \log(n!_K) &= n \log n - (\gamma_K - \gamma + 1)n + \frac{r_1 + r_2 - 1}{2} \log(2\pi n) + \frac{1}{2}C_K + \frac{1}{2}B_K(n) \\ &\quad - \sum_{|\operatorname{Im} \rho| < T_0} \zeta(\rho) \frac{n^\rho}{\rho} - \frac{1}{2\pi i} \int_{-b-iT_0}^{-b+iT_0} \frac{\zeta'_K(s)}{\zeta_K(s)} \zeta(s) \frac{n^s}{s} ds \\ &\quad + O_{c,K} \left( \frac{n^{1+c}}{T_0} \right) + O_{b,c,\varepsilon,K} \left( \frac{n^c}{T_0^{1/2-b-\varepsilon}} \right), \end{aligned}$$

for  $T_0$  chosen appropriately using Corollary 2.2.2. We want to replace  $T_0$  with  $T$  in the last four terms. The last two terms are easy: since  $T < T_0 < T + 1$ , we can use  $O(1/T_0) = O(1/T)$  in the third term. Similarly we have  $O(1/T_0^{1/2-b-\varepsilon}) = O(1/T^{1/2-b-\varepsilon})$  in the fourth term. To replace  $T_0$  in the sum, we use the following lemma.

**Lemma 2.2.4.** *For  $n \in \mathbb{Z}^+$ ,  $T > 1$ , and  $T < T_0 < T + 1$ ,*

$$\sum_{|\operatorname{Im} \rho| < T_0} \zeta(\rho) \frac{n^\rho}{\rho} = \sum_{|\operatorname{Im} \rho| < T} \zeta(\rho) \frac{n^\rho}{\rho} + O_K \left( \frac{n^{\beta_K(T)}}{T^{1/4}} \right), \quad (4.2.1)$$

where the sums are over nontrivial zeros  $\rho$  of  $\zeta_K(s)$  counted with multiplicity and  $\beta_K(T) = 1 - \lambda_K / \log(T + 3)$  for some constant  $\lambda_K > 0$  depending only on  $K$ . Under GRH for  $\zeta_K(s)$ , we have  $\beta_K(T) = \frac{1}{2}$  for all  $T$ .

*Proof.* The left sum in (4.2.1) minus the right sum has absolute value

$$\begin{aligned} \left| \sum_{T \leq |\operatorname{Im} \rho| < T_0} \zeta(\rho) \frac{n^\rho}{\rho} \right| &\leq \sum_{T \leq |\operatorname{Im} \rho| \leq T_0} |\zeta(\rho)| \frac{n^{\operatorname{Re} \rho}}{|\rho|} \\ &\leq \frac{1}{T} \sum_{T \leq |\operatorname{Im} \rho| \leq T_0} |\zeta(\rho)| n^{\beta_K(T)}, \end{aligned}$$

where the second inequality follows by Theorem 3.2.1 and  $1/|\rho| \leq 1/T$ . Furthermore,  $|\operatorname{Im} \rho| \leq T_0$  and  $T < T_0 < T + 1$  imply  $|\operatorname{Im} \rho| = O(T)$ . By [9, p. 23], for each  $\delta > 0$  it follows that  $|\zeta(\rho)| = O_\delta(T^{1/2+\delta/2})$ . Thus

$$\frac{1}{T} \sum_{T \leq |\operatorname{Im} \rho| \leq T_0} |\zeta(\rho)| n^{\beta_K(T)} = O_\delta \left( \frac{N_K(T, T_0) T^{1/2+\delta/2} n^{\beta_K(T)}}{T} \right),$$

where  $N_K(T, T_0)$  is the number of zeros satisfying  $T \leq |\operatorname{Im} \rho| \leq T_0$  counted with multiplicity. Since  $N_K(T, T_0) = O_K(\log T)$  by Theorem 2.2.1 and  $\log T = O_\delta(T^{\delta/2})$ ,

$$\left| \sum_{T \leq |\operatorname{Im} \rho| < T_0} \zeta(\rho) \frac{n^\rho}{\rho} \right| = O_{\delta, K} \left( \frac{n^{\beta_K(T)}}{T^{1/2-\delta}} \right). \quad (4.2.2)$$

Using a  $\delta$  satisfying  $0 < \delta < \frac{1}{2}$ , the  $O$ -estimate in (4.2.2) tends to 0 as  $T \rightarrow \infty$ .

We choose  $\delta = \frac{1}{4}$  for concreteness. Assuming GRH for  $\zeta_K(s)$ , the estimate in (4.2.2)

becomes

$$\left| \sum_{T \leq |\operatorname{Im} \rho| < T_0} \zeta(\rho) \frac{n^\rho}{\rho} \right| = O_K \left( \frac{\sqrt{n}}{T^{1/4}} \right). \quad (4.2.3)$$

■

**Lemma 2.2.5.** *For  $0 < b < \frac{1}{2}$ ,  $n \in \mathbb{Z}^+$ ,  $T > 1$ ,  $T < T_0 < T + 1$ , and all  $\varepsilon > 0$ , we*

have

$$\int_{-b-iT_0}^{-b+iT_0} \frac{\zeta'_K(s)}{\zeta_K(s)} \zeta(s) \frac{n^s}{s} ds = \int_{-b-iT}^{-b+iT} \frac{\zeta'_K(s)}{\zeta_K(s)} \zeta(s) \frac{n^s}{s} ds + O_{b,\varepsilon,K} \left( \frac{1}{T^{1/2-b-\varepsilon}} \right).$$

*Proof.* The difference between the two integrals is a sum of complex-conjugate integrals along  $[-b+iT, -b+iT_0]$  and  $[-b-iT_0, -b-iT]$ . See Figure 4.1.

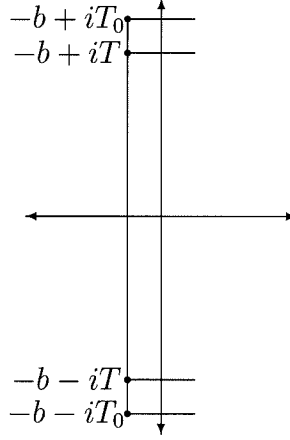


FIGURE 4.1: Adjusting the height of the left contour.

We will show for any  $\varepsilon > 0$  that

$$\left| \int_{-b-iT}^{-b+iT_0} \frac{\zeta'_K(s)}{\zeta_K(s)} \zeta(s) \frac{n^s}{s} ds \right| = \left| \int_{-b-iT_0}^{-b-iT} \frac{\zeta'_K(s)}{\zeta_K(s)} \zeta(s) \frac{n^s}{s} ds \right| = O_{b,\varepsilon,K} \left( \frac{1}{T^{1/2-b-\varepsilon}} \right).$$

First we write

$$\left| \int_{-b-iT}^{-b+iT_0} \frac{\zeta'_K(s)}{\zeta_K(s)} \zeta(s) \frac{n^s}{s} ds \right| \leq \int_T^{T_0} \left| \frac{\zeta'_K(-b+iy)}{\zeta_K(-b+iy)} \right| |\zeta(-b+iy)| \frac{n^{-b}}{y} dy.$$

Since  $T \leq y \leq T_0 < T + 1$ , by (2.2.7)

$$\left| \frac{\zeta'_K(-b + iy)}{\zeta_K(-b + iy)} \right| = O_{b,K}(\log^2 T),$$

and by (2.2.9) we have

$$|\zeta(-b + iy)| = O_{b,\varepsilon}(y^{1/2+b+\varepsilon/2}) = O_{b,\varepsilon}(T^{1/2+b+\varepsilon/2}).$$

Since  $[T, T_0]$  has length at most 1 and  $n^b > 1$ ,

$$\begin{aligned} \int_T^{T_0} \left| \frac{\zeta'_K(-b + iy)}{\zeta_K(-b + iy)} \right| |\zeta(-b + iy)| \frac{n^{-b}}{y} dy &= O_{b,\varepsilon,K} \left( \frac{(\log^2 T) T^{1/2+b+\varepsilon/2}}{T} \right) \\ &= O_{b,\varepsilon,K} \left( \frac{1}{T^{1/2-b-\varepsilon}} \right), \end{aligned}$$

where the last estimate uses  $\log^2 T = O_\varepsilon(T^{\varepsilon/2})$  for all  $\varepsilon > 0$ . Applying complex conjugation throughout this process yields the same result for the integral from  $-b - iT_0$  to  $-b - iT$ . ■

# Chapter 5

## Non-analogues between $n!$ and $n!_K$

### 5.1 Recursive Formula

The classical factorial sequence satisfies  $(n+1)! = (n+1)n!$  for nonnegative integers  $n$ , but computing a recursion for  $n!_K$  when  $K \neq \mathbb{Q}$  is more complicated. Here is a formula for quadratic number fields.

**Theorem 5.1.1.** *Assume  $[K : \mathbb{Q}] = 2$ . For nonnegative integers  $n$ ,*

$$\frac{(n+1)!_K}{n!_K} = (n+1) \prod_{\substack{p|(n+1) \\ p \text{ splits in } \mathcal{O}_K}} p^{\text{ord}_p(n+1)} \prod_{\substack{p|(n+1) \\ p \text{ inert in } \mathcal{O}_K}} \frac{1}{p^{a_p(n)}},$$

where  $a_p(n) = 1$  if  $\text{ord}_p(n+1) \equiv 1 \pmod{2}$  and  $a_p(n) = 0$  if  $\text{ord}_p(n+1) \equiv 0 \pmod{2}$ .

*Proof.* By definition

$$\frac{(n+1)!_K}{n!_K} = \frac{\prod_{\mathfrak{p}} \text{N}\mathfrak{p}^{\sum_{k \geq 1} [(n+1)/\text{N}\mathfrak{p}^k]}}{\prod_{\mathfrak{p}} \text{N}\mathfrak{p}^{\sum_{k \geq 1} [n/\text{N}\mathfrak{p}^k]}} = \prod_{\text{N}\mathfrak{p} \leq n+1} \text{N}\mathfrak{p}^{\sum_{k \geq 1} ([ (n+1)/\text{N}\mathfrak{p}^k ] - [n/\text{N}\mathfrak{p}^k ])}.$$

Since

$$\left[ \frac{n+1}{N\mathfrak{p}^k} \right] - \left[ \frac{n}{N\mathfrak{p}^k} \right] = \begin{cases} 1, & \text{if } n+1 \equiv 0 \pmod{N\mathfrak{p}^k}, \\ 0, & \text{otherwise,} \end{cases}$$

it follows that

$$\prod_{N\mathfrak{p} \leq n+1} N\mathfrak{p}^{\sum_{k \geq 1} ((n+1)/N\mathfrak{p}^k) - [n/N\mathfrak{p}^k]} = \prod_{N\mathfrak{p} | (n+1)} N\mathfrak{p}^{\#\{k \geq 1 : N\mathfrak{p}^k | (n+1)\}}. \quad (5.1.1)$$

Writing this as a product over primes in  $\mathbb{Z}$ , (5.1.1) becomes

$$\prod_{p|(n+1)} \prod_{\mathfrak{p}|p} p^{f_{\mathfrak{p}} \#\{k \geq 1 : N\mathfrak{p}^k | (n+1)\}} = \prod_{p|(n+1)} p^{r_p f_p [\text{ord}_p(n+1)/f_p]}, \quad (5.1.2)$$

where  $f_p$  is the residue field degree of  $p$  in  $\mathcal{O}_K$  and  $r_p$  is the number of primes in  $\mathcal{O}_K$  above  $p$ . If we arrange the factors of the product in (5.1.2) based on how each prime behaves in  $\mathcal{O}_K$ , we get

$$\prod_{\substack{p|(n+1) \\ p \text{ ramifies in } \mathcal{O}_K}} p^{\text{ord}_p(n+1)} \prod_{\substack{p|(n+1) \\ p \text{ splits in } \mathcal{O}_K}} p^{2 \text{ord}_p(n+1)} \prod_{\substack{p|(n+1) \\ p \text{ inert in } \mathcal{O}_K}} p^{2[\text{ord}_p(n+1)/2]}. \quad (5.1.3)$$

The third exponent in (5.1.3) is

$$2 \left\lfloor \frac{\text{ord}_p(n+1)}{2} \right\rfloor = \text{ord}_p(n+1) - a_p(n),$$

where  $a_p(n) = 1$  if  $\text{ord}_p(n+1) \equiv 1 \pmod{2}$  and  $a_p(n) = 0$  if  $\text{ord}_p(n+1) \equiv 0 \pmod{2}$ .



By collecting a factor of  $p^{\text{ord}_p(n+1)}$  for each prime  $p$ , (5.1.3) implies

$$\frac{(n+1)!_K}{n!_K} = (n+1) \prod_{\substack{p|(n+1) \\ p \text{ splits in } \mathcal{O}_K}} p^{\text{ord}_p(n+1)} \prod_{\substack{p|(n+1) \\ p \text{ inert in } \mathcal{O}_K}} \frac{1}{p^{a_p(n)}}.$$

■

**Remark 5.1.2.** Let  $K$  be a cyclic extension of  $\mathbb{Q}$  of prime degree  $l$ . Theorem 5.1.1 holds for  $n!_K$  if we define  $a_p(n)$  by  $0 \leq a_p(n) \leq l-1$  and  $a_p(n) = \text{ord}_p(n+1) \bmod l$ .

## 5.2 Factorials as Moments

Discussing a generalization of the factorial naturally leads to the question: can we also generalize the Gamma function? We know that the factorial function, defined on the nonnegative integers, has an integral representation

$$n! = \int_0^\infty x^n e^{-x} dx.$$

To determine if  $n!_K$  has an integral representation of the form

$$n!_K = \int_0^\infty x^n d\mu(x)$$

for some Borel measure  $\mu$ , we use necessary and sufficient conditions given by the following theorem of Stieltjes [4].

**Theorem 5.2.1.** *For  $n \geq 0$ , let  $\{a_n\}$  be a sequence of positive real numbers, and*

define the  $(n+1) \times (n+1)$  matrices

$$A_n = \begin{bmatrix} a_0 & a_1 & \dots & a_n \\ a_1 & a_2 & \dots & a_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ a_n & a_{n+1} & \dots & a_{2n} \end{bmatrix} \quad A'_n = \begin{bmatrix} a_1 & a_2 & \dots & a_{n+1} \\ a_2 & a_3 & \dots & a_{n+2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+1} & a_{n+2} & \dots & a_{2n+1} \end{bmatrix}.$$

There exists a Borel measure  $\mu$  on  $(0, \infty)$  satisfying  $a_n = \int_0^\infty x^n d\mu(x)$  for all  $n \geq 0$  if and only if  $\det A_n$  and  $\det A'_n$  are positive for all  $n$ .

Below is a table of values of  $\det A_n$  when  $a_n = n!_K$  for several choices of  $K$ .

$K$	$n = 1$	2	3	4	5	6
$\mathbb{Q}$	1	4	144	82944	$1.19 \times 10^9$	$6.19 \times 10^{14}$
$\mathbb{Q}(i)$	1	4	-36976	$1.08 \times 10^9$	$1.47 \times 10^{15}$	$1.06 \times 10^{23}$
$\mathbb{Q}(\sqrt{2})$	1	4	-112	$-3.05 \times 10^6$	$1.61 \times 10^{12}$	$1.31 \times 10^{19}$
$\mathbb{Q}(\sqrt{5})$	0	0	-27	53824	$1.75 \times 10^8$	$-9.90 \times 10^{15}$
$\mathbb{Q}(\sqrt{10})$	1	-188	-900720	$6.26 \times 10^{10}$	$5.12 \times 10^{18}$	$1.18 \times 10^{29}$
$\mathbb{Q}(\sqrt{-2})$	1	-188	-320112	$9.20 \times 10^8$	$2.03 \times 10^{17}$	$1.00 \times 10^{27}$
$\mathbb{Q}(\sqrt{-10})$	1	4	-1136	$-7.26 \times 10^7$	$9.67 \times 10^{13}$	$6.90 \times 10^{24}$
$\mathbb{Q}(\zeta_7 + \zeta_7^{-1})$	0	0	0	0	0	-46656
$\mathbb{Q}(\zeta_9 + \zeta_9^{-1})$	0	-4	0	0	$1.00 \times 10^6$	$-4.27 \times 10^{11}$
$\mathbb{Q}(\sqrt[3]{2})$	1	4	144	$-5.96 \times 10^7$	$1.24 \times 10^{13}$	$5.02 \times 10^{21}$
$\mathbb{Q}(\sqrt{2}, \sqrt{3})$	1	4	-112	-7168	$1.16 \times 10^8$	$4.10 \times 10^{12}$

TABLE 5.1: Values of  $\det A_n$  when  $a_n = n!_K$  for different number fields  $K$ .

For each  $K \neq \mathbb{Q}$  listed in Table 5.1,  $\det A_n < 0$  for at least one  $n \geq 1$ . Similar results were found for  $\det A'_n$  when  $a_n = n!_K$ , as indicated in Table 5.2.

$K$	$n = 1$	2	3	4	5	6
$\mathbb{Q}$	2	24	3456	$9.95 \times 10^6$	$8.60 \times 10^{11}$	$3.12 \times 10^{18}$
$\mathbb{Q}(i)$	-2	-408	-5912704	$6.73 \times 10^{10}$	$1.11 \times 10^{19}$	$8.81 \times 10^{27}$
$\mathbb{Q}(\sqrt{2})$	-2	-24	-17536	$1.13 \times 10^9$	$6.54 \times 10^{15}$	$-5.59 \times 10^{23}$
$\mathbb{Q}(\sqrt{5})$	0	-9	-3364	$-1.07 \times 10^6$	$-4.34 \times 10^{11}$	$2.63 \times 10^{20}$
$\mathbb{Q}(\sqrt{10})$	14	-792	-298318464	$-3.76 \times 10^{14}$	$8.74 \times 10^{23}$	$1.70 \times 10^{35}$
$\mathbb{Q}(\sqrt{-2})$	14	-4824	20062080	$1.58 \times 10^{13}$	$4.99 \times 10^{21}$	$-9.29 \times 10^{32}$
$\mathbb{Q}(\sqrt{-10})$	-2	-88	-339584	$5.83 \times 10^{10}$	$1.87 \times 10^{19}$	$1.38 \times 10^{31}$
$\mathbb{Q}(\zeta_7 + \zeta_7^{-1})$	0	0	0	0	7776	$-1.14 \times 10^{10}$
$\mathbb{Q}(\zeta_9 + \zeta_9^{-1})$	2	-12	0	47628	$2.33 \times 10^9$	$1.19 \times 10^{14}$
$\mathbb{Q}(\sqrt[3]{2})$	2	24	-100224	$6.73 \times 10^{10}$	$5.26 \times 10^{17}$	$-4.34 \times 10^{26}$
$\mathbb{Q}(\sqrt{2}, \sqrt{3})$	-2	-24	896	$1.07 \times 10^6$	$2.58 \times 10^{10}$	$-3.66 \times 10^{15}$

TABLE 5.2: Values of  $\det A'_n$  when  $a_n = n!_K$  for different number fields  $K$ .

Thus if there were an integral formula for  $a_n = n!_K$  when  $K \neq \mathbb{Q}$ , it would not have the form  $a_n = \int_0^\infty x^n d\mu(x)$  for any  $K$  that we have checked. This makes sense: for  $K = \mathbb{Q}$ , the proof of the integral formula relies on the recursive definition of  $n!$  and repeated integration by parts, and there is no simple recursive formula for  $n!_K$ . See Theorem 5.1.1 for  $[K : \mathbb{Q}] = 2$ .

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