

Spring 5-1-2018

Harmonic Functions and Harmonic Measure

David McDonald
descripen@gmail.com

Follow this and additional works at: https://opencommons.uconn.edu/srhonors_theses

 Part of the [Partial Differential Equations Commons](#)

Recommended Citation

McDonald, David, "Harmonic Functions and Harmonic Measure" (2018). *Honors Scholar Theses*. 558.
https://opencommons.uconn.edu/srhonors_theses/558

Harmonic Functions and Harmonic Measure

David McDonald

B.S., Applied Mathematics

An Undergraduate Honors Thesis
Submitted in Partial Fulfillment of the
Requirements for the Degree of
Bachelor of Science
at the
University of Connecticut

May 2018

Copyright by

David McDonald

May 2018

APPROVAL PAGE

Bachelor of Science Honors Thesis

**Harmonic Functions and Harmonic
Measure**

Presented by

David McDonald, B.S. Applied Mathematics

Honors Major Advisor _____
HONORS ADVISOR NAME

Honors Thesis Advisor _____
THESIS ADVISOR NAME

University of Connecticut

May 2018

ACKNOWLEDGMENTS

I would like to thank Professor Murat Akman for guiding me through a subject that, prior to this year, I was unfamiliar with but interested in. I also would like to thank Professor Keith Conrad for sharing the tex files and guiding me for the process of writing this honors thesis. I'd also like to thank the University of Connecticut for giving me the opportunity to write this thesis, as well as my high school calculus teacher Mrs. Zackeo for teaching me how to do math correctly. Finally I'd like to thank my friends and family for always being there to support me; especially Leo for making sure his thesis is less interesting than mine, Will for raving to SHM with me, Chris Pratt for encouraging me to follow my dreams, Alicia for her assistance in the researching of Ja-Fire, Jacob for always having so much more work than me, and Kelly for helping me continue my streak of never losing at trivia.

Harmonic Functions and Harmonic Measure

David McDonald, B.S.

University of Connecticut, May 2018

ABSTRACT

The purpose of this thesis is to give a brief introduction to the field of harmonic measure. In order to do this we first introduce a few important properties of harmonic functions and show how to find a Green's function for a given domain. Following this we calculate the harmonic measure for some easy cases and end by examining the connection between harmonic measure and Brownian motion.

Contents

Introduction	1
Ch. 1. Harmonic Functions	2
1.1 Mean Value Property	4
1.1.1 Mean Value Property for Volume	5
1.2 The Maximum Principle	6
1.3 The Poisson Integral and Green's function	8
1.3.1 Green's Function for the Upper Half Plane	15
1.3.2 Green's Function for the Unit Ball	16
1.3.3 Green's Function for the First Quadrant	18
1.3.4 Poisson's Formula for the Ball	19
Ch. 2. Harmonic Measure	26
2.1 Examples	28
2.1.1 The Upper Half Plane	28
2.1.2 The Unit Disk	30
2.2 Brownian Motion	32
2.2.1 Harmonic Measure in Brownian Motion	33
2.2.2 Multiply Connected Domains	35

INTRODUCTION

Harmonic functions are those which satisfy Laplace's equation. They have a number of convenient properties including mean value properties and the Maximum Principle that make them easy to work with. Understanding harmonic functions is the first step to calculating the harmonic measure. The next step is finding Green's functions; in section 1.3 we'll introduce the Dirichlet problem and show how to solve a Green's function. To make it less abstract there are a few examples: the upper half plane, the unit ball, and the first quadrant. Then, using the Green's function for the unit ball, Poisson's formula for the Ball can be found.

These pieces are all integral to understanding and being able to find the harmonic measure. In probability theory, the harmonic measure ω of a particle in some domain Ω is the probability that it will exit Ω within some E along $\partial\Omega$. At the beginning of chapter two we'll calculate the harmonic measure for both the ball of radius r and the upper half plane as well as prove its uniqueness and existence in all cases. Harmonic measure has had quite a few recent theorems published, including Makarov's theorem which was quite a breakthrough at the time, so we'll conclude by going over those.

Chapter 1

Harmonic Functions

Before diving into harmonic measure, it is important to first have some knowledge of what a harmonic function is and the properties that it has. A *harmonic function* is any function that is twice continuously differentiable and satisfies the Laplace partial differential equation, that is, in \mathbb{R}^n , $n \geq 2$,

$$\Delta u = \sum_{i=1}^n u_{x_i x_i} = \frac{\partial^2 u}{\partial x_1^2} + \dots + \frac{\partial^2 u}{\partial x_n^2} = 0.$$

Here Δ is known as the *Laplacian* and denotes the sum of the second partial derivatives for each respective variable. One fundamental property of harmonic functions is that they are invariant under both rotations and translations. That is to say, both rotations and translations of harmonic functions are also harmonic.

Example 1.0.1. We shall see that $v(x) = |x|^{2-n} = (x_1^2 + x_2^2 \dots + x_n^2)^{(2-n)/2}$ is harmonic in $\mathbb{R}^n \setminus \{0\}$ when $n > 2$ and $v(x) = \ln|x| = \ln(x_1^2 + x_2^2)^{1/2}$ is harmonic in $\mathbb{R}^2 \setminus \{0\}$.

Let us start with $n > 2$, in this case, for $i = 1, 2, \dots, n$, we have

$$v_{x_i} = (2 - n)x_i(x_1^2 + \dots + x_n^2)^{(-n)/2}$$

and

$$v_{x_i x_i} = (2 - n)[(x_1^2 + \dots + x_n^2)^{(-n)/2} - nx_i^2(x_1^2 + \dots + x_n^2)^{(-n-2)/2}].$$

Now

$$\begin{aligned} \sum_{i=1}^n v_{x_i x_i} &= (2 - n) \sum_{i=1}^n [(x_1^2 + \dots + x_n^2)^{(-n)/2} - nx_i^2(x_1^2 + \dots + x_n^2)^{(-n-2)/2}] \\ &= (2 - n)[n|x|^{-n} - n|x|^{(-n-2)/2} \sum_{i=1}^n x_i^2] \\ &= (2 - n)n(|x|^{-n} - |x|^{-n}) = 0. \end{aligned}$$

Hence v is harmonic in \mathbb{R}^n .

When $n = 2$, i.e when $v(x) = \ln(x_1^2 + x_2^2)^{1/2}$ we have

$$v_{x_i} = \frac{x_i}{x_1^2 + x_2^2} \quad \text{and} \quad v_{x_i x_i} = \frac{1}{x_1^2 + x_2^2} - \frac{2x_i^2}{(x_1^2 + x_2^2)^2}.$$

From this we get

$$\begin{aligned} \sum_{i=1}^2 v_{x_i x_i} &= \sum_{i=1}^2 \left[\frac{1}{x_1^2 + x_2^2} - \frac{2x_i^2}{(x_1^2 + x_2^2)^2} \right] \\ &= \frac{2}{x_1^2 + x_2^2} - \frac{2}{(x_1^2 + x_2^2)^2} \sum_{i=1}^2 x_i^2 \\ &= \frac{2}{x_1^2 + x_2^2} - \frac{2}{x_1^2 + x_2^2} = 0. \end{aligned}$$

1.1 Mean Value Property

Another important property of harmonic functions is called the *mean value property*. The mean value property states that, given u harmonic on the closed ball centered at a with radius r , $\bar{B}(a, r)$, u is equal to the average of u over $\partial B(a, r)$. To be more precise, we have

$$u(a) = \int_S u(a + r\zeta) d\sigma(\zeta)$$

where S is the unit sphere and σ is the Borel probability measure on the unit sphere so that $\sigma(S) = 1$.

Proof. First assume that $n > 2$ and take $B(a, r) = B$ and $\epsilon \in (0, 1)$. Then using Green's identity, it follows that

$$\int_{\Omega} (u\Delta v - v\Delta u) dV = \int_{\partial\Omega} (uD_nv - vD_nu) ds$$

with $\{\Omega = x \in \mathbb{R}^n : \epsilon < |x| < 1\}$ and $v(x) = |x|^{2-n}$. This results in

$$0 = (2-n) \int_S u ds - (2-n)\epsilon^{1-n} \int_{\epsilon S} u ds - \int_S D_n u ds - \epsilon^{2-n} \int_{\epsilon S} D_n u ds \quad (1.1.1)$$

Here D_n is the differentiation with respect to the outward unit normal, $D_n(\xi) = \langle \nabla u(\xi), \mathbf{n}(\xi) \rangle$. Now since u is harmonic and $v \equiv 1$ we can apply another form of Green's identity,

$$\int_{\partial\Omega} D_n u ds = 0.$$

By the identity above, the last two terms of (1.1.1) are 0, so we are left with

$$\int_S u ds = \epsilon^{1-n} \int_{\epsilon S} u ds$$

which is just

$$\int_S u d\sigma = \int_S u(\epsilon\zeta) d\sigma(\zeta).$$

This is what we are looking to prove with $\epsilon \rightarrow 0$ with u continuous at 0. For $n = 2$ the proof is exactly the same, with the exception that $|x|^{2-n}$ be replaced with $\log|x|$. □

1.1.1 Mean Value Property for Volume

A similar property exists with respect to the volume of the ball rather than the surface, and its proof utilizes the polar coordinates formula for integration on \mathbb{R}^n ,

$$\frac{1}{nV(B)} \int_{\mathbb{R}^n} f dV = \int_0^\infty r^{n-1} \int_S f(r\zeta) d\sigma(\zeta) dr \quad (1.1.2)$$

The mean value property for volume states that if u is harmonic on $\bar{B}(a, r)$, then $u(a)$ is equal to the average of u over $B(a, r)$, or

$$u(a) = \frac{1}{V(B(a, r))} \int_{B(a, r)} u dV.$$

Proof. Once again take $B(a, r) = B$ and apply equation (1.1.2) setting f to be u multiplied by the characteristic function ξ of B . Following this, simply apply the standard mean value Property to get the desired result. \square

From the mean value property we gain an important insight about the singularities of harmonic functions.

Corollary 1.1.1. *Any zeros of a real-valued harmonic function are not isolated.*

Proof. Take u to be real-valued and harmonic on the open connected set Ω with $a \in \Omega$ and $u(a) = 0$. Now consider the closed ball $\bar{B}(a, r) \subset \Omega$ with $r > 0$; since the average of the harmonic function u over the boundary of the ball is 0, u must either be identically 0 or it must take both positive and negative values on $\partial B(a, r)$. Either way, this implies that u has a zero on $\partial B(a, r)$. Therefor since u must have a zero on the boundary of any ball $r > 0$ centered at a . Hence a is not an isolated zero of u . \square

1.2 The Maximum Principle

Another important property of Harmonic Functions which stems from the mean value Property is called the *maximum principle*. The maximum principle states that for any real valued harmonic function u on Ω , where Ω is connected, if u attains a maximum or minimum in Ω then u must be constant.

Proof. For some $a \in \Omega$, Consider $u(a)$ to be a point where u attains its maximum. Now choose $r > 0$ such that $\bar{B}(a, r) \subset \Omega$. If at any point of $B(a, r)$, u were less than $u(a)$, then, by continuity, the average of u over $B(a, r)$ would be less than $u(a)$, which contradicts the mean value property. Therefore u must be constant on $B(a, r)$, so the set where u attains its maximum is open in Ω . However the set is also closed in Ω because of u 's continuity, so the set must be all of Ω . So u must be constant on Ω . A similar argument can be made if u attains a minimum in Ω . \square

This principle leads to the following corollaries about real-valued harmonic functions.

Corollary 1.2.1. *Suppose Ω is bounded and u is a continuous real valued function on $\bar{\Omega}$ that is harmonic on Ω . Then u must attain its maximum and minimum values over $\bar{\Omega}$ on $\partial\Omega$.*

Therefore a harmonic function on a bounded domain is determined by its own boundary values. So for u and v continuous on $\bar{\Omega}$ and harmonic on Ω with bounded Ω . Provided $u = v$ on $\partial\Omega$, then $u = v$ on Ω . This is only guaranteed for a bounded domain however, and can fail in the unbounded case. For Ω unbounded, or when u is not continuous on $\bar{\Omega}$ we have a different version of the maximum principle.

Corollary 1.2.2. *For u real valued and harmonic on Ω , suppose*

$$\limsup_{k \rightarrow \infty} u(a_k) \leq M$$

is true for every sequence a_k in Ω converging to either a point in $\partial\Omega$ or ∞ . Then $u \leq M$ on Ω .

Proof. Let $M' = \sup u(x) : x \in \Omega$ and consider a sequence b_k in Ω such that $u(b_k) \rightarrow M'$. If there exists a subsequence of b_k that converges to a point $b \in \Omega$ then $u(b) = M'$, and, by the maximum principle, u must be constant on the component of Ω that contains b . Therefore there must be a sequence a_k converging to a boundary point of Ω or ∞ where $u = M'$, so $M' \leq M$. If no subsequence of b_k converges to a point in Ω , then there must exist a subsequence of b_k converging to either a boundary point of Ω or ∞ . Regardless, $M' \leq M$ for this case as well. \square

Since Corollaries 0.3 and 0.4 only apply to real functions, we need another corollary about the maximum principle for complex functions.

Corollary 1.2.3. *Let Ω be connected with u harmonic on Ω . If $|u|$ attains a maximum on Ω then u must be constant.*

Proof. Suppose $|u|$ attains a maximum value M at a point $a \in \Omega$. Choose $\lambda \in \mathbb{C}$ so that $|\lambda| = 1$ and $\lambda u(a) = M$. Now the real part of the harmonic function u attains a maximum value M at a ; and by the maximum principle, $\operatorname{Re} \lambda u \equiv M$ on Ω . And since $|\lambda u| = |u| \leq M$, the imaginary part of the harmonic function u , $\operatorname{Im} \lambda u$ is equivalent to 0. Therefore λu and u are constant on Ω . \square

1.3 The Poisson Integral and Green's function

In this section we want to solve the following Dirichlet problem. Let Ω be an open and bounded subset of \mathbb{R}^n

$$\begin{cases} -\Delta u = f & x \in \Omega \subset \mathbb{R}^n \\ u = g & x \in \partial\Omega. \end{cases} \quad (1.3.1)$$

We will use Green's function to solve this problem and for this we shall first give a motivation. Consider the following problem:

$$\begin{cases} -\Delta_y G(x, y) = \delta_x & y \in \Omega \\ G(x, y) = 0 & y \in \partial\Omega \end{cases} \quad (1.3.2)$$

Our first aim is to find such G . To this end, formally we have,

$$\begin{aligned} u(x) &= \int_{\Omega} \delta_x u(y) dy \\ &= - \int_{\Omega} \Delta_y G(x, y) u(y) dy \\ &= \int_{\Omega} \langle \nabla_y G(x, y), \nabla_y u(y) \rangle dy - \int_{\Omega} \frac{\partial G}{\partial \nu}(x, y) u(y) dS(y) \\ &= - \int_{\Omega} G(x, y) \Delta_y u(y) dy + \int_{\partial\Omega} G(x, y) \frac{\partial u}{\partial \nu}(y) dS(y) - \int_{\Omega} \frac{\partial G}{\partial \nu}(x, y) u(y) dS(y) \\ &= \int_{\Omega} G(x, y) f(y) dy - \int_{\Omega} \frac{\partial G}{\partial \nu}(x, y) g(y) dS(y). \end{aligned}$$

where we have used $-\Delta u = f$ in Ω , $u = g$ on $\partial\Omega$ and $G(x, y) = 0$ on $\partial\Omega$. Here ν is the outer unit normal to Ω . Since we also know that

$$\begin{aligned} -\Delta_y \Phi(y) &= \delta_0 \\ -\Delta_y \Phi(x - y) &= \delta_x \end{aligned}$$

where Φ is the fundamental solution of the Laplace equation,

$$\Phi(x) = \begin{cases} -\frac{1}{2\pi} \ln |x| & n = 2 \\ \frac{1}{n(n-2)\alpha(n)|x|^{n-2}} & n \geq 3. \end{cases}$$

Unfortunately $\Phi(x - y)$ doesn't satisfy the given boundary conditions, but it can

still be used to help find a solution for (1.0.4). Integrating

$$\int_{\Omega} \Phi(x-y)\Delta u(y) dy$$

by parts is a bit tricky because $\Phi(x-y)$ has a singularity at $x=y$. To take care of this, fixed Ω , let $\epsilon > 0$ such that $\text{dist}(x, \partial\Omega) < \epsilon$, where $\text{dist}(x, \partial\Omega)$ is the distance between x and $\partial\Omega$, so that $B(x, \epsilon) \subset \Omega$. Suppose for the moment that $u \in C^2(\Omega)$ and using Divergence theorem and integration by parts we get

$$\begin{aligned} \int_{V_\epsilon} \Phi(x-y)\Delta u(y)dy &= \int_{\partial V_\epsilon} \Phi(x-y)\frac{\partial u}{\partial \nu}(y)dS(y) - \int_{V_\epsilon} \langle \nabla_y \Phi(y-x), \nabla_y u(y) \rangle dy \\ &= \int_{\partial V_\epsilon} \Phi(x-y)\frac{\partial u}{\partial \nu}(y)dS(y) - \int_{\partial V_\epsilon} \frac{\partial \Phi(x-y)}{\partial \nu} u(y)dS(y) + \int_{V_\epsilon} \Delta_y \Phi(x-y)u(y)dy \end{aligned} \quad (1.3.3)$$

where $V_\epsilon = \Omega \setminus B(x, \epsilon)$ and once again ν is the outer unit normal to ∂V_ϵ . Since $\Delta_y \Phi(x-y) = 0$ in V_ϵ we have

$$\int_{V_\epsilon} \Phi(x-y)\Delta u(y)dy = \int_{\partial V_\epsilon} \Phi(x-y)\frac{\partial u}{\partial \nu}(y)dS(y) - \int_{\partial V_\epsilon} \frac{\partial \Phi(x-y)}{\partial \nu} u(y)dS(y). \quad (1.3.4)$$

As $\epsilon \rightarrow 0$ $V_\epsilon \rightarrow \Omega$, we want to send $\epsilon \rightarrow 0$ in the above identity and obtain

$$\int_{\Omega} \Phi(x-y)\Delta u(y)dy = \int_{\partial\Omega} \Phi(x-y)\frac{\partial u}{\partial \nu}(y)dS(y) - \int_{\partial\Omega} \frac{\partial \Phi(x-y)}{\partial \nu} u(y)dS(y). \quad (1.3.5)$$

To this end, first observe on the left-hand side of (1.3.4) converges to left-hand side

of (1.3.5)

$$\lim_{\epsilon \rightarrow 0^+} \int_{V_\epsilon} \Phi(x-y) \Delta u(y) dy = \int_{\Omega} \Phi(x-y) \Delta u(y) dy.$$

It remains to show that the first and the second integral in (1.3.4) converges to first and the second integral in (1.3.5) respectively

$$\lim_{\epsilon \rightarrow 0^+} \int_{\partial V_\epsilon} \Phi(x-y) \frac{\partial u}{\partial \nu}(y) dS(y) = \int_{\partial \Omega} \Phi(x-y) \frac{\partial u}{\partial \nu}(y) dS(y) \quad (1.3.6)$$

and

$$\lim_{\epsilon \rightarrow 0^+} \int_{\partial V_\epsilon} \frac{\partial \Phi(x-y)}{\partial \nu} u(y) dS(y) = \int_{\partial \Omega} \frac{\partial \Phi(x-y)}{\partial \nu} u(y) dS(y) + u(x). \quad (1.3.7)$$

This will finish (1.3.5). Let us start with (1.3.6). Notice that $\partial V_\epsilon = \partial \Omega \cup \partial B(x, \epsilon)$.

Therefore,

$$\int_{\partial V_\epsilon} \Phi(x-y) \frac{\partial u}{\partial \nu}(y) dS(y) = \int_{\partial \Omega} \Phi(x-y) \frac{\partial u}{\partial \nu}(y) dS(y) + \int_{\partial B(x, \epsilon)} \Phi(x-y) \frac{\partial u}{\partial \nu}(y) dS(y).$$

To show (1.3.6), we need to show

$$\left| \int_{\partial B(x, \epsilon)} \Phi(x-y) \frac{\partial u}{\partial \nu}(y) dS(y) \right| \rightarrow 0$$

as $\epsilon \rightarrow 0^+$. To this end, using definition of Φ we get (we just do this when $n \geq 3$ and

$n = 2$ is the same argument)

$$\begin{aligned}
\left| \int_{\partial B(x,\epsilon)} \Phi(x-y) \frac{\partial u}{\partial \nu}(y) dS(y) \right| &\leq \frac{1}{n(n-2)\alpha(n)} |\nabla u|_{L^\infty(B(x,\epsilon))} \int_{\partial B(x,\epsilon)} \frac{1}{|x-y|^{n-2}} dS(y) \\
&= \frac{1}{n(n-2)\alpha(n)\epsilon^{n-2}} |\nabla u|_{L^\infty(B(x,\epsilon))} \int_{\partial B(x,\epsilon)} dS(y) \\
&= C\epsilon
\end{aligned}$$

which clearly goes to zero as $\epsilon \rightarrow 0^+$. Hence this finishes the proof of (1.3.6). Now we turn to (1.3.7). Using $\partial V_\epsilon = \partial\Omega \cup \partial B(x,\epsilon)$ and on $\partial B(x,\epsilon)$, the inner unit normal at $y \in \partial B(x,\epsilon)$ is $\nu = -\frac{y-x}{|y-x|}$. Moreover,

$$\nabla_y \Phi(y-x) = -\frac{1}{n\alpha(n)} \frac{y-x}{|y-x|^n}$$

Then

$$\frac{\partial \Phi(x-y)}{\partial \nu} = \langle \nabla \Phi(x-y), \nu \rangle = -\frac{1}{n\alpha(n)} \left\langle \frac{y-x}{|y-x|^n}, -\frac{y-x}{|y-x|} \right\rangle = \frac{1}{n\alpha(n)} \frac{1}{|y-x|^{n-1}}.$$

Hence using these in (1.3.7) we get

$$\begin{aligned}
\int_{\partial V_\epsilon} \frac{\partial \Phi(x-y)}{\partial \nu} u(y) dS(y) &= \int_{\partial\Omega} \frac{\partial \Phi(x-y)}{\partial \nu} u(y) dS(y) + \int_{\partial B(x,\epsilon)} \frac{\partial \Phi(x-y)}{\partial \nu} u(y) dS(y) \\
&= \int_{\partial\Omega} \frac{\partial \Phi(x-y)}{\partial \nu} u(y) dS(y) + \frac{1}{n\alpha(n)} \int_{\partial B(x,\epsilon)} \frac{1}{|y-x|^{n-1}} u(y) dS(y) \\
&= \int_{\partial\Omega} \frac{\partial \Phi(x-y)}{\partial \nu} u(y) dS(y) + \frac{1}{n\alpha(n)\epsilon^{n-1}} \int_{\partial B(x,\epsilon)} u(y) dS(y).
\end{aligned}$$

If we let $\epsilon \rightarrow 0^+$ and use the mean value property for u we get

$$\frac{1}{n\alpha(n)\epsilon^{n-1}} \int_{\partial B(x,\epsilon)} u(y) dS(y) \rightarrow u(x).$$

This finishes the proof of (1.3.7). Combining these we get

$$u(x) = \int_{\partial\Omega} \left[\Phi(y-x) \frac{\partial u}{\partial \nu}(y) - \frac{\partial \Phi}{\partial \nu}(y-x) u(y) \right] dS(y) - \int_{\Omega} \Phi(y-x) \Delta u(y) dy. \quad (1.3.8)$$

We still cannot use this to solve (1.0.3), however, because we do not know $\frac{\partial u}{\partial \nu}$ on $\partial\Omega$.

Because of this, we have to introduce a corrector function, $h^x(y)$ which satisfies

$$\begin{cases} \Delta_y h^x = 0 & y \in \Omega \\ h^x(y) = \Phi(y-x) & y \in \partial\Omega. \end{cases}$$

Suppose we can find a smooth function h^x to satisfy these conditions. Then,

$$\begin{aligned} \int_{\Omega} h^x(y) \Delta u(y) dy &= \int_{\partial\Omega} h^x(y) \frac{\partial u}{\partial \nu} dS(y) - \int_{\Omega} \nabla_y h^x(y) \cdot \nabla_y u(y) dy \\ &= \int_{\partial\Omega} h^x(y) \frac{\partial u}{\partial \nu}(y) dS(y) \\ &\quad - \int_{\partial\Omega} \frac{\partial h^x}{\partial \nu}(y) u(y) dS(y) + \int_{\Omega} \Delta_y h^x(y) u(y) dy \end{aligned}$$

and since h^x is a solution to the given boundary conditions,

$$0 = \int_{\partial\Omega} \left[\Phi(y-x) \frac{\partial u}{\partial \nu}(y) - \frac{\partial h^x}{\partial \nu}(y) u(y) \right] dS(y) - \int_{\Omega} h^x \Delta u(y) dy. \quad (1.3.9)$$

Subtracting equation (1.0.5) from (1.0.6) and allowing

$$G(x, y) = \Phi(y - x) - h^x(y)$$

results in

$$u(x) = - \int_{\partial\Omega} \frac{\partial G}{\partial \nu}(x, y) u(y) dS(y) - \int_{\Omega} G(x, y) \Delta u(y) dy. \quad (1.3.10)$$

The function G is known as Green's function for Ω , it is formally defined as

$$G(x, y) = \Phi(y - x) - h^x(y) \quad x, y \in \Omega, x \neq y.$$

Therefor, when u is a smooth solution of a problem of the type (1.3.1) and $G(x, y)$ is Green's function then

$$u(x) = - \int_{\partial\Omega} \frac{\partial G}{\partial \nu}(x, y) g(y) dS(y) + \int_{\Omega} G(x, y) f(y) dy. \quad (1.3.11)$$

Let us state this as a theorem.

Theorem 1.3.1. *Let f, g be continuous function and if $u \in C^2(\bar{\Omega})$ is a solution of*

$$\begin{cases} -\Delta u = f & x \in \Omega \subset \mathbb{R}^n \\ u = g & x \in \partial\Omega \end{cases} \quad (1.3.12)$$

then for $x \in \Omega$ one has

$$u(x) = - \int_{\partial\Omega} \frac{\partial G}{\partial \nu}(x, y) g(y) dS(y) + \int_{\Omega} G(x, y) f(y) dy. \quad (1.3.13)$$

where $G(x, y)$ is the Green's function for Ω .

We have an immediate corollary of this theorem.

Corollary 1.3.2. *If u is a harmonic function in Ω with $u = g$ on $\partial\Omega$ then*

$$u(x) = - \int_{\partial\Omega} \frac{\partial G}{\partial \nu}(x, y) g(y) dS(y).$$

1.3.1 Green's Function for the Upper Half Plane

For our first example, we'll look to find the Green's function for \mathbb{R}_+^2 . In order to do this, we'll need a corrector function h^x for each $x \in \mathbb{R}_+^2$ such that

$$\begin{cases} \Delta_y h^x(y) = 0 & y \in \mathbb{R}_+^2 \\ h_x(y) = \Phi(y - x) & y \in \partial\mathbb{R}_+^2 \end{cases} \quad (1.3.14)$$

First fix $x \in \mathbb{R}_+^2$. Since $\Delta_y \Phi(y - x) = 0$ for all $y \neq x$ we choose $z \notin \Omega$, so that $\Delta_y \Phi(y - z) = 0$ for all $y \in \Omega$. Now all that's left is to choose $z = z(x)$, $z \notin \Omega$ such that $\Phi(y - z) = \Phi(y - x)$ for $y \in \partial\Omega$, and let $h^x(y) = \Phi(y - z(x))$, we've found a corrector function. Since, for this example, $n = 2$,

$$\Phi(y - z) = -\frac{1}{2\pi} \ln |y - z|.$$

This tells us that $\Phi(y - z)$ is a function of $|y - z|$, and so for $x = (x_1, x_2) \in \mathbb{R}_+^2$, for $y \in \partial\mathbb{R}_+^2$,

$$|y - x| = |(y_1, 0) - (x_1, x_2)| = |(y_1, 0) - (x_1, -x_2)| = |y - \tilde{x}|.$$

Where \tilde{x} is the reflection of x in the \mathbb{R}^2 plane.

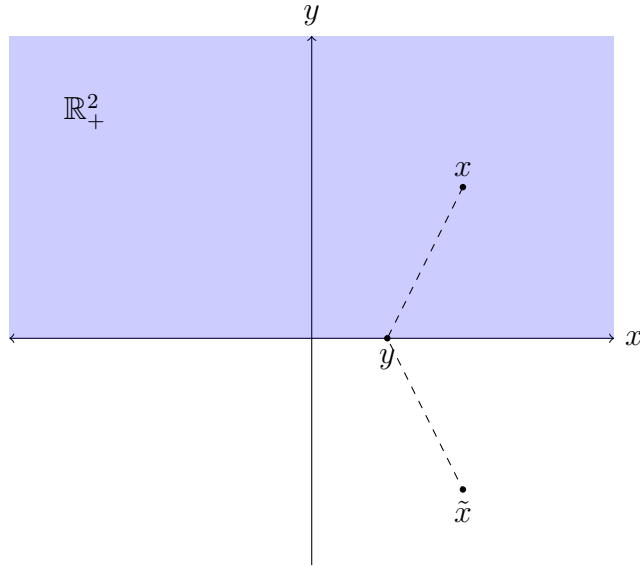


FIGURE 1.3.1: Location of \tilde{x} in the plane

This means that our corrector function for \mathbb{R}_+^2 is $h^x(y) = \Phi(y - \tilde{x})$. Furthermore, a Green's function for \mathbb{R}_+^2 is

$$G(x, y) = \Phi(y - x) - \Phi(y - \tilde{x}) = -\frac{1}{2\pi}(\ln |y - x| - \ln |y - \tilde{x}|).$$

A Green's function for the upper half space in \mathbb{R}^n is still $G(x, y) = \Phi(y - x) - \Phi(y - \tilde{x})$ where $\Phi(x) = \Phi(y - x) - \Phi(y - \tilde{x})$ is the fundamental solution when $n \geq 3$, however instead of having $\tilde{x} \equiv (x_1, -x_2)$, $\tilde{x} \equiv (x_1, \dots, x_{n-1}, -x_n)$.

1.3.2 Green's Function for the Unit Ball

Finding the Green's function for the unit ball is a bit more difficult, let

$$B_2(0, 1) \equiv \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 < 1\},$$

and fix $x \in B_2(0, 1)$. Since we are again working in \mathbb{R}^2 ,

$$\Phi(y - x) = -\frac{1}{2\pi} \ln |y - x|.$$

Like before, this means that $\Phi(y - x)$ is a function of $|y - x|$. Now we need to find $h_x(y) = \Phi(y - x)$ for all $y \in \partial B_2(0, 1)$, or all y such that $|y| = 1$. So for these y ,

$$\begin{aligned} |y - x|^2 &= (y - x) \cdot (y - x) \\ &= |y|^2 - 2y \cdot x + |x|^2 \\ &= |x|^2 - 2x \cdot y + 1 \\ &= |x|^2 |y|^2 - 2x \cdot y + 1 \\ &= |x|^2 \left(|y|^2 - \frac{2x \cdot y}{|x|^2} + \frac{1}{|x|^2} \right) \\ &= |x|^2 \left(|y|^2 - 2y \cdot \frac{x}{|x|^2} + \frac{|x|^2}{|x|^4} \right) \\ &= |x|^2 |y - x^*|^2 \end{aligned}$$

with $x^* = \frac{x}{|x|^2}$ being known as the point dual to x .

Also, because x^* is not in $B_2(0, 1)$, we know that $\Phi(|x|(y - x^*))$ is harmonic for all y in Ω . This $\Phi(|x|(y - x^*))$ is also our corrector function for the unit ball because $\Phi(|x|(y - x^*)) = \Phi(y - x)$ for all $y \in \partial B_2(0, 1)$. Therefor the Green's function for the unit ball in \mathbb{R}^2 is

$$G(x, y) = \Phi(y - x) - \Phi(|x|(y - x^*)) = -\frac{1}{2\pi} (\ln |y - x| - \ln ||x||y - x^*|).$$

The Green's function for the unit ball in \mathbb{R}^n is still $G(x, y) = \Phi(y - x) - \Phi(|x|(y - x^*))$,

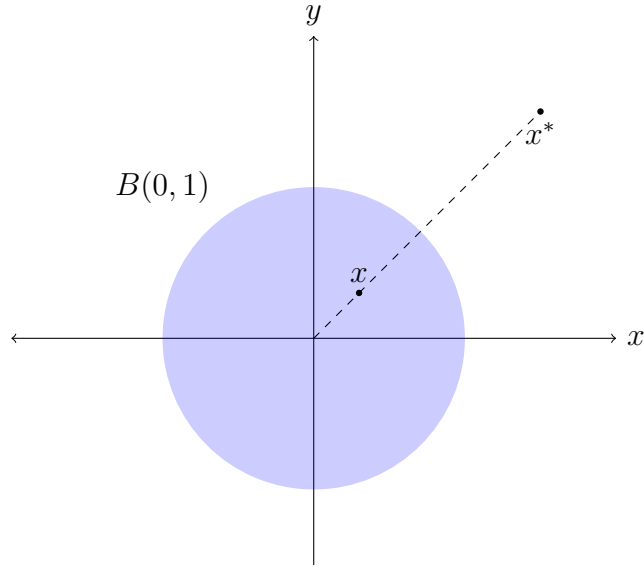


FIGURE 1.3.2: Location of x^* in the plane

where $\Phi(x) = \Phi(y - x) - \Phi(|x|(y - x^*))$ is the fundamental solution when $n \geq 3$, however rather than $B_2(0, 1) \equiv \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 < 1\}$,

$$B_n(0, 1) \equiv \{(x_1, \dots, x_n) : x_1^2 + x_2^2 + \dots + x_n^2 = 1\}$$

1.3.3 Green's Function for the First Quadrant

We shall also find the Green's function for the first quadrant $\{(x, y) \in \mathbb{R}^2, x > 0, y > 0\}$. Let $x^* = (x_1, -x_2)$ if $x = (x_1, x_2)$. In this case the Green's function is

$$G(x, y) = -\frac{1}{2\pi} \ln |x - y| + \frac{1}{2\pi} \ln |x^* - y| - \frac{1}{2\pi} \ln |x + y| + \frac{1}{2\pi} \ln |x^* + y|$$

for the first quadrant.

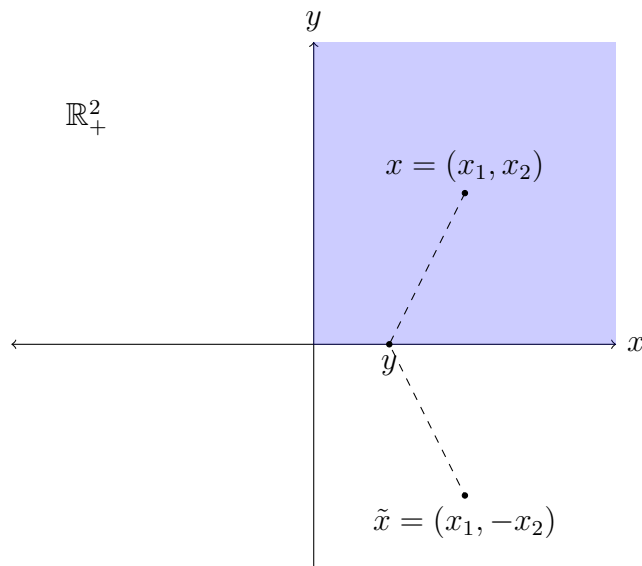


FIGURE 1.3.3: Location of \tilde{x} in the plane

1.3.4 Poisson's Formula for the Ball

Consider the unit ball in \mathbb{R}^n , we are looking for a solution to Laplace's equation in $B_n(0, 1)$ with boundary conditions

$$\begin{cases} \Delta u = 0 & x \in B_n(0, 1) \\ u = g & x \in \partial B_n(0, 1). \end{cases} \quad (1.3.15)$$

We know from Corollary 1.3.2 that u has the form

$$u(x) = - \int_{\partial B_n(0,1)} \frac{\partial G}{\partial \nu}(x, y) g(y) dS(y).$$

We also know from the previous example that the Green's function for a unit ball on \mathbb{R}^n is

$$G(x, y) = \Phi(y - x) - \Phi(|x|(y - x^*))$$

and that

$$\Phi(y) = \frac{1}{n\alpha(n)} \frac{1}{y^{n-2}} \quad (1.3.16)$$

for $n \geq 3$. Working with (1.3.16), we see that

$$\Phi(|x|(y - x^*)) = \frac{1}{n\alpha(n)} \frac{1}{|x|(y - x^*)|^{n-2}} \quad (1.3.17)$$

$$= \frac{1}{|x|^{n-2}} \Phi(y - x^*). \quad (1.3.18)$$

Beginning again with (1.3.16), we find

$$\nabla \Phi(y) = -\frac{y}{n\alpha(n)|y|^n}$$

and

$$\nabla_y \Phi(y - x) = -\frac{y - x}{n\alpha(n)|y - x|^n} \quad (1.3.19)$$

Combining (1.3.18) with (1.3.19) we see that

$$\begin{aligned} \nabla_y \Phi(|x|(y - x^*)) &= -\frac{1}{|x|^{n-2}} \frac{y - x^*}{n\alpha(n)|y - x^*|^n} \\ &= -\frac{y|x|^2 - x}{n\alpha(n)||x|(y - x^*)|^n} \\ &= -\frac{y|x|^2 - x}{n\alpha(n)|y - x|^n} \end{aligned}$$

Since we are working with the unit ball, the unit normal vector to $B_n(0, 1)$ is just

$$\nu = \frac{y}{|y|} = y.$$

Thus, we take the normal derivative of $G(x, \cdot)$ on $\partial B_n(0, 1)$,

$$\begin{aligned}
\frac{\partial G}{\partial \nu} &= \frac{\partial \Phi}{\partial \nu}(y - x) - \frac{\partial \Phi}{\partial \nu}(|x|(y - x^*)) \\
&= -\frac{y - x}{n\alpha(n)|y - x|^n} \cdot y + \frac{|x|^2 - x}{n\alpha(n)|y - x|^n} \cdot y \\
&= \frac{\|y\|^2 + x \cdot y + |y|^2|x|^2 - x \cdot y}{n\alpha(n)|y - x|^n} \\
&= \frac{|y|^2(|x|^2 - 1)}{n\alpha(n)|y - x|^n} \\
&= \frac{|x|^2 - 1}{n\alpha(n)|y - x|^n}.
\end{aligned}$$

Now utilizing the form of $u(x)$ given by Corollary 1.3.2, we know that

$$u(x) = \frac{1 - |x|^2}{n\alpha(n)} \int_{\partial B_n(0,1)} \frac{g(y)}{|y - x|^n} dS(y). \quad (1.3.20)$$

This formula is a solution for the unit ball, so now we want to find a solution formula for the ball of radius r , this time with boundary conditions

$$\begin{cases} \Delta u = 0 & x \in B_n(0, r) \\ u = g & x \in \partial B_n(0, r). \end{cases} \quad (1.3.21)$$

To do this, we take u to be a solution to (1.3.21), and define $\tilde{u}(x) = u(rx)$,

$\tilde{g}(x) = g(rx)$, and $\tilde{y} = ry$. Then use these in (1.3.20) to obtain

$$\begin{aligned}
\tilde{u}(x) &= \frac{1 - |x|^2}{n\alpha(n)} \int_{\partial B_n(0,1)} \frac{\tilde{g}(y)}{|y - x|^n} dS(y) \\
&= \frac{1 - |x|^2}{n\alpha(n)} \int_{\partial B_n(0,r)} \frac{g(\tilde{y})}{|\tilde{y}/r - x|^n} dS(\tilde{y}) \\
&= \frac{r^n(1 - |x|^2)}{n\alpha(n)} \int_{\partial B_n(0,r)} \frac{g(\tilde{y})}{|\tilde{y} - rx|^n} dS(\tilde{y}) \\
&= \frac{r^2 - |rx|^2}{n\alpha(n)r} \int_{\partial B_n(0,r)} \frac{g(y)}{|\tilde{y} - rx|^n} dS(\tilde{y})
\end{aligned}$$

So a solution for the given boundary conditions would be

$$u(x) = \frac{r^2 - |x|^2}{n\alpha(n)r} \int_{\partial B_n(0,r)} \frac{g(y)}{|y - x|^n} dS(y). \quad (1.3.22)$$

This equation is known as Poisson's formula for the ball. In order to show that (1.3.22) is a solution to Laplace's equation and satisfies the boundary conditions given by (1.3.1), we need to make use of the following theorem.

Theorem 1.3.3. *For $g \in C(\partial B_n(0, r))$ and u as defined by Poisson's formula for the ball, u satisfies*

1. $u \in C^\infty(B_n(0, r))$
2. $\Delta u = 0$ for $x \in B_n(0, r)$
3. for $x \in B_n(0, r)$ and all $x_0 \in \partial B_n(0, r)$, $\lim_{x \rightarrow x_0} u(x) = g(x_0)$

To prove (1.3.3) we first need to prove that Green's functions are symmetric,

Lemma 1.3.4. *For all $x, y \in \Omega, x \neq y$*

$$G(x, y) = G(y, x)$$

Proof. Begin by fixing $x, y \in \Omega, x \neq y$, and let $v(z) \equiv G(x, z)$ and $w(z) \equiv G(y, z)$. We need to show that $v(y) = w(x)$ in order to show that the functions are symmetric. Remember that Green's function is

$$G(x, y) = \Phi(y - x) - h^x(y)$$

where $h^x(y)$ satisfies boundary conditions

$$\begin{cases} \Delta_y h^x = 0 & y \in \Omega \\ h^x(y) = \Phi(y - x) & y \in \partial\Omega. \end{cases}$$

Therefore both $G(x, z) = 0$ and $G(y, z) = 0$. Since v and w are smooth everywhere except when $z = x$ and $z = y$, respectively, we should define a new region V_ϵ such that $V_\epsilon = \Omega - [B(x, \epsilon) \cup B(y, \epsilon)]$. Since both v and w are smooth on this new region, we can integrate by parts,

$$\begin{aligned} \int_{V_\epsilon} \Delta v w \, dz &= - \int_{V_\epsilon} \nabla v \cdot \nabla w \, dz + \int_{\partial V_\epsilon} \frac{\partial v}{\partial \nu} w \, dS(z) \\ &= \int_{V_\epsilon} v \Delta w \, dz - \int_{\partial V_\epsilon} \frac{\partial w}{\partial \nu} v \, dS(z) + \int_{\partial V_\epsilon} \frac{\partial v}{\partial \nu} w \, dS(z) \end{aligned}$$

Since $\Delta w = \Delta v = 0$ the first term on the right and the term on the left both vanish

$$0 = - \int_{\partial V_\epsilon} \frac{\partial w}{\partial \nu} v \, dS(z) + \int_{\partial V_\epsilon} \frac{\partial v}{\partial \nu} w \, dS(z)$$

and thus,

$$\int_{\partial V_\epsilon} \frac{\partial w}{\partial \nu} v \, dS(z) = \int_{\partial V_\epsilon} \frac{\partial v}{\partial \nu} w \, dS(z)$$

Since $v = w = 0$ on $\partial\Omega$, we can rewrite this equality as

$$\int_{\partial B(x,\epsilon)} \left[\frac{\partial v}{\partial \nu} w - \frac{\partial w}{\partial \nu} v \right] dS(z) = \int_{\partial B(y,\epsilon)} \left[\frac{\partial w}{\partial \nu} v - \frac{\partial v}{\partial \nu} w \right] dS(z) \quad (1.3.23)$$

Now we must show that as ϵ approaches 0, the left and right sides converge to $v(x)$ and $w(y)$ respectively. Beginning with the left side,

$$\begin{aligned} \int_{\partial B(x,\epsilon)} \left[\frac{\partial v}{\partial \nu} w - \frac{\partial w}{\partial \nu} v \right] dS(z) &= \int_{\partial B(x,\epsilon)} \frac{\partial v}{\partial \nu} w dS(z) - \int_{\partial B(x,\epsilon)} \frac{\partial w}{\partial \nu} v dS(z) \\ &= \int_{\partial B(x,\epsilon)} \left[\frac{\partial \Phi}{\partial \nu}(z-x) - \frac{\partial h^x}{\partial \nu}(z) \right] w dS(z) \\ &\quad - \int_{\partial B(x,\epsilon)} \frac{\partial w}{\partial \nu} v dS(z) \\ &= \int_{\partial B(x,\epsilon)} \frac{\partial \Phi}{\partial \nu}(z-x) w(z) dS(z) - \int_{\partial B(x,\epsilon)} \frac{\partial h^x}{\partial \nu}(z) w dS(z) \\ &\quad - \int_{\partial B(x,\epsilon)} \frac{\partial w}{\partial \nu} v dS(z) \end{aligned}$$

We will show that the last two terms converge to zero while the first term converges to $w(x)$. Beginning with the final term, since w is smooth near x , we know that $\frac{\partial w}{\partial \nu}$ is bounded near $\partial B(x, \epsilon)$, this means that $v(z) = \Phi(z-x) - h^x(z)$ and

$$\left| \int_{\partial B(x,\epsilon)} \frac{\partial w}{\partial \nu} v dS(z) \right| \leq C \sup_{\partial B(x,\epsilon)} |v| \int_{\partial B(x,\epsilon)} dS(z) = C\epsilon^{n-1} \sup_{\partial B(x,\epsilon)} |v| = O(\epsilon)$$

which converges to 0 as $\epsilon \rightarrow 0$. The second term follows in a similar manner,

$$\left| \int_{\partial B(x,\epsilon)} \frac{\partial h^x}{\partial \nu}(z) w dS(z) \right| \leq C \int_{\partial B(x,\epsilon)} dS(z) \leq C\epsilon^{n-1}.$$

Which again converges to 0 as $\epsilon \rightarrow 0$. Finally for the first term,

$$\int_{\partial B(x,\epsilon)} \frac{\partial \Phi}{\partial \nu} (z-x)w(z) dS(z) = \frac{1}{n\alpha(n)\epsilon^{n-1}} \int_{\partial B(x,\epsilon)} w(z) dS(z) \rightarrow w(x) \text{ as } \epsilon \rightarrow 0.$$

Thus the left side converges to $w(x)$; the right side converges to $v(y)$ in a similar manner. □

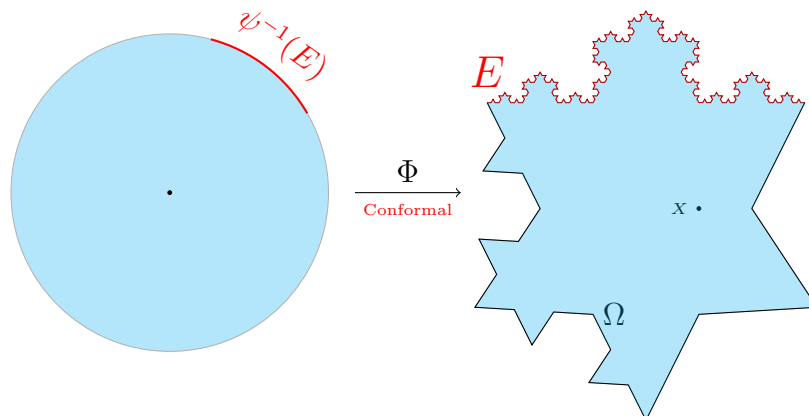
Chapter 2

Harmonic Measure

Harmonic measure stems from the Dirichlet problem and can be used to measure the growth of harmonic functions. The harmonic measure is denoted by $\omega(z, \Omega, E)$, and measures the value of a harmonic function ω at a point z with a boundary limit of 1 at E and 0 at $\partial\Omega \setminus E$, where Ω is a domain in the complex plane \mathbb{C} consisting of finitely many closed curves called *Jordan Curves*. In this setting, the harmonic measure always exists, is always unique, and is a conformal invariant; its resolvability is highly dependent on the simplicity of the boundary of the domain, Ω .

Theorem 2.0.1. *The harmonic measure, $\omega(z, \Omega, E)$, of a harmonic function, u , always exists and is always unique.*

Proof. Fix a Jordan domain Ω . Since for any Ω there exists a conformal mapping $\Phi : \Omega \rightarrow U$ where U is a domain bounded by finitely many circles. Now take F such that $F \subset \partial\Omega$. Since F consists of finitely many circular arcs, its harmonic measure exists and is equal to the Poisson integral of F 's characteristic function. We know from Carathéodory's Theorem that the extension of Φ to its boundaries is a



homeomorphism of the closures, so therefore we can use Φ to show that the harmonic Measure of Ω exists in all cases.

Uniqueness for bounded Ω follows from the maximum principle, but for unbounded Ω we need to use an extension called Lindelöf's maximum principle. It states that for any domain Ω whose boundary is not a finite set, let u be a real valued harmonic function on Ω and take $M > 0$ such that

$$u(z) \leq M \quad \text{for all } z \in \Omega.$$

Now take m to be a real valued constant such that

$$\limsup_{z \rightarrow \zeta} u(z) \leq m$$

for all except finitely many points of $\zeta \in \Omega$. Then $u(z) \leq m$ for all $z \in \Omega$. With this extension of the maximum principle we cover all cases for Ω bounded and unbounded, so the harmonic measure is always unique. \square

Remark 2.0.2. When domains are sufficiently smooth, then Green's theorem implies

harmonic measure is given by the normal derivative of Green's function times surface measure on the boundary. Therefore, to find a harmonic measure of a domain, the key thing is to find the gradient of Greens function.

2.1 Examples

2.1.1 The Upper Half Plane

For the first example, choose Ω to be the upper half of the complex plane, and E to be any interval $[-T, T]$ on the real axis. Given that the Poisson kernel for Ω in this case is

$$P(x, y) = \frac{1}{\pi} \cdot \frac{x^2}{x^2 + y^2}.$$

We can easily get this by using the Green's function for the upper half space. That is, for fixed $x + iy \in \mathbb{R}_+^2$, i.e., $y > 0$, from our earlier work we have

$$G(x, y, \xi, \eta) = \frac{1}{4\pi} \ln \frac{(\xi - z)^2 + (\eta - y)^2}{(\xi - x)^2 + (\eta + y)^2}.$$

Hence on the boundary of \mathbb{R}_+^2 , i.e., $\eta = 0$ we find

$$\frac{\partial G}{\partial \nu} = \langle \nabla G, \nu \rangle = -\frac{\partial G}{\partial \eta}.$$

Now if we do the algebra we get

$$-\frac{\partial G}{\partial \eta} = \frac{1}{\pi} \cdot \frac{x^2}{x^2 + y^2}.$$

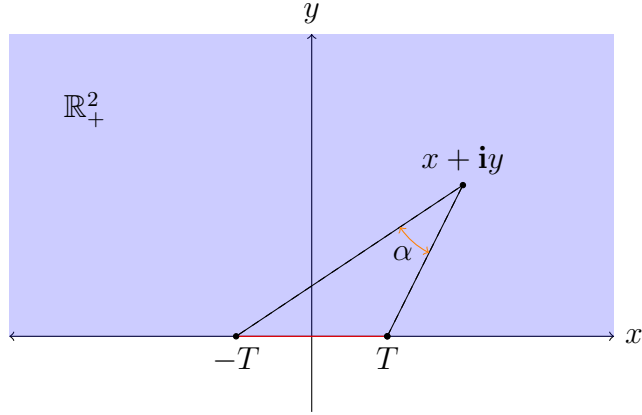


FIGURE 2.1.1: Harmonic measure of upper half space for $[-T, T]$

As we want to find the harmonic measure of $[-T, T]$ with fixed $x + iy$ we need to solve the following Dirichlet problem

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}_+^2, \\ u = 1 & \text{on } y = 0 \text{ and } -T \leq x \leq T, \\ u = 0 & \text{on } y = 0 \text{ and } x \in \mathbb{R} \setminus [-T, T]. \end{cases}$$

If we let $\omega(x + iy, \mathbb{R}_+^2, [-T, T]) = u(x, y)$ we know that the solution is given he harmonic function is given by

$$\begin{aligned} u(x, y) = \omega(x + iy, \mathbb{R}_+^2, [-T, T]) &= \frac{1}{\pi} \int_{-\infty}^{\infty} \chi_{[-T, T]}(t) \frac{y}{(x-t)^2 + y^2} dt \\ &= \frac{1}{\pi} \int_{-T}^T \frac{y}{(x-t)^2 + y^2} dt. \end{aligned} \tag{2.1.1}$$

This can be rewritten in terms of $\omega(x + iy, \Omega, E)$ and simplified to yield

$$\omega(x + iy, \Omega, E) = \frac{1}{\pi} \tan^{-1} \left(\frac{x+T}{y} \right) - \frac{1}{\pi} \tan^{-1} \left(\frac{x-T}{y} \right)$$

Where \tan^{-1} takes values from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$. This implies that $\omega(x + iy, \Omega, E)$ is equal to $\frac{\alpha}{\pi}$, where α is the angle subtended by the interval $[-T, T]$ at z . In only Cartesian coordinates, this is

$$\omega(z, \Omega, E) = \frac{\partial G}{\partial n} = -\frac{1}{\pi} \frac{y_0}{(x - x_0)^2 + y_0^2}.$$

2.1.2 The Unit Disk

For another example, rather than having E be a line segment, let it be the arc of a circle in \mathbb{C} from $-i$ to i such that

$$\omega(z, D, E) = \frac{2\theta - \alpha}{2\pi}$$

Here α is the central angle of the circle and θ is the angle subtended by E at the point z . Just like in the previous example we calculate the Poisson integral of the curve. Because our curve is part of a circle, the Poisson kernel is

$$P_r(\theta) = \frac{1 - r^2}{1 - 2r \cos(\theta) + r^2}$$

And the full Poisson integral is

$$\begin{aligned} \omega(z, D, E) &= \frac{1}{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1 - r^2}{1 - 2r \cos(\lambda - t) + r^2} dt \\ &= \frac{1 - r^2}{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{dt}{(1 + r^2) - 2r \cos(\lambda - t) + r^2} \end{aligned} \tag{2.1.2}$$

using the identity

$$\int \frac{dx}{a + b \cos(x)} = \frac{2}{\sqrt{a^2 - b^2}} \tan^{-1} \left(\sqrt{\frac{a-b}{a+b}} \tan \frac{x}{2} \right)$$

the integral simplifies to

$$\omega(z, D, E) = \frac{1}{\pi} \left[\tan^{-1} \left(\frac{1+r}{1-r} \tan \frac{\lambda + \pi/2}{2} \right) - \tan^{-1} \left(\frac{1+r}{1-r} \tan \frac{\lambda - \pi/2}{2} \right) \right]$$

$$\omega(z, D, E) = \frac{1}{\pi} \left[\tan^{-1} \left(\left(\frac{1+r}{1-r} \right) \cdot \left(\frac{\sin \lambda + 1}{\cos \lambda} \right) \right) - \tan^{-1} \left(\left(\frac{1+r}{1-r} \right) \cdot \left(\frac{\sin \lambda - 1}{\cos \lambda} \right) \right) \right]$$

then applying the identity

$$\tan^{-1} a - \tan^{-1} b = \tan^{-1} \left(\frac{a-b}{1+ab} \right)$$

one can simplify the equation to

$$\omega(z, D, E) = \frac{1}{\pi} \tan^{-1} \left(\frac{1-r^2}{2r \cos \lambda} \right).$$

This can also be written as

$$\omega(z, D, E) = \frac{1}{2\pi r} \frac{r^2 - \rho^2}{r^2 + \rho^2 - 2r\rho \cos(\theta - \phi)}$$

Where our starting point is defined by $z = (\rho \cos \phi, \rho \sin \phi)$ and ϕ is the angle at center of point z with respect to the x axis.

2.2 Brownian Motion

A common problem that harmonic measure is used to solve is detecting an exit point in Brownian motion. That is, given some particle in a domain $\Omega \in \mathbb{R}$ that moves in accordance with Brownian motion, we are looking to find the point at which the particle first hits the boundary. In order to do this, we would place circular detectors of some radius r along $\partial\Omega$ which would detect when the particle hits.

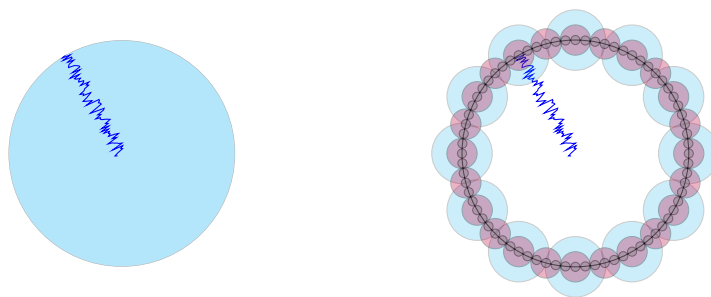


FIGURE 2.2.1: Brownian motion inside ball and covering the boundary with balls

Given that these detectors have a cost of $\phi(r)$ we have to consider that, given a finite budget, we wouldn't be able to find the exit point of certain Ω such as the unit disk, because the detectors would have to be so small that their radii sum to infinity. However if $\phi(r)/r \rightarrow 0$ as $r \rightarrow 0$ then covering the boundary with n_k balls of radius $1/n_k$ would result in us having a finite cost.

Now consider $\partial\Omega$ to be the von Koch snowflake. It would take 4^n balls of size 3^{-n} to cover the boundary, so we can accomplish this on a finite budget. Though it would seem more expensive, it is actually cheaper to detect exit points on this fractal because all points of the boundary are not equally likely to be exit points like they were on the disk. We do this using the cost function $\phi(r) = r$.

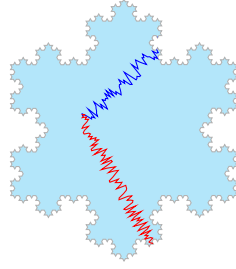


FIGURE 2.2.2: Brownian motion when boundary is the von Koch snowflake

2.2.1 Harmonic Measure in Brownian Motion

Given $z \in \Omega \subset \mathbb{R}^2$ and $E \subset \partial\Omega$ the harmonic measure of E in Ω with respect to z is the probability that a Brownian motion started at z exits Ω somewhere in E . For fixed E this is a harmonic function with values in $[0,1]$. The minimum principle tells us that if the function vanishes anywhere on Ω then it vanishes everywhere. Therefore sets where $\omega(z, E, \Omega) = 0$ are not dependent on z . So we're looking for conditions that would force the harmonic measure to be zero. Two of these conditions is the F. and M. Riesz Theorem (2.2.1) and Dahlberg's Theorem:

Theorem 2.2.1. *If $\Omega \in \mathbb{R}^2$ is a simply connected planar domain curve such that $H_1(\partial\Omega) < \infty$, then harmonic measure is mutually absolutely continuous with respect to arc length. That is,*

$$\text{for } E \subset \partial\Omega, \omega(z, E, \Omega) = 0 \iff H_1(E) = 0$$

Where H_1 is the one dimensional *Hausdorff Measure* and is equal to the length of the curve.

Theorem 2.2.2 (Dahlberg, [7]). *For a bounded Lipschitz' domain $\Omega \in \mathbb{R}^n$ then both $(n-1)$ -dimensional Hausdorff measure and harmonic measure are mutually absolutely*

continuous.

$$E \in \partial\Omega, \omega(z, E, \Omega) = 0 \iff H^{n-1}(E) = 0$$

Another theorem helpful theorem here is Makarov's Theorem.

Theorem 2.2.3 (Makarov, [3]). *If Ω is simply connected, then $\dim(\omega) = 1$.*

Makarov's theorem, published in 1985, was a huge jump in the field. He proved an even more precise version in the same year. This more precise version is known as Makarov's Law of the Iterated Logarithm (LIL) for ω .

Theorem 2.2.4 (Makarov, [3]). *Let*

$$\phi_C(t) = t \exp\left(C\sqrt{\log\frac{1}{t} \log\log\log\frac{1}{t}}\right).$$

Then there is a value A so that $\omega \ll H_{\phi_A}$ for every simply connected domain and a value B so that $\omega \perp H_{\phi_B}$ for some simply connected domain.

The LIL has a very useful reduction when applied in a Bloch Space. A *Bloch Space* \mathfrak{B} is the space of holomorphic functions defined on the unit disc in the complex plane such that

$$\|f\|_{\mathfrak{B}} = \sup_{z \in \mathbb{D}} |f'(z)|(1 - |z|) < \infty.$$

With the knowledge that $\|\log f'\|_{\mathfrak{B}} \leq 2 \implies f$ is conformal $\implies \|\log f'\|_{\mathfrak{B}} \leq 6$, and using

$$\text{diam}(J) = \text{diam}(f(I)) \approx |f'(z_I)|\text{diam}(I) \approx |f'(z_I)|\omega(J),$$

where I is an interval on \mathbb{T} and $J = f(I) \subset \partial\Omega$, we find that the LIL reduces to

$$\limsup_{r \rightarrow 1} \frac{|g(re^{i\theta})|}{\sqrt{\log \frac{1}{1-r} \log \log \log \frac{1}{1-r}}} = O(\|g\|_{\mathfrak{B}}) \text{ for a.e. } \theta \text{ when } g \in \mathfrak{B}. \quad (2.2.1)$$

2.2.2 Multiply Connected Domains

Up until now, we have only dealt with simply connected domains. For a multiply connected domain all of $\dim(\partial\Omega)$ could be less than one, and so we wouldn't have $\omega \ll H_\alpha$ for all $\alpha < 1$. For the multiply connected domains, the equation

$$\int_{\partial\Omega} \log \frac{\partial G}{\partial \nu} d\omega = C + \sum_{z: \nabla G(z)=0} G(z) \quad (2.2.2)$$

is useful. In (2.2.2) C is the Robin constant of $\partial\Omega$ and G is the Green's function for Ω . Since $G > 0$ both sides are bounded below, and since $d\omega = \frac{\partial G}{\partial \nu} ds$ $d\omega$ has density bounded below compared to arclength and ω is part of a one dimensional set. If $\partial\Omega$ has "many, well separated" components then $\dim(\omega) < 1$.

Bibliography

- [1] Sheldon Axler, Paul Bourdon, and Wade Ramey. *Harmonic function theory*, volume 137 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1992.
- [2] John B. Garnett and Donald E. Marshall. *Harmonic Measure*, volume 2 of *New Mathematical Monographs*. Cambridge University Press, Cambridge, 2008.
- [3] Nikolai G. Makarov. On the distortion of boundary sets under conformal mappings. *Proc. London Math. Soc. (3)*, 51(2):369–384, 1985. [34](#)
- [4] Christopher J. Bishop. Book Review of *Harmonic Measure*. Volume 44, Number 2, pages 267-276 of the Bulletin of the American Mathematics Society, 2006.
- [5] Kakutani, Shizuo. *Two-dimensional Brownian motion and harmonic functions*. Proc. Imp. Acad. 20 (1944), no. 10, 706–714. doi:10.3792/pia/1195572706. <https://projecteuclid.org/euclid.pja/1195572706>
- [6] Steven G. Krantz *The Theory and Practice of Conformal Geometry*. 169-189, Dover Publications, 2016.
- [7] Björn E. J. Dahlberg. On the absolute continuity of elliptic measures. *Amer. J. Math.*, 108(5):1119–1138, 1986. [33](#)

- [8] Julie Levandosky, *Green's Functions* Lecture Notes. Stanford University, Math 220b, 2003, <https://web.stanford.edu/class/math220b/handouts/greensfcns.pdf>