September 2008

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Department of Economics Working Paper Series

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Working Paper 2008-37

September 2008

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Abstract

This paper shows how one can infer the nature of local returns to scale at the input- or output-oriented efficient projection of a technically inefficient input-output bundle, when the input- and output-oriented measures of efficiency differ.

Journal of Economic Literature Classification: D2, C6

Keywords: Most Productive Scale Size; Convex Technologies, Nonparametric Efficiency Analysis

COMPARING INPUT- AND OUTPUT-ORIENTED MEASURES OF TECHNICAL EFFICIENCY TO DETERMINE LOCAL RETURNS TO SCALE IN DEA MODELS

It is well known in the nonparametric Data Envelopment Analysis (DEA) literature that when the technology exhibits constant returns to scale (CRS) globally, input- and output oriented radial measures of technical efficiency are identical. By implication, if this equality does not hold for every input-output bundle, the technology is characterized by variable returns to scale (VRS).

There are three alternative approaches to identifying the nature of local returns to scale at any given input-output bundle on the frontier of the production possibility set: (i) a primal approach due to Banker (1984), (ii) a dual approach due to Banker, Charnes, and Cooper (1984), and (iii) a "nesting approach" due to Färe, Grosskopf, and Lovell (1985)[1].

The objective of this methodological note is to show how a comparison of the two different efficiency measures reveals the nature of (local) returns to scale at the input-oriented projection of an inefficient input-output bundle on to the frontier of the production possibility set.

The paper unfolds as follows. We start with some basic concepts and definitions. Next we prove a lemma showing that if the production possibility set is convex, increasing returns to scale (IRS) cannot follow diminishing returns to scale. Next we prove the main result for both single-output single input and multi-output multi-input technologies.

Basic Concepts and Definitions:

The production technology faced by firms in an industry producing output vectors (y) from input vectors (x) can be described by the *production possibility set*

$$T = \{(x, y): x \in R_n^+; y \in R_m^+; y \text{ can be produced from } x\}. \qquad (1)$$

An input-output bundle *(x, y)* is considered feasible if and only if $(x, y) \in T$. The *frontier* of the production possibility set (also known as the *graph of the technology*) is

$$G = \{(x, y) : (x, y) \in T; \alpha > 1 \Rightarrow (x, \alpha y) \notin T; \beta < 1 \Rightarrow (\beta x, y) \in T\}. \quad (2)$$

The input-oriented technical efficiency of a feasible input-output bundle *(x, y)* is

$$\tau_x = \theta^* = \min \theta : (\theta x, y) \in G. \qquad (2)$$

Similarly, the output-oriented technical efficiency of the same bundle is

[1] See Ray (2004; chapter 3) for an exposition of the alternative approaches and their equivalence.

$\tau_y = \frac{1}{\varphi^*}$, where

$\varphi^* = \max \varphi : (x, \varphi y) \in G.$ (3)

Obviously, $\theta^* \leq 1$ and $\varphi^* \geq 1$.

Banker (1984) generalized Frisch's (1963) concept of the *technically optimal production scale* to define a *most productive scale size* (MPSS) in the context of multiple-input multiple-output technology. An input-output bundle $(x^*, y^*) \in G$ is a most productive scale size if, for any non-negative scalars α and β such that $(\beta x^*, \alpha y^*) \in G, \frac{\alpha}{\beta} < 1$. Thus, (x^*, y^*) is a point on the frontier of the production possibility set where average productivity attains a maximum in the single-output single-input case or *ray average productivity* reaches a maximum in the multiple-output multiple-input case.

Locally *constant returns to scale* holds at an input-output bundle $(\bar{x}, \bar{y}) \in G$, if there is a real number $\varepsilon > 0$ (however small) such that

$(\beta\bar{x}, \alpha\bar{y}) \in G$ and $\alpha = \beta$, where $\beta = 1 \pm \varepsilon$.

Locally increasing *returns to scale* holds if

$(\beta\bar{x}, \alpha\bar{y}) \in G$ and $\alpha > \beta$, where $\beta = 1 + \varepsilon$.

Similarly, locally *diminishing returns to scale* holds if

$(\beta\bar{x}, \alpha\bar{y}) \in G$ and $\alpha < \beta$, where $\beta = 1 + \varepsilon$.

Banker (1984) has shown that locally constant returns to scale holds at an MPSS. It is possible, however, that the maximum (ray) average productivity is attained at multiple levels (scales) of input. For such technologies, locally constant returns to scale holds at every input (bundle) within this range. In such cases, of particular interest are the smallest and the largest MPSS bundles[2].

(Insert Figure 1 here)

In Figure 1, the two axes measure the levels of input and output in the one input-one output case and the graph is merely the production function. In the multiple-output multiple-input case, the two axes show the input and output *scale* for some given input- and output-mix. The broken line *ABCDEF* in Figure 1 shows the graph of some hypothetical technology. All points between (and including *C* and *D*) represent an MPSS. In this figure, *C* represents the smallest and *D* the largest MPSS. All points to the left of *C* exhibit locally increasing returns to scale and locally diminishing returns to scale holds at any point to the right of *D*. For the inefficient input-output bundle shown by the point *R*, R_x is its input-oriented technically efficient projection on to the graph. Similarly, the point R_y is its output oriented projection.

We now prove the following lemma to show that locally increasing returns holds at every scale smaller than the smallest MPSS and locally diminishing returns hold s at very scale greater than the largest MPSS.

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[2] See also Banker and Thrall (1992) and Banker *et al.* (2004)

Lemma: *For any convex productivity possibility set T, if there exist non-negative scalars α and β such that α >β >1, and both* $(\bar{x}, \bar{y})$ *and* $(\beta\bar{x}, \alpha\bar{y}) \in G$, *then for every γ and δ such that 1<δ<β and* $(\delta\bar{x}, \gamma\bar{y}) \in G, \gamma > \delta.$

Proof: Because $(\bar{x}, \bar{y})$ and $(\beta\bar{x}, \alpha\bar{y})$ are both feasible, by convexity of *T*, for every $\lambda \in (0,1), ((\lambda + (1-\lambda)\beta)\bar{x}, (\lambda + (1-\lambda)\alpha)\bar{y})$ is also feasible. Now select λ such that $\lambda + (1-\lambda)\beta = \delta.$ Further, define $\mu = \lambda + (1-\lambda)\alpha.$ Using these notations, $(\delta\bar{x}, \mu\bar{y}) \in T.$ But, because $(\delta\bar{x}, \gamma\bar{y}) \in G, \gamma \geq \mu.$ However, because $\alpha > \beta, \mu > \delta.$ Hence, $\gamma > \delta.$

An implication of this lemma is that, when the production possibility set is convex, if the technology exhibits locally diminishing returns to scale at smaller input scale, it cannot exhibit increasing returns at a bigger input scale. This is easily understood in the single-input ingle-output case. When both *x* and *y* are scalars, average productivity at $(\bar{x}, \bar{y})$ is $\frac{\bar{y}}{\bar{x}}$ and at $(\beta\bar{x}, \alpha\bar{y})$ it is $\frac{\alpha}{\beta}\frac{\bar{y}}{\bar{x}}$. Thus, when $\alpha > \beta,$ average productivity has increased. The above lemma implies that for every input level *x* in between $\bar{x}$ and $\beta\bar{x}$, average productivity is greater than $\frac{\bar{y}}{\bar{x}}$. Thus, average productivity could not first decline and then increase as the input level increased from $\bar{x}$ to $\beta\bar{x}$.

Two results follow immediately. First, locally increasing returns to scale holds at every input-output bundle $(x, y) \in G$ that is smaller than the smallest MPSS. Second, locally diminishing returns to scale holds at every input-output bundle $(x, y) \in G$ that is greater than the largest MPSS. To see this, let $x = bx^*$ and $y = ay^*$, where (x^*, y^*) is the smallest MPSS for the given input and output mix. Because (x, y) is not an MPSS, $\frac{a}{b} < 1.$ Further, assume that $b < 1$. Define $\beta = \frac{1}{b}(> 1)$ and $\alpha = \frac{1}{a}$. Then $(x^*, y^*) = (\beta x, \alpha y)$ and $\frac{\alpha}{\beta} > 1.$ Because ray average productivity is higher at a larger input scale, by virtue of the lemma, locally increasing returns to scale holds at $(x$, y). Next assume that $b > 1$. Again, because (x, y) is not an MPSS, $\frac{a}{b}<1$. That is ray average productivity has fallen as the input scale is increased from x^* to $x = bx^*$. Then, by virtue to of the lemma, ray average product could not be any higher than $\frac{a}{b}$ at a slightly greater input scale, $\bar{\bar{x}} = (1+\varepsilon)x.$ But, because (x, y) is not an MPSS, ray average product cannot remain constant as the input scale is slightly increased. Hence, ray average product must fall as the input scale is slight increased from *x*. Hence, locally diminishing returns to scale holds at every $(x, y) \in G,$ when *x* is larger than the largest MPSS.

We may now present the main result of this note relating the input- and output-oriented technical efficiencies of an inefficient bundle (x^0, y^0).

Theorem 1: If the input-oriented technical efficiency is greater than the output-oriented technical efficiency, then locally increasing returns to scale holds at the efficient input-oriented projection of (x^0, y^0).

Proof: The input-oriented projection of the bundle (x^0, y^0) on to G is $(\theta^* x^0, y^0)$ where θ^* is the measured level of input-oriented technical efficiency (τ_x). Similarly, the output-oriented projection is $(x^0, \varphi^* y^0)$ and the output-oriented technical efficiency is $\tau_y = \frac{1}{\varphi^*}$. Define the input bundle $\bar{x}^0 = \theta^* x^0$ and the output bundle $\bar{y}^0 = \varphi^* y^0$. Note that both the input-output bundles $(\bar{x}^0, y^0)$ and $(x^0, \bar{y}^0)$ are in G. Further, $(x^0, \bar{y}^0)$ can also be expressed as $(\beta\bar{x}^0, \alpha y^0)$ where $\beta = \frac{1}{\theta^*}$ and $\alpha = \varphi^*$. Now, $\tau_x = \theta^* > \tau_y = \frac{1}{\varphi^*}$ implies $\alpha = \varphi^* > \beta = \frac{1}{\theta^*}$. Note that, by construction, $\theta^* < 1$ and $\beta > 1$. Thus, ray average productivity is higher At the bigger input scale $\beta\bar{x}^0$. Hence, by virtue of the lemma, locally increasing returns to scale holds at the input-oriented efficient projection $(\bar{x}^0, y^0)$.

Theorem 2: If the output-oriented technical efficiency is greater than the input-oriented technical efficiency, then locally diminishing returns to scale holds at the efficient output-oriented projection of (x^0, y^0).

Proof: Now suppose that $\tau_y > \tau_x$. That is, $\frac{1}{\varphi^*} > \theta^*$ implying that $\alpha = \varphi^* < \beta = \frac{1}{\theta^*}$. In other words, ray average productivity is lower at the output-oriented projection $(x^0, \varphi^* y^0)$ than at the input-oriented projection $(\theta^* x^0, y^0)$. Hence, locally diminishing returns to scale holds at $(x^0, \varphi^* y^0)$.

Conclusion: An implication of the above is that when input-oriented technical efficiency is higher than the output-oriented, the firm would need to increase its output scale in order to attain the most productive scale size, once input-inefficiency is eliminated. Similarly, if output-efficiency is higher, the firm needs to scale down after eliminating output inefficiency[3]. One limitation of the methodology proposed here, however, is that it can be applied only when $\tau_x \neq \tau_y$.

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[3] This was pointed out to me by Subal Kumbhakar.

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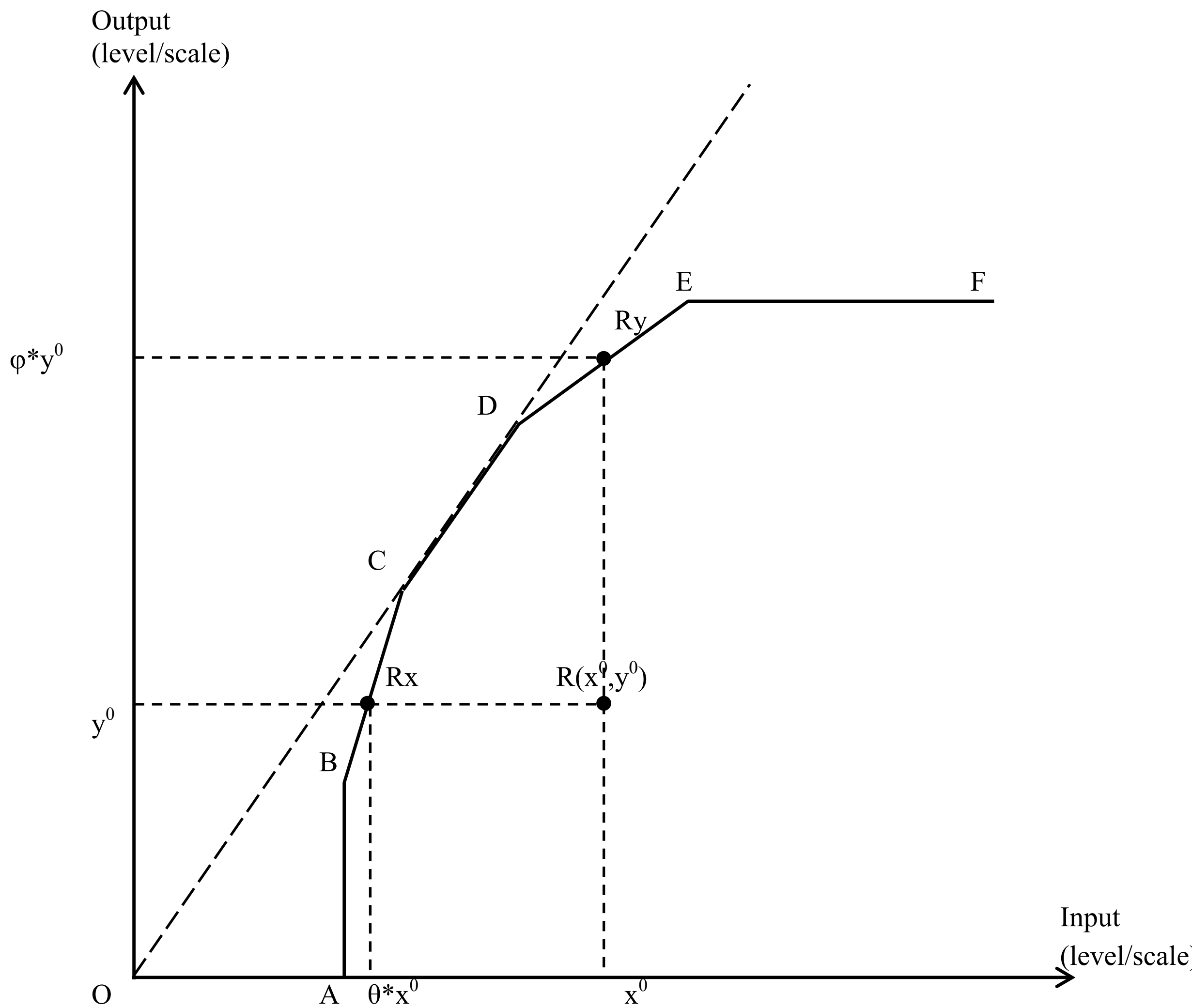

Figure 1. Graph of the Technology and MPSS