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# Explicit and Implicit Methods In Solving Differential Equations

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# Explicit and Implicit Methods In Solving Differential Equations

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**Abstract:**

Differential equations are equations that involve an unknown function and derivatives. There will be times when solving the exact solution for the equation may be unavailable or the means to solve it will be unavailable. At these times and most of the time explicit and implicit methods will be used in place of exact solution. In the simpler cases, ordinary differential equations or ODEs, the forward Euler's method and backward Euler's method are efficient methods to yield fairly accurate approximations of the actual solutions. By manipulating such methods, one can find ways to provide good approximations compared to the exact solution of parabolic partial differential equations and nonlinear parabolic differential equations. Further, the experimental results show that smaller h values result in reduction in error which is true in both cases of differential equations.

## Introduction

Differential equations are equations that involve an unknown function and derivatives. Braun, Golubitsky, Sirovich and Jager (1992) defined differential equation as the *equation relates a function to its derivatives in such a way that the function itself can be determin*ed. Vrabie (2004) indicated that mathematicians had realized that many differential equations cannot be solved using explicitly. The Euler Implicit method was identified as a useful method to approximate the solution. In other cases, ordinary differential equations or ODEs, the forward Euler's method and backward Euler's method are also efficient methods to yield fairly accurate approximations of the actual solutions. This study attempts to show that by manipulating explicit and implicit methods, one can find ways to provide good approximations compared to the exact solution of parabolic partial differential equations and nonlinear parabolic differential equations. Furthermore, the result of h values, step size, is also part of the discussion in error reduction in both cases of differential equations.

**Explicit and Implicit Methods in Solving Differential Equations**

A differential equation is also considered an ordinary differential equation (ODE) if the unknown function depends only on one independent variable. Frequently exact solutions to differential equations are unavailable and numerical methods become necessary to yield fairly accurate approximations of the actual solutions. Bronson and Costa (2006) discussed the concept of qualitative methods regarding differential equations; that is, techniques which are used when analytical solutions are difficult or virtually impossible to obtain.

Let us take as an example an initial value problem in ODE

$$y_t = f(t, y), \;\; y(t_0) = y_0$$

where f is a given smooth function. There are numerical methods that provide quantitative information about solutions even if formulas or the exact solution cannot be found. Most of the time the work done by using numerical methods can be performed by the machine. However numerical investigations possess limitations in that only approximations can be obtained and a finite number of initial conditions can be experimented. Groisman (2005) took a similar numerical approximation approach and utilized totally discrete explicit and semi-implicit Euler methods to explore problem in several space dimensions.

The forward Euler's method is one such numerical method and is explicit. Explicit methods calculate the state of the system at a later time from the state of the system at the current time without the need to solve algebraic equations. For the forward (from this point on forward Euler's method will be known as forward) method, we begin by choosing a step size or $\Delta t$. The size of $\Delta t$ determines the accuracy of the approximate solutions as well as the number of computations . Graphically this method produces a series of line segments, which thereby approximates the solution curve.

Let $t_k$, $k = 0, 1, 2, \ldots$, be a sequence in time with

$$t_{k+1} = t_k + \Delta t.$$

Let $y_k$ and $Y_k$ be the exact and the approximate solution at $t = t_k$, respectively. To obtain $Y_{k+1}$ from $(t_k, Y_k)$, we use the differential equation. Since the slope of the solution to the equation $y_t = f(t, y)$ at the point $(t_k, y_k)$ is $f(t_k, y_k)$, the Euler method determines the point $(t_{k+1}, Y_{k+1})$ by assuming that it lies on the line through $(t_k, Y_k)$ with the slope $f(t_k, Y_k)$. Hence the formula for the slope of a line gives

$$\frac{Y_{k+1} - Y_k}{\Delta t} = f(t_k, Y_k)$$

or

$$Y_{k+1} = Y_k + f(t_k, Y_k)\Delta t.$$

As the step size or $\Delta t$ decreases then the error between the actual and approximation is reduced. Roughly speaking we halve the error by halving the step size in this case. However, halving the $\Delta t$ doubles the amount of computation.

The backward Euler's method is an implicit one which contrary to explicit methods finds the solution by solving an equation involving the current state of the system and the later one.

More precisely we have

$$Y_{k+1} = Y_k + f(t_k, Y_{k+1})\Delta t.$$

This disadvantage to using this method is the time it takes to solve this equation. However, advantages to this method include that they are usually more numerically stable for solving a stiff equation a larger step size $\Delta t$ can be used.

Let us take following initial value problem

$$y' + 2y = 2 - e^{-4t}, \quad y(0) = 1, \qquad 0 \le t \le 0.5,$$

we will use forward and the backward Euler's method to approximate the solution to this problem and these approximations to the exact solution

$$y(t) = 1 + 0.5 * e^{-4t} - 0.5 * e^{-2t}.$$

In both methods we let $\Delta t = 0.1$ and the final time $t_f = 0.5$.

**Table** 1a. Forward and Backward Euler's Method Compared To Exact Solution

| $t_n$ (time) | $Y_n$, forward Euler's approximation | $Y_n$, backward Euler's approximation | $y_n$, exact | $\lvert e_i \rvert$ error forward | $\lvert e_i \rvert$ error backward |
|---|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 0 | 0 |
| 0.1 | 0.9 | 0.9441 | 0.925795 | 0.025795 | 0.018305 |
| 0.2 | 0.853 | 0.916 | 0.889504 | 0.036504 | 0.026496 |
| 0.3 | 0.8374 | 0.9049 | 0.876191 | 0.03791 | 0.028709 |
| 0.4 | 0.8398 | 0.9039 | 0.876191 | 0.036391 | 0.027709 |
| 0.5 | 0.8517 | 0.9086 | 0.883728 | 0.032028 | 0.024872 |

The $|e_i|$ error averages were also computed for both methods and the result was for the average error for forward Euler's method was 0.028105 and the average error for the backward Euler's method was 0.021015. As it can be seen in both the chart above and the $|e_i|$ error averages that the backward Euler's method seems to be the more accurate between the methods.

The solution at $t_n = 0.5$ was approximated using the forward Euler's method (Fi. A) and backward Euler's method (Fig. B) with $\Delta t = 0.1$, $\Delta t = 0.05$, $\Delta t = 0.0125$, $\Delta t = 0.00625$. The solutions for these four conditions varying h were compared by taking the absolute difference against the exact solution at that point.

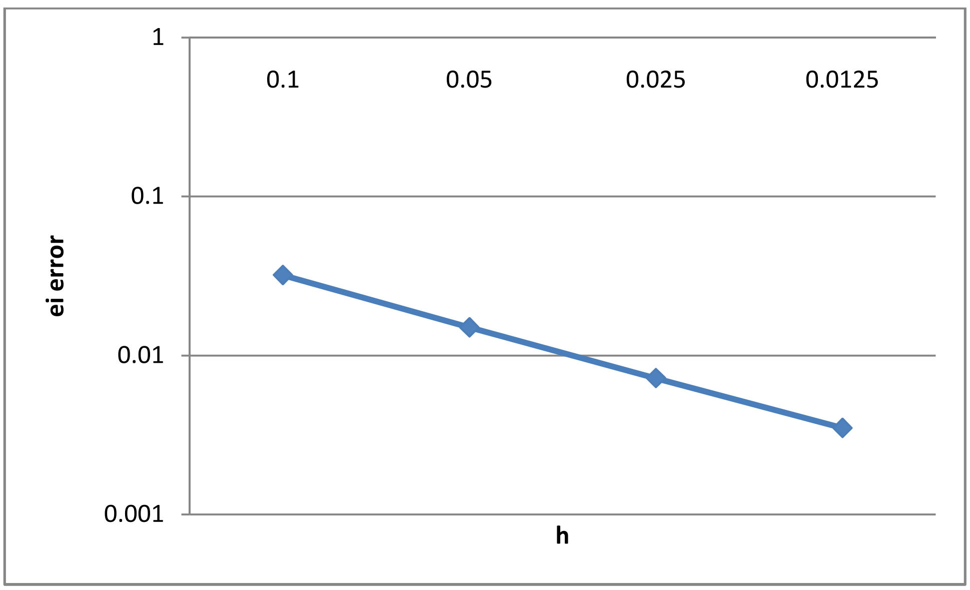


**Figure A**. Plot of $e_i$ error varying h at =0.5 and $t_n$=0.5 with the Forward Euler's method

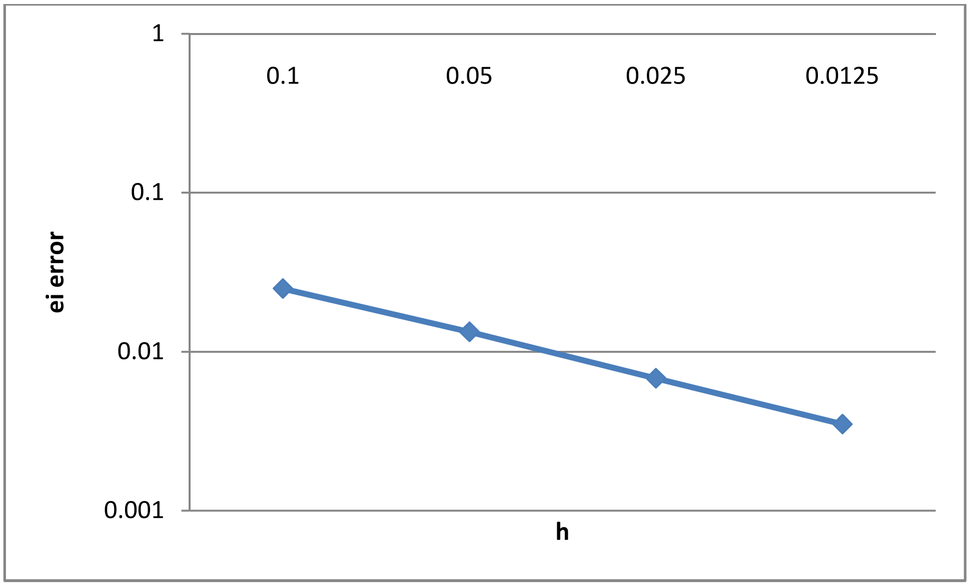


**Figure B**. Plot of $e_i$ error varying h at =0.5 and $t_n$=0.5 with the Backward Euler's method

The slope of Fig. A is approximately 1.06, which means that as the $\Delta t$ decreases by half the accuracy of the analytical solution of the forward method changes by that factor. The slope of Fig. B is approximately 0.946; as $\Delta t$ is halved in the case of the backward method the accuracy of the solution is improved by a smaller degree than that of the forward method.

Next we tackle the parabolic partial-differential equation. The simplest example is the heat conduction equation given by

$$\frac{\partial u}{\partial t}(x,t) = \alpha^2 \frac{\partial^2 u}{\partial x^2}(x,t), \quad 0 < x < l, \quad t > 0, \quad (a)$$

with boundary conditions

$$u(0,t) = 0, \quad u(l,t) = 0, \quad t > 0,$$

And initial conditions

$$u(x,0) = f(x), \quad 0 \le x \le l.$$

One approach used to solve such a problem involves finite differences. First select a spatial mesh constant h and a time step size k with stipulation that $m = l/h$ being an integer. Hence a typical point in the domain is given by $(x_i, t_j) = (i_h, j_k)$

We obtain the difference method by using the Taylor series in $t$ to form the difference quotient

$$\frac{\partial u}{\partial t}(x_i, t_j) = \frac{u(x_i, t_j + k) - u(x_i, t_j)}{k} - \frac{k}{2}\frac{\partial^2}{\partial t^2}u(x_i, \mu_j), \qquad (b)$$

for some $\mu_j \in (t_j, t_{j+1})$, and the Taylor series in $x$ to form the difference quotient

$$\frac{\partial^2 u}{\partial x^2}(x_i, t_j) = \frac{u(x_i + h, t_j) - 2u(x_i, t_j) + u(x_i - h, t_j)}{h^2}$$

$$-\frac{h^2}{12}\frac{\partial^4 u}{\partial x^4}(\xi_i, t_j), \qquad (c)$$

where $\xi_i \epsilon (x_{i-1}, x_{i+1})$. Thus we employ

$$\frac{w_{i,j+1} - w_{i,j}}{k} - \alpha^2 \frac{w_{i+1,j} - 2w_{i,j} + w_{i-1,j}}{h^2} = 0, \ \ (d)$$

as a numerical scheme to solve equation (a), where $w_{ij}$ approximates $u(x_i, t_j)$. We note that $w_{i,0}$ is given by the initial condition $f(ih)$, and $w_{0,j} = w_{m,j} = 0$ due to the boundary conditions.

Which simplifies to

$$w_{i,j+1} = \left(1 - \frac{2\alpha^2 k}{h^2}\right) w_{i,j} + \alpha^2 \frac{k}{h^2}\left(w_{i+1,j} + w_{i-1,j}\right), \quad (e)$$

for each $i = 1, 2, \ldots, (m-1)$ and $j = 1, 2, \ldots$. With known initial and boundary conditions for w, one can solve for $w_{i,j}$ for successively larger j.

If we let $\lambda = \alpha^2(\frac{k}{h^2})$,

$$\boldsymbol{w}^{(0)} = (f(x_1), f(x_2), \ldots, f(x_{m-1}))^t,$$

$$\boldsymbol{w}^{(j)} = (w_{1,j}, w_{2,j}, \ldots, w_{m-1,j})^t,$$

for each $j = 1, 2, \ldots$, and $A = \begin{bmatrix} (1-2\lambda) & \lambda & 0 \ldots\ldots\ldots. & 0 \\ \lambda & (1-2\lambda) & \lambda \ldots\ldots\ldots & \ldots\ldots. \\ 0 & .. & .. & 0 \\ .. & .. & .. & \lambda \\ 0 & 0 & \lambda & (1-2\lambda) \end{bmatrix}$, then (e) is equivalent to

$$\boldsymbol{w}^{(j)} = A\boldsymbol{w}^{(j-1)},$$

for each $j = 1, 2, \ldots$. There is no need to solve a system of an algebraic equation for $\vec{w}^{(j)}$ as this forward difference method is explicit. The forward difference method is the result of a modification to the Forward Euler's method. There are stability requirements that must be met in order for method to yield accurate solutions, namely

$$\lambda = \alpha^2 \frac{k}{h^2} = \alpha^2 \frac{\Delta t}{\Delta x^2} \leq \frac{1}{2}$$

As long as this inequality holds, this method is numerically stable and solution $\vec{w}^{(j)}$ remains bounded even when $\Delta t$ and $\Delta x$ go to zero. Once this stability condition is met, $w_{i_k,j_k}$ converges to $u(x_i, t_j)$ with a note of convergence $O(k + h^2)$.

Forward-difference method was tested using the following example. Consider the example with $\alpha^2 = 1$, $l = 1$ and $u(x, 0) = \sin \pi x$. The exact solution is given by $u(x,t) = e^{-\pi^2 t} \sin(\pi x)$.

The solution at $t = 0.5$ was then approximated using the forward difference method first with $h = 0.1$, $h = 0.05$, $h = 0.0125$, $h = 0.00625$ successively while keeping $\lambda = 0.05$ by adjusting $k$. The solutions for these four conditions varying h were compared at $x_i = 0.5$ by taking the absolute difference against the exact solution at that point.

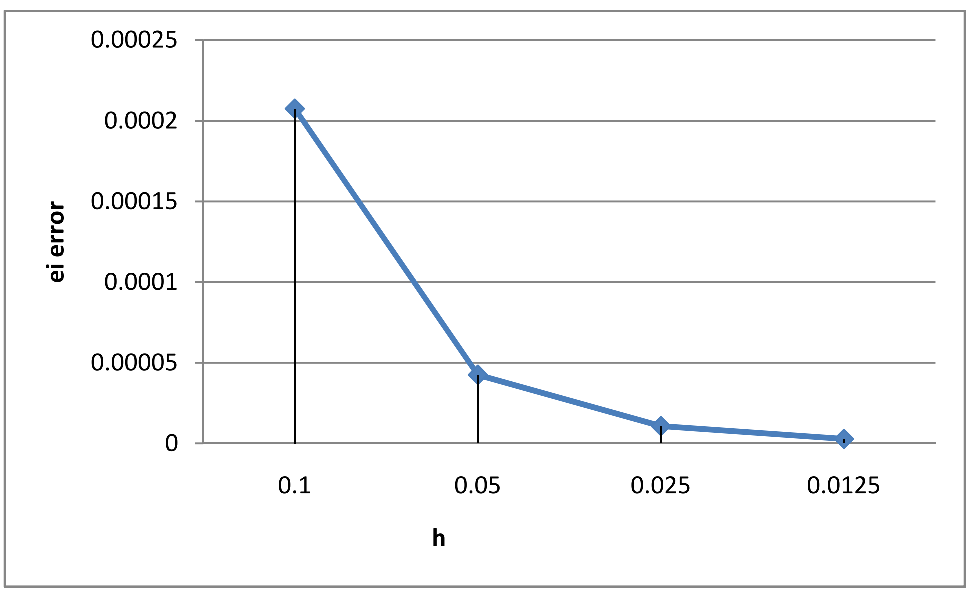


**Figure 1**. Plot of $e_i$ error varying h at xi=0.5 and t=0.5 with the Forward-Difference method

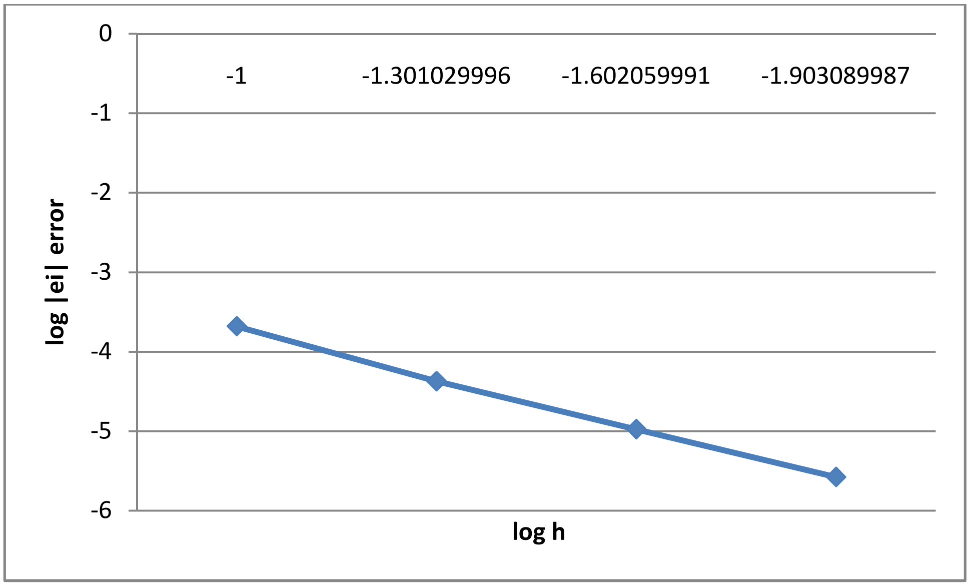


**Figure 2**. Logarithmic plot of Figure 1

A straight line plot can be observed in Fig. 2 with a slope of approximately 2. As the value of h decreases so does the error.

Another way to compare the data is to take the $l2$ error, which is $\sqrt{\frac{1}{m}\sum_{i=1}^{m-1} e_i^2}$.

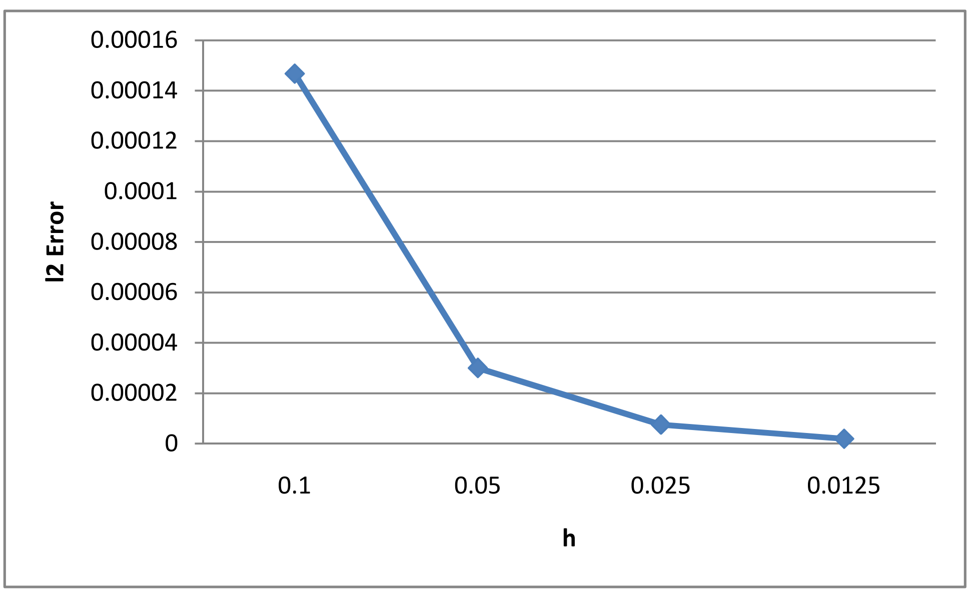


**Figure 3**. Plot of value of h with respect to L2 error with Forward-Difference method

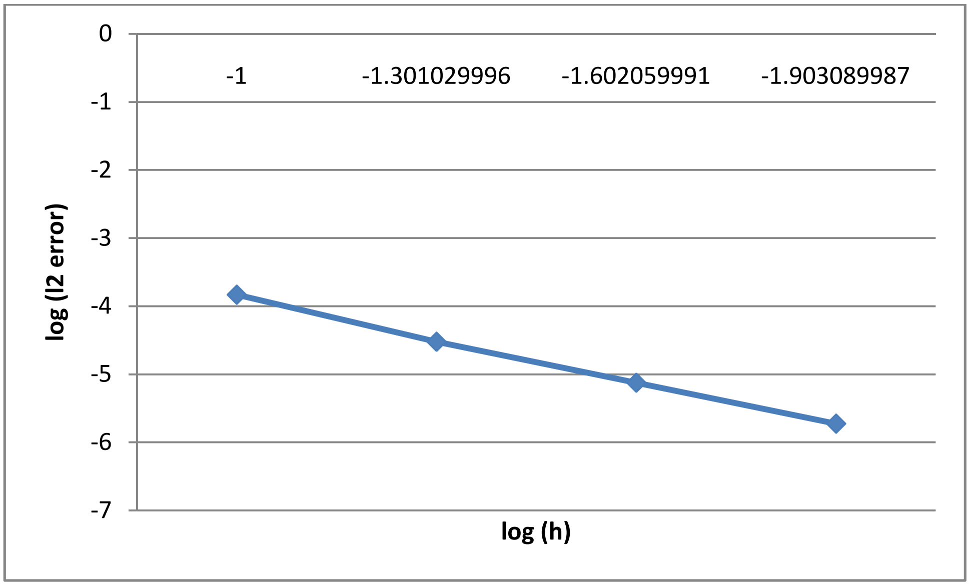


**Figure 4**. Logarithmic plot of Figure 3

The slope of the line featured in Fig. 4 is also about 2.

The forward-difference method limits the conditions we put on $\Delta t$ and $\Delta x$, so we need another method that is unconditionally stable. Consider an implicit-difference method which uses a backward-difference quotient for $\left(\frac{\partial u}{\partial t}\right)(x_i, t_j)$ in the form

$$\frac{\partial u}{\partial t}(x_i, t_j) = \frac{u(x_i, t_j) - u(x_i, t_{j-1})}{k} + \frac{k}{2}\frac{\partial^2 u}{\partial t^2}(x_i, \mu_j),$$

where $\mu_j$ is in $(t_{j-1}, t_j)$. Substituting this equation, together with equation (c) for $\frac{\partial^2 u}{\partial x^2}$, into the partial differential equation yields the following:

$$\frac{u(x_i,t_j)-u(x_i,t_{j-1})}{k}-\alpha^2\frac{u(x_{i+1},t_j)-2u(x_i,t_j)+u(x_{i-1},t_j)}{h^2}$$

$$=-\frac{k}{2}\frac{\partial^2 u}{\partial t^2}(x_i,\mu_j)-\alpha^2\frac{h^2}{12}\frac{\partial^4 u}{\partial x^4}(\xi_i,t_j),$$

for some $\xi_i\epsilon(x_{i-1},x_{i+1})$. All these calculations, suggests a numerical scheme

$$\frac{w_{i,j}-w_{i,j-1}}{k}-\alpha^2\frac{w_{i+1,j}-2w_{i,j}+w_{i-1,j}}{h^2}=0$$

for each $i=1,2,\dots,m-1$ and $j=1,2,\dots$, where $w_{i,j}$ is an approximation to $u(x_i,t_j)$.

As we did with the forward-difference method, let $\lambda=\alpha^2\frac{k}{h^2}$, the backward difference method becomes

$$(1+2\lambda)w_{i,j}-\lambda w_{i+1,j}-\lambda w_{i-1,j}=w_{i,j-1},$$

for each $i=1,2,\dots,m-1$ and $j=1,2,\dots$. Knowing that $w_{i,0}=f(x_i)$, for each $i=1,2,\dots,m-1$ and $w_{m,j}=w_{0,j}=0$, for each $j=1,2,\dots$, this difference method, just like the forward-difference, also has a matrix representation:

$$\begin{bmatrix}(1+2\lambda) & -\lambda & 0\dots & 0\\ -\lambda & \dots & \dots & .\\ 0 & \dots & \dots & 0\\ . & \dots & \dots & -\lambda\\ 0 & \dots & -\lambda & (1+2\lambda)\end{bmatrix}\begin{bmatrix}w_{1,j}\\ w_{2,j}\\ .\\ w_{m-1,j}\end{bmatrix}=\begin{bmatrix}w_{1,j-1}\\ w_{2,j-1}\\ .\\ w_{m-1,j-1}\end{bmatrix},\quad (f)$$

or

$$B\boldsymbol{w}^{(j)}=\boldsymbol{w}^{(j-1)}, for\ each\ j=1,2,\dots.$$

We now test this method using the same example as used previously with the forward-difference method.

Let us consider again the heat equation

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0, \;\; 0 < x < 1, \;\; 0 \le t,$$

with the boundary conditions

$$u(0,t) = u(1,t) = 0, \;\; 0 < t,$$

and the initial conditions

$$u(x,0) = \sin(\pi x)\,, \;\; 0 \le x \le 1.$$

The solution at $t = 0.5$ was approximated using the backward-difference method with $h = 0.1$, $h = 0.05$, $h = 0.0125$, $h = 0.00625$; just as the example was tested with the Forward-Difference method, $\lambda$ was kept constant at 0.05. The solutions for these four cases were compared at $x_i = 0.5$ by taking the absolute difference against the exact solution at that point.

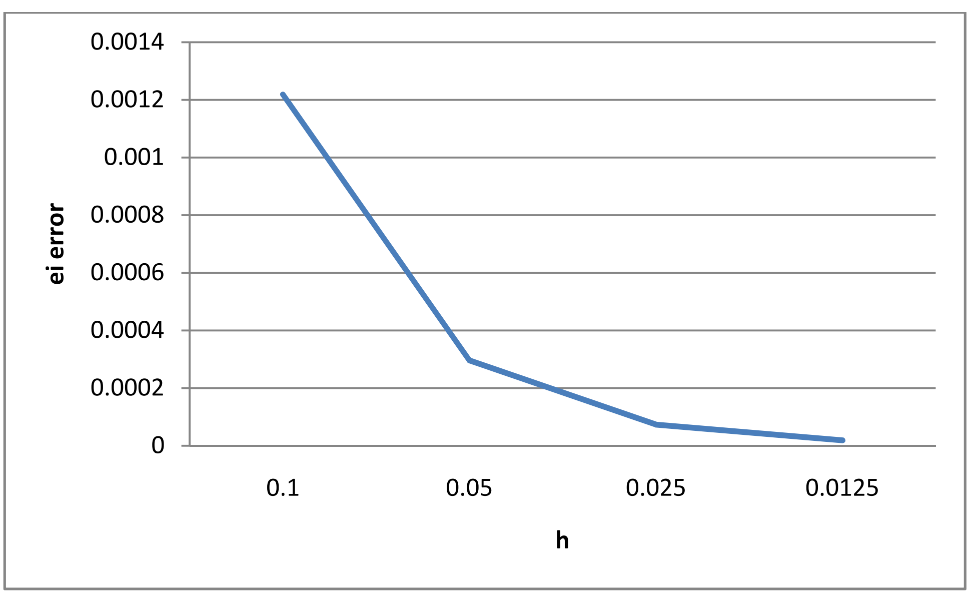

**Figure 5**. Plot of $e_i$ error at xi=0.5 and t=0.5 for various h using backward-difference method

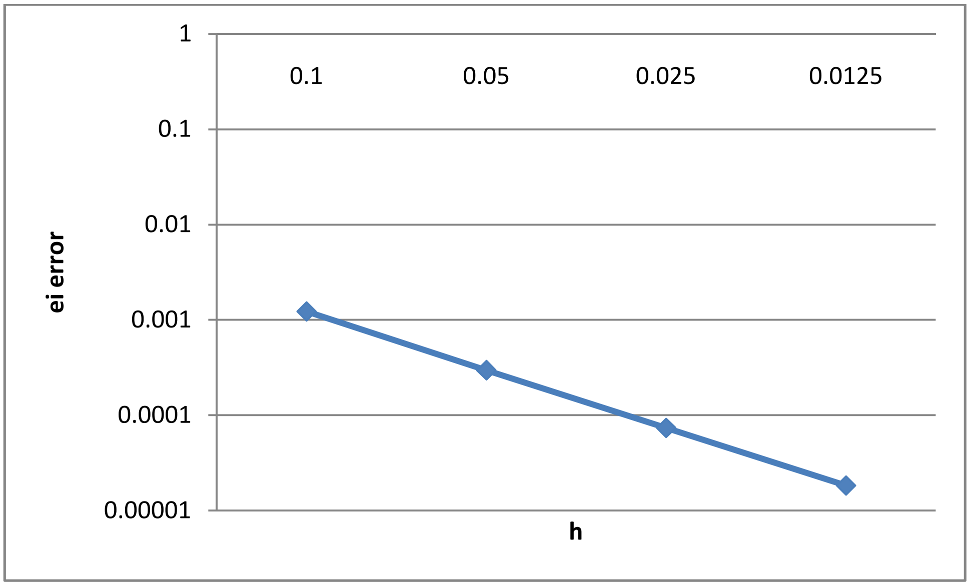


**Figure 6**. Logarithmic plot of Figure 5

The logarithmic plot of the $e_i$ error (Fig. 6) shows that a slope of about 2; this value is approximately close to the value of the slope in Fig. 2. So similarly to the forward-difference method, the backward-difference decreases in error as h decreases.

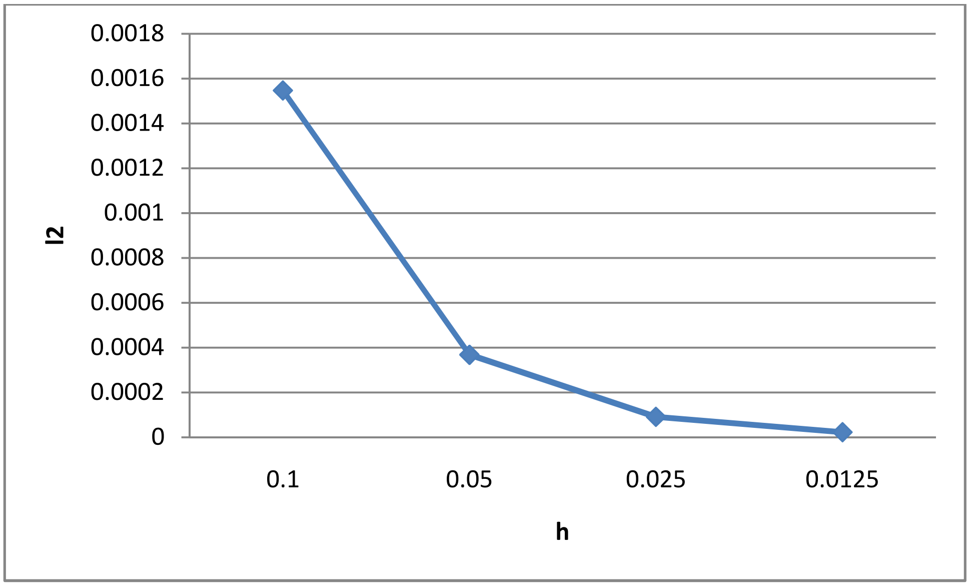


**Figure 7**. Plot of L2 error varying h at xi=0.5 and t=0.5 with Backward-Difference method

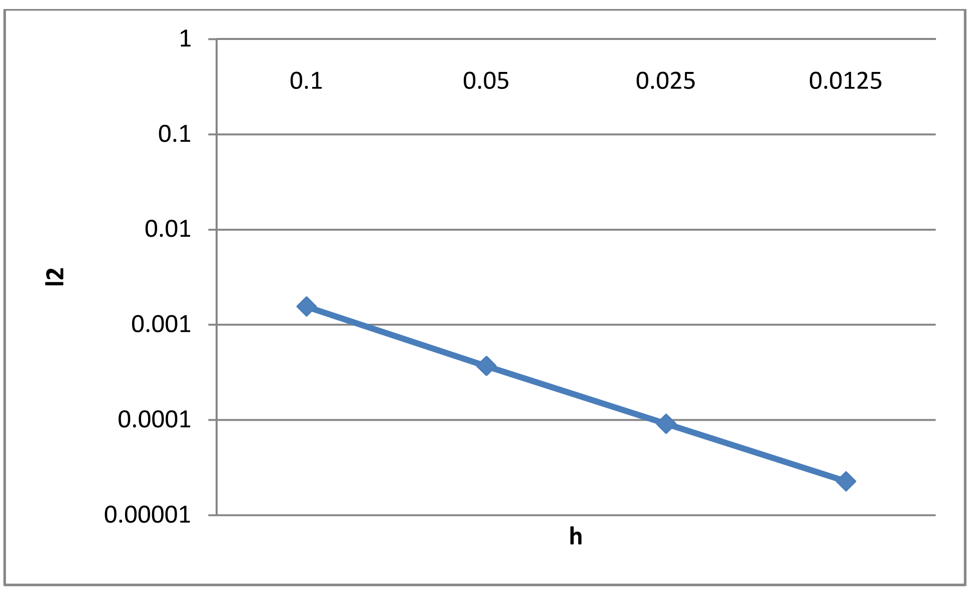


**Figure 8**. Logarithmic plot of Figure 7

Both line plots Figure 6 and Figure 8, which are plots of $e_i$ error and l2 error respectively have slope of approximately 2. The slopes are comparatively similar to the plots constructed to analyze the forward difference method, so as the value of h decreases so does the error.

In many applications the governing parabolic differential equations are nonlinear. The numerical methods for solving such equations will be based on that for the linear equations. Let us consider now the nonlinear parabolic differential equation:

$$u_t = u_{xx} + u(1-u) + g(x,t)$$

with boundary conditions $u(0,t) = u(1,t) = 0$ and some given initial conditions.

Using forward Euler's method as a guide, if $w_i^k \cong u(i\Delta x, k\Delta t)$, then we employ

$$\frac{w_i^{k+1} - w_i^k}{\Delta t} = \frac{w_{i+1}^k - 2w_i^k + w_{i-1}^k}{(\Delta x)^2} + g(x_i, t_k) + w_i^k\left(1 - w_i^k\right).$$

Next, we can rearrange the terms and utilize the tridiagonal A matrix as previously defined before to rewrite the equation as:

$$A\boldsymbol{w}^{(k+1)} = \boldsymbol{w}^{(k)} + \boldsymbol{g}^{(k)} + \begin{pmatrix} \vdots \\ u^k(1-u^k) \\ \vdots \end{pmatrix}$$

Where $g^{(k)} = \begin{pmatrix} g(x_1, t_k) \\ \vdots \\ g(x_{n-1}, t_k) \end{pmatrix}$. Let us now apply this to an example, in which we know the exact solution. Consider the following example:

$$u_t = u_{xx} + x(1-x)(e^t - 1)[1 - x(1-x)], \quad u(0,t) = u(1,t) = 0$$

with the boundary conditions and initial condition $u(x,0) = 0$. We vary h or $\Delta\mathrm{x}$ and choose $\Delta t$ such that $\lambda = \frac{\Delta t}{(\Delta \mathrm{x})^2} = \frac{1}{4}$. The numerical solution at $t = 0.5$ were obtained when $h = 0.1$, $h = 0.05$, and $h = 0.0125$. These solutions were compared also with the actual solution which can be calculated using:

$$u_{exact} = x(1-x)(e^t - 1).$$

The results are reported in the following table:

**Table** 2. A comparison of the exact and the numerical solution $w$ at $(x, t) = (0.5, 0.5)$

| h= | w | $u_{exact}$ | $e_i$ error |
|---|---|---|---|
| 0.1 | 0.461279 | 0.42957 | 0.031709 |
| 0.05 | 0.459834 | 0.42957 | 0.030264 |
| 0.025 | 0.459473 | 0.42957 | 0.029903 |

The results in the Table 2 show that as smallest h step used results in a smallest ei error compared to the other h steps.

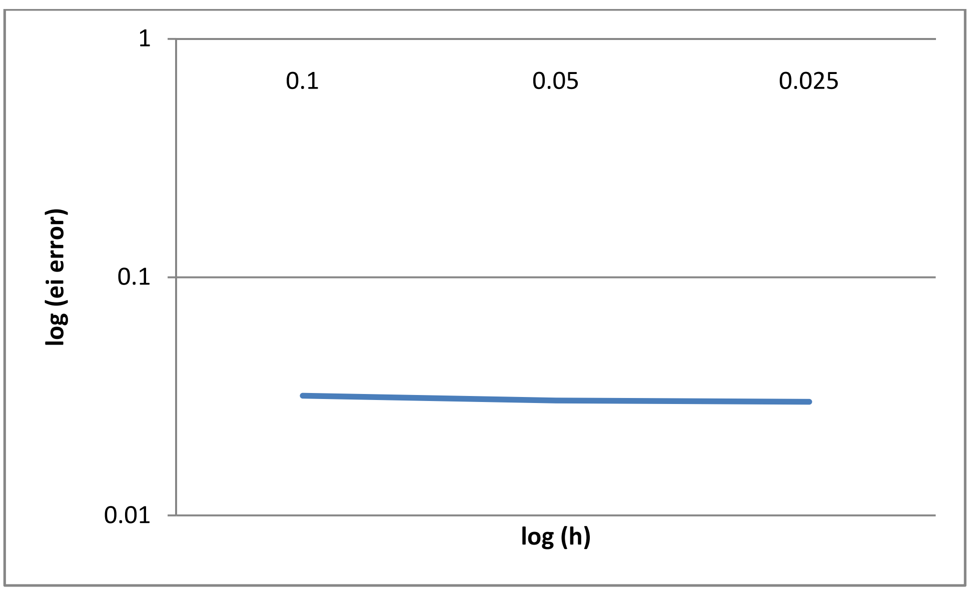


**Figure 9**. Logarithmic Plot of $e_i$ error for various h using the nonlinear forward-difference method

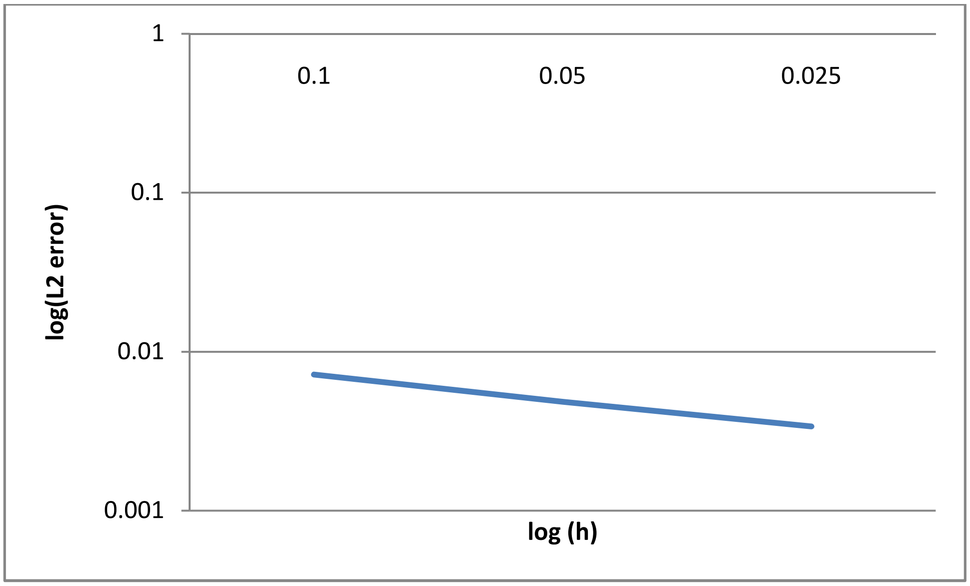


**Figure 10**. Logarithmic Plot of l2 error for various h using the nonlinear forward-difference method

The slope in Fig. 9 is about 0.04 and in Fig. 10 about 0.05. It shows that halving the h value does little to affect the error whether it is $e_i$ or L2. Strangely enough the error does not decrease at a faster rate when the h step is halved as seen with the parabolic partial differential equations or the ordinary differential equations.

## Conclusion/Future Work

This study showed that by manipulating explicit and implicit methods, one can successfully provide good approximations compared to the exact solution of parabolic partial differential equations and nonlinear parabolic differential equations. MATLAB software was used as an analytical aid and tool to manipulate the data. Robinson (2004) pointed out that useful of MATLAB commands to visualize and solve a variety of differential equations. Data from MATLAB was imported to Microsoft Excel where numerical and graphical approximations were examined. Bronson and Costa (2006) discussed about the choice of constant h, step size, and the accuracy of the approximation solution. In this study, the smaller the values of h are preferred to minimize the error in both cases of differential equations. As shown by the data produced by each method, both explicit and implicit methods of solving differential equations produce solutions fairly close to the exact solutions. As $\Delta t$ and h go to zero while the stability requirement is satisfied, the methods show convergence to the exact solution. From our numerical experiments for solving the linear parabolic equations, while keeping $\frac{\Delta t}{h^2}$ constant, by halving the h-step, pointwise error $e_i$ and the l2 error decreases roughly by a factor of two. However, the forward-difference does possess a stability criterion

$$\lambda = \alpha^2 \frac{k}{h^2} = \frac{\Delta t}{\Delta x^2} \leq \frac{1}{2}$$

otherwise the solution will not be stable. There is an unconditionally stable method called the backward-difference method. This method does require more calculation per time step compared to the forward-difference method. Both have similar rates of error reduction for pointwise error $e_i$: halving the step size h cuts the error in half. However for L2, error reduction for Backward-Difference is by a factor of 2 and Forward-Difference is by 2. For nonlinear parabolic equations, both exact solutions and error analysis are difficult to obtain. However, our experiment does show a convergence by using a slight modification of the forward difference method. Further investigation is needed to explore the depth of nonlinear parabolic equations, as the data extracted in this research is far from conclusive. Data and calculations to support the findings are located in the Appendix A.

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# APPENDIX

## Data extrapolated from MATLAB (software version 7.7.0.471) and Calculated in Excel

### Euler:

| | | | |
|---|---|---|---|
| 0.1 | 0.05 | 0.025 | 0.0125 |
| 0.007399 | 0.291731 | 0.734684 | 0.925798 |
| | | | |
| 0.002287 | 0.045637 | 0.03125 | 0.007813 |
| 0.004349 | 0.09015 | 0.057643 | 0.036347 |
| 0.005986 | 0.132443 | 0.11493 | 0.072637 |
| 0.007037 | 0.171475 | 0.171509 | 0.108816 |
| 0.007399 | 0.206285 | 0.22703 | 0.144827 |
| | 0.236015 | 0.281152 | 0.180614 |
| | 0.259934 | 0.33354 | 0.216123 |
| | 0.277452 | 0.383872 | 0.251299 |
| | 0.288139 | 0.431837 | 0.286087 |
| | 0.291731 | 0.477139 | 0.320434 |
| | | 0.5195 | 0.354288 |
| | | 0.558658 | 0.387594 |
| | | 0.594372 | 0.420303 |
| | | 0.626421 | 0.452365 |
| | | 0.654609 | 0.483728 |
| | | 0.67876 | 0.514346 |
| | | 0.698726 | 0.54417 |
| | | 0.714385 | 0.573156 |
| | | 0.725639 | 0.601258 |
| | | 0.73242 | 0.628432 |
| | | 0.734684 | 0.654638 |
| | | | 0.679834 |
| | | | 0.703982 |
| | | | 0.727045 |
| | | | 0.748986 |
| | | | 0.769773 |
| | | | 0.789372 |
| | | | 0.807755 |
| | | | 0.824892 |
| | | | 0.840757 |
| | | | 0.855326 |
| | | | 0.868576 |

0.880486
0.891039
0.900218
0.908009
0.9144
0.919381
0.922944
0.925084
0.925798

```
% Input:
t_0=0;          % initial time
w_0=[0.9 0];          % initial condition for the solution
t_final=1;      % final time
h=0.05;              % time steps

% Count flops
%flops(0)

% Call Euler's or Runge Kunta's method
% Output is   result, which contain [t w]
method=input('method to be employed: (Euler=1, RK4=2) ')
if method==1,
   euler
else
   rk4
end

%Euler's method
t=t_0; w=w_0; result=[t w];

while t < t_final

   w_new=w+h*euler_fcn(t,w);

   result=[result; t+h, w_new];
   w=w_new; t=t+h;

end

function slope=euler_fcn(t,y)
slope(1)=y(2);
slope(2)=y(1)^2-1;

%function t=t(n,t0,t1,y0)
function y=y(n,t0,t1,y0)
h=(t1-t0)/n;
t(1)=t0;
y(1)=y0;
for i=1:n
t(i+1)=t(i)+h;
y(i+1)=y(i)+h*ex(t(i),y(i));
end;
```

```
V=[t',y']
plot(t,y)
title('satya')
```

**Backward Euler**:

| dx=dt | 0.10 | 0.05 | 0.025 | 0.0125 |
|---|---|---|---|---|
| exact | 0.01 | 0.005 | 0.0025 | 0.00125 |
| | | | | |
| 0.002287 | 0.010183 | 0.002853 | 0.000955 | 0.000374 |
| 0.004349 | 0.01937 | 0.005636 | 0.001904 | 0.000748 |
| 0.005986 | 0.026661 | 0.008279 | 0.002842 | 0.001121 |
| 0.007037 | 0.031342 | 0.010719 | 0.003761 | 0.001492 |
| 0.007399 | 0.032954 | 0.012896 | 0.004658 | 0.00186 |
| 0.007037 | 0.031342 | 0.014754 | 0.005526 | 0.002226 |
| 0.005986 | 0.026661 | 0.016249 | 0.00636 | 0.002588 |
| 0.004349 | 0.01937 | 0.017344 | 0.007155 | 0.002947 |
| 0.002287 | 0.010183 | 0.018013 | 0.007905 | 0.0033 |
| | | 0.018237 | 0.008607 | 0.003649 |
| | | 0.018013 | 0.009256 | 0.003992 |
| | | 0.017344 | 0.009847 | 0.004329 |
| | | 0.016249 | 0.010378 | 0.004659 |
| | | 0.014754 | 0.010845 | 0.004982 |
| | | 0.012896 | 0.011246 | 0.005297 |
| | | 0.010719 | 0.011576 | 0.005605 |
| | | 0.008279 | 0.011836 | 0.005903 |
| | | 0.005636 | 0.012022 | 0.006193 |
| | | 0.002853 | 0.012135 | 0.006473 |
| | | | 0.012172 | 0.006742 |
| | | | 0.012135 | 0.007002 |
| | | | 0.012022 | 0.007251 |
| | | | 0.011836 | 0.007488 |
| | | | 0.011576 | 0.007714 |
| | | | 0.011246 | 0.007928 |
| | | | 0.010845 | 0.00813 |
| | | | 0.010378 | 0.008319 |
| | | | 0.009847 | 0.008496 |
| | | | 0.009256 | 0.008659 |
| | | | 0.008607 | 0.008809 |
| | | | 0.007905 | 0.008946 |
| | | | 0.007155 | 0.009069 |
| | | | 0.00636 | 0.009177 |
| | | | 0.005526 | 0.009272 |
| | | | 0.004658 | 0.009352 |
| | | | 0.003761 | 0.009418 |
| | | | 0.002842 | 0.009469 |
| | | | 0.001904 | 0.009506 |
| | | | 0.000955 | 0.009528 |

| | | | |
|---|---|---|---|
| | | | 0.009535 |
| | | | 0.009528 |
| | | | 0.009506 |
| | | | 0.009469 |
| | | | 0.009418 |
| | | | 0.009352 |
| | | | 0.009272 |
| | | | 0.009177 |
| | | | 0.009069 |
| | | | 0.008946 |
| | | | 0.008809 |
| | | | 0.008659 |
| | | | 0.008496 |
| | | | 0.008319 |
| | | | 0.00813 |
| | | | 0.007928 |
| | | | 0.007714 |
| | | | 0.007488 |
| | | | 0.007251 |
| | | | 0.007002 |
| | | | 0.006742 |
| | | | 0.006473 |
| | | | 0.006193 |
| | | | 0.005903 |
| | | | 0.005605 |
| | | | 0.005297 |
| | | | 0.004982 |
| | | | 0.004659 |
| | | | 0.004329 |
| | | | 0.003992 |
| | | | 0.003649 |
| | | | 0.0033 |
| | | | 0.002947 |
| | | | 0.002588 |
| | | | 0.002226 |
| | | | 0.00186 |
| | | | 0.001492 |
| | | | 0.001121 |
| | | | 0.000748 |
| | | | 0.000374 |
| 0.1 | -2.3025 | -3.65883 | |
| 0.05 | -2.9957 | -4.50576 | |
| 0.025 | -3.6888 | -5.30228 | |

| | | |
|---|---|---|
| 0.0125 | -4.3820 | -6.05617 |
| | | 1.152502 |

| | | | | | |
|---|---|---|---|---|---|
| 0.007192 | 0.009378 | 0.002186299 | 0.01 | -4.6051 | -6.12555 |
| 0.007192 | 0.007712 | 0.000519869 | 0.0025 | -5.9914 | -7.56193 |
| 0.007192 | 0.00732 | 0.000128276 | 0.000625 | -7.3776 | -8.96133 |
| 0.007192 | 0.007224 | 3.19651E-05 | 0.000156 | -8.7645 | -10.3509 |
| | | | | | 1.015322 |

| dt=dx^2 | 0.01 | 0.0025 | 0.000625 | 0.000156 |
|---|---|---|---|---|
| | | | | |
| 0.002898 | 0.002898 | 0.001206 | 0.000574 | 0.000284 |
| 0.005512 | 0.005512 | 0.002383 | 0.001145 | 0.000567 |
| 0.007587 | 0.007587 | 0.003501 | 0.001709 | 0.000849 |
| 0.008919 | 0.008919 | 0.004533 | 0.002262 | 0.00113 |
| 0.009378 | 0.009378 | 0.005453 | 0.002801 | 0.001409 |
| 0.891918 | 0.008919 | 0.006239 | 0.003323 | 0.001686 |
| 0.007587 | 0.007587 | 0.006871 | 0.003825 | 0.001961 |
| 0.005512 | 0.005512 | 0.007334 | 0.004303 | 0.002232 |
| 0.002898 | 0.002898 | 0.007617 | 0.004754 | 0.0025 |
| | | 0.007712 | 0.005176 | 0.002764 |
| | | 0.007617 | 0.005566 | 0.003024 |
| | | 0.007334 | 0.005922 | 0.00328 |
| | | 0.006871 | 0.006241 | 0.00353 |
| | | 0.006239 | 0.006522 | 0.003774 |
| | | 0.005453 | 0.006763 | 0.004013 |
| | | 0.004533 | 0.006962 | 0.004246 |
| | | 0.003501 | 0.007118 | 0.004472 |
| | | 0.002383 | 0.00723 | 0.004692 |
| | | 0.001206 | 0.007298 | 0.004904 |
| | | | 0.00732 | 0.005108 |
| | | | 0.007298 | 0.005305 |
| | | | 0.00723 | 0.005493 |
| | | | 0.007118 | 0.005673 |
| | | | 0.006962 | 0.005844 |
| | | | 0.006763 | 0.006006 |
| | | | 0.006522 | 0.006159 |
| | | | 0.006241 | 0.006303 |
| | | | 0.005922 | 0.006436 |
| | | | 0.005566 | 0.00656 |
| | | | 0.005176 | 0.006674 |
| | | | 0.004754 | 0.006777 |
| | | | 0.004303 | 0.00687 |

| | |
|---|---|
| 0.003825 | 0.006953 |
| 0.003323 | 0.007024 |
| 0.002801 | 0.007085 |
| 0.002262 | 0.007135 |
| 0.001709 | 0.007174 |
| 0.001145 | 0.007202 |
| 0.000574 | 0.007218 |
| | 0.007224 |
| | 0.007218 |
| | 0.007202 |
| | 0.007174 |
| | 0.007135 |
| | 0.007085 |
| | 0.007024 |
| | 0.006953 |
| | 0.00687 |
| | 0.006777 |
| | 0.006674 |
| | 0.00656 |
| | 0.006436 |
| | 0.006303 |
| | 0.006159 |
| | 0.006006 |
| | 0.005844 |
| | 0.005673 |
| | 0.005493 |
| | 0.005305 |
| | 0.005108 |
| | 0.004904 |
| | 0.004692 |
| | 0.004472 |
| | 0.004246 |
| | 0.004013 |
| | 0.003774 |
| | 0.00353 |
| | 0.00328 |
| | 0.003024 |
| | 0.002764 |
| | 0.0025 |
| | 0.002232 |
| | 0.001961 |
| | 0.001686 |
| | 0.001409 |

| |
|---|
| 0.00113 |
| 0.000849 |
| 0.000567 |
| 0.000284 |

| | |
|---|---|
| 0.000376 | 1.41672E-07 |
| 0.000716 | 5.12567E-07 |
| 0.000985 | 9.71005E-07 |
| 0.001158 | 1.3419E-06 |
| 0.001218 | 1.48358E-06 |
| 0.001158 | 1.3419E-06 |
| 0.000985 | 9.71005E-07 |
| 0.000716 | 5.12567E-07 |
| 0.000376 | 1.41672E-07 |
| | 7.41787E-06 |
| | 0.002723577 |
| | 0.000272358 |

0.05

| | | | |
|---|---|---|---|
| 0.001171 | 0.001125 | 4.62E-05 | 2.13049E-09 |
| 0.002314 | 0.002222 | 9.12E-05 | 8.31342E-09 |
| 0.003399 | 0.003265 | 0.000134 | 1.79436E-08 |
| 0.004401 | 0.004227 | 0.000173 | 3.00782E-08 |
| 0.005294 | 0.005085 | 0.000209 | 4.35296E-08 |
| 0.006057 | 0.005818 | 0.000239 | 5.6981E-08 |
| 0.006671 | 0.006408 | 0.000263 | 6.91157E-08 |
| 0.007121 | 0.00684 | 0.000281 | 7.87459E-08 |
| 0.007395 | 0.007103 | 0.000291 | 8.49288E-08 |
| 0.007487 | 0.007192 | 0.000295 | 8.70593E-08 |
| 0.007395 | 0.007103 | 0.000291 | 8.49288E-08 |
| 0.007121 | 0.00684 | 0.000281 | 7.87459E-08 |
| 0.006671 | 0.006408 | 0.000263 | 6.91157E-08 |
| 0.006057 | 0.005818 | 0.000239 | 5.6981E-08 |
| 0.005294 | 0.005085 | 0.000209 | 4.35296E-08 |
| 0.004401 | 0.004227 | 0.000173 | 3.00782E-08 |
| 0.003399 | 0.003265 | 0.000134 | 1.79436E-08 |
| 0.002314 | 0.002222 | 9.12E-05 | 8.31342E-09 |
| 0.001171 | 0.001125 | 4.62E-05 | 2.13049E-09 |
| | | | 3.91767E-07 |
| | | | 0.000625913 |
| | | | 3.12956E-05 |

0.03

| | | | |
|---|---|---|---|
| 0.00057 | 5.64E-04 | 5.74E-06 | 3.29593E-11 |
| 0.001137 | 0.001125 | 1.14E-05 | 1.31026E-10 |
| 0.001696 | 0.001679 | 1.71E-05 | 2.91784E-10 |

| | | | |
|---|---|---|---|
| 0.002245 | 0.002222 | 2.26E-05 | 5.11276E-10 |
| 0.00278 | 0.002752 | 2.8E-05 | 7.84098E-10 |
| 0.003298 | 0.003265 | 3.32E-05 | 1.10353E-09 |
| 0.003796 | 0.003758 | 3.82E-05 | 1.46171E-09 |
| 0.00427 | 0.004227 | 4.3E-05 | 1.84982E-09 |
| 0.004718 | 0.004671 | 4.75E-05 | 2.25829E-09 |
| 0.005137 | 0.005085 | 5.17E-05 | 2.67708E-09 |
| 0.005524 | 0.005469 | 5.56E-05 | 3.09586E-09 |
| 0.005878 | 0.005818 | 5.92E-05 | 3.50434E-09 |
| 0.006194 | 0.006132 | 6.24E-05 | 3.89245E-09 |
| 0.006473 | 0.006408 | 6.52E-05 | 4.25062E-09 |
| 0.006712 | 0.006644 | 6.76E-05 | 4.57006E-09 |
| 0.006909 | 0.00684 | 6.96E-05 | 4.84288E-09 |
| 0.007064 | 0.006993 | 7.12E-05 | 5.06237E-09 |
| 0.007176 | 0.007103 | 7.23E-05 | 5.22313E-09 |
| 0.007243 | 0.00717 | 7.29E-05 | 5.3212E-09 |
| 0.007265 | 0.007192 | 7.32E-05 | 5.35416E-09 |
| 0.007243 | 0.00717 | 7.29E-05 | 5.3212E-09 |
| 0.007176 | 0.007103 | 7.23E-05 | 5.22313E-09 |
| 0.007064 | 0.006993 | 7.12E-05 | 5.06237E-09 |
| 0.006909 | 0.00684 | 6.96E-05 | 4.84288E-09 |
| 0.006712 | 0.006644 | 6.76E-05 | 4.57006E-09 |
| 0.006473 | 0.006408 | 6.52E-05 | 4.25062E-09 |
| 0.006194 | 0.006132 | 6.24E-05 | 3.89245E-09 |
| 0.005878 | 0.005818 | 5.92E-05 | 3.50434E-09 |
| 0.005524 | 0.005469 | 5.56E-05 | 3.09586E-09 |
| 0.005137 | 0.005085 | 5.17E-05 | 2.67703E-09 |
| 0.004718 | 0.004671 | 4.75E-05 | 2.25829E-09 |
| 0.00427 | 0.004227 | 4.3E-05 | 1.84982E-09 |
| 0.003796 | 0.003758 | 3.82E-05 | 1.46171E-09 |
| 0.003298 | 0.003265 | 3.32E-05 | 1.10353E-09 |
| 0.00278 | 0.002752 | 2.8E-05 | 7.84098E-10 |
| 0.002245 | 0.002222 | 2.26E-05 | 5.11276E-10 |
| 0.001696 | 0.001679 | 1.71E-05 | 2.91784E-10 |
| 0.001137 | 0.001125 | 1.14E-05 | 1.31026E-10 |
| 0.00057 | 5.64E-04 | 5.74E-06 | 3.29593E-11 |
| | | | 8.42449E-09 |
| | | | 9.1785E-05 |
| | | | 2.29463E-06 |
| 0.01 | | | |
| 0.000283 | 2.82E-04 | 7.17E-07 | 5.13695E-13 |
| 0.000566 | 5.64E-04 | 1.43E-06 | 2.05162E-12 |
| 0.000847 | 8.45E-04 | 2.15E-06 | 4.60428E-12 |

| | | | |
|---|---|---|---|
| 0.001128 | 1.13E-03 | 2.86E-06 | 8.15595E-12 |
| 0.001407 | 1.40E-03 | 3.56E-06 | 1.26847E-11 |
| 0.001683 | 1.68E-03 | 4.26E-06 | 1.81627E-11 |
| 0.001957 | 1.95E-03 | 4.96E-06 | 2.45561E-11 |
| 0.002228 | 2.22E-03 | 5.64E-06 | 3.18254E-11 |
| 0.002496 | 2.49E-03 | 6.32E-06 | 3.9926E-11 |
| 0.002759 | 2.75E-03 | 6.99E-06 | 4.88078E-11 |
| 0.003019 | 3.01E-03 | 7.64E-06 | 5.84161E-11 |
| 0.003273 | 3.27E-03 | 8.29E-06 | 6.86915E-11 |
| 0.003523 | 3.51E-03 | 8.92E-06 | 7.95709E-11 |
| 0.003767 | 3.76E-03 | 9.54E-06 | 9.09871E-11 |
| 0.004006 | 4.00E-03 | 1.01E-05 | 1.0287E-10 |
| 0.004238 | 4.23E-03 | 1.07E-05 | 1.15146E-10 |
| 0.004464 | 4.45E-03 | 1.13E-05 | 1.27739E-10 |
| 0.004683 | 4.67E-03 | 1.19E-05 | 1.40572E-10 |
| 0.004894 | 4.88E-03 | 1.24E-05 | 1.53566E-10 |
| 0.005098 | 5.09E-03 | 1.29E-05 | 1.6664E-10 |
| 0.005295 | 5.28E-03 | 1.34E-05 | 1.79715E-10 |
| 0.005483 | 5.47E-03 | 1.39E-05 | 1.92708E-10 |
| 0.005662 | 5.65E-03 | 1.43E-05 | 2.05542E-10 |
| 0.005833 | 5.82E-03 | 1.48E-05 | 2.18135E-10 |
| 0.005995 | 5.98E-03 | 1.52E-05 | 2.30411E-10 |
| 0.006148 | 6.13E-03 | 1.56E-05 | 2.42293E-10 |
| 0.006291 | 6.27E-03 | 1.59E-05 | 2.53709E-10 |
| 0.006424 | 6.41E-03 | 1.63E-05 | 2.64589E-10 |
| 0.006548 | 6.53E-03 | 1.66E-05 | 2.74864E-10 |
| 0.006661 | 6.64E-03 | 1.69E-05 | 2.84473E-10 |
| 0.006764 | 6.75E-03 | 1.71E-05 | 2.93354E-10 |
| 0.006857 | 6.84E-03 | 1.74E-05 | 3.01455E-10 |
| 0.006939 | 6.92E-03 | 1.78E-05 | 3.16071E-10 |
| 0.007011 | 6.99E-03 | 1.78E-05 | 3.15118E-10 |
| 0.007072 | 7.05E-03 | 1.79E-05 | 3.20596E-10 |
| 0.007121 | 7.10E-03 | 1.8E-05 | 3.25124E-10 |
| 0.00716 | 7.14E-03 | 1.81E-05 | 3.28676E-10 |
| 0.007188 | 7.17E-03 | 1.82E-05 | 3.31229E-10 |
| 0.007205 | 7.19E-03 | 1.82E-05 | 3.32767E-10 |
| 0.00721 | 7.19E-03 | 1.83E-05 | 3.3328E-10 |
| 0.007205 | 7.19E-03 | 1.82E-05 | 3.32767E-10 |
| 0.007188 | 7.17E-03 | 1.82E-05 | 3.31229E-10 |
| 0.00716 | 7.14E-03 | 1.81E-05 | 3.28676E-10 |
| 0.007121 | 7.10E-03 | 1.8E-05 | 3.25124E-10 |
| 0.007072 | 7.05E-03 | 1.79E-05 | 3.20596E-10 |
| 0.007011 | 6.99E-03 | 1.78E-05 | 3.15118E-10 |

| | | | |
|---|---|---|---|
| 0.006939 | 6.92E-03 | 1.76E-05 | 3.08724E-10 |
| 0.006857 | 6.84E-03 | 1.74E-05 | 3.01455E-10 |
| 0.006764 | 6.75E-03 | 1.71E-05 | 2.93354E-10 |
| 0.006661 | 6.64E-03 | 1.69E-05 | 2.84473E-10 |
| 0.006548 | 6.53E-03 | 1.66E-05 | 2.74864E-10 |
| 0.006424 | 6.41E-03 | 1.63E-05 | 2.64589E-10 |
| 0.006291 | 6.27E-03 | 1.59E-05 | 2.53709E-10 |
| 0.006148 | 6.13E-03 | 1.56E-05 | 2.42293E-10 |
| 0.005995 | 5.98E-03 | 1.52E-05 | 2.30411E-10 |
| 0.005833 | 5.82E-03 | 1.48E-05 | 2.18135E-10 |
| 0.005662 | 5.65E-03 | 1.43E-05 | 2.05542E-10 |
| 0.005483 | 5.47E-03 | 1.39E-05 | 1.92708E-10 |
| 0.005295 | 5.28E-03 | 1.34E-05 | 1.79715E-10 |
| 0.005098 | 5.09E-03 | 1.29E-05 | 1.6664E-10 |
| 0.004894 | 4.88E-03 | 1.24E-05 | 1.53566E-10 |
| 0.004683 | 4.67E-03 | 1.19E-05 | 1.40572E-10 |
| 0.004464 | 4.45E-03 | 1.13E-05 | 1.27739E-10 |
| 0.004238 | 4.23E-03 | 1.07E-05 | 1.15146E-10 |
| 0.004006 | 4.00E-03 | 1.01E-05 | 1.0287E-10 |
| 0.003767 | 3.76E-03 | 9.54E-06 | 9.09871E-11 |
| 0.003523 | 3.51E-03 | 8.92E-06 | 7.95709E-11 |
| 0.003273 | 3.27E-03 | 8.29E-06 | 6.86915E-11 |
| 0.003019 | 3.01E-03 | 7.64E-06 | 5.84161E-11 |
| 0.002759 | 2.75E-03 | 6.99E-06 | 4.88078E-11 |
| 0.002496 | 2.49E-03 | 6.32E-06 | 3.9926E-11 |
| 0.002228 | 2.22E-03 | 5.64E-06 | 3.18254E-11 |
| 0.001957 | 1.95E-03 | 4.96E-06 | 2.45561E-11 |
| 0.001683 | 1.68E-03 | 4.26E-06 | 1.81627E-11 |
| 0.001407 | 1.40E-03 | 3.56E-06 | 1.26847E-11 |
| 0.001128 | 1.13E-03 | 2.86E-06 | 8.15595E-12 |
| 0.000847 | 8.45E-04 | 2.15E-06 | 4.60428E-12 |
| 0.000566 | 5.64E-04 | 1.43E-06 | 2.05162E-12 |
| 0.000283 | 2.82E-04 | 7.17E-07 | 5.13695E-13 |
| | | | 1.4248E-10 |
| | | | 1.19365E-05 |
| | | | 1.49206E-07 |

| | | | | | |
|---|---|---|---|---|---|
| 0.1 | 0.00841 | 0.007192 | 0.001218 | -1 | -2.91434 |
| 0.05 | 0.007487 | 0.007192 | 0.000295 | -1.3010 | -3.53009 |
| 0.025 | 0.007265 | 0.007192 | 7.32E-05 | -1.6020 | -4.13563 |
| 0.0125 | 0.00721 | 0.007192 | 1.83E-05 | -1.9030 | -4.73852 |
| | | | | | 2.019088 |

| | | | | |
|---|---|---|---|---|
| 0.1 | 0.000272 | 0.099728 | -1 | -1.00118 |
| 0.05 | 3.13E-05 | 0.049969 | -1.3010 | -1.3013 |

| | | | | | |
|---|---|---|---|---|---|
| 0.025 | 2.29E-06 | 0.024998 | | -1.6020 | -1.6021 |
| 0.0125 | 1.49E-07 | 0.0125 | | -1.9030 | -1.9031 |
| | | | | | 0.998748 |

## **Forward Euler:**

```
global A
% Input:
mm=(1/.0005);
m=mm-1;
h1=1/mm;
x=[h1:h1:1-h1];
t_0=0;          % initial time
w_0=[sin(pi*x)];          % initial condition for the solution
t_final=0.5;      % final time
h=.1;              % time steps

e=ones(m,1);
A=spdiags([e -2*e e],-1:1,m,m);
A=A/(h1*h1);


%Euler's method
t=t_0; w=w_0;

while t < t_final

   w_new=w+h*slope_fcn(t,w);

   w=w_new; t=t+h;

end

function slope=slope_fcn(t,y)
global A
slope=A*y;
```

| |
|---|
| 4.11019E-09 |
| 1.48696E-08 |
| 2.81666E-08 |
| 3.89263E-08 |
| 4.30383E-08 |
| 3.89263E-08 |
| 2.81666E-08 |
| 1.48696E-08 |
| 4.11019E-09 |
| 2.15184E-07 |
| 0.000463879 |
| 4.63879E-05 |

| | | | |
|---|---|---|---|
| 0.001132 | 0.001125 | 6.6263E-06 | 4.39082E-11 |
| 0.002236 | 0.002222 | 1.3089E-05 | 1.71335E-10 |
| 0.003284 | 0.003265 | 1.9230E-05 | 3.69806E-10 |

| | | | |
|---|---|---|---|
| 0.004252 | 0.004227 | 2.4897E-05 | 6.19895E-10 |
| 0.005115 | 0.005085 | 2.9952E-05 | 8.9712E-10 |
| 0.005853 | 0.005818 | 3.4268E-05 | 1.17435E-09 |
| 0.006446 | 0.006408 | 3.7741E-05 | 1.42443E-09 |
| 0.00688 | 0.00684 | 4.0285E-05 | 1.62291E-09 |
| 0.007145 | 0.007103 | 4.1837E-05 | 1.75033E-09 |
| 0.007234 | 0.007192 | 4.2358E-05 | 1.79424E-09 |
| 0.007145 | 0.007103 | 4.1837E-05 | 1.75033E-09 |
| 0.00688 | 0.00684 | 4.0285E-05 | 1.62291E-09 |
| 0.006446 | 0.006408 | 3.7741E-05 | 1.42443E-09 |
| 0.005853 | 0.005818 | 3.4268E-05 | 1.17435E-09 |
| 0.005115 | 0.005085 | 2.9952E-05 | 8.9712E-10 |
| 0.004252 | 0.004227 | 2.4897E-05 | 6.19895E-10 |
| 0.003284 | 0.003265 | 1.9230E-05 | 3.69806E-10 |
| 0.002236 | 0.002222 | 1.3089E-05 | 1.71335E-10 |
| 0.001132 | 0.001125 | 6.6263E-06 | 4.39082E-11 |
| | | | 1.79424E-08 |
| | | | 0.000133949 |
| | | | 6.69746E-06 |
| 0.000565 | 5.64E-04 | 8.29E-07 | 6.86675E-13 |
| 0.001127 | 0.001125 | 1.65E-06 | 2.7298E-12 |
| 0.001681 | 0.001679 | 2.47E-06 | 6.07905E-12 |
| 0.002226 | 0.002222 | 3.26E-06 | 1.0652E-11 |
| 0.002756 | 0.002752 | 4.04E-06 | 1.63359E-11 |
| 0.00327 | 0.003265 | 4.79E-06 | 2.2991E-11 |
| 0.003763 | 0.003758 | 5.52E-06 | 3.04534E-11 |
| 0.004233 | 0.004227 | 6.21E-06 | 3.85392E-11 |
| 0.004678 | 0.004671 | 6.86E-06 | 4.70494E-11 |
| 0.005093 | 0.005085 | 7.47E-06 | 5.57744E-11 |
| 0.005477 | 0.005469 | 8.03E-06 | 6.44995E-11 |
| 0.005827 | 0.005818 | 8.54E-06 | 7.30097E-11 |
| 0.006141 | 0.006132 | 9.01E-06 | 8.10955E-11 |
| 0.006417 | 0.006408 | 9.41E-06 | 8.85578E-11 |
| 0.006654 | 0.006644 | 9.76E-06 | 9.52129E-11 |
| 0.00685 | 0.00684 | 1.00E-05 | 1.00897E-10 |
| 0.007003 | 0.006993 | 1.03E-05 | 1.0547E-10 |
| 0.007114 | 0.007103 | 1.04E-05 | 1.08819E-10 |
| 0.00718 | 0.00717 | 1.05E-05 | 1.10862E-10 |
| 0.007202 | 0.007192 | 1.06E-05 | 1.11549E-10 |
| 0.00718 | 0.00717 | 1.05E-05 | 1.10862E-10 |
| 0.007114 | 0.007103 | 1.04E-05 | 1.08819E-10 |
| 0.007003 | 0.006993 | 1.03E-05 | 1.0547E-10 |
| 0.00685 | 0.00684 | 1.00E-05 | 1.00897E-10 |

| | | | |
|---|---|---|---|
| 0.006654 | 0.006644 | 9.76E-06 | 9.52129E-11 |
| 0.006417 | 0.006408 | 9.41E-06 | 8.85578E-11 |
| 0.006141 | 0.006132 | 9.01E-06 | 8.10955E-11 |
| 0.005827 | 0.005818 | 8.54E-06 | 7.30097E-11 |
| 0.005477 | 0.005469 | 8.03E-06 | 6.44995E-11 |
| 0.005093 | 0.005085 | 7.47E-06 | 5.57676E-11 |
| 0.004678 | 0.004671 | 6.86E-06 | 4.70494E-11 |
| 0.004233 | 0.004227 | 6.21E-06 | 3.85392E-11 |
| 0.003763 | 0.003758 | 5.52E-06 | 3.04534E-11 |
| 0.00327 | 0.003265 | 4.79E-06 | 2.2991E-11 |
| 0.002756 | 0.002752 | 4.04E-06 | 1.63359E-11 |
| 0.002226 | 0.002222 | 3.26E-06 | 1.0652E-11 |
| 0.001681 | 0.001679 | 2.47E-06 | 6.07905E-12 |
| 0.001127 | 0.001125 | 1.65E-06 | 2.7298E-12 |
| 0.000565 | 5.64E-04 | 8.29E-07 | 6.86675E-13 |
| | | | 2.23097E-09 |
| | | | 4.72331E-05 |
| | | | 1.18083E-06 |
| 0.000282 | 2.82E-04 | 1.04E-07 | 1.07316E-14 |
| 0.000564 | 5.64E-04 | 2.07E-07 | 4.28604E-14 |
| 0.000846 | 8.45E-04 | 3.10E-07 | 9.61889E-14 |
| 0.001125 | 1.13E-03 | 4.13E-07 | 1.70387E-13 |
| 0.001404 | 1.40E-03 | 5.15E-07 | 2.64998E-13 |
| 0.00168 | 1.68E-03 | 6.16E-07 | 3.79439E-13 |
| 0.001953 | 1.95E-03 | 7.16E-07 | 5.13005E-13 |
| 0.002223 | 2.22E-03 | 8.15E-07 | 6.64868E-13 |
| 0.00249 | 2.49E-03 | 9.13E-07 | 8.34099E-13 |
| 0.002753 | 2.75E-03 | 1.01E-06 | 1.01965E-12 |
| 0.003012 | 3.01E-03 | 1.10E-06 | 1.22038E-12 |
| 0.003266 | 3.27E-03 | 1.20E-06 | 1.43504E-12 |
| 0.003515 | 3.51E-03 | 1.29E-06 | 1.66233E-12 |
| 0.003759 | 3.76E-03 | 1.38E-06 | 1.90082E-12 |
| 0.003997 | 4.00E-03 | 1.47E-06 | 2.14906E-12 |
| 0.004229 | 4.23E-03 | 1.55E-06 | 2.40552E-12 |
| 0.004454 | 4.45E-03 | 1.63E-06 | 2.66861E-12 |
| 0.004672 | 4.67E-03 | 1.71E-06 | 2.93671E-12 |
| 0.004884 | 4.88E-03 | 1.79E-06 | 3.20816E-12 |
| 0.005087 | 5.09E-03 | 1.87E-06 | 3.4813E-12 |
| 0.005283 | 5.28E-03 | 1.94E-06 | 3.75444E-12 |
| 0.005471 | 5.47E-03 | 2.01E-06 | 4.02589E-12 |
| 0.00565 | 5.65E-03 | 2.07E-06 | 4.29399E-12 |
| 0.00582 | 5.82E-03 | 2.13E-06 | 4.55708E-12 |
| 0.005982 | 5.98E-03 | 2.19E-06 | 4.81353E-12 |

| | | | |
|---|---|---|---|
| 0.006134 | 6.13E-03 | 2.25E-06 | 5.06178E-12 |
| 0.006277 | 6.27E-03 | 2.30E-06 | 5.30028E-12 |
| 0.00641 | 6.41E-03 | 2.35E-06 | 5.52756E-12 |
| 0.006534 | 6.53E-03 | 2.40E-06 | 5.74222E-12 |
| 0.006647 | 6.64E-03 | 2.44E-06 | 5.94295E-12 |
| 0.00675 | 6.75E-03 | 2.48E-06 | 6.1285E-12 |
| 0.006842 | 6.84E-03 | 2.51E-06 | 6.29773E-12 |
| 0.006924 | 6.92E-03 | 2.75E-06 | 7.54843E-12 |
| 0.006996 | 6.99E-03 | 2.57E-06 | 6.58316E-12 |
| 0.007056 | 7.05E-03 | 2.59E-06 | 6.6976E-12 |
| 0.007106 | 7.10E-03 | 2.61E-06 | 6.79222E-12 |
| 0.007145 | 7.14E-03 | 2.62E-06 | 6.86641E-12 |
| 0.007172 | 7.17E-03 | 2.63E-06 | 6.91974E-12 |
| 0.007189 | 7.19E-03 | 2.64E-06 | 6.95187E-12 |
| 0.007195 | 7.19E-03 | 2.64E-06 | 6.9626E-12 |
| 0.007189 | 7.19E-03 | 2.64E-06 | 6.95187E-12 |
| 0.007172 | 7.17E-03 | 2.63E-06 | 6.91974E-12 |
| 0.007145 | 7.14E-03 | 2.62E-06 | 6.86641E-12 |
| 0.007106 | 7.10E-03 | 2.61E-06 | 6.79222E-12 |
| 0.007056 | 7.05E-03 | 2.59E-06 | 6.6976E-12 |
| 0.006996 | 6.99E-03 | 2.57E-06 | 6.58316E-12 |
| 0.006924 | 6.92E-03 | 2.54E-06 | 6.4496E-12 |
| 0.006842 | 6.84E-03 | 2.51E-06 | 6.29773E-12 |
| 0.00675 | 6.75E-03 | 2.48E-06 | 6.1285E-12 |
| 0.006647 | 6.64E-03 | 2.44E-06 | 5.94295E-12 |
| 0.006534 | 6.53E-03 | 2.40E-06 | 5.74222E-12 |
| 0.00641 | 6.41E-03 | 2.35E-06 | 5.52756E-12 |
| 0.006277 | 6.27E-03 | 2.30E-06 | 5.30028E-12 |
| 0.006134 | 6.13E-03 | 2.25E-06 | 5.06178E-12 |
| 0.005982 | 5.98E-03 | 2.19E-06 | 4.81353E-12 |
| 0.00582 | 5.82E-03 | 2.13E-06 | 4.55708E-12 |
| 0.00565 | 5.65E-03 | 2.07E-06 | 4.29399E-12 |
| 0.005471 | 5.47E-03 | 2.01E-06 | 4.02589E-12 |
| 0.005283 | 5.28E-03 | 1.94E-06 | 3.75444E-12 |
| 0.005087 | 5.09E-03 | 1.87E-06 | 3.4813E-12 |
| 0.004884 | 4.88E-03 | 1.79E-06 | 3.20816E-12 |
| 0.004672 | 4.67E-03 | 1.71E-06 | 2.93671E-12 |
| 0.004454 | 4.45E-03 | 1.63E-06 | 2.66861E-12 |
| 0.004229 | 4.23E-03 | 1.55E-06 | 2.40552E-12 |
| 0.003997 | 4.00E-03 | 1.47E-06 | 2.14906E-12 |
| 0.003759 | 3.76E-03 | 1.38E-06 | 1.90082E-12 |
| 0.003515 | 3.51E-03 | 1.29E-06 | 1.66233E-12 |
| 0.003266 | 3.27E-03 | 1.20E-06 | 1.43504E-12 |

| | | | |
|---|---|---|---|
| 0.003012 | 3.01E-03 | 1.10E-06 | 1.22038E-12 |
| 0.002753 | 2.75E-03 | 1.01E-06 | 1.01965E-12 |
| 0.00249 | 2.49E-03 | 9.13E-07 | 8.34099E-13 |
| 0.002223 | 2.22E-03 | 8.15E-07 | 6.64868E-13 |
| 0.001953 | 1.95E-03 | 7.16E-07 | 5.13005E-13 |
| 0.00168 | 1.68E-03 | 6.16E-07 | 3.79439E-13 |
| 0.001404 | 1.40E-03 | 5.15E-07 | 2.64998E-13 |
| 0.001125 | 1.13E-03 | 4.13E-07 | 1.70387E-13 |
| 0.000846 | 8.45E-04 | 3.10E-07 | 9.61889E-14 |
| 0.000564 | 5.64E-04 | 2.07E-07 | 4.28604E-14 |
| 0.000282 | 2.82E-04 | 1.04E-07 | 1.07316E-14 |
| | | | 2.79603E-10 |
| | | | 1.67213E-05 |
| | | | 2.09017E-07 |

| dx | at 0.5 | difference | log dx | log error |
|---|---|---|---|---|
| 0.1 | 0.007399 | 0.000207 | -1 | -3.68307 |
| 0.05 | 0.007234 | 4.24E-05 | -1.3010 | -4.37303 |
| 0.025 | 0.007202 | 1.06E-05 | -1.6020 | -4.97613 |
| 0.0125 | 0.007195 | 2.64E-06 | -1.9030 | -5.57806 |
| | | | | 2.088853 |

| dx | L2 error | | | |
|---|---|---|---|---|
| 0.1 | 2.1E-08 | 4.6E-05 | -1 | -4.3336 |
| 0.05 | 8.9E-10 | 6.7E-06 | -1.3010 | -5.17409 |
| 0.025 | 5.5E-11 | 1.1E-06 | -1.6020 | -5.92781 |
| 0.0125 | 3.5E-12 | 2.0E-07 | -1.9030 | -6.67982 |
| | | | | 2.588577 |

# Nonlinear-Forward Euler:

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| solving nonlinear forward euler | | | | tinit=0 | tfinal=0.5 | | | |
| let uexact= x(1-x)((e^t)-1) | | | | dx=.1 | dt=.0025 | | | |
| I.C= | | | | | | | | |
| 0 | | refining the time step: dx=.05, .025 | | | | | | |
| x= | exact | fwnon | | | dx=.05 | fwnon | | |
| 0.1 | 0.05838 | 0.061664 | 0.003284 | | 0.025 | 0.032582 | | |
| 0.2 | 0.1038 | 0.109922 | 0.006122 | | 0.05 | 0.061849 | | |
| 0.3 | 0.13623 | 0.144522 | 0.008292 | | 0.075 | 0.087753 | | |
| 0.4 | 0.15569 | 0.165324 | 0.009634 | | 0.1 | 0.110259 | | |
| 0.5 | 0.16218 | 0.172265 | 0.010085 | | 0.125 | 0.129338 | | |
| | | | | | 0.15 | 0.14497 | | |
| | | | | | 0.175 | 0.157142 | | |
| | | | | | 0.2 | 0.165842 | | |
| | | | | | | 0.171064 | | |
| | | | | | | 0.172806 | 0.16218 | 0.010626 |

| h= | uestimate | uexact | ei error |
|---|---|---|---|
| 0.1 | 0.461279 | 0.42957 | 0.031709 |
| 0.05 | 0.459834 | 0.42957 | 0.030264 |
| 0.025 | 0.459473 | 0.42957 | 0.029903 |

```
global A
global U
% Input:
mm=40;
m=mm-1;
h1=1/mm;
x=[h1:h1:1-h1];
t_0=0;          % initial time
w_0=zeros(1,m);          % initial condition for the solution

t_final=1;      % final time
h=0.25*(1/mm)^(2);              % time steps

e=ones(m,1);
A=spdiags([e -2*e e],-1:1,m,m);
A=A/(h1*h1);

t=t_0; w=w_0; result=[t w];

while t < t_final

   z_0=exp(t).*x.*(1-x)+2.*(exp(t)-1)+x.*(1-x).*(exp(t)-1).*(1-x.*(1-x));
   %U=w_0*(1-w_0);
   w_new=w+h*z_0+h*fnon_fcn(t,w);

   result=[result; t+h, w_new];
   w=w_new; t=t+h;
   end
```

## BACKWARD/FORWARD EULER:

```
global A
% Input:

%mm=1000;
%m=mm-1;
%h1=1/mm;
%x=[h1:h1:1-h1];
t_0=0;          % initial time
l1=-1;
l2=-100;
l3=l1*0.5;
l4=l2*0.5;
w_0=[0;1];          % initial condition for the solution
t_final=.29;      % final time
h=0.02;             % time steps

A=[(l3+l4) (l3-l4);
    (l3-l4) (l3+l4)];

%e=ones(m,1);
%A=spdiags([e -2*e e],-1:1,m,m);
%A=A/(h1*h1);

method=input('method to be employed: (newfwrd=1, newbwrd=2) ')
if method==1,
   newfwrd
else
   newbwrd
end
```

| parameters | fwrd sol'n | bkwd sol'n | l1 | -1 |
|---|---|---|---|---|
| t init=0 | 0.3025 | 0.304 | l2 | -100 |
| t finl=0.5 | 0.3025 | 0.304 | | |
| w_0=[0;1] | | | | |
| h=.01 | | | | |
| dx=.01 | | | | |
| | | | | |
| t init=0 | 0.8017 | 0.3048 | | |
| t finl=0.5 | -0.1983 | 0.3048 | | |
| w_0=[0;1] | | | | |
| h=.02 | | | | |
| dx=.01 | | | | |
| | | | | |
| tinit=0 | 0.183 | 0.1849 | | |
| tfinl=1 | 0.183 | 0.1849 | | |
| w_0=[0;1] | | | | |

h=.01
dx=.01

| | | |
|---|---|---|
| tinit=0 | -0.3179 | 0.1858 |
| tfinl=1 | 0.6821 | 0.1858 |

w_0=[0;1]
h=.02
dx=.01

| t= | x | y |
|---|---|---|
| 0.01 | 0.31109 | 0.67896 |
| 0.02 | 0.42243 | 0.55777 |
| 0.03 | 0.46033 | 0.51012 |
| 0.04 | 0.47124 | 0.48955 |
| 0.05 | 0.47225 | 0.47898 |
| 0.06 | 0.46964 | 0.47212 |
| 0.07 | 0.46574 | 0.46665 |
| 0.08 | 0.46139 | 0.46173 |
| 0.09 | 0.4569 | 0.45703 |
| 0.1 | 0.4524 | 0.45244 |
| 0.11 | 0.44791 | 0.44793 |
| 0.12 | 0.44346 | 0.44346 |
| 0.13 | 0.43905 | 0.43905 |
| 0.14 | 0.43468 | 0.43468 |
| 0.15 | 0.43035 | 0.43035 |
| 0.16 | 0.42607 | 0.44793 |
| 0.17 | 0.42183 | 0.42607 |
| 0.18 | 0.41764 | 0.41764 |
| 0.19 | 0.41348 | 0.41348 |
| 0.2 | 0.40937 | 0.40937 |
| 0.21 | 0.40529 | 0.40529 |
| 0.22 | 0.40126 | 0.40126 |
| 0.23 | 0.39727 | 0.39727 |
| 0.24 | 0.39331 | 0.39331 |
| 0.25 | 0.3894 | 0.3894 |
| 0.26 | 0.38553 | 0.38553 |
| 0.27 | 0.38169 | 0.38169 |
| 0.28 | 0.37789 | 0.37789 |
| 0.29 | 0.37416 | 0.37416 |
| 0.3 | 0.37041 | 0.37041 |
| 0.31 | 0.36672 | 0.36672 |
| 0.32 | 0.36307 | 0.36307 |
| 0.33 | 0.35946 | 0.35946 |
| 0.34 | 0.35589 | 0.35589 |

| | | | |
|---|---|---|---|
| 0.35 | 0.35234 | 0.35234 | |
| 0.36 | 0.34884 | 0.34884 | |
| 0.37 | 0.34537 | 0.34537 | |
| 0.38 | 0.34193 | 0.34193 | |
| 0.39 | 0.33853 | 0.33853 | |
| 0.4 | 0.33516 | 0.33516 | |
| 0.41 | 0.33183 | 0.33183 | |
| 0.42 | 0.32852 | 0.32852 | |
| 0.43 | 0.32525 | 0.32525 | |
| 0.44 | 0.32202 | 0.32202 | |
| 0.45 | 0.31881 | 0.31881 | |
| 0.46 | 0.31564 | 0.31564 | |
| 0.47 | 0.3125 | 0.3125 | |
| 0.48 | 0.30939 | 0.30939 | |
| 0.49 | 0.30631 | 0.30631 | |
| 0.5 | 0.30327 | 0.30327 | |
| t= | x | bw | fw |
| 0.01 | 0.31109 | 0.24505 | 0.495 |
| 0.02 | 0.42243 | 0.365148 | 0.49005 |
| 0.03 | 0.46033 | 0.422795 | 0.48515 |
| 0.04 | 0.47124 | 0.44924 | 0.480298 |
| 0.05 | 0.47225 | 0.460108 | 0.475495 |
| 0.06 | 0.46964 | 0.46321 | 0.47074 |
| 0.07 | 0.46574 | 0.462453 | 0.466033 |
| 0.08 | 0.46139 | 0.459788 | 0.461372 |
| 0.09 | 0.4569 | 0.456193 | 0.456759 |
| 0.1 | 0.4524 | 0.452155 | 0.447669 |
| 0.11 | 0.44791 | 0.447918 | 0.443192 |
| 0.12 | 0.44346 | 0.443603 | 0.438761 |
| 0.13 | 0.43905 | 0.43927 | 0.434373 |
| 0.14 | 0.43468 | 0.434951 | 0.430029 |
| 0.15 | 0.43035 | 0.430659 | 0.430029 |
| 0.16 | 0.42607 | 0.426403 | 0.425729 |
| 0.17 | 0.42183 | 0.422185 | 0.421472 |
| 0.18 | 0.41764 | 0.418007 | 0.417257 |
| 0.19 | 0.41348 | 0.413869 | 0.413084 |
| 0.2 | 0.40937 | 0.409772 | 0.408953 |
| 0.21 | 0.40529 | 0.405715 | 0.404864 |
| 0.22 | 0.40126 | 0.401698 | 0.400815 |
| 0.23 | 0.39727 | 0.397721 | 0.396807 |
| 0.24 | 0.39331 | 0.393783 | 0.392839 |
| 0.25 | 0.3894 | 0.394246 | 0.388911 |
| 0.26 | 0.38553 | 0.386516 | 0.394246 |

| | | | |
|---|---|---|---|
| 0.27 | 0.38169 | 0.386516 | 0.386516 |
| 0.28 | 0.37789 | 0.378937 | 0.386516 |
| 0.29 | 0.37416 | 0.378937 | 0.378937 |
| 0.3 | 0.37041 | 0.371507 | 0.378937 |
| 0.31 | 0.36672 | 0.371507 | 0.371507 |
| 0.32 | 0.36307 | 0.364223 | 0.371507 |
| 0.33 | 0.35946 | 0.364223 | 0.364223 |
| 0.34 | 0.35589 | 0.357081 | 0.364223 |
| 0.35 | 0.35234 | 0.357081 | 0.357081 |
| 0.36 | 0.34884 | 0.35008 | 0.357081 |
| 0.37 | 0.34537 | 0.35008 | 0.35008 |
| 0.38 | 0.34193 | 0.343215 | 0.35008 |
| 0.39 | 0.33853 | 0.343215 | 0.343215 |
| 0.4 | 0.33516 | 0.336486 | 0.343215 |
| 0.41 | 0.33183 | 0.336486 | 0.336486 |
| 0.42 | 0.32852 | 0.329888 | 0.336486 |
| 0.43 | 0.32525 | 0.329888 | 0.329888 |
| 0.44 | 0.32202 | 0.32342 | 0.329888 |
| 0.45 | 0.31881 | 0.32342 | 0.32342 |
| 0.46 | 0.31564 | 0.317078 | 0.32342 |
| 0.47 | 0.3125 | 0.317078 | 0.317078 |
| 0.48 | 0.30939 | 0.310861 | 0.317078 |
| 0.49 | 0.30631 | 0.310861 | 0.310861 |
| 0.5 | 0.30327 | 0.304765 | 0.310861 |