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Getting to the "Why": Teacher Practices that Support Mathematically Sound Student Justifications

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Abstract

Though the mathematical and education communities both value justification and argumentation in the middle grades classroom, teachers have historically found these practices difficult to support. This article discusses teaching practices that are associated with high levels of mathematically acceptable argumentation by students. Data were collected on seven committed teachers who explored justification and then implemented the same justification task over two years. Thus, the data reflected fourteen different implementations of the same task, allowing us to compare lessons directly. The findings describe how teachers' *Focusing Students' Mathematics* and *Providing Scaffolding Questions* are consistently associated with high levels of justification, while *Leveraging a Critical Classroom Community* and *Providing Task Specific Tools* are only sometimes associated an increased level of justification in a classroom. There are implications for teachers wishing to implement their own justification tasks, and researchers wishing to further study justification at the middle school level.

Background

Introduction

Teaching mathematics at the middle school level involves imparting both mathematical content and mathematical dispositions. One such overarching disposition is reasoning. The National Council of Teachers of Mathematics (NCTM, 2000) listed several principles and standards for teaching mathematics at all grade levels. NCTM standards include "reasoning and proof" as a theme to be explored in the middle school mathematics classroom. Their goal is to have every middle school student be able to examine patterns, to construct and evaluate mathematical arguments, and to understand how fundamental proof is to mathematics as a discipline (NCTM, 2000). A standard mathematical practice, as set by the Common Core State Standards (which have been accepted by most of the states in the United States) is for students to "construct viable arguments and critique the reasoning of others" (CCSS, 2010). Thus, mathematical reasoning is currently a valued part of teaching mathematics at the middle school level.

One classroom practice that can be used to build reasoning skills is justification. Justification can be thought of in two ways: as either a process or an end goal. When teachers ask students to explain the mathematical reasoning behind their explanations, they are employing justification as a process. In using this process, the teachers use justification to help students learn the mathematics. A justification can also be a product produced by the students, similar to, or a precursor to, a proof. These justifications are a logical linking of mathematical ideas that together provide a reason for a solution. This kind of justification is valued by the mathematical community, and can be seen as a valuable product in its own right.

Historically, teachers have had trouble implementing tasks that call for reasoning and proof in their classrooms (Kazemi & Stipek, 2001; Stein, Grover, & Henningsen, 1996). Teachers may not know how to implement justification tasks or how to elicit justifications from their students. In order to help teachers make mathematical reasoning a common practice in their classroom, it is critical to help them understand what teaching practices best support justification. To do this, researchers must first uncover what teachers who are currently committed to justification are doing in their classrooms to promote this mathematical practice. In this paper, I explore the question "what teaching practices are associated with high levels of student justification?"

What Kinds of Justification Are Productive?

This paper emphasizes the term "justification" over the more formalized "proof." For our purpose, proof is seen as a subset of justification practices. This distinction is made because there is some question in the mathematical community about whether or not students at the middle school level should be expected to prove their mathematical conjectures, and debate about what would constitute a proof. Also, research on justification and proving has shown highly contrasting results on whether proof methods themselves lead to understanding. Martin, Soucy McCrone, Wallace Bower, and Dindyal (2005) presented a case study on an honors geometry teacher who, despite developing quite sophisticated proof methods, seemed unable to develop mathematical understanding with his students. In this study, the teacher, Mr. Drummond, provided a space for students to make conjectures, and coached their proving in an axiomatic system. Mr. Drummond also modeled the deductive proof system for his students. The students developed an ability to construct formal proofs in this complex axiomatic system, using the theorems and definitions. Even though their proofs were mathematically sound, the students were not often convinced of the generality of the proofs that they provided. Martin and colleagues (2005) considered this evidence that the students did not have a good conceptual understanding of the proofs that they were writing. Though the students could provide a deductive proof, in this instance, the proof did not help them to understand the underlying geometry.

In contrast, Maher and Martino (1996) explored over five years how one young student can develop quite sophisticated methods of proof in mathematics. This case study illustrated how "Stephanie" at age ten developed a proof that showed how many ways towers of two different colored blocks could be stacked. Maher and Martino wrote that Stephanie's teachers provided her with many spaces to develop multiple representations of the problem and discussed aspects of the problem with her at length. This approach contrasts with Mr. Drummond's method of proof teaching in that her (Stephanie's) teachers focused not on deductive proof but understanding of the mathematical concepts. This demonstrates two different ways to focus on proof. It can either be thought of as a highly formalized method of communication, or as a method of understanding the underlying mathematics. This study will focus on the latter definition, and use the term "justification" rather than proof to emphasize that our focus is conceptual understandings, not the formal processes expected of older students. In this context, the production of a justification would indicate that a student has conceptual understanding as the student has shown both the conclusion and the mathematical reasoning behind it.

The Difficulty with Justification and Other Reform Methods

As mentioned previously, the Standards set by the National Council of Teachers of Mathematics (2000) state that reasoning and proof is a valuable part of learning mathematics. This was also the case in the 1989 Standards set by NCTM. However, an analysis of data collected during the Third International Mathematics and Science Study (TIMSS) in 1999 showed that teaching for reasoning and proof does not occur with most teachers on a regular basis. This study collected data on sampled classrooms in which teachers were told not to change their teaching practices. Over half of the sampled American teachers stated that they were knowledgeable or very knowledgeable about current standards, (in this case, the 1989 NCTM standards) and 86% said that the filmed lesson was in line with current standards. Nevertheless, no lesson had students or teachers engaged in justification (Jacobs et al., 2006).

The TIMSS study contrasts with many other studies that have examined justification in the classroom in that it sampled an average classroom day. In many other studies, teachers were purposely chosen because they were exceptional (e.g., Staples, 2007), or were given specific tasks to do (e.g., Kazemi & Stipek, 2001). Data collected from such purposefully selected samples or intervention studies show us what justification can look like when the teacher specifically values the process and has training and support for it. The TIMSS study collected random samples, and thus gave a snapshot of how the standards are implemented in the United States on a daily basis. The TIMSS evidence indicated that teachers believed their teaching to be in line with current practices, but their actual teaching practices lacked activities that required reasoning or proof.

Even if a teacher values reasoning and knows how to foster this ability in her students, it is possible that secondary teachers are not being adequately prepared to teach justification and other proof methods. To be able to teach their students how to produce proofs, teachers must understand what kinds of proofs would be acceptable in a mathematical community. In spite of this, many secondary mathematics teachers lack understanding of proof tasks (Knuth, 2002). Teachers use proof as a method of communication and verification, but they do not see the power of proof as a method of understanding. Some teachers are also unable to recognize what constitutes a formal or acceptable mathematics proof (Knuth, 2002, Stylianides & Ball, 2008). It is important that teachers understand what kind of general reasoning is acceptable in the mathematical community, so that they can teach their students how to differentiate between good reasons and poor reasons

Student-Centered Vs. Math-centered

Beyond understanding proof, teaching for and with justification comes with some of the same challenges that other teaching practices pose. Thus, it makes sense to examine the challenges of student-centered teaching in general and also teaching for justification specifically. Teachers can use discourse and sociomathematical norms to promote justification in their classrooms, but these techniques are complex. First, there is a critical difference between a classroom that has student talk and a classroom that has mathematical student talk. Thus, simply encouraging students to speak does not guarantee that they will give justifications. Next, there is a conflict between allowing students' ideas to be at the front and keeping the focus on sound mathematics. It is helpful to think about a mathematics class as being on a spectrum (Figure 1). Classrooms can be completely focused on either formal mathematics or on student contributions. In the next sections I will explore the difficulties in teaching at either end of the spectrum.

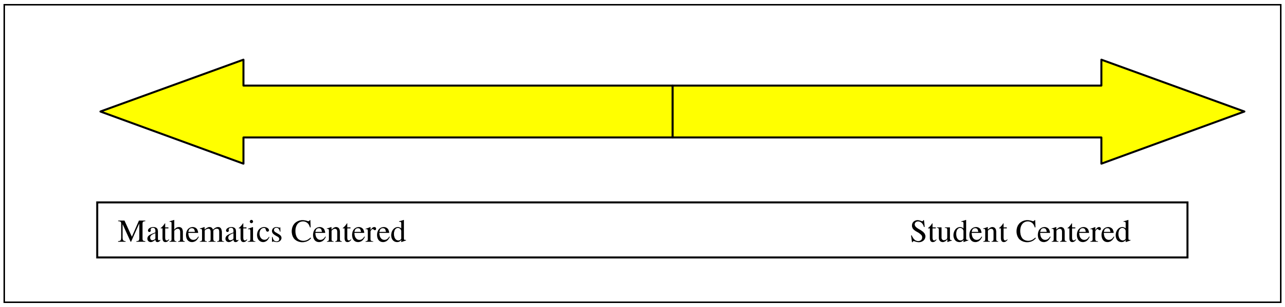

Figure 1: *The Math/Student Continuum.*

Eliciting Mathematical Student Ideas

Many teachers and researchers are familiar with the classic "IRF" pattern of discourse. In this pattern described by Edwards and Mercer (1987), a teacher "initiates" a conversation, a student "responds" and then the teacher "follows up." This pattern can become routine in the classroom, and students can often find clues to the correct response in the way that a conversation is initiated, or in the teacher's follow-up remarks. The authors describe how a teacher can use this elicitation pattern to directly influence a student's thinking, even when it appears that the student is experimenting and hypothesizing independently. In one episode, the focal teacher for their study initiates conversation with two different questions about a pendulum's swing: "Is it going faster or slower" and "does it change?" The first question implied that there is a change, even if the students had not noted one. The second question, as it did not offer choices for how the swing changed implied that there was no change. Thus the students' responses were limited. Edwards and Mercer noted that although the teacher wanted the students to generate the ideas and learn from experience, these ideas are pre-determined and the teacher controlled what student ideas are taken up. It is important to note that this discourse pattern is controlled by the teacher – there is little room for student-student talk

Teachers who value justification may believe that they are pushing for mathematical reasoning, but there are dramatic differences between teachers who emphasize reasoning in general and those who emphasize mathematical reasoning (Kazemi & Stipek, 2001; Wood, Williams, & McNeal 2006). Several studies have examined the kinds of questions teachers ask as they strive to encourage mathematical reasoning; these studies have demonstrated important findings regarding the relationships between teacher questions and the level of reasoning that emerges in student responses. In their case study of four upper elementary classrooms, Kazemi and Stipek (2001) classified extended mathematical conversations as either "High Press" or

"Low Press." In both kinds of conversation the students are engaged in sharing their work with the class. The main difference is that "High Press" interactions are those during which teachers ask students to focus on the concepts behind the mathematics. During "Low Press" interactions, the focus is primarily on the method behind the mathematics. These are two very different kinds of reasoning. Students who were engaged in a "Low Press" conversation were more likely to summarize mathematical answers, as opposed to providing mathematical reasoning. Even though teachers are encouraging student participation in both kinds of conversation, one kind of conversation has students primarily engaged in fact recall, as opposed to building and examining mathematics, a much less trivial activity. It is critical that researchers continue to characterize what kinds of discourse result in mathematical reasoning as opposed to fact recall as this will provide information that teachers can use to improve their teaching practices.

The differences between a conventional classroom and one that is committed to reform practices were demonstrated further by Wood and colleagues (2006). They found that teachers who followed a conventional curriculum were likely to ask that their students recall factual information. In response, their students were not likely to engage in any mathematical reasoning above "recall." An interesting pattern of interaction they noted is the "funnel" pattern, in which a teacher asks a directed line of questions that guide the students to the answer, but remove the burden of the heavy mathematical thinking from the students. Wood (1998) described this pattern of discourse in more depth. She described an interaction in which a teacher engaged in a question/response dialogue after the student said that 9+7 = 14. This dialogue involved a complex mathematical process in which the teacher asked the student to add "7+7" and then add one to the result twice to get to the answer "9+7." This process gets at the fundamental concept of decomposing numbers and recombining them. It seems that the student understood how to decompose nine into seven plus one plus one, but in reality the student only had to add one to their previous answer. The teacher stopped this line of questioning when the student gave the answer "16." Thus, even though the student found an answer, he did not at any point engage in mathematical thinking. There is little evidence to show that the student understood the mathematical process in the same way as the teacher; the student may not see that 7+7+1+1 = 7+9, but rather "the student may have interpreted the task as simply 'add one to the number given by the teacher'" (p. 171).

In contrast, Wood et al. (2006) found that lessons that focused on inquiry and argument had high levels of mathematical reasoning displayed by the students. This is similar to the findings of Kazemi and Stipek (2001) in that there is a relationship between the ways that a teacher prompts for mathematical solutions and the types of mathematical reasoning in which their students engage. Wood (1998) offered a description of "focusing" discourse through which students are encouraged and guided in their attempts at mathematical discourse. Wood described how a teacher who "focuses" student interaction allows a student to present a solution, and give an explanation. Then the teacher, upon noticing that other students may not understand, directs the class' attention to either the important parts or the difficult parts of the explanation, and allows the student who is presenting to re-voice his or her solution. Thus a focusing move facilitates sharing and having students understand each other's mathematics. The difference between the "funneling" and the "focusing" patterns of interaction is that the teacher allows the student to have control over the solution, rather than giving a quick error-correction and imposing an explanation.

Justification: A Sociomathematical Norm

Standards for middle school mathematics focus on practices that emphasize discussion, risk taking, and exploration of mathematical ideas (Stein et al., 1996; NCTM, 2000). These dispositions are critical to making mathematical discoveries and to understanding mathematics. Related to the idea of mathematical dispositions is the idea of sociomathematical norms. As defined by Kazemi and Stipek (2001), sociomathematical norms are those that emphasize the mathematical process in a classroom. These are different from common classroom norms, which might include respect, participation, and hard work. When Yackel and Cobb (1996) defined a sociomathematical norm, they gave the example "what counts as an acceptable mathematical explanation and justification" (p. 461). These sociomathematical norms set expectations for the students about what they are to produce in the mathematical classroom, and they help students to evaluate their own answers to questions to see if their reasoning is mathematically acceptable.

Justification is one of many potential sociomathematical norms that a teacher could emphasize in the classroom. Yackel and Cobb (1996) found that teachers who develop a sociomathematical norm where students understand what constitutes a justification help their students to take the responsibility for deciding how well or how much their answer explains the underlying mathematics. These researchers noted that in many classrooms the acceptance of an answer was often based on social status – because the teacher had the highest social status in the room, he or she was expected to have the answers and the explanations. However, students in classrooms in which justification was a sociomathematical norm took responsibility for providing their own personal mathematical explanations. Bieda (2010) also noted that the development of a community that can critique justifications is an important step in helping students to build justification skills. Thus, when a teacher emphasizes justification in the whole classroom over time, the individual students learn to analyze their own and each other's work for mathematical soundness.

Goos (2004) elaborated on how teachers can promote reasoning and sense making. In her article she described how an upper school teacher encouraged students to engage in mathematics in a meaningful way. She noted several assumptions about mathematics, and paired them with teacher and student actions that help make these assumptions a normative part of the classroom. The assumptions included that "mathematical thinking is an act of sense-making and rests on the processes of specializing, generalizing, conjecturing and convincing." This is paired with the idea that if a teacher "models mathematical thinking" and "invites students to take ownership of the lesson content by providing intermediate or final steps in solutions or arguments" that the students will "begin to offer conjectures and justifications without the teacher's prompting." (p. 270). Goos then showed how the teacher repeated the actions over an entire year, and the students slowly began to exhibit the desired response.

It is important to note that these classroom practices may take time to develop. Over a period of months or years, teachers can build systems of discourse and language in their classroom that they will be able use to direct their students' thinking toward the mathematical ideas (Seymour & Lehrer, 2006). Students too, must practice justification tasks over time. Justification can be thought of as a "habit of mind" that needs to be developed in many different areas over all grade levels (NCTM, 2000). However, there is a need for research on how justification specifically can be encouraged over time.

Keeping a Student Centered Class Focused on Sound Mathematics

It is possible to see how sociomathematical norms can promote justification. For instance, building a critical classroom community is an important step in providing students with a motivation to justify their answers (Bieda, 2010; Yackel & Cobb, 1996). However, this does not mean that impressing a sociomathematical norm of justification upon students implies that students will begin to justify. It is also critical to note that this does not mean that the teacher should remove herself from the conversation.

Nathan and Knuth (2003) describe how one math teacher attempted to encourage more student-student interaction in her classroom. In the first of the two years examined, the students spent very little time talking to each other, and it was clear that the mathematics was directed by the teacher (Ann). In the second year, Ann asked the students to comment on each others' work, and waited until they would do so. This dramatically increased student to student interaction in her classroom. However, both she and the researchers noted that the mathematics was not as strong in the second classroom. At one point the student's took a vote to decide who was correct during a mathematical debate. This article seems to make the point that if the teacher removes herself from the classroom that the mathematics may be lost. However, teachers can support student talk without this effect.

Two articles have evaluated how a teacher can direct the mathematics while still encouraging students to build concepts for themselves. Rasmussen and Marrongelle (2006) explored how a student-centered class could be built in a college-level differential equations class. Their article described how during one lesson, the class decided the amount of salt being added to a mixed solution could be represented as "2t" in a differential equation. Up to this point, the class had debated whether they should use "2t" or simply "2" in their formula, and had justified their responses. However, some students had justified the incorrect response so well that they erroneously convinced the whole class that "2t" represented the amount of salt being added to the solution. A reader familiar with the classic differential equations problem will recognize that this value actually represents the amount of salt that has already been added to the solution, not the amount currently being added. At this point the professor had to step in and re-question the students to direct their thinking toward a more mathematically valid answer.

Lobato, Clarke and Ellis (2005) described how teachers can "tell" mathematics without overly controlling the class. They illustrated a situation where a student, "Clarissa", doesn't understand a partitioning model of division. The teacher asked Clarissa about what division means, and about situations in which Clarissa had previously used division. The teacher also set up models, and asked Clarissa to partition them. Through this direction of the mathematics, Clarissa built the ability to fluently partition time and length segments. The authors make an important distinction between what has historically been considered "telling" – giving specific information on a procedure, and what they re-define telling as – providing conceptual information and ideas. The critical distinction is that the former promotes a rote method of learning mathematics through operations and key examples that can be followed. The latter promotes conceptual understanding and helps students to develop their own (correct) meaning of the ideas.

So far we have discussed how teachers allow student ideas to become a part of the class. In some cases, this is at the expense of the mathematics (e.g. Nathan and Knuth, 2003). In this case, the classroom is on the student-centered side of the continuum (Figure 1). More centered on the continuum are the teachers who allow student input without allowing incorrect ideas to override sound mathematics (e.g. Rasmussen & Marrongelle, 2006, Lobato, Clarke & Ellis, 2005). A key difference may be what teachers do when students offer ideas.

Responding to Student Ideas

Teachers can respond to student ideas in different ways, and some of these responses might be more productive to furthering student thinking. Pierson (2008) distinguished different types of response moves that a teacher can make. The first is low level response, where the teacher does not respond to the student's idea. Other levels include high level give moves, where a teacher provides conceptual information and high level demand moves, where a teacher asks for something else (maybe a justification, an example or a connection) from the students. Pierson showed that high level responses as she defines them are correlated with student learning. Thus the way that a teacher responds to a student idea seems to be important and I will explore these responses in this section.

Bieda (2010) examined how a specific curriculum (the Connected Math Program or CMP), which was designed for middle school students with a focus on proof, was implemented in seven middle school classrooms (6^{th}, 7^{th}, and 8^{th} grades). The author observed several lessons focused on proof in each classroom. In contrast with the TIMSS study, Bieda found that 71% of the lessons observed included a "proving opportunity" in which a student provided a conjecture that could lead to a proof. These conjectures were offered without a justification 46% of the time. In response to those with no justification, the teachers did one of three things: 42% of the time the teacher gave no feedback, 34% of the time teacher sanctioned the conjecture, and 24% of the time the teacher asked for class input. None of these teacher responses helped students to begin to give mathematically sound justifications for their work, as they did not press for mathematical justifications. Bieda noted several reasons for this, including two key causes. First, many teachers did not believe that their students were capable of the high level of reasoning required to formally prove their mathematics. Some of the teachers, when interviewed, believed that a justification that relied on numbers was better than a justification that relied on manipulatives. Because reasoning with numbers may be more difficult than reasoning with manipulatives, this may lead the teachers to believe that the students cannot provide mathematical reasoning at all. The second reason that Bieda gave for a lack of press for justification is that teachers often did not have enough time to dedicate to the task. The limit on time kept students from being able to think deeply about the mathematics at hand, and implied that teachers believe that encouraging justification takes more time than it is worth.

A special skill is necessary for teachers in order that they might respond to student's' ideas – the skill of noticing that the student has said something mathematically relevant. Jacobs, Lamb and Phillip (2010) showed that teachers with very little training had more trouble identifying potentially important parts or flaws in a student's answer than teachers with professional development that focused on students' mathematics. This study showed that the less-trained teachers could not differentiate between an arithmetic error and a conceptual misunderstanding. When asked what steps they would next take with the students, the teachers with more training responded with specific questions that they might pose, with reasoning that pointed to an inconsistency in the student's work, or exploring how the student might tackle a similar problem. This has implications for teaching with justification: if a teacher does not notice the mathematics that a student is trying to impart, they will have much trouble building on that student's mathematics.

Justification: Not Just for Gifted Students

There is a special need for the enriching experience that justification tasks provide to students who have fallen behind in mathematics. These students may believe that they are just not good at mathematics and as a result may not enjoy mathematics (Jorgensen & Niesche, 2008). Justification tasks are used to give students the opportunity to establish their own sense of the material, and to build number senses that will provide a solid foundation for higher mathematics (Boaler & Staples, 2008; Stylianides & Ball, 2008). In one study on "Railside

School," a high school serving a racially diverse and largely economically disadvantaged population, reform practices (practices which de-emphasize rote-learning in favor of sense making) made a great difference on students' mathematics performance. During a period of only two years, these students went from drastically underperforming their peers to dramatically outperforming them. Teachers, students, and researchers all reported that justification played a major part in the everyday mathematical activities (Boaler & Staples, 2008). Staples (2007) described a similar effect at a different school, showing that students who would traditionally be placed in lower tracks can benefit from collaborative inquiry. In this case study, Staples described a classroom teacher, Mrs. Nelson, who worked with the lowest track (a pre-algebra class) of ninth- grade students. The teacher valued collaborative work (closely related to argumentation) in her students. In both of these schools, advances were made in students' mathematical self-efficacy, and in Railside School there were marked increases in achievement.

Railside School is an exceptional school in that the department of mathematics and the school as a whole worked together to create the changes that were seen. The students at Railside School were in deliberately heterogeneous classes, meaning that teachers had a wide range of abilities to draw from (Boaler & Staples, 2008). However, not every teacher works with the same supportive coworkers and work environment. Staples (2007) reported in her case study that Mrs. Nelson did not work under extraordinary circumstances. Her classroom included students who were not prepared for algebra, and any students with much higher ability would have been tracked into a different class. This did not keep Mrs. Nelson from assigning cognitively demanding tasks to her students, and it did not keep her students from being successful at these tasks. Thus, it is important to know what teachers can do, even when they are in a traditional school setting, to encourage mathematical argumentation in their classrooms.

Conclusion

There is a great need for more research on justification practices at the middle school level. Research has primarily focused either on case studies of teachers' pedagogical strategies that are small in scale and do not emphasize justification, or on broad studies that simply show that reasoning methods are not being taught in the classroom. The first type of study does show that some teachers put a lot of effort into teaching reasoning in the classroom, and begins to discuss what qualities of teaching help to promote justification. On the other hand, some studies show that teachers do not understand how to use, or value these teaching practices (e.g. Bieda, 2010, Kazemi & Stipek 2001).

Thus, the research must move forward in the following way. There is a need for understanding how teachers who are committed to teaching justification in their classroom do so, and how they improve their own teaching. It is also important that these teachers work in varied settings, to provide information on what diverse methods can be used. The following project studied several teachers who had been given one task to implement over two years. We were able to see how teachers developed their practices and developed in the way that they teach for justification. Because each teacher in the study taught the same material, their methods of teaching can be directly compared, alongside their student work. These comparisons will help future teachers to see how they can implement justification tasks in their middle school classrooms and to learn what they can expect from their middle school students with regards to reasoning. Ultimately, teachers across the United States will be able to develop these practices in their own classrooms.

Methodology

This chapter details the methods used in data collection and analysis during the course of this study.

The General Perspective

This is a collective longitudinal case study. Each teacher operates as an intrinsic case study – worth examining because of unique characteristics. As a whole, the case studies are instrumental in that they are used to illustrate the issue of teaching for justification in the classroom (Creswell, 1998, p. 62). A collective case study design allowed me to compare and contrast the practices of several different teachers as they engaged their classrooms in justification, and then to aggregate the data and perform a holistic analysis of the cases together. The design is used to answer descriptive questions such as "what are teachers doing" and other qualitative questions. The longitudinal aspect of this case study allowed the researchers to examine the development of teachers' practices over time. This study is small enough in scale that the data can be examined on a minute level, but large enough that substantive conclusions can be drawn.

The original JAGUAR (Justification and Argumentation: Growing Understanding of Algebraic Reasoning) study is based in grounded theory, whereby theories are constructed from the data and guide further data collection. This is also a characteristic of constant comparison research design (Merriam, 1998). As data were collected, emergent themes were used to guide further steps and analysis.

The Research Context

Data were collected as part of a larger NSF funded project: Justification and Argumentation: Growing Understanding of Algebraic Reasoning (JAGUAR). As part of this study, twelve teachers taught four justification tasks each year, three of which were prescribed, attended three working sessions per year, and participated in a summer institute focused on developing an understanding of justification and of the tasks they were implementing.

This study focuses on one of the tasks entitled the Number Trick Task. Each teacher taught this in two consecutive years to a different class each year. After preliminary notes were examined, seven teachers were selected from the group of twelve for explanatory and interesting characteristics, which will be described in the analysis section.

The Number Trick Task

Each teacher was given the same task (see the task in Figure 2) to implement in year one of the project and in year two. Some teachers chose to modify this task to suit their classroom's needs (for instance, Kelly had a group of students in the 8^{th} grade in the second year that had seen the task during the previous year when they were in 7^{th} grade. As such, she asked the students to generalize the task to more situations.) This allowed the researchers to compare teaching and questioning styles when the classrooms were working on the same material.

Jessie discovers a number trick. She thinks of a number between one and ten, adds four to it, and doubles the result. She then writes the answer down. She goes back to the number she first thought of, she doubles it and then adds eight and writes this answer down. Will Jessie get the same answer for both methods every time?

Here is an example:
Jessie thinks of a number: 5
She adds 4 to her number: 5+4 = 9
She doubles the result: 9 x 2 = 18
She writes down her answer: 18

Jessie goes back to the first number she thought of: 5
She doubles her number: 5 x 2 = 10
She adds 8 to the result: 10 + 8 =18
She writes down her answer: 18

Figure 2. The number trick task. This task was given to teachers to implement in their classrooms.

All of the tasks in the original JAGUAR project were chosen for their canonical nature and their ability to prompt student thinking. Out of the original tasks, the number trick was chosen because it is deceptively difficult. During the working sessions, teachers noted that students had more trouble with the task than they anticipated. The mathematics is subtle, as this is presented as a word problem. Also, some students had not yet had exposure to the distributive property. Appendix B shows a range of student responses.

The Summer Institutes and Working Sessions

The summer institutes and working sessions were designed to challenge and extend the teachers' thinking around their tasks and about justification more generally. Thus, the teachers were intentionally exposed to the ideas of the other teachers and of the researchers, in an attempt to help them improve their practices. After each cycle, teachers reflected on how they wanted to modify or develop tasks to encourage justification. During these workshops teachers also developed criteria for what constitutes as a "good" justification and criteria for assessing justification. The author of this paper was involved in filming these summer institutes, although she did not regularly participate in the conversations.

The Research Participants

The seven selected teachers came from varied backgrounds. They were purposively selected based on pedagogy of 'strategy-reporting' or 'inquiry-argument' (Wood, Williams & McNeal, 2006). All but one of the teachers had been teaching for at least three years. The teachers selected the class that was to be filmed. Classes were either 7th or 8th grade, and no class was an Algebra class**.** Participants were either recruited through previous participation in research projects or directly by the researchers. Most teachers taught heterogeneous classes. See Table 1 for specifics.

Table 1

School Profiles by Teacher

Teacher	Percent of English Language Learners	Percent of Students Qualifying for Free or Reduced Lunch	Percent of Minority Students in School	Course Taught
Audrey	3%	23%	12%	Year 1: 8th Grade Year 2: 7th grade
Bruce	50%	78%	74%	8th Grade Math
Cynthia	1%	20%	10%	7th Grade Math
Irene	1%	20%	10%	8th Grade Math
Joan	26%	32%	71%	Pre-Algebra (7th Grade)
Kelly	0%	13%	8%	8th Grade Math
Paige	26%	32%	71%	Pre-Algebra (7th grade)

Data Sources

Many sources of data were collected as part of the overall JAGUAR project. However, only a subset of these data was used for this study. These data included videos and video transcripts, teacher reflections and student work. This author filmed some of the classroom implementations and also transcribed, while other tasks were transcribed by other researchers and graduate and undergraduate students. Each lesson lasted between 1 and 3 days, with 2 days being the norm. Most classes were regular 45-minute periods, while Kelly and Bruce taught 80-minute periods.

Videos and Video Transcripts

Each lesson was filmed by a researcher. The lessons were filmed for as many days as the teacher spent on the task (between one and three days). During filming, the teacher wore a personal microphone, and another shotgun microphone captured the students' voices. Field notes with general impressions and important activities were recorded for some lessons. After filming, all task-related activity was transcribed using transcription software.

Teacher Pre and Post Lesson Reflections

Before and after each lesson, the teachers completed a reflection (see reflection questions in Appendix A). These reflections were systematically used to identify common themes and gain insight into the teachers' thinking. The pre-lesson questions clarified what the teacher wanted from the class and what he or she expected from the lesson. The post-lesson questions gave the researchers insight into very specific instances of justification that the teacher valued. These questions also allowed the researchers to develop an idea of what the teachers thought of justification in general. The teachers' written responses also provided triangulation for data collected during the video transcripts – an important part of checking validity in qualitative research (Fraenkel & Wallen, 2000).

Student Work

All written student work that was completed during the lessons was collected and scanned. This included work written on any worksheets handed out by the teacher, and any scrap paper that the students worked on. This gave researchers a picture of what each individual student was doing during the lesson. This contrasts with the video, which shows only small parts of the individual student work, and the oral work, which was done as a whole class. Examining this work again provides triangulation, and thus greater validity for the research study (Fraenkel

& Wallen, 2000). It also provided a copy of the writing when the video did not show what a student was referencing.

Procedures Used for Data Analysis

Data were analyzed primarily through the use of coding and through searches for evidence of specific teaching practices. There were three main phases of data analysis. First, summaries of each transcript were written with notes on the teacher's reflections and general impressions were recorded. The transcripts were coded according to a coding scheme. In phase two, these data were used to identify successful practices at a fine grained level. Third, the coding data was again used to identify successful practices at the lesson level.

Phase One: Coding

The coding scheme operated at two levels, the utterance level and the episode level. At each level, codes included *"answer/statement," "explanation," "reasoning," and "justification."* Codes such as *"build," "refute," "agree,"* and *"disagree"* captured student-student interaction. These codes were designed identify high-level episodes and to identify patterns of interaction. This coding scheme was used Appendix C shows an example of the coding scheme. Table 2 gives a description of the codes. Transcripts were coded by two or three researchers with an inter-rater reliability of 76% over the lessons.

After the student utterances were coded, researchers examined episodes. An episode was considered one interaction between teacher and/or students where the participants explored one topic. This topic might be an individual student's justification or it might be a sub-question to the task. For example, an episode might be centered on finding a formula to express the number trick.

Analysis: Phase Two

Initially, all of the videos were viewed, and transcripts were coded. Successful and non-successful episodes were identified, and common themes among these episodes were recorded. The following teacher practices emerged as successful practices at the episode level.

Following students' mathematics
Keeping cognitive demand high
Building a critical classroom community
Giving students methods of communication
Holding students accountable for each other's ideas
Giving students background knowledge

As more data were analyzed, it informed the theory development (Simon & Tzur, 1999) Some of these practices such as "giving students background knowledge" turned out to be more local – they worked at the episode level, but were either not strongly demonstrated over several classes, or were unproductive in other instances. The practice of "keeping cognitive demand high" turned out not to be distinct from other practices; we could see when a teacher lowered the cognitive demand, but when they did not do this, the practice was better classified as "following students' mathematics" or "giving students methods of communication". Thus the practices collapsed down in phase two.

Analysis: Phase Three

The results from the student codes were used to make the Table 4. The number of distinct justifications (at the turn level) was calculated for each class and then used to find the number of justification per hour. Implementations were classified as high justification if there was more than one distinct justification, medium justification if there was exactly one justification and low justification if there were no justifications. Then the number of justifications per hour was calculated, and the implementations were listed from highest to lowest justifications per hour within their category.

The amount of justification were used to classify successful lessons as a whole (those with more than one instance of justification), and to identify practices that were successful at the lesson level (as opposed to the episode level). This table was also used to identify explanatory cases that would illustrate the productive practices, and to refine practices identified in phase two. Paige, Cynthia and Irene all clearly exhibited some of the initial practices. However, none of the initial practices explained Kelly's great leap in level of justification. Thus the practice of

“Providing scaffolding questions” was added to the list of practices. A similar, yet qualitatively different practice “Providing Task-Specific tools” was also added at the whole task level.

Table 2

Student utterance codes

Code	Description	Example
Answer/Statement	Used when a student made a direct statement, with no explanation of his ideas. Often used when a student answered a direct question from the teacher.	“Seven works”,
Agree or Disagree	These two codes were used when students agreed or disagreed with another student without elaboration.	“I think Claire’s right”
Explanation	Used when a student described a procedure that they used to find a solution.	“I added four and four.”
Reasoning	Used when a student attempted to describe why his or her solution is the correct one, but the proof was incorrect or incomplete.	“Then go back and add eight on this one, and this one's plus two fours. They both have eight.”
Justification	Used when a student gave a complete justification for why their answer was correct. This was sometimes a response to a specific teacher question. This reason was mathematically valid and acceptable.	“it’s the same thing as uh, five x plus five y because if You count All of the number of y's and x's You get five x's and five y's”
Build/Refute	Used when a student offered something new based on what a peer had said or refutes a peers’ idea. This was always double-coded with Explanation, Reasoning or Justification.	“If your variable was something different than 5 then it would not be 9.”
Question	Used when a student asked a question of a teacher or peer.	“Why did you put plus two?”

Table 3

Episode level codes

Code	Description
Meta	Used when the discussion in the episode was primarily about justification in general, how to justify or what constituted a complete justification.
Answering	Used when the students contributed very little beyond surface level observations. An “Answering” episode was characterized by answer/statement codes.
Explaining	Used when the students attempted to explain their work. Explaining episodes did not contain any “justification” or “reasoning” codes.
Reasoning	Used when students reasoned about the mathematics but did not provide a complete justification.
Justifying	Used when students justified sub-questions to the task (in response to a teacher question) but did not justify the task as a whole. The justifications were complete and had valid warrants
Justifying Completely	Used when a student gave a complete reason as to why the number trick task worked for every number. This justification was complete and had a valid warrant.

Table 4

Classes by Amount of Justification Episodes

High		Medium		Low	
Teacher	Justifications per Hour	Teacher	Justifications per hour	Teacher	Justifications per Hour
Paige, Y2	12	Irene, Y2	1.2	Audrey, Y2	0
Kelly, Y2	3.9	Joan, Y2	1.1	Cynthia, Y1	0
Cynthia, Y2	3.5	Joan, Y1	0.9	Irene, Y1	0
Audrey, Y1	3.4			Kelly, Y1	0
Paige, Y1	2.5				
Bruce, Y1	2.3				
Bruce, Y2	1.6				

After refinement of themes, each lesson was examined to find instances of the following five practices:

1. Providing Scaffolding questions
2. Following students mathematics
3. Helping students to communicate their mathematics
4. Leveraging a critical classroom community
5. Providing task-specific tools

During this process, it was found that “following students’ mathematics” and “helping students to communicate their mathematics” were not easily distinguishable – even if they theoretically had different definitions. Thus they were re-constituted into a theme called “focusing students’ mathematics.”

Defining the Practices

As I defined the practices (Table 5) I also noted that they worked at different “levels” of the task. The smallest level is the episode level – practices that happen in the moment-to-moment interactions between teachers and students. Though these may be repeated practices, they are specific to the students and the situation that the teacher is working with at the time. The next level is the task level. This includes things that teachers do specifically for the task at hand, but generally with the whole class. Classroom community is something that is built over time with the same group of people. It is neither task specific nor student specific. Thus it is the broadest level. Table 6 gives a description of how each theme was measured.

Table 5

Definitions of the Practices

Theme	Level	Definition
Focusing students' mathematics	Episode Level	The teacher asked a specific question about a student's mathematics (referred to as a "focusing question.") The question called for elaboration or for consideration of something that the student did not note. The teacher did not evaluate or imply an answer. Examples: "How did you know where to put the parenthesis?" "Why is this picture the same as this picture?" "How are you showing that this also has a 2x in it?" Non-examples: "What do you mean by that?" "I like how you saw an increasing pattern of two."
Providing scaffolding questions	Task Level	The teacher developed "go-to" questions that are asked to most or all of the groups, or added scaffolding questions to the task. These questions were not specific to what the students are saying, but may have been used when a student is stuck. These questions were not equivalent to the task itself, but rather were sub questions. However, they brought attention to a critical part of the mathematics. Examples: "Write a number sentence for Jessie's rules." "What happens if you change the four?" Non-examples: "Why did you write "plus eight"" "Make a table." "Can you draw a picture?"
Providing task specific tools	Task Level	The teacher guided students toward a specific method of justification at any point in the lesson. This could take the form of an oral outline where the teacher asked a directed line of questions. This is different from scaffolding because the method would result in a complete justification.
Leveraging a critical classroom community	Classroom Level	The students in the classroom questioned each others' answers and refuted incorrect answers. Students also built on each other's responses. Though we could not measure how this was built over time, we could see that teachers prompted for this behavior.

Table 6

Measuring the Practices

Theme	Level	Measurement
Focusing students' mathematics	Episode Level	As this theme operated at the episode level, it was measured at the episode level. Only episodes higher than "explaining" were examined as it was assumed that these were the only episodes with mathematical content that a teacher could focus on. Each episode was counted as either having a focusing move or not having a focusing move. The percent of focusing episodes at each episode was calculated and the percent of focusing episodes for each individual implementation was calculated. (Tables 7 and 8).
Providing scaffolding questions	Task Level	This theme was measured as a presence or absence theme. If a teacher added specific scaffolding questions to their task, then the lesson was automatically classified as having scaffolding questions, and the question(s) were recorded. If the teacher did not add more questions to the task, but asked similar scaffolding questions of the whole class or to many individual groups then the lesson was classified as having scaffolding questions and the questions were recorded.
Providing task specific tools	Task Level	A lesson was classified as providing task specific tools if the teacher guided students toward a specific method of justification at any point in the lesson. The specific method was recorded.
Building a critical classroom community	Classroom Level	This element was measured at the turn level by the number of *build, refute, agree, disagree, and restate* codes that the students had. The number of episodes with one of these codes was counted and the percentage of such episodes was calculated for each implementation and also by episode type.

Results and Discussion

Each lesson was classified on the four practices. The results are in table 7. They are listed from highest level of justification to lowest.

Table 7

Measures of Each Practice, by Lesson

Level of Justification	Lesson	Focusing Mathematics, % of Episodes	Scaffolding Questions	Providing Specific Tools	Classroom Community, % of Episodes
High Justification	Paige Y2	65%	Present	Present	9%
	Kelly Y2	55%	Present		33%
	Cynthia Y2	67%		Present	14%
	Audrey, Y1	50%			3%
	Paige Y1	50%			25%
	Bruce Y1	47%	Present		33%
	Bruce Y2	44%	Present		50%
Medium Justification	Irene Y2	20%		Present	10%
	Joan Y2	56%	Present		0%
	Joan Y1	50%			0%
Low Justification (zero justifications)	Irene Y1	29%			15%
	Cynthia Y1	42%			10%
	Audrey Y2	29%			31%
	Kelly Y1	20%			7%

The percent of focusing students' mathematics episodes for the high justification classes was between 50% and 67% while the percent of focusing students' mathematics episodes for low justification classes was between 20% and 42%. Low justification lessons had no scaffolding questions, nor did the teachers give specific justification tools. The highest level of classroom community codes was in Cynthia's year one lesson, which was a low justification lesson.

Since each episode was individually classified as either having focusing questions or not, the episodes themselves could be grouped by code and the percentage with focusing questions found, as shown in Table 8. In the same way, each episode could be classified as either having classroom community codes or not, as shown in Table 9.

Table 8

Focusing Students' Mathematics

Code	Number of episodes	Percent showing focusing moves
Explanation	148	34%
Reasoning	164	45%
Justification*	70	77%

*Complete and sub-question justifications were grouped

Table 9

Critical Classroom Community Codes

Code	Number of episodes	Percent with Critical Classroom Community Codes
Explanation	148	19%
Reasoning	164	24%
Justification*	70	37%

*Complete and sub-question justifications were grouped

During this phase of analysis, it was found that two themes worked productively in some instances and not others. These two themes, *Leveraging a critical classroom community* and *Providing task-specific tools* were classified as "variable" productive practices, in contrast to the other two, which I classified as "constant" productive practices. I will elaborate on these themes as it is counter-intuitive that they would not be directly associated with student justification. It is also important to clarify what happens when the variable practices are productive. All of the practices were also seen as belonging on the continuum shown in Figure 1. Providing task specific tools was considered the most mathematically centered, while leveraging a critical classroom community was considered the most student centered. Figure 3 shows the classifications below.

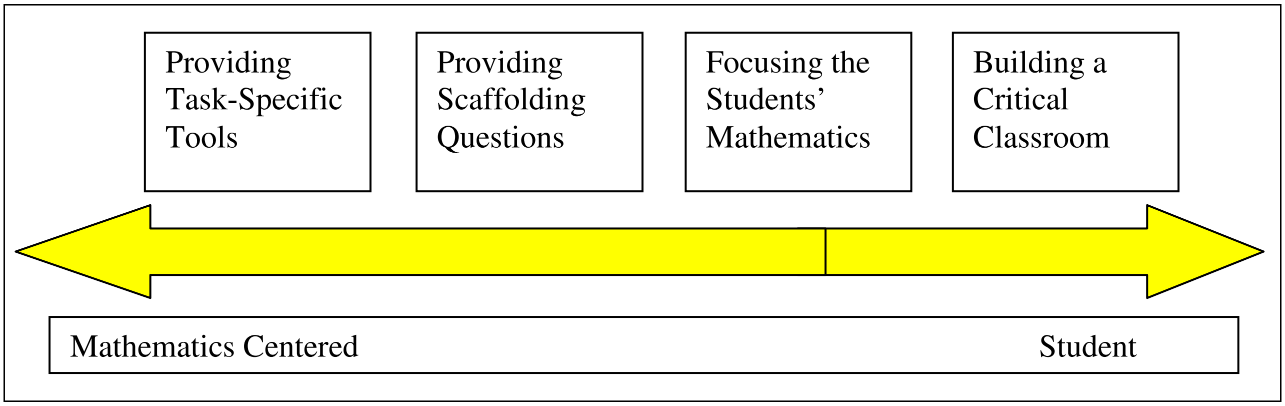

Figure 3: *Identified Practices on the Math/Student Continuum.*

In the next sections, I will discuss how each of the individual practices operated, and give examples of some teachers implemented the task.

Focusing Students' Mathematics

Focusing student's mathematics is one of the consistently productive practices. Table 8 shows that episodes which are coded justification are associated with the presence of focusing moves. In year two, Paige put effort into focusing students' ideas. She was an exemplary case of this teaching practice in that she asked specific questions which helped students to clarify their own justification. She avoided interpreting for the students, preferring instead to call the student to the front. As shown in Figure 4, Paige asked Rebecca to clarify why they add eight in the second method. When Rebecca gave an unsatisfactory response, Paige asked her to point at the writing that shows where the eight is. Rebecca was then able to articulate her justification much more completely than before. Paige continued this pattern of interaction with all of her students; using focusing questions in 65% of the episodes (see Table 8). This lesson also had the highest level of student justification (see Table 5).

In contrast, Irene's year one lesson had no complete justifications. In her journal, she expressed frustration that she was unable to help her students produce a justification of the number trick that did not rely on a specific example. For instance, one student had drawn boxes, showing how the number trick would operate if the input was 1. Danny was able to point to how there were two sets of four boxes which matched up to eight boxes in the second part, and two boxes in the first part that matched to two boxes in the second part (see Figure 5). Irene had trouble extending Danny' justification to apply it to every number (not just n=1). Danny was able to show multiplication as repeated addition through a picture, an idea that could lead directly to a proof of the distributive property. However, although Irene knew that Danny's work did not constitute a complete justification, she did not ask any specific questions to show him what he needed to add. On line 5, she took Danny's explanation and explained it using a numerical example.

01. Rebecca: ok on part 1 it has 4 first and then multiplies it by 2, and part 2 it multiplies by
02. Paige: So you're saying it adds 4 and then multiply by 2. Ok, and then the other time?
03. Rebecca: And then for part 2 it is first multiplying by 2 and then adding 8 because it's multiplying by 2 first you have to multiply 4 by 2 and add 8 because since you're multiplying by 2.
04. Paige: Ok, so you're saying since you multiply by 2 first then you have to add 8 because why?
05. Rebecca: Because normally you'd be adding 4 first and then multiplying by 2 so you have to multiply 4 by 2.
06. Paige: Ok, I don't understand what you're saying. **Can you come up maybe and point to these numbers in one of the examples so everyone else can understand?** Ok, so make sure you're paying attention to where Rebecca is talking about.
07. Rebecca : (goes up to board). Ok, so um in, since this one (pointing to equation of part 1) you add 4 first and then multiply by 2 and in this one (pointing to equation of part 2) you're multiplying by 2 first you have to add 8 because 4 times 2 is 8. I don't know how to explain it, but, like, since this one you're multiplying second the 4 is being doubled. So you have to add 8 because 4 doubled is 8.

Figure 4. Paige, Year Two. An example of Focusing Students' Mathematics

[on the board was a drawing of two rows of five boxes]

01. Danny: So, I started put with this number is one, so it's like one plus one, plus four. Yeah. It says to double the one, so I did, it says to double the one, but then here's the eight. So on this one, here's the two one's there are right here (Danny pointing with a pencil on the overhead display). Then there are four plus four, which is eight, see how four, plus four, is eight, and then I use the two extra one's.

02. Irene: So, question or understanding?

03. Irene: Ralph?

04. Irene: Ralph, question or understanding?

05. Irene: I'm going to actually so, Danny is that an example of a specific one, So let's do rule one with your example. And then rule two (Irene writes on the flipchart). So this is an example, with the number one, is that what you're saying? So, the rule one, one plus four, and then double it, one plus four plus one plus four. Rule two, would take one, double it, and then eight. And both ways you get ten.

06. Irene: So, your picture goes with these numbers, right? (Irene pointing towards the flipchart). So can you explain again? This is rule one, right here, (Irene pointing to Danny's work projected on the whiteboard and writing Rule 1 and Rule 2 on the whiteboard) And then right here, is rule two. Okay, so go ahead.

07. Danny: You already explained it. Well. Well this one starts out with just doubling it. This doubles one and there's two. So then there's two (Number trick rule #2), then it says to add eight. This one is two (rule 2), this box and this box right here, then go back and add eight on this one, and this one's plus two fours. They both have eight.

Figure 5. Irene, Year Two. A Non-Example of Focusing.

Asking focusing question can be a demanding task. The teacher must first recognize what is mathematically relevant in the student's idea. The skill of "noticing" has been documented as lacking in some teachers – they simply do not notice the mathematics behind what the students are saying (Jacobs Lamb, & Phillips, 2010). Then the teacher must establish what might be missing from the justification, and ask the student to elaborate. This question cannot imply the answer – yes/no questions do not help the student to complete their justification; rather they explicitly tell the student what is missing.

When a good focusing question has been asked, the result is that the student is able to develop their own ideas. A question that re-directs a student away from what they were already thinking pulls the basis out from under a student's ideas, and forces the student to completely rethink what they had been saying. This is less productive for student justification than when a teacher uses what a student had already been working on, and helps the student to make the connection to justifying why the number trick works. The process of answering these specific questions also potentially helps students to learn what is necessary for a complete justification, and helps them to become better justifiers in general.

Providing Scaffolding Questions

Kelly decided to develop her task in year two by changing what the students started with. This is a direct way of providing scaffolding questions – the students had sub questions to the whole task that they could work on if they were stuck with the task in general. After discussion with the researchers, Kelly added the following three questions to her number trick task:

Q1: If Mary **triples** instead of doubling above, would her results still be equal? **Explain your thinking and tell why or why not. If your results are not equal, how could you change it so that it does work?**

Q2: Suppose Mary chose the number N, added 5 to it, and then multiplied by the sum by 4. If Mary writes this out, what does it look like? Would it be equal if she chose the number N, multiplied it by 4 and then added 5 to it? **Be sure to justify your thinking and demonstrate why you think they are or are not equal.**

Q3: Mary chose 2 numbers X and Y and she multiplied their sum by 5. She was able to write this expression. She said that 5(x+y) = 5x +5y. **Can you draw a picture to show that Mary is correct? Is there another way that you could show that Mary is correct?** (Emphasis in original.)

These three questions prompted thinking for her students. The students were now asked to examine examples and counterexamples of the distributive property, much in the way that Lakatos (1976) described how the mathematical community builds theorems by using the method of proofs and refutations. The counterexample in question 1 keeps students from thinking that two rules with addition and multiplication by the same multiplicand will have the same result. It was expected that the students might pick a number, try it and see that this trick does not work, and to avoid this low level of thinking, Kelly added the prompted "how could you change it so that it does work?" This was successful in getting the students to recognize that the distributive property did not apply to this case. They suggested applying the distributive property to 3(x+5), and quickly saw that the 10 must also be changed to 15.

The second question prompted the students to write an algebraic expression for the trick. For those well-versed in mathematics, this seems like an obvious first step for tackling this problem. However, many students across all of the classes had trouble turning the word problem into an algebraic expression. Kelly was not the only teacher who chose to add this scaffolding question to her task. Joan and Paige also added this question in year two. In all cases, this scaffolding question helped the students to realize that this is an important step in justifying the number trick task for all numbers.

Kelly's third question prompted a generalization of the number trick task. Kelly was one of only two teachers to ask for a justification of a more general distributive property, and she was "blown away" by what the students came up with – including a proof using the perimeter of a pentagon. Among the groups, there were four distinct visual justifications of the final hypothesis. Figure 6 gives an example of one of the visual justifications shared in this class. Kelly's scaffolding set up different logical ways of thinking that the students could explore.

01. Jacob: First we drew a pentagon, and then I decided to put lines between each one like this. [Jacob draws a pentagon, and splits each side in two parts with a perpendicular line] to show that this is a, and each one is a different length, so this line is x, and the other one is y, so you had to do that for all the other sides. [Jacob labels one part of one side with x, and the other part of the same side with y, and repeats on all sides] And it works because for both equations because if you look at it, its x plus y that makes a side, and if you times it by five to find the perimeter, so that is the same thing as five times x plus y... I mean x plus y times five

02. [Jacob's Partner writes 5x +5y and 5(x+5)]

03. Jacob: and it also, it's the same thing as uh, five x plus five y because if you count all of the number of y's and x's you get five x's and five y's

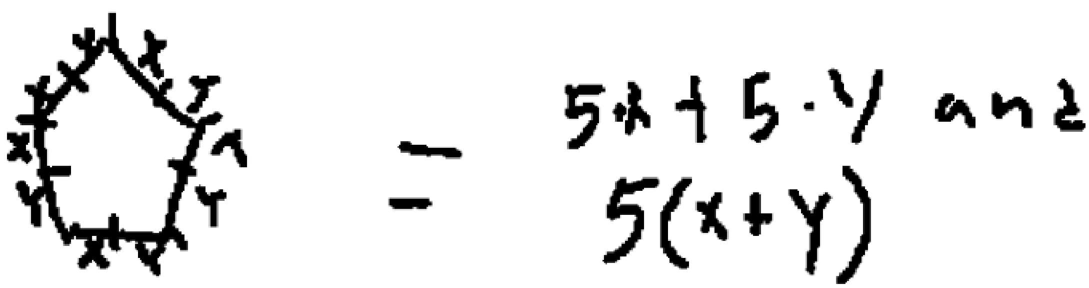

Figure 6: Kelly, Year Two, Student Justification After Scaffolding Questions Were Provided.

Finding good scaffolding questions can be almost as difficult as developing good focusing questions. Developing these questions involves predicting what students might have trouble with, or what they might need more direction on. The teacher must be careful that the scaffolding question not be equivalent to the main question. If it is, then the teacher has probably taken away some of the cognitive demands of the task, not broken it down into smaller, yet still demanding parts. Similarly to asking focusing questions, the teacher must also be careful that the scaffolding questions do not imply an answer. This would also remove the challenge from the task.

Providing Task-Specific Tools

An identified dimension similar to scaffolding, but different in key ways, is “providing task-specific tools.” This was characterized by the teacher providing an outline that was specific to the number trick task. The students were then left to fill in the outline. Depending on the situation, these tools could be productive for student justification or less so. One of the factors that changed how productive these tools were was that they could be used in several ways. The teacher could provide the outline at the beginning of the class, the teacher could provide it when it seemed as if the students were stuck, or the teacher could use it as a wrap-up at the end of class. These pedagogical discussions, and other factors, affect how much student justification occurs.

Irene expressed frustration after her first year’s task, so she decided to change her approach to the number trick task. She positioned this task in the middle of her linear equations unit, and spent some time at the beginning of class reviewing how to make a table for, graph, and write out a linear equation. She used graphing as a specific tool to help students justify. Most of her students followed this example in their small groups, some on their own, and some with express encouragement from Irene (see Figure 5). In a significant move, Irene re-focused all of the students’ ideas back to the graphing. The students had little room to explore other ideas. This could be why this lesson had a medium amount of justification – more than none, but not high amounts. .

It is interesting that this lesson was more productive for Irene than her year one lesson (see Table 5). One student provided a justification response, possibly because they were comfortable with the graphing tool. However, this lesson was not high on justification partially because the students did not generate their own justifications of the number trick. It is possible that this lesson had increased justification over year one because Irene has trouble with following student ideas (see Figure 5). In year 2, when Irene directed the acceptable method more closely, she had a better developed understanding of what kinds of things the students should say and how she should respond. This is seen in Line 2 of Figure 7. The student responded with reasoning about the graph of Jessie’s method. Then Irene was able to prompt the student for what would make up a complete justification. Later on, a student provided a complete justification of the trick.

01. Irene: Ok so maybe you guys should try to graph then. Because if that's not going to be able to justify then why don't you try a graph? Ok so how will this graph show? How does this graph show whether or not the y output will be the same for every input? Or does this show it just from zero to 10?
02. Amy: yours does show it to 10, but it would probably it would probably just go on forever, if you just extended it.
03. Irene: So do you know how to show it on a graph? You can put an arrow on the end of that line, and that means it will continue. But that part about it being linear, and it's going to continue on forever that would be some words that you would probably add.

Figure 7: Irene, Year Two, Providing a Task-Specific Tool

Cynthia provided a tool in a different way. At the end of her third day of teaching in year two, she focused her students on "drawing a picture." When her students were confused, she introduced the idea of representing variables as "smilies (☺)" and units as stars. Then she helped the students model both of Jessie's methods using smilies and stars. The students quickly saw that there was an equal number of smilies and stars on both sides of the equation.

There were two differences between Cynthia's and Irene's use of these methods. First, Cynthia positioned her method at the end of her lesson. At that point, her students had already spent two days working on the number trick task. This was where the bulk of the justification episodes were located. Thus, it was not actually this episode that bolstered the levels of justification. However, this tool helped the students to see a different way of justifying. The other difference between Cynthia's and Irene's use of a justification tool was that Irene's was not a mathematically sound method of justification. Irene only asked that her students graph the two lines and show that they were the same line. However, this rests on the fact that both of the methods produce continuous linear functions – something that should not be taken for granted in a justification task.

Thus providing a task-specific tool can turn out to be somewhat productive or not productive at all. In order for it to be productive, several things must occur. First, the students should be given time to do their own reasoning before a tool is given to them. Second, when the tool is provided, it should be a mathematically sound tool. Third, the tool should be left as open to interpretation as possible. Irene's tool dictated a step-by-step method, while Cynthia's tool simply invited the students to represent the situation in a new way. All of these factors change the productivity of this practice.

Leveraging a Critical Classroom Community

Leveraging a Critical Classroom Community was classified as variably productive. Table 9 shows that "justification" episodes were the most likely to have a Critical Classroom codes, though Table 7 shows that some implementations that were classified as high justification had few classroom community codes. Cynthia's two implementations were very illuminative in describing high levels of student justification. In year one, her students spent significant amounts of classroom time co-constructing acceptable ways to write out the algebraic expression for both parts of the number trick (see Figure 8). On the surface, it seemed as if the students had formed a collaborative community where they question their peer's answers, and provide their own reasoning. For instance, in the highlighted lines Robert and Alice negotiated acceptable ways to algebraically represent the numeric manipulations that are being performed. The data (see Table 7) also showed that this lesson is the highest in classroom community. Cynthia said very little outside of eliciting more students' questions and ideas, as in lines 2 and 4. Specifically she never told the students that certain representations are more efficient than others, preferring to allow the students to work these details out on their own. This was a classic example of discussion in a reform classroom, which has been thought to promote student learning, and it is clear that students in this classroom are able to reason through abstract representations. However, this class never completely justified why the number trick works. Thus, students are able to think and reason through parts of the problem, but may need the teacher to push them towards a complete justification.

[Written on the board is n+4+9, along with other variable expressions]

01. Alice: Wouldn't the second one (n+4+9) not work because if your variable was something different than 5 then it would not be 9.

02. Cynthia: Ooh what do you guys think about that? Ooh Alice what makes you think that, come up here and show us what you mean because that's a good point. And you guys, ask any questions on this, but explain it to us Alice.

03. Alice: So wouldn't you have to add 4 every time, if this was 6 then that's going to be 10 not 9

04. Cynthia: Questions for her, questions Alice you got people. Okay do you have more questions?

[Alice calls on Robert]

05. Robert: What um, didn't she say n=5. So wouldn't she know that n is supposed to be 5?

06. Alice: Yeah but like if it was any other number, for every number.

07. Robert: So are you saying that if that n=5 and was not there that couldn't be any other number but is that what you are saying? So you are saying that if see how she wrote n=5, so you are saying since you will always know what n equals if that wasn't there then it could be any number, except wouldn't 9 minus 4 equal 5?

08. Alice: Yeah, like 9 wouldn't work in that spot for any number.

09. Jenn: I agree with that because of you wanted to do the n+4+ 9 you would have to put like you would almost have to put like parentheses in and more like variables and it could become like really confusing.

Figure 8: Cynthia, Year One, A Critical Classroom Community

In year two, Cynthia stepped into the conversation more often. She introduced ideas for how to tackle the problem and she focused students' attention on important or missing parts. Cynthia did not abandon her collaborative focus. She still asked for students to comment on each other, and still allowed students time and space to do so. However, when a justification was mathematically incomplete, she stepped in and asks the student to clarify their work (e.g. "how do you see the 8 in that?"). These types of focusing questions helped the students to make their work mathematically acceptable without dismissing the work that they had already done. Rather than letting her students decide when a justification is complete, Cynthia pressed on what the students offer until she had determined that it was mathematically complete.

It is important to note this dimension, because it is counter-intuitive that it would not directly correlate with student justification. Much "reform" mathematics focuses on building a classroom of learners who engage each other, yet this practice is not always sufficient for high levels of mathematical justification. The critical difference between these two lessons is how close the students remain to the end goal – justifying the number trick task. It is theoretically possible that a community justify the whole task with no input from a mathematical authority simply by working together and making sense of the mathematics. However, in practice, it seems that middle school students need some guidance from a teacher who can keep their discourse focused on the main point of the lesson. Without this guidance, the students get wrapped up in parts of the task - in this case writing a rule.

Teachers did note that a collaborative community was an important part of supporting justification. During working sessions, Kelly specifically noted that having students work together helped them to come up with justifications. She arranged her class into groups for both of her lessons. The students who explain their work in Figure 6 built the ideas together with little input from Kelly – besides the scaffolding questions that she provided.

Examining the Practices on the Math/Student Continuum

A glance at the Math/Student continuum (Figure 3) shows that the constantly productive practices are in the middle – they bring together student ideas and strong mathematics. The variable practices are those on the sides of the continuum. When the variable practices are successful, they are used in a way such that they are located closer to the middle of the continuum. For instance, providing specific tools is more productive when the teacher provides a way to conceptually represent the task, but allows the students to do it themselves. On the other end, leveraging a critical classroom community is productive when the teacher ensures that this community is focused on the mathematics. Thus supporting student justification involves emphasizing both math and student ideas.

Summary and Discussion

In this chapter, I present a summary of my thesis and discuss implications of the study.

Summary of Results

This thesis was written to answer the question “what teacher practices are associated with high levels of student justification?” Because this is a qualitative question, research methods included filming classes, collecting student work and reading teacher reflections. Even though this thesis is part of a larger study, I bounded it by examining only the work that seven teachers did on the “number trick task” implemented in two consecutive years. I then used a grounded theory approach to develop and closely define practices that were associated with student justification.

My results describe two practices that were consistently associated with high levels of student justification and two that were variably associated with high levels of student justification. The consistent practices are “focusing students’ mathematics” and “providing scaffolding questions.” Both of these practices bring together students ideas and the field of mathematics to support sound justification. The variable practices are “providing task-specific tools” and “leveraging a critical classroom community.” The first emphasizes the mathematics behind the task and the second emphasizes the students’ contributions. When the emphasis is too far to one side, these become unproductive.

Discussion

Although these practices were productive in most cases, some cases exhibited these practices and yet did not achieve high levels of justification. This could be due to several factors. First, there is variability in how much a teacher uses a specific practice. A teacher may focus a student’s idea only sometimes, or may focus only some of the students’ ideas. It is also possible that some teachers continue this practice outside of the filming sessions while others do not. These teachers would normalize the practice of justification in their classroom, and their students would find it easier to justify on the spot. Within the classroom context, a teacher must also make a choice between focusing a student’s mathematics and asking the class for their thoughts on a student’s ideas. The second option would be a move to leverage a classroom community, but it would be at the expense of asking a focusing question. Some teachers are able to both focus the student's ideas and leverage a classroom community, but some teachers err on the side of one or the other.

Teachers also debated when and how to use scaffolding questions during the working session. Some teachers held the opinion that adding these scaffolding questions takes away the chance for their students to experiment, and made the pedagogical choice not to add these scaffolding questions. It may be that helping students with justifications inherently improves the level of justification, but it may not be the kind of justification that a student would produce without the help.

Along the same lines, teachers debated whether or not to give students the task-specific tools. They understood that ensuing justifications would not come totally from the students. However, in giving these tools they helped students to see what the mathematical community at large would value in a proof. This may be critical for helping students to transition into higher level mathematics where having an idea is not enough – one must also communicate it in accepted ways.

We cannot measure how teachers built their critical communities at all; rather we can only see how it plays out in the classroom. Thus, although this paper states that having a critical community can help in producing student justification, it does not outline how such a community could be built. Since we cannot see this, it is possible that teachers with a critical classroom community may simply have skeptical students – this could increase how much students challenge each other, and how much they justify their work without influence from the teacher.

This research is in line with much of the literature around reform teaching practices and student-centered classrooms. However, it focuses specifically on the mathematical practice of justification. At this point, additional research is needed on finding a causal relationship between teacher actions and student justification. Additional research is also needed to show that justification is, beyond being a mathematical practice, productive for student learning in general. This research might address the differences between justification in general and the formal proof that are often introduced in high school geometry.

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Appendix A: Teacher Pre-and Post Reflections

Before each lesson, the teachers completed a reflection with the following questions:

Q1: What were the primary goals of this lesson?
Q2: What role will justification play in this lesson?
Q3: What kinds of justifications do you expect students might produce as they work on the task for this lesson?

After each lesson, the teachers completed a reflection with the following questions:

Q4: Write down some examples of student justification. Did these justifications align with what you thought would happen?
Q5: Pick one instance involving student justification that you thought was productive and/or went well. Explain what you valued about this instance.
Q6: Pick one instance involving student justification that didn't go as well as it could have. What would you do next time with this exact situation to make it go better? Explain why you revised it as you did?
Q7: What, if anything do you think the students understood about the math content that was a result of, or strengthened by, justification? What, if anything, do you think students learned about justification as a result of the lesson?
Q8: What, if anything, did you learn about teaching to promote student justification as a result of this lesson?
Q9: Given where your students are with justification now, what goals do you have for your students with respect to justification? What will you do next time to move towards these goals?
Q10: Anything else on your mind with respect to justification or your students' work with justification?

Appendix B: Student Attempts at Justification with the Number Trick Task

John: I tried it with an even number, odd number, and a negative number. So that's pretty much covered most types of numbers. I just thought that it will never not work out.

Tony: She alternated the places in which you multiply.

Dave: So we had four, like four we cut in half, we have two. And then this one, we cut it in half and got four.

Dorothy: because if you add four to this, and then you have to multiply it by two and it would be like eight because we did four by, and that equals eight. So that would come out to that.

Lester: Well, I did a table and a graph. I did the graph, and then there's arrows all over the sides because no matter how far you put the arrows, it's always going to be a straight line. That's why it always works.

Paula: Yeah. It's the numbers. I mean the rules are exactly the same except they're just written differently. And so, if they were different then you would have two lines.

William: Okay so what I was thinking was if you have in this case 5+4+n, since you don't know what that equals, 5+4, if you are not given the information where the letter could be from anywhere like from 1 to like a billion or like 1 to a thousand, I am not even sure you could do that or not. But if that were the case it would take like a really really, really long time because you have to be like 5+4+1 and then say what that equals and then 5+4 +2 and see what that equals.

Helene: I disagree with that because if you didn't know what to do, if this was a variable x, um and you had this problem and you didn't have the rule then you wouldn't know what to do with this number, you would need to know what the number was.

Albert: So, the pattern that I saw was every time we add a number, it went by two.

Justina: I don't know why, it's kind of confusing. Because you double the numbers I guess. And, like, you double those numbers. And, with this one when you double it, it's higher, and with that one, when you double it, it's lower.

Gennie: And then show me how to do thirty, and then she showed me that thirty four minus thirty is four and then four is half of eight, and that's what you're adding.

Ray: Because you multi- you add four. And you multiply by two, which is like multiplying the four by two which is eight. So then, you do the same thing over here, but first you multiply by two and then you add eight... I don't know what I just said.

Appendix C: An Example of Coding

1. James: Well, in Matthew's, he showed the two of the equations with each other. They're just the same - 2x plus 8 - because you distribute in the parentheses. And so there would be no point in drawing another one because it would just end up looking the same as that one. ***Build/Justification***

2. Audrey: Rudy, do you have a comment or a question about that?

3. Rudy: Well, I'm, just from Matthew's and James', I'm convinced that they are the, it'll work no matter what because I also tried, on my table, I did a negative, zero, a normal number, a large number, and a decimal, and they all worked. So. ***Explanation***

4. Audrey: Erin, do you have any comments or questions for Royce?

5. Erin: That makes total sense to me. ***Answer/Statement***

6. Audrey: How does it make sense?

7. Erin: Just because they're exactly the same thing, pretty much. It's just because you're adding 4, but you're adding 8 down here, and you're timesing, like. It's the exact same thing, what he's talking about, the two problems, the two things. Like, if you take one number and then you do one thing with the first equation with it, and you do the same number with the second equation, they're the exact same thing... ***Justification***

8. Audrey: Clinton, why is there only one line there?

9. Clinton: Oh, because they're the same. ***Answer/Statement***

10. Audrey: What do you mean they're the same?

11. Clinton: I just heard you guys say that. ***Answer/Statement***

12. Audrey: Okay, somebody explain why they're the same. Katy, can you give a go. Why is there only one line up there?

13. Katy: Because it goes at a constant rate. ***Reasoning***