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Subhash Ray
University of Connecticut

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Subhash C. Ray
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Abstract

Lovell and Rouse (LR) have recently proposed a modification of the standard DEA model that overcomes the infeasibility problem often encountered in computing super-efficiency. In the LR procedure one appropriately scales up the observed input vector (scale down the output vector) of the relevant super-efficient firm thereby usually creating its inefficient surrogate. An alternative procedure proposed in this paper uses the directional distance function introduced by Chambers, Chung, and Fre and the resulting Nerlove-Luenberger (NL) measure of super-efficiency. The fact that the directional distance function combines features of both an input-oriented and an output-oriented model, generally leads to a more complete ranking of the observations than either of the oriented models. An added advantage of this approach is that the NL super-efficiency measure is unique and does not depend on any arbitrary choice of a scaling parameter. A data set on international airlines from Coelli, Perelman, and Griffel-Tatje (2002) is utilized in an illustrative empirical application.

THE DIRECTIONAL DISTANCE FUNCTION AND MEASUREMENTOF SUPER-EFFICIENCY: AN APPLICATION TO AIRLINES DATA

Data Envelopment Analysis (DEA) provides an objective basis for ranking firms in an industry in order of their measured technical efficiency scores. This, however, is not possible for the sub-group of firms that lie on the graph of the technology and are all rated at 100% technical efficiency. A procedure first proposed by Andersen and Petersen (1993) uses the *super-efficiency* measures of these efficient firms to resolve this problem. A firm is regarded as super-efficient if its DEA efficiency score exceeds 100% when measured against a production possibility set constructed from the input-output data of *all other firms* in the sample. While this modified DEA procedure is quite useful in most cases, the relevant linear programming problem for measuring the super-efficiency score may not have any feasible solution in certain situations. Chavas and Cox (1999) point out that Shephard (1970) made an *input attainability* assumption that ensures that all output bundles can be produced from the rescaling of any non-zero input bundle. Similarly, by the *output attainability* assumption, every input bundle is feasible in the production of any rescaled non-zero output bundle. The problem of feasibility arises in an output-oriented (input-oriented) super-efficiency model when an efficient input-output bundle fails to satisfy the input (output) attainability assumption with respect to the modified production possibility set. Several authors (e.g., Thrall (1996), Zhu (1996), Dula and Hickman (1997), Seiford and Zhu (1999), and Harker and Xue (2002)) have noted various necessary and sufficient conditions for infeasibility in super-efficiency DEA models. Lovell and Rouse (LR) (2003) have recently proposed a modification of the standard DEA model that overcomes the infeasibility problem. The essence of the LR procedure is to appropriately scale up the observed input vector (scale down the output vector) of the relevant super-efficient firm thereby usually creating its inefficient surrogate[1]. Because an inefficient firm plays no role in defining the frontier, it would not make any difference whatsoever if the firm with the revised input/output data is retained in the reference set. An alternative procedure proposed in this paper uses the directional distance function introduced by Chambers, Chung, and Färe (1996) and the resulting Nerlove-Luenberger (NL) measure of super-efficiency. The paper unfolds as follows. Section 2 describes the directional distance function and the associated NL efficiency measure. Section 3 presents the standard Debreu-Farrell super-efficiency model along with the NL super-efficiency model. Section 4 addresses the infeasibility problem that sometimes arises in a super-efficiency model and compares the LR approach with the one presented in this paper in solving the problem. A data set on international airlines from Coelli, Perelman, and Griffel-Tatje (2002) is utilized in an illustrative empirical application presented in section 5. The main conclusions are summarized in section 6.

2. The Directional Distance Function and Nerlove-Luenberger Efficiency

Chambers, Chung, and Färe (1996) introduced the *directional distance function* based on Luenberger's *benefit function* to obtain a measure of technical efficiency reflecting the potential for increasing outputs

[1] As discussed below, in some cases with 0 values of individual outputs or inputs, even the surrogate remains efficient.

while reducing inputs simultaneously. Consider some input-output bundle (x^0, y^0) and a reference input-output bundle (g^x, g^y). Then, with reference to some production possibility set, *T,* the directional distance function can be defined as:

$$\vec{D}(x^0, y^0; g^x, g^y) = \max\ \beta : (x^0 + \beta g^x, y^0 + \beta g^y) \in T. \quad (1)$$

Clearly, the directional distance function evaluated at any specific input-output bundle will depend on (g^x, g^y) as well as on the reference technology. The bundle (g^x, g^y) defines the direction along which the observed bundle, if it is an interior point, is projected on to the efficient frontier of the production possibility set. In (1) above, the bundle (g^x, g^y) is chosen quite arbitrarily. As suggested by Chambers, Chung, and Färe, we may select $(-x^0, y^0)$ for (g^x, g^y) and in that case the directional distance function becomes

$$\vec{D}(x^0, y^0) = \max\ \beta : ((1-\beta)x^0, (1+\beta)y^0) \in T. \quad (2)$$

In other words, we seek to increase the output and reduce the input simultaneously by the proportion β. For example, if β equals 10%, we expand all outputs by 10%, while at the same time reducing all inputs by 10%.

Under the standard assumptions of convexity and free disposability of inputs and outputs, the production possibility set constructed from a set of *N* observed input-output bundles (x^j, y^j) $(j = 1, 2, \ldots, N)$ is

$$T = \{(x, y) : x \geq \sum_1^N \lambda_j x^j ; y \leq \sum_1^N \lambda_j y^j ; \sum_1^N \lambda_j = 1; \lambda_j \geq 0; (j = 1,2,\ldots, N)\}. \quad (3)$$

The VRS DEA formulation for the directional distance function for this production possibility set is:

$$\begin{aligned}
&\max\ \beta \\
\text{s.t.}\quad &\sum_1^N \lambda_j y^j - \beta y^0 \geq y^0; \\
&\sum_1^N \lambda_j x^j + \beta x^0 \leq x^0; \qquad (4) \\
&\sum_1^N \lambda_j = 1; \\
&\lambda_j \geq 0 (j = 1,2,\ldots, N); \beta \text{ unrestricted.}
\end{aligned}$$

This is a straightforward LP problem and can be solved quite easily. The factor β is the Nerlove-Luenberger measure of technical *inefficiency* of the firm. By implication, its *efficiency* equals *(1-β).*

3. Debreu-Farrell and Nerlove-Luenberger Super-efficiency Measures

The standard DEA models – both the CCR model for CRS and the BCC model for VRS – provide measures of technical efficiency of a firm relative to the others within the same sample. Firms that are found to be technically inefficient can be ranked in order of their measured levels of efficiency. Firms that are found to be efficient are, however, all ranked equally by this criterion. Andersen and Petersen (1993) suggest a criterion that permits one to rank order firms that all found to be at 100% technical efficiency by DEA. The underlying idea behind this criterion is quite simple. Consider the single-input, single-output case. Suppose that a firm with input-output (x_0, y_0) has been found to be technically efficient in an output-oriented DEA problem. Obviously, if its output had been any larger than y_0 it would have remained efficient. But a small reduction in its output will not necessarily lower its technical efficiency rating from 100%. In that sense, this firm may allow some deterioration in its performance without becoming inefficient. In other words, its observed output exceeds what is necessary for this firm to be considered efficient relative to other firms in the sample. In that case, the firm may be regarded as *super-efficient.* Naturally, between two firms both of which are technically efficient, the one with a greater room for reducing its output without becoming inefficient is, in a sense, more *super-efficient* than the other.

For any individual firm *k*, the modified production possibility set can be constructed as

$$T_k^- = \{(x, y) : x \geq \sum_{\substack{j=1 \\ j \neq k}}^{N} \lambda_j x^j ; y \leq \sum_{\substack{j=1 \\ j \neq k}}^{N} \lambda_j y^j ; \sum_{\substack{j=1 \\ j \neq k}}^{N} \lambda_j = 1; (j = 1,2,..., N; j \neq k)\}. \qquad (5)$$

In the general case of *N* firms with the observed input-output bundle (x^j, y^j) for firm *j (=1,2,…,N),* for each technically efficient firm *k,* we solve the following DEA problem for measuring the usual Debreu-Farrell super-efficiency of firm *k*:

$$\phi_k^- = \max \ \phi$$

$$\text{s.t.} \sum_{j \neq k} \lambda_j y^j \geq \phi y^k ;$$

$$\sum_{j \neq k} \lambda_j x^j \leq x^k ; \qquad (6)$$

$$\sum_{j \neq k} \lambda_j = 1; \lambda_j \geq 0 (j = 1,2,.., N; j \neq k).$$

The output bundle $y_k^- = \phi_k^- y^k$ is what the firm *k* needs to produce from the input bundle x^k in order to remain (output-oriented) technically efficient relative to the other firms in the sample. Thus, $\frac{1}{\varphi_k^-}$ is a measure of its *super-efficiency.* Two things may be noted. First, as noted before, if firm *k* is found to be technically inefficient in a conventional BCC DEA model, its exclusion from the reference set has no impact on its measured efficiency level. Thus, its super-efficiency measure is the same as its standard efficiency measure. Second, if the firm *k* is "extreme efficient" as defined by Charnes, Cooper, and Thrall (1994), $\varphi_k^- < 1$ and its super-efficiency is greater than 1. Hence, between two firms *i* and *j*, both technically efficient by the standard measure, *j* is ranked above *i,* if $\phi_j^- < \phi_i^-$.

The directional distance function for firm *k* with reference to the modified production possibility set is

$$\beta_k^- = \max \beta : ((1+\beta) y^k, (1-\beta) x^k) \in T_k^- . \qquad (7).$$

The relevant DEA model for computing the Nerlove-Luenberger super-efficiency of firm *k* is:

$$\begin{aligned} & \max \ \beta \\ \text{s.t.} \quad & \sum_{\substack{j=1 \\ j \neq k}}^{N} \lambda_j y^j - \beta y^k \geq y^k ; \\ & \sum_{\substack{j=1 \\ j \neq k}}^{N} \lambda_j x^j + \beta x^k \leq x^k ; \qquad (8) \\ & \sum_{\substack{j=1 \\ j \neq 1}}^{N} \lambda_j = 1; \\ & \lambda_j \geq 0 (j = 1,2,..., N; j \neq k); \beta \ \text{unrestricted.} \end{aligned}$$

If firm *k* is Nerlove-Luenberger super-efficient, $\beta_k^- < 0$ implying that the output bundle of the firm has to be scaled down while its input bundle is scaled up in order to get an attainable input-output bundle in the modified production possibility set. Between two firms both Nerlove-Luenberger super-efficient, the one with a lower (i.e., more pronouncedly negative) value of β is ranked higher in terms of super-efficiency.

4. The Problem of Infeasibility in Super-efficiency Models

It is apparent that this super-efficiency DEA problem in (6) is infeasible if all possible convex combinations of the input vectors of the remaining firms are weakly greater than x^k. A special case of this is one where any one of the element of the input bundle x^k is strictly smaller than the corresponding element of the input bundles of all the other firms in the sample.[2]

Note, however, that in most cases this does not pose any problem in a VRS DEA model of Nerlove-Luenberger super-efficiency. By selecting a negative value of β one can scale down the output bundle and scale up the input bundle of the firm under evaluation. Typically a negative optimal value of β yields a projection of (x^k, y^k) on to a non-negative input-output bundle that lies on the frontier of the modified production possibility set T_k^-. There are two exceptions, however.

First, if

$$2x^k < \sum_{\substack{j=1 \\ j \neq k}}^{N} \lambda_j x^j \text{ for all } \lambda_j\text{s satisfying}$$

$$\sum_{\substack{j=1 \\ j \neq k}}^{N} \lambda_j = 1; \lambda_j \geq 0 (j = 1,2,..., N; j \neq k),$$

one must set β at a value lower than –1 (i.e., more than double the input bundle x^k). But in the process the output bundle is rendered negative. Although, the relative magnitude of the optimal β, whether positive or negative, remains a valid criterion for ranking firms in terms of their super-efficiency, a negative output bundle at the efficient projection of (x^k, y^k) creates a conceptual problem.

The other case is one where at least one element of the input bundle of firm *k* is 0 and all other firms in the sample use strictly positive quantities of that input. For every β (whether positive negative, or zero) the corresponding element of the bundle $(1-\beta)x^k$ remains zero and the relevant input constraint in the problem (7) remains infeasible.

For reasons explained below, the method proposed by LR, by contrast, always yields a feasible solution of the relevant LP problem. This is true even when the firm under review is the only one in the sample with a 0 input of any factor. This, however, is a mere artifact of the way the model is constructed and the resulting super-efficiency measure has no economic meaning. Moreover, even though it does provide a super-efficiency measure of each firm, the problematic firms are all tied at the top.

The Lovell-Rouse Method:

Consider again the output-oriented DEA problem for Debreu-Farrell super-efficiency in (6) above. Now replace the output bundle of firm *k* by $\bar{y}^k = \delta y^k$ where $0 < \delta < 1$ is a scale factor that is determined in light of the sample data. Next solve the following conventional output-oriented BCC DEA problem for the surrogate of firm k with input-output bundles $(x^k, \bar{y}^k)$:

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[2] Analogous conditions for infeasibility of input-oriented BCC super-efficiency models can be found in the relevant literature (e.g., Harker and Xue (2002)).

$$\bar{\phi}_k = \max\ \phi$$

$$\text{s.t.} \sum_{j \neq k} \lambda_j y^j + \lambda_k \bar{y}^k \geq \phi \bar{y}^k;$$

$$\sum_{j \neq k} \lambda_j x^j + \lambda_k x^k \leq x^k; \qquad (9)$$

$$\sum_{j \neq k} \lambda_j + \lambda_k = 1; \lambda_j \geq 0 (j = 1,2,.., N); \phi \text{ unrestricted.}$$

In most situations, by selecting an appropriately small value of δ one can ensure that $(x^k, \bar{y}^k)$ is an inefficient input-output combination and, hence, λ_k equals 0 at the optimal solution of (9). Thus, $\bar{\phi}$ provides a measure of the (inverse) of the output-oriented Debreu-Farrell super-efficiency of a firm producing the output bundle $\bar{y}^k$ from the input bundle x^k. Hence, a measure of the corresponding output-oriented super-efficiency of the observed input-output pair is

$$\tau_O = \frac{\delta}{\bar{\phi}}. \qquad (10)$$

Note that the only time λ_k will be strictly positive at the optimal solution of (9) is when all convex combinations of the input vectors of the other firms in the sample are weakly greater than the observed input bundle of firm k. In this case, $\bar{\phi}$ equals unity and

$$\tau_O = \delta.$$

But this is the case where the input-attainability assumption fails for the output bundle y^k with reference to the modified production possibility set constructed from the input-output bundles of the sample firms other than k and the conventional DEA problem shown in (6) would be infeasible. As noted by LR, the optimal solution for both the standard output-oriented BCC DEA and the modified super-efficiency DEA for firm k would be unity and, as defined by Harker and Xue (2002), firm k is strongly super-efficient in this case.

For the comparable input-oriented problem we define the revised input vector $\bar{x}^k = \alpha x^k$ where $\alpha>1$ is an arbitrary scale factor. Next we solve the DEA problem:

$$\bar{\theta}_k = \min\ \theta$$

$$\text{s.t.} \sum_{j \neq k} \lambda_j y^j + \lambda_k y^k \geq y^k;$$

$$\sum_{j \neq k} \lambda_j x^j + \lambda_k \bar{x}^k \leq \theta \bar{x}^k; \qquad (11)$$

$$\sum_{j \neq k} \lambda_j + \lambda_k = 1; \lambda_j \geq 0 (j = 1,2,.., N); \ \theta \text{ unrestricted.}$$

The corresponding input-oriented super-efficiency measure is

$$\tau_I = \bar{\theta}\alpha. \qquad (12)$$

LR suggest selecting

$$\alpha = \max i\{\max x_{ij} / \min x_{ij}\} + 1$$

for the input-oriented model. Similarly, for the output-oriented approach, one may set

$$\delta = \left[\max r\{\max y_{rj} / \min y_{rj}\} + 1\right]^{-1}.$$

An advantage of this procedure is that because the bundle $(x^k, \bar{y}^k)$ is in the reference set of the problem in (9),a solution with $\bar{\phi}$ equal to unity will always be feasible for this problem. This is true, even when firm *k* is the only firm with a 0 level of any one input. Similarly, $(\bar{x}^k, y^k)$ is in the reference set and $\bar{\theta}$ equal to unity is always a feasible solution for the problem in (11). By contrast, as noted earlier, the Luenberger-Nerlove super-efficiency problem in (8) will not have a feasible solution in such cases.

There are two problems with this approach, however. First, whenever for any firm $\bar{\phi}$ equals unity, its output-oriented Debreu-Shephard super-efficiency equals $\frac{1}{\delta}$. Similarly, whenever $\bar{\theta}$ equals unity, the corresponding input-oriented super-efficiency equals α. Thus, this approach fails to provide a ranking of these strongly super-efficient firms. Secondly, and just as important, the super-efficiency measure is entirely determined by the arbitrarily chosen value of δ or α and has no meaningful economic interpretation.

A Comparison of the Two Approaches:

We use the following data from Seiford and Zhu (19978; Table 7) to illustrate the difference between the modified super-efficiency model due to RL and the Nerlove-Luenberger super-efficiency model proposed here.

Table 1. Hypothetical Input-output Data for 10 Firms

Name	x1	x2	x3	y1	y2
D1	182	237	468	5008	5303
D2	74	82	148	1857	2336
D3	160	195	400	4041	5001
D4	183	150	339	2779	2418
D5	133	155	329	3506	3602
D6	106	120	138	1306	956
D7	109	110	188	1515	2282
D8	240	243	806	7763	9601
D9	276	188	574	4577	6493
D10	191	117	466	3322	4233

Source: Seiford and Zhu (1998; Table 7)

The various super-efficiency measures for the firms shown in Table 1 are reported below in Table 2.

Table 2. Alternative Measures of Super Efficiency

name	Super-E_I	Super-E_O	MsuperE$_1$	MSuper-E_O	NL
D1	1.0626	1.0551	1.0626	1.0551	1.02847
D2	1.5277	infeasible	1.5277	11	1.44299
D3	0.9765	0.9796	0.9765	0.9796	0.98894
D4	0.7354	0.7617	0.7354	0.7617	0.85662
D5	0.9752	0.9777	0.9752	0.9777	0.98812
D6	1.0725	infeasible	1.0725	11	1.07246
D7	0.7852	0.8216	0.7852	0.8216	0.88626
D8	infeasible	1.6223	6	1.6223	1.38359
D9	0.9246	0.9224	0.9246	0.9224	0.95808
D10	1.0642	1.0811	1.0642	1.0811	1.03339

The column identified as Super-E_I shows the input-oriented Debreu-Shephard super-efficiency measures of the individual firms. Firms D1, D2, D6, and D10 are super-efficient whereas firm D8 without a feasible solution for the conventional input-oriented super-efficiency DEA problem is strongly super-efficient. MSuper-E_I shows the modified input-oriented super-efficiency obtained from the solution of problem (11). For this problem, we set α equal to 6. Note that, for the strong super-efficient firm D8, this measure equals 6. Super-E_O shows the conventional super-efficiency measures of the same 10 firms. Three firms, D1, D8, and D10 are super-efficient while firms D2 and D6 without feasible solutions for the conventional output-oriented super-efficiency DEA problem are strongly efficient. Both of these firms are assigned an output-oriented super-efficiency value 11 (equal to the inverse of the value chosen for δ) in the column showing the modified super-efficiency (MSuper-E_O). Finally, NL shows the levels of Nerlove-Luenberger super-efficiency (1 –β) for the different firms. Note that for the super-efficient firms, the optimal value of β is negative leading to measured super-efficiency levels exceeding unity. Unlike the modified super-efficiency measures (either input or output-oriented), however, the NL measures are not identical for the strongly super-efficient firms. For example, firms D2 and D6 cannot be ranked in order of output-oriented super-efficiency although D2 ranks way above D6 in terms of input-oriented super-efficiency. The Nerlove-Luenberger directional super-efficiency measure clearly ranks D2 higher than D6.

The more important point to note is that the unlike the RL modified super-efficiency measures, the NL directional super-efficiency measures can be easily interpreted. For example, firm D2 could increase all of its inputs and at the same time reduce all of its outputs by about 44.3% without becoming inefficient relative to the other firms in the sample. Firm D6 could similarly scale up its input bundle and scale down its output bundle by 7% and still remain efficient. Their, modified output-oriented super-efficiency rating of 1100% does not mean that they actually produce 11 times what would be minimally required for them to retain an output-oriented technical efficiency of 100%. A different choice of the scale factor β would yield a different super-efficiency rating.

5. An Application to Airlines Data:

This example considers the performance of 28 international airlines from North America, Europe, and Asia-Australia during the year 1990. The data set is taken from Coelli, Grifell-Tatje, and Perelman (2002, Table 1). Each firm produces two outputs: (a) passenger-kilometers flown *(PASS)* and (b) freight tonne-kilometers flown *(CARGO)* . Inputs used are: (i) number of employees *(LAB)* , (ii) fuel measured in millions of gallons) (*FUEL)* , (iii) other inputs (millions of U.S. dollar equivalent) consisting of operating and maintenance expenses excluding labor and fuel expenses, *(MATERIAL)* and (iv) capital (sum of the maximum take-off weights of all aircrafts flown multiplied by the number of days flown) *(CAP)* . The input-output data set is shown in Table 3. Various super-efficiency measures are reported for these firms in Table 4. For the modified input-oriented problem we set α equal to 39.4. The optimal value of $\overline{\theta}$ equals unity for 3 airlines (*LUFTHANSA, AMERICAN,* and *UNITED*). Each of these strongly super-efficient firms is assigned a LR input-oriented super-efficiency score of 39.4. From the entries in the column identified as "LR-inp", we find that 12 other firms (*JAL, SAUDIA, SINGAPORE, AUSTRIAN, FINNAIR, SWISSAIR, PORTUGAL, NORTHWEST, PANAM,* and *TWA*) are also super-efficient. Note that while these 13 firms can be ranked in order of super-efficiency, the earlier 3 are all tied at 39.4. For the output-oriented LR

Table 3. Input-Output Data from Selected Airlines for the year 1990

Obs	NAME	PASS	CARGO	LAB	FUEL	MATL	CAP
1	NIPPON	35261	614	12222	860	2008	6074
2	CATHAY	23388	1580	12214	456	1492	4174
3	GARUDA	14074	539	10428	304	3171	3305
4	JAL	57290	3781	21430	1351	2536	17932
5	MALAYSIA	12891	599	15156	279	1246	2258
6	QUANTAS	28991	1330	17997	393	1474	4784
7	SAUDIA	18969	760	24708	235	806	6819
8	SINGAPORE	32404	1902	10864	523	1512	4479
9	AUSTRIA	2943	65	4067	62	241	587
10	BRITISH	67364	2618	51802	1294	4276	12161
11	FINNAIR	9925	157	8630	185	303	1482
12	IBERIA	23312	845	30140	499	1238	3771
13	LUFTHANSA	50989	5346	45514	1078	3314	9004
14	SAS	20799	619	22180	377	1234	3119
15	SWISSAIR	20092	1375	19985	392	964	2929
16	PORTUGAL	8961	234	10520	121	831	1117
17	AIR CANADA	27676	998	22766	626	1197	4829
18	AM. WEST	18378	169	11914	309	611	2124
19	AMERICAN	133796	1838	80627	2381	5149	18624
20	CANADIAN	24372	625	16613	513	1051	3358
21	CONTINENTAL	69050	1090	35661	1285	2835	9960
22	DELTA	96540	1300	61675	1997	3972	14063
23	EASTERN	29050	245	21350	580	1498	4459
24	NORTHWEST	85744	2513	42989	1762	3678	13698
25	PANAM	54054	1382	28638	991	2193	7131
26	TWA	62345	1119	35783	1118	2389	8704
27	UNITED	131905	2326	73902	2246	5678	18204
28	USAIR	59001	392	53557	1252	3030	8952

Source: Coelli, Griffel-Tatje, and Perelman (2002), Table 1.

Table 4. Alternative Measures of Super-efficiency

Obs	NAME		LR-inp		LR-out	beta	NL
1	NIPPON	0.0251	0.9888	83.917	0.993	0.00431	0.99569
2	CATHAY	0.02337	0.9209	91.977	0.906	0.04486	0.95514
3	GARUDA	0.01883	0.7419	117.943	0.7066	0.15919	0.84081
4	JAL	0.04432	1.7463	60.028	1.3882	-0.20343	1.20343
5	MALAYSIA	0.01965	0.7741	109.681	0.7598	0.13417	0.86583
6	QUANTAS	0.02876	1.133	74.184	1.1233	-0.06009	1.06009
7	SAUDIA	0.02954	1.1638	70.362	1.1844	-0.07982	1.07982
8	SINGAPORE	0.03692	1.4546	56.174	1.4835	-0.17072	1.17072
9	AUSTRIAN	0.0617	2.4309	1	83.3333	-1.4309	2.4309
10	BRITISH AIR	0.02263	0.8915	91.925	0.9065	0.05287	0.94713
11	FINNAIR	0.03519	1.3865	46.007	1.8113	-0.20747	1.20747
12	IBERIA	0.02008	0.7912	103.563	0.8047	0.11307	0.88693
13	LUFTHANSA	1	39.4	39.5	2.1097	-0.42583	1.42583
14	SAS	0.02218	0.874	94.455	0.8823	0.06481	0.93519
15	SWISSAIR	0.02763	1.0885	75.567	1.1028	-0.04539	1.04539
16	PORTUGAL	0.03115	1.2275	65.33	1.2756	-0.11081	1.11081
17	AIR CANADA	0.02348	0.9252	89.583	0.9302	0.03745	0.96255
18	AMER WEST	0.03031	1.1943	68.972	1.2082	-0.09132	1.09132
19	AMERICAN	1	39.4	75.325	1.1063	-0.05345	1.05345
20	CANADIAN	0.023	0.9061	91.155	0.9142	0.04693	0.95307
21	CONTINENTAL	0.02542	1.0015	83.22	1.0014	-0.00071	1.00071
22	DELTA	0.02394	0.9433	88.165	0.9452	0.02867	0.97133
23	EASTERN	0.02118	0.8344	98.992	0.8418	0.08802	0.91198
24	NORTHWEST	0.02582	1.0175	82.134	1.0146	-0.00789	1.00789
25	PANAM	0.02612	1.029	81.124	1.0272	-0.01384	1.01384
26	TWA	0.02561	1.0088	82.618	1.0087	-0.00436	1.00436
27	UNITED	1	39.4	77.882	1.07	-0.05154	1.05154
28	USAIR	0.02221	0.8752	94.429	0.8825	0.06442	0.93558

problem, we set δ equal to 0.012. This time, only one airline (*AUSTRIAN*) was strongly super-efficient and was assigned an LR super-efficiency score of 83.3333. Of the rest , 15 airlines (*JAL, QUANTAS, SAUDIA, SINGAPORE, FINNAIR, LUFTHANSA, SWISSAIR, PORTUGAL, AMER WEST, AMERICAN, CONTINENTAL, NORTHWEST< PANAM, TWA,* and *UNITED*) were super-efficient. The modified super-efficiency scores are reported in the column "LR-out". Finally, the column "beta" reports the directional distance function values and the Nerlove-Luenberger super-efficiency measures are shown in the column "NL". For one airline (*AUSTRIAN)* the directional distance function (β) is lower than -1. As a result, at the projection of the observed input-output bundle onto the modified frontier, the output bundle would be negative. This clearly, is problematic. For the other airlines, however, the results are quite sensible and can be easily interpreted. The NL super-efficiency of 16 firms permits a completed ordering of all of these firms. Note that while the RL input-oriented measure results in a tie for *LUFTHANSA, AMERICAN,* and *UNITED,* the NL measure ranks *LUFTHANSA* way above the other two firms while *AMERICAN* barely dominates *UNITED.* It is interesting to note that the LR-output oriented super-efficiency would generate the same ranking of these firms. The fact that the directional distance function combines features of both an input-oriented and an output-oriented model, generally leads to a more complete ranking of the observations than either of the oriented models. An added advantage of this approach is that the NL super-efficiency measure is unique and does not depend on any arbitrary choice of a scaling parameter.

6. Conclusion:

A radial DEA models of super-efficiency has no feasible solution if Shephard's input (output) attainability assumption is violated. The method proposed by LR ensures a feasible solution of the appropriately modified problem. However, even under this procedure firms that are strongly super-efficient are all tied at the maximum score. Moreover, the super-efficiency score of these firms depends on the arbitrary choice of the scaling parameter and cannot be interpreted. By contrast, the NL super-efficiency scores obtained from the direction distance function are unique, easily interpreted, and yield a complete ranking of firms in the sample. There are two limitations of this approach, however. First, no feasible solution is obtained if the firm under evaluation has any input at the zero level. Second, when the NL super-efficiency score exceeds 2, the projected point in the input-output space involves negative output quantities.

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