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Abstract

A problem with a practical application of Varian.s Weak Axiom of Cost Minimization is that an observed violation may be due to random variation in the output quantities produced by firms rather than due to inefficiency on the part of the firm. In this paper, unlike in Varian (1985), the output rather than the input quantities are treated as random and an alternative statistical test of the violation of WACM is proposed. We assume that there is no technical inefficiency and provide a test of the hypothesis that an observed violation of WACM is merely due to random variations in the output levels of the firms being compared.. We suggest an intuitive approach for specifying a value of the variance of the noise term that is needed for the test. The paper includes an illustrative example utilizing a data set relating to a number of U.S. airlines.

This paper was written while the author was visiting the Indian Statistical Institute, Calcutta.

A SIMPLE STATISTICAL TEST OF VIOLATION OF THE *WEAK AXIOM OF COST MINIMIZATION*

Varian (1984) introduced the *Weak Axiom of Cost Minimization (WACM)* as a simple non-parametric test[1] of cost minimizing behavior by a firm facing given input prices. Any violation of *WACM* implies that the actual input bundle chosen by the firm is not cost minimizing and the firm is inefficient. A problem with its practical application is that an observed violation may be due to random variation in the output quantities produced by firms rather than due to inefficiency on the part of the firm. Varian (1985) addressed the problem of measurement errors in the observed input quantities and proposed a chi-squared based test of *WACM*. Unfortunately, the quadratic programming procedure proposed by Varian is computationally quite demanding. In this paper, the output rather than the input quantities are treated as random and an alternative statistical test of the violation of *WACM* is proposed. Specifically, we assume that there is no technical inefficiency and provide a test of the hypothesis that an observed violation of *WACM* is merely due to random variations in the output levels of the firms being compared. As in the case of Varian's test, here also one needs a prior value of the variance of the noise term in order to perform the proposed test. We do, however, suggest an intuitive approach for specifying this value. The rest of this paper is organized as follows. Section 2 explains the nonparametric methodology behind the *WACM* and describes the proposed statistical test. Section 3 includes an illustrative example utilizing a data set relating to a number of U.S. airlines. Section 4 summarizes.

2. The methodology

2.1 Weak Axiom of Cost Minimization

Consider an industry producing a scalar output (y) from a bundle of n inputs x. The underlying technology can be characterized by the production possibility set

$$T = \{(x, y) : y \text{ can be produced from } x \}. \qquad (1)$$

[1] Varian's paper builds on earlier work by Afriat (1972), Hanoch and Rothschild (1972), and Diewert and Parkan (1983).

One can define the production function[2] as

$$f(x) = \max\ y : (x, y) \in T. \qquad (2)$$

For any output level y, the input set $V(y)$ consists of all input vectors that can produce y. Thus,

$$V(y) = \{x: (x, y) \in T\}. \qquad (3)$$

Alternatively,

$$V(y) = \{ x : f(x) \geq y \}. \qquad (4)$$

Next consider an n-element input price vector w. The minimum cost of producing output y at price w is

$$C(w, y) = \min\ w'x : x \in V(y). \qquad (5)$$

Clearly, the cost function $C(w, y)$ will depend on the specification of the technology. In parametric analysis one typically estimates an explicitly specified form of the cost function. A problem with this approach, however, is that the validity of the findings depends on the appropriateness of the specified form. In a nonparametric analysis, by contrast, one leaves the technology unspecified beyond a few general assumptions and examines whether there exists *any* reference technology satisfying those assumptions for which the observed data would be consistent with cost-minimization.Consider a data set relating to N firms from an industry. For any individual firm i $(i = 1,2,\ldots,N)$ let y_i denote its scalar output, x^i its actual input vector, and w^i the vector of input prices paid by this firm. Thus its actual cost is $C_i = w^{i\prime}x^i$. The question is whether the firm is producing its output using the least cost input bundle.

We make only the following two assumptions:

- All actually observed input-output combinations are feasible. That is $(x^j, y_j) \in T$ for $j = 1, 2,\ldots,N$. Alternatively, $x^j \in V(y_j)$ for each $j = 1,2,\ldots,N$.
- Output is freely disposable. Thus, if $y_1 > y_0$, then $x \in V(y_1) \Rightarrow x \in V(y_0)$.

[2] We do not assume that the function is continuous or differentiable.

Now suppose that the observations are rearranged in ascending order of the output quantities produced. Thus, $j \geq i$ implies $y_j \geq y_i$ .Now, if there is some firm $j \geq i$ such that $w^{i}{}'x^{j} < w^{i}{}'x^{i}$, then firm i cannot be minimizing cost. The intuition behind this test is quite straightforward. Note that x^j actually produces y_j. Hence, by free disposability of output, x^j can also produce y_i. That is, $x^j \in V(y_i)$. Therefore, if $w^{i}{}'x^{j} < w^{i}{}'x^{i}$, obviously x^i is not the least cost bundle in the input requirement set of output y_i. That is, firm i is not minimizing cost. This is a remarkably powerful test that can be carried out with the very little computation.

Varian formalized this test as the *Weak Axiom of Cost Minimization* that can be stated as follows.

(WACM):For an observed data set to be consistent with competitive cost minimizing hypothesis, we must have $w^{i}{}'x^{i} \leq w^{i}{}'x^{j}$ *for all* $i=1,2,\ldots,N$, *and* $j \geq i$.

It may be noted that in deriving *WACM* it was not necessary to assume convexity of the input requirement set. The relation between *WACM* and the standard DEA model for cost minimization under VRS can be best understood by considering the following mixed integer programming problem[3]:

$$
\begin{aligned}
& \min\ w^{i}{}'x \\
\text{subject to}\quad & \sum_{j=1}^{N} \lambda_j x^j \leq x; \\
& \sum_{j=1}^{N} \lambda_j y_j \geq y_i; \qquad\qquad (6) \\
& \sum_{j=1}^{N} \lambda_j = 1; \\
& x \geq 0; \lambda_j \in \{0,1\} (j = 1,2,\ldots,N).
\end{aligned}
$$

Note that the constraints on the λ_js ensure that only one λ_j will take the value *1* while all others will be *0* at the optimal solution. Further, the output constraint requires $j \geq i$. Clearly, there will not

[3] Ray (2004) discusses the relation between *WACM* and Fee Disposal Hull (FDH) analysis.

be any input slack in the optimal bundle x^*. That means that x^* will be the observed input bundle of some firm j satisfying $j \geq i$. In other words, applying *WACM* to test for cost minimizing behavior on the part of firm i is equivalent to solving the mixed integer programming problem *(6).* This is a restricted version of the standard DEA LP model for cost minimization under the VRS assumption where the λ_js are allowed to take *any non-negative value* so long as they add up to unity.

2.2 Stochastic Output and WACM

In the foregoing analysis output is treated as deterministic and a firm fails to minimize cost because it selects a wrong input mix and/or produces less than the maximum output feasible from the input bundle used. Varian (1985) allowed random noise in the data in the form of measurement errors in the input quantities but the output quantities were treated as deterministic. In the present paper we follow the convention in production function analysis and regard the output rather than the input quantities as random. While the input quantities may differ from their optimal cost-minimizing levels, such deviations result from allocative inefficiency rather than from random shocks.

Consider the stochastic production function

$$y = f(x).e^{u - \tau} \qquad (7)$$

where $\tau \geq 0$ represents technical inefficiency while u is a two-sided disturbance term capturing favorable as well as unfavorable random shocks shifting the frontier. Following Aigner, Lovell, and Schmidt (1977) one may specify a half-normal distribution for τ and the usual normal distribution $N(0, \sigma^2)$ for u. In the present case we assume away any technical inefficiency and set τ equal to zero for all firms. By implication, cost inefficiency is caused only by the choice of an inappropriate input mix.

Define the planned output as

$$y^* = f(x). \qquad (8)$$

In the presence of random variation in the output levels, a test of violation of the *WACM* should be conducted in terms of the planned output (y^*) rather than the realized output (y). It is possible

that *WACM* holds in terms of the planned levels of output even when the realized output levels imply a violation. A practical problem with this, of course, is that the planned output levels are not observed and one must use the realized output levels for the test.

Consider the firm i using the input bundle x^i at input price (vector) w^i and producing the output level y_i. Next consider the input-output combination (x^j, y_j) observed for each firm j $(j=1,2,\ldots,N)$ in the sample. Define the index set

$$L(i) = \{ j: w^{i\prime}x^i > w^{i\prime}x^j \} \qquad (9)$$

and a dummy variable C_{ij} that takes the value 1 if $j \in L(i)$ and the value 0 otherwise. Thus, C_{ij} equal to 1 implies that, at the input price (vector) w^i, the input bundle of firm j would cost strictly less than the input bundle actually used by firm i. Define another dummy variable D_{ij} that takes the value 1 if $y_i^* > y_j^*$ and the value 0 otherwise. In a pair wise comparison of the firms, a violation of *WACM* occurs if C_{ij} equals 1 and D_{ij} equals 0 at the same time. Two things need to be highlighted. First, because input prices and quantities are non-random, C_{ij} is also non-random and takes the value 0 or 1 with certainty in any specific instance. Second, the planned output levels (y_i^* and y_j^*) are functions of non-random input bundles and, although unobserved, are also non-random at least conceptually. Thus, D_{ij} is either 1 ($y_i^* > y_j^*$) or 0 ($y_i^* \le y_j^*$). Hence, D_{ij} also would have a degenerate distribution at either 0 or 1. In that sense, failure of *WACM* is a binary outcome that is either true or false. In reality, however, D_{ij} is not observed and violation of *WACM* cannot be verified by simply looking at the data. We may, however, take a different approach. In a Bayesian fashion we look at the *posterior* probability that D_{ij} equals 1 given the observed values of y_i and y_j and the knowledge that C_{ij} equals unity.

Now, $D_{ij} = 1 \Leftrightarrow y_i^* > y_j^* \Leftrightarrow \ln y_i^* > \ln y_j^*$. But, from (6), $\ln y = \ln y^* + u$. Hence,

$$Pr\{D_{ij} = 1\} = Pr\{u_i - u_j < \ln y_i - \ln y_j\}.$$

Define the variable

$$\varepsilon_{ij} = u_i - u_j.$$

Now recall that u_i and u_j have identical and independent Normal distributions with mean 0 and variance σ^2.Hence ε_{ij} has the Normal distribution with mean 0 and variance $2\sigma^2$. Therefore, the variable

$$z_{ij} = \frac{\varepsilon_{ij}}{\sigma\sqrt{2}} = \frac{u_i - u_j}{\sigma\sqrt{2}}$$ has the standard Normal distribution.

Hence,

$$Pr\ \{D_{ij} = 1\} = \Pr\left\{z_{ij} < \frac{\ln y_i - \ln y_j}{\sigma\sqrt{2}}\right\}. \qquad (10)$$

Clearly, this probability would be less than 5% if

$$\frac{\ln y_i - \ln y_j}{\sigma\sqrt{2}} < -1.64\,. \qquad (11)$$

Thus, even when C_{ij} equals 1, a statistically significant violation of the *WACM* is not detected unless

$$\ln y_j \geq \ln y_i + (1.64)\sqrt{2}\sigma\,. \qquad (12)$$

This inequality does not provide a critical value for hypothesis testing, however, unless one specifies a numerical value of σ.

Faced with a similar problem of having to specify a value of σ, Varian (1985) proceeded by setting an upper bound on σ for *WACM* to hold. In the present paper, we take a different approach and derive a plausible value of σ from what we regard as a reasonable range of variation in the frontier output due to random shocks. Although an exact value of the variance of u_i may not be available, in most cases we have a prior belief about the variability of the realized output from a given input bundle around its norm. For example, in farming where the output is greatly influenced by rainfall, variance due to random factors will be large. But even there, we may hold the belief that the output will exceed twice its normal level or fall below half of the normal level with probability no more than 5%. Suppose that we are able to stipulate the probability

$$\Pr\left\{\tfrac{1}{\beta} \leq \frac{y_t}{y_t^*} \leq \beta\right\} = 1-\alpha \text{ for some pair } (\alpha, \beta). \qquad (13)$$

This is equivalent to

$$\Pr\{-\ln\beta \leq u_t \leq \ln\beta\} = 1-\alpha. \qquad (14)$$

Alternatively,

$$\Pr\left\{-\frac{\ln\beta}{\sigma} \leq \frac{u_t}{\sigma} \leq \frac{\ln\beta}{\sigma}\right\} = 1-\alpha. \qquad (15)$$

Given the Normal distribution for *u,* this is equivalent to

$$\Phi\left(\frac{\ln\beta}{\sigma}\right) = 1-\tfrac{1}{2}\alpha, \qquad (16)$$

where $\Phi(.)$ is the cumulative standard Normal distribution function. Thus,

$$\ln \beta = \sigma.\Phi^{-1}(1-\tfrac{\alpha}{2}). \qquad (17)$$

For the conventional significance level of 5%, α equals 0.05.In that case,

$$\ln \beta = 1.96\sigma. \qquad (18)$$

This yields

$$\sigma = \frac{\ln \beta}{1.96}. \qquad (19)$$

Substituting this value in *(10)* we obtain

$$Pr\ \{D_{ij} = 1\} = \Pr\left\{ z_{ij} < \frac{(1.96)(\ln y_i - \ln y_j)}{(\ln \beta)\sqrt{2}} \right\}. \qquad (20)$$

We may now define the critical value

$$z_{ij}^* = \frac{(1.96)(\ln y_i - \ln y_j)}{(\ln \beta)\sqrt{2}} \qquad (21)$$

Clearly, if $z_{ij}^* < -1.64$, $Pr\ \{ D_{ij} = 1\}$ is less than 5%. An implication of this, of course, is that even when y_i is smaller than y_j while $w^{i\prime}x^j$ is less than $w^{i\prime}x^i$, violation of *WACM* is not significant at the 5% level unless z_{ij}^* falls below -1.64. In any empirical application one may parametrically vary the value of β within a given range and statistically test the violation of *WACM* for alternative values of β.

3. An Application to U.S. Airlines Data

In this section we present an application of the proposed test using a data set for a number of U.S. airlines from the year 1984. A single output, five-input technology is considered. The output is a quantity index (QYI) constructed from the numbers of revenue passenger miles flown, ton-miles of cargo flown, and ton-miles of mail flown. The inputs are quantity indexes of labor (QLI), fuel (QFI), materials (QMI), flight equipment (QFLI), and ground equipment (QGRI). The corresponding input prices for the different inputs were constructed indirectly for the individual airlines by dividing the total expenditure incurred on any input by the quantity index. The data form a subset of a larger data set constructed by Caves, Christensen, and Trethaway (1984) and are reported in Tables 1a – 1b.

Table 2 reports the summary findings from the statistical test of significance of any violation of *WACM* observed in a pair wise comparison of firms. We considered four different values of β. They were $\beta_1 = 1.5$, $\beta_2 = 1.75$, $\beta_3 = 2.0$, and $\beta_4 = 2.5$. The corresponding values of σ

are $\sigma_1 = 0.20687$, $\sigma_2 = 0.28552$, $\sigma_3 = 0.35365$, and $\sigma_4 = 0.4675$. Thus, a higher value of β allows a greater degree of random variation in the realized output. Clearly, violation of *WACM* for a higher value of β in itself implies a violation for any smaller value of β as well.

Naturally, the test could not be performed for the airline with the largest output quantity (United (UN)). For 7 of the remaining 20 airlines no violation was observed in any relevant pair wise comparison. These airlines were New York Air (NYA), Frontier (FR), Peoples Express (PE), Western (WE), Continental (CO), Northwest (NW), PanAm (PA), and American (AM). For 6 other airlines (namely, Midway (MI), Muse (MU), Piedmont (PI), US Air (USA), TWA, and Eastern (EA)) none of the observed violations was statistically significant even for the smallest value of β considered here. The remaining 7 airlines showed significant violation although in one case (Southwest (SW)) a violation was observed only for the lowest value of β but not for the higher values. Of the 210 possible pair wise comparisons involving the 20 airlines, violation was observed in 48 cases but 19 of them were not statistically significant. Four airlines (Air California (AC), Ozark (OZ), Pacific South (PS), and Republic Hughes Air (RHA)) account for the bulk of the significant violations. Out of the 35 violations observed for these four airlines 22 were significant (7 for AC, 6 for OZ, 5 for PS, and 4 for RHA) even for β equal to 2.5. The evidence is fairly conclusive that these airlines fail to minimize cost.

The statistical tests show that for reasonable values of β (i.e., for plausible degrees of random variation in the output) about 40% of the observed violation of *WACM* in pair wise comparison of firms can be ascribed to chance variation in output rather than to inefficiency.

Summary

In the presence of random variation in output an observed violation of *WACM* may be caused by chance factors. One may, however, perform a simple statistical test of significance of the violation. As shown above, the value of σ needed for performing the test can be obtained in a rather intuitive manner and can be varied parametrically to examine the robustness of the findings from the test. The principal appeal of the procedure proposed here lies in the fact that it retains the computational simplicity that is so appealing about the *WACM*.

Table 1a: Output and Input Quantities

NAME	YI	QLI	QFI	QMI	QFLI	QGRI
MIDWAY (MI)	0.037	0.1164	0.0456	0.1275	0.0791	0.0233
MUSE (MU)	0.0439	0.1128	0.0395	0.0893	0.0774	0.0323
NEW YORK AIR (NYA)	0.0458	0.0833	0.0459	0.0766	0.0672	0.0339
AIR CALIFORNIA (AC)	0.0816	1.2518	0.0702	1.2631	1.2579	0.0784
OZARK (OZ)	0.1387	1.2552	0.1236	1.5153	1.149	0.1266
PACIFIC SOUTH (PS)	0.155	1.478	0.1168	1.5579	1.2602	0.2274
SOUTHWEST (SW)	0.1997	0.1703	0.1806	0.177	0.1387	0.1587
FRONTIER (FR)	0.2133	0.121	0.1524	0.1095	0.0859	0.1961
PIEDMONT (PI)	0.304	0.0813	0.3004	0.0778	0.0611	0.2591
PEOPLES EXPRESS (PE)	0.3277	0.0632	0.2154	0.0645	0.0545	0.2064
US AIR (USA)	0.4214	0.3209	0.374	0.3812	0.2898	0.4883
REPUBLIC HUGHES AIR (RHA)	0.4332	1.6912	0.4369	1.56	1.7614	0.3107
WESTERN (WE)	0.4933	0.2892	0.3547	0.3677	0.3239	0.3141
CONTINENTAL (CO)	0.5455	0.2778	0.3906	0.3431	0.3012	0.4303
NORTHWEST (NW)	1.2485	0.1291	0.7906	0.1674	0.1071	0.6194
DELTA (DE)	1.3897	0.6984	1.123	0.6272	0.6006	1.7945
TWA (TWA)	1.5134	0.2983	0.9349	0.3558	0.3177	1.5457
EASTERN (EA)	1.5157	1.1117	1.1765	1.1327	0.9668	1.444
PANAM (PA)	1.5685	0.1045	0.9764	0.069	0.067	1.2589
AMERICAN (AM)	1.9365	0.3344	1.3036	0.3931	0.3273	2.1644
UNITED (UN)	2.4424	1.2481	1.5965	1.283	1.2726	2.7084

Table 1b. Input Prices

NAME	PL	PF	PM	PFL	PGR
MIDWAY	348559.6	855663.1	284983.6	131301.8	86121.06
MUSE	348071.4	790689.5	284858.6	129331.4	86281.44
NEW YORK AIR	333140.4	824780.8	285004.2	128801.2	86168.56
AIR CALIFORNIA	383249.2	894019.3	283203.3	194992.5	86208
OZARK	359895.3	819571.5	283820.2	204325.9	86178.56
PACIFIC SOUTH	331526.8	834531.1	283208.1	202811.9	86145.75
SOUTHWEST	351176.8	808326.4	284839.6	129386.5	86165.5
FRONTIER	336276.9	844862.8	284970.8	137560.9	86159.31
PIEDMONT	338440.1	819421.1	284863.8	129227.5	86157.63
PEOPLES	308486.3	825227.8	284879.6	128931.4	86165.94
US AIR	355947.4	831339.5	297321.8	186006.2	86151.44
REPUBLIC HUGHES AIR	391499.4	828228.9	283208.6	194647.2	86161
WESTERN	382666	844183.7	297305.6	172419.9	86144.31
CONTINENTAL	353894	843951.3	283779.5	175142.4	86153.19
NORTHWEST	366434.6	860378.3	283768.3	196795.7	86149.94
DELTA	345456	821327.6	283797	186283	86154.19
TWA	346849.4	835999.5	283792.7	198661.9	86155.25
EASTERN	338026.2	814236.9	283207.4	169123.6	86154.06
PANAM	308360.2	867325.3	285030.9	136297.9	86154.19
AMERICAN	344729.9	823391	283819.6	199609.4	86158.31
UNITED	414678.7	831375.1	296689	184200.6	86155.88

Table 2 Test of Violation of WACM

NAME	Observed and significant violations of WACM
MIDWAY (MI)	MU[1], NYA[1]
MUSE (MU)	NYA[1]
NEW YOR AIR (NYA)	None
AIR CALIFORNIA (AC)	SW[4], FR[4], PI[5], PE[5], USA[5], WE[5], CO[5], NW[5], PA[5]
OZARK(OZ)	SW[1], FR[1], PI[3], PE[4], USA[5], WE[5], CO[5], NW[5], TWA[45], PA[5]
PACIFIC SOUTH (PS)	SW[1], FR[1], PI[3], PE[3], USA[4], WE[5], CO[5], NW[5], TWA[5], PA[5]
SOUTHWEST (SW)	FR[1], PE[2]
FRONTIER (FR)	None
PIEDMONT (PI)	PE[1]
PEOPLES EXPRESS (PE)	None
US AIR (USA)	WE[1], CO[1]
REPUBLIC HUGHES AIR (RHA)	WE[1], CE[1], NW[4], DE[5], TWA[5], PA[5], AM[6]
WESTERN (WE)	None
CONTINENTAL (CO)	None
NORTHWEST (NW)	None
DELTA (DE)	TWA[1], PA[1], AM[1]
TWA (TWA)	PA[1]
EASTERN (EA)	PA[1], AM[1]
PANAM (PA)	None
AMERICAN (AM)	None
UNITED (UN)	Not testable

Notes: 1.Not significant for $\beta \geq 1.5$.
2. Significant for $\beta \leq 1.5$.
3. Significant for $\beta \leq 1.75$.
4. Significant for $\beta \leq 2.0$.
5. Significant for $\beta \leq 2.5$.

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