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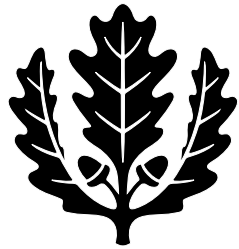
Finite Characterizations and Paretian Preferences

Vicki Knoblauch
University of Connecticut

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Finite Characterizations and Paretian Preferences

Vicki Knoblauch
University of Connecticut

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341 Mansfield Road, Unit 1063
Storrs, CT 06269-1063
Phone: (860) 486-3022
Fax: (860) 486-4463
<http://www.econ.uconn.edu/>

Abstract

A characterization of a property of binary relations is of finite type if it is stated in terms of ordered T -tuples of alternatives for some positive integer T . A characterization of finite type can be used to determine in polynomial time whether a binary relation over a finite set has the property characterized. Unfortunately, Pareto representability in R^2 has no characterization of finite type (Knoblauch, 2002). This result is generalized below R^l , l larger than 2. The method of proof is applied to other properties of binary relations.

Journal of Economic Literature Classification: D11, D70

Keywords: preferences, Pareto order, voting

1. Introduction.

It is in general useful to have a utility representation for the preferences of an individual or group. When those preferences cannot be so represented, a useful alternative is representation via a preference preserving function from the set of alternatives to a Euclidean space ordered by the Pareto relation or the lexicographic order. Any such representation provides a conceptual handle for preferences and, more concretely, can serve as the first step in the transformation of a formless list of preferences into a more useful object, such as a demand function.

Of course, before beginning a search for a particular form of representation for a particular set of preferences, it is helpful to know whether or not that set of preferences *has* a representation of that form. That is one of the roles of a characterization. We introduce characterizations in a more general setting by considering *any* property possessed by some binary relations and not by others, rather than limiting ourselves to preference representations.

A characterization of a property of binary relations is a set of criteria that allow one to determine easily whether a given binary relation has the property being characterized. Perhaps the example best known to economists of a characterization of a property of binary relations is Debreu's (1964) and Birkhoff's (Theorem 2, Chapter III, 1948) characterization of representability by a utility function,¹ which, for finite sets of alternatives, can be stated in the following form: a binary relation \succ on a finite set X can be represented by a utility function if and only if, for all $x, y, z \in X$, $x \succ y$ implies $x \succsim y$ and $x \succsim y \succsim z$ implies $x \succsim z$. (The weak preference relation \succsim is defined by $a \succsim b$ if and only if not($b \succ a$)).

Knoblauch (2002) singled out for study characterizations of finite type. A *characterization of finite type* is a characterization that can be stated in terms of ordered T -tuples of alternatives for some fixed positive integer T and such that conditions on ordered T -tuples do not reference arbitrarily large integers (see Section 2 for counterexamples). Debreu's and Birkhoff's characterization of utility representability stated above is a characterization of type 3. A characterization of finite type has at least two desirable features. First, a

¹ It has been pointed out to me by Juan Dubra that Birkhoff deserves but is rarely given partial credit for this characterization.

characterization of finite type enhances our general understanding of the property characterized. For example, before Debreu's characterization, economists often assumed the existence of a utility function without a second thought. Now most know that a representability test exists and some know that existence of a utility function representing a binary relation \succ requires asymmetry of \succ and transitivity of \succsim . Second, a characterization of finite type can be used to determine *in polynomial time* whether a binary relation on a finite set has the property characterized: if \succ is a binary relation on a set X of N elements and C is a characterization of finite type T for property Q , then, when using C to determine whether \succ has property Q , one needs to consider only ordered T -tuples, of which there are N^T . There is a polynomial A such that for each ordered T -tuple $A(N)$ steps are sufficient to construct a table for \succ restricted to elements of that ordered T -tuple. For each T -tuple C can be checked directly from the table constructed for that T -tuple (as is the case for the Debreu-Birkhoff characterization) without referring to arbitrarily large integers. Therefore, there is an integer B such that for each ordered T -tuple, C requires the carrying out of a procedure the maximal number of whose steps is B . The whole procedure takes at most $N^T(A(N) + B)$ steps, a polynomial in N . The existence of such a polynomial is desirable since a polynomial $P(N)$ grows with N much more slowly than, for example, an exponential function of N . If the number of steps required to test a binary relation is exponential in N , then as N grows the time required by even the most powerful computer to carry out the test grows quickly from seconds to days to centuries.

Unfortunately, Knoblauch (2002) showed that there is no characterization of finite type for Pareto representability ² in \mathfrak{R}^2 . The proof revolves around the construction of binary relations that are almost but not quite Pareto representable in \mathfrak{R}^2 . More precisely, the proof contains a construction, for arbitrarily large N , of a binary relation \succ on a set X of N alternatives such that \succ is not Pareto representable in \mathfrak{R}^2 , but \succ restricted to S is Pareto representable in \mathfrak{R}^2 for every proper subset S of X . That proof does not generalize to Pareto representability in \mathfrak{R}^l , $l \geq 2$. The proof below that there is no characterization of finite type for Pareto representability in \mathfrak{R}^l again revolves around simply constructed

² A Pareto representation in \mathfrak{R}^l for a binary relation \succ on X is a function $v: X \rightarrow \mathfrak{R}^l$ such that $x \succ y$ if and only if $v(x) \succ v(y)$, where $v(x) \succ v(y)$ if $v_i(x) \geq v_i(y)$ for $1 \leq i \leq l$ and $v(x) \neq v(y)$.

binary relations, but the proof that the binary relations constructed are “almost but not quite Pareto representable in \mathfrak{R}^l ” is more difficult and the words in quotes take on a different meaning than they had in the proof for the case $l = 2$.

Sprumont (2001) and Aizerman and Aleskerov (1995) provide characterizations of Pareto representability in \mathfrak{R}^2 and in \mathfrak{R}^l respectively. Sprumont’s characterization involves intermediateness conditions, which are of finite type, and regularity and continuity conditions, which are not. Earlier studies by Dilworth (1950), Hiraguchi (1955) and Leclerc (1976) give upper bounds on l (in terms of $|X|$ and properties of \succ) beyond which Pareto representability in \mathfrak{R}^l is guaranteed for a transitive, asymmetric binary relation \succ on a set X . Of course none of these studies provides a characterization *of finite type* for Pareto representability in \mathfrak{R}^l for a fixed $l \geq 2$.

The rest of the paper is organized as follows. A clearer (than that in Knoblauch, 2002) definition of a characterization of finite type is given in Section 2. Section 3 contains statements of preliminary results and the statement and proof of our main result. Three applications of the method of proof of our main result are given in Section 4. The preliminary results are proven in Section 5. Section 6 contains a voting paradox and final remarks.

2. Preliminaries; Characterizations of Finite Type.

A *binary relation* \succ on a set X is a subset of $X \times X$. For convenience $\langle x, y \rangle \in \succ$ will be written $x \succ y$. Associated with binary relation \succ on X are binary relations \succsim and \sim on X defined by $x \succsim y$ if not($y \succ x$) and $x \sim y$ if $x \succsim y \succsim x$. If \succ on X and $Y \subseteq X$, the *restriction* of \succ to Y is defined by $\succ|_Y = \succ \cap (Y \times Y)$. If \succ on X and $x \in X$, let $W(x) = \{z \in X: x \succ z\}$ and $B(x) = \{z \in X: z \succ x\}$.

The symbol \oplus will be used for addition mod n on the set $\{1, 2, \dots, n\}$; that is, for $i, j \in \{1, 2, \dots, n\}$,

$$i \oplus j = \begin{cases} i + j & \text{if } i + j \leq n \\ i + j - n & \text{if } i + j > n \end{cases}$$

While addition mod n is usually defined on $\{0, 1, \dots, n-1\}$, we use $\{1, 2, \dots, n\}$ because it is more compatible with the rest of our notation.

For a set X , $|X|$ denotes the cardinality of X .

The *Pareto relation* $>$ on \mathbb{R}^l is defined by $r > s$ if $r_i \geq s_i$ for all $i \in \{1, 2, \dots, l\}$ and at least one inequality is strict. A *Pareto representation in* \mathbb{R}^l for \succ on X is a function $v: X \rightarrow \mathbb{R}^l$ such that for all $x, y \in X$, $x \succ y$ if and only if $v(x) > v(y)$.

The definition of a characterization of finite type is due to Knoblauch (2002). We will try to make the definition clearer here. For positive integer T , a *characterization of type* T of a property Q of binary relations is a true statement of the form “A binary relation \succ on X satisfies property Q if and only if for every ordered T -tuple $\langle x^1, x^2, \dots, x^T \rangle$ of elements of X , ...” where the ellipses represent conditions that can be checked using only a table that lists the preferences $\succ|_{\{x^1, x^2, \dots, x^T\}}$. In addition these conditions must not reference arbitrarily large integers. For example, the Debreu-Birkhoff characterization is a characterization of type 3, but the two characterizations below are not characterizations of finite type; although each places conditions only on ordered pairs, the first references arbitrarily large integers through reference to inherent properties of elements of X and the second references arbitrarily large integers through reference to $|X|$:

Characterization 1: \succ on X satisfies property Q if and only if for every ordered pair $\langle x^1, x^2 \rangle$ of elements of X , $x^1 \succ x^2$ implies x^1 and x^2 are integers and $x^1 > x^2$.

Characterization 2: \succ on X satisfies property Q if and only if for every ordered pair $\langle x^1, x^2 \rangle$ of elements of X , $x^1 \succ x^2$ if $|X|$ is a prime number.

The key points of the definition are that a characterization of type T must not put conditions on ordered tuples of length greater than T , and conditions must not reference arbitrarily large integers.

For the sake of completeness we could allow in the definition finitely many occurrences of “... for every ordered T -tuple ...” and finitely many occurrences of “...*there exists* an ordered T -tuple ...” all tied together with “and”s and “or”s. There are three reasons for staying with the definition as stated. The first is simplicity. Second, the change in definition wouldn’t affect our results. Third, many known characterizations of preference representations take the form of our definition.

3. Nonexistence of Characterizations of Finite Type for Pareto Representability in \mathfrak{R}^l .

The following example is central to the proof of the main result of this section.

Example 1. Suppose n and k are positive integers with $n > k$. Let $X^n = \{x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n\}$. Define binary relation $\succ_{n,k}$ on X^n as follows: for $w, z \in X^n$,

$$w \succ_{n,k} z \text{ if and only if } w = x^j, z = y^{j'} \text{ and } j' \in \{j \oplus 1, j \oplus 2, \dots, j \oplus k\} \quad (1)$$

Also, for $w, z \in X^n$ recall that, as defined in Section 2,

$$w \succsim_{n,k} z \text{ if and only if not } (z \succ_{n,k} w) \quad (2)$$

For each $l \geq 3$, Propositions 1 and 2 establish the existence of a binary relation that is *not* Pareto representable in \mathfrak{R}^l , but whose restriction to any subset of alternatives up to a certain size limit *is* Pareto representable in \mathfrak{R}^l .

Proposition 1. *If $l \geq 3$, $k > \frac{l(l-1)}{2}$, $n > k$ and $(l-2)n < l(k-1)$, then $\succ_{n,k}$ is not Pareto representable in \mathfrak{R}^l .*

See the Appendix for the proofs of Propositions 1 and 2.

Notice that for each l there are arbitrarily large values of n and k that satisfy the hypotheses of Proposition 1.

Proposition 2. *Suppose $l \geq 3$, $k > \frac{l(l-1)}{2}$, $k-1$ is divisible by $l-2$, $n = \frac{l(k-1)}{l-2} - l$ and $n \geq 2l^2$. If $S \subseteq X^n$ and $|S| < \frac{n-(l-1)}{l}$, then $\succ_{n,k} \upharpoonright_S$ is Pareto representable in \mathfrak{R}^l .*

Notice that for each l there are arbitrarily large values of n and k that satisfy the hypotheses of Proposition 1 and Proposition 2.

Theorem 1. *There is no characterization of finite type for Pareto representability in \mathfrak{R}^l , $l \geq 2$.*

Proof. For $l = 2$ see Knoblauch (2002). Suppose $l \geq 3$, T is a positive integer and C is a characterization of type T for Pareto representability in \mathfrak{R}^l . Choose n and k such that n, k , and l satisfy the hypotheses of Proposition 2 and $n > l^2 T + l^2 - l$. Let

$\langle z^1, z^2, \dots, z^T \rangle$ be any ordered T -tuple of elements of X^n (which was constructed in Example 1). Let S be any subset of X^n such that $\{z^1, z^2, \dots, z^T\} \subseteq S$ and $|S| \leq T$. Then $|S| < \frac{n-(l-1)}{l}$. By Proposition 2, $\succ_{n,k} \upharpoonright_S$ is Pareto representable in \mathfrak{R}^l . Since C is a characterization of type T for Pareto representability in \mathfrak{R}^l , $\succ_{n,k} \upharpoonright_S$ is Pareto representable in \mathfrak{R}^l and $\langle z^1, z^2, \dots, z^T \rangle$ is an ordered T -tuple of elements of S , $\succ_{n,k} \upharpoonright_{\{z^1, z^2, \dots, z^T\}}$ satisfies the conditions of characterization C . Since $\succ_{n,k} \upharpoonright_{\{z^1, z^2, \dots, z^T\}}$ satisfies the conditions of C , $\langle z^1, z^2, \dots, z^T \rangle$ was an arbitrary ordered T -tuple of elements of X^n , and C is a characterization of type T for Pareto representability in \mathfrak{R}^l , $\succ_{n,k}$ is Pareto representable in \mathfrak{R}^l , contradicting Proposition 1.

The assumption that Pareto representability in \mathfrak{R}^l has a characterization of type T for some $l \geq 3$ has led to a contradiction. ■

4. Applications of the Method of Proof for Theorem 1.

In this section the method of proof used in Section 3 will be applied to three examples.

Example 2. Property Ext: binary relation \succ on X can be extended to a linear order on X .

A *linear order* is an asymmetric, transitive binary relation such that $x \sim y$ implies $x = y$. An *extension* of a binary relation \succ on X is a binary relation \succ' on X such that $\succ \subseteq \succ'$. Let $X^N = \{x^1, x^2, \dots, x^N\}$ and define \succ^N on X^N by $x^1 \succ x^2 \succ \dots \succ x^N \succ x^1$. Then \succ^N cannot be extended to a linear order, since for any transitive extension \succ' of \succ^N , $x^1 \succ' x^1$. However, for any proper subset S of X^N , the transitive closure of $\succ^N \upharpoonright_S$ is a linear order on S .

For each positive integer N we have produced a binary relation \succ^N on X^N such that \succ^N does not have Property Ext but $\succ^N \upharpoonright_S$ does have property Ext for every proper subset of X^N . From this it follows, as in the proof of Theorem 1, that there is no characterization of finite type for Property Ext.

Example 3. It was pointed out in Section 1 that the Debreu-Birkhoff characterization of utility representability for binary relations on finite sets is a characterization of type 3. The Debreu-Birkhoff characterization for utility representability of binary relations *on any set*

contains a condition called separability (see Debreu, 1964) that is not of finite type. The method of proof of Theorem 1 above can be used to prove that there is *no* characterization of finite type for utility representability for binary relations on any set. Let \succ_L be the lexicographic order on \mathfrak{R}^2 . Then \succ_L is not representable by a utility function, since \succ_L is not separable. However, $\succ_L|_S$ can be represented by a utility function if $S \subseteq \mathfrak{R}^2$ is finite. Therefore the method of proof of Theorem 1 can be used to show that utility representability has no characterization of finite type.

It should be pointed out that we have proven in Theorem 1 that there is no characterization of finite type even for Pareto representability of binary relations *on finite sets*.

Example 4. Property Incl: For \succ on X , there exists a finite set A and a bijection $f: X \rightarrow P(A)$, the set of subsets of A , such that, for $x, y \in X$ $x \succ y$ if and only if $f(x) \subseteq f(y)$.

Let X^N be a set of 2^N elements, $N > 1$, $A = \{1, 2, \dots, N\}$ and $f: X^N \rightarrow P(A)$ be a bijection. Define \succ^N on X^N by $x \succ y$ if $f(x) \subseteq f(y)$. By definition \succ^N has Property Incl. However, for each $x \in X$, $\succ^N|_{X^N - \{x\}}$ does not have property Incl since $|X^N - \{x\}|$ is not a power of 2.

Then by the method of proof of Theorem 1, Property Incl has no characterization of finite type. Actually a slight variation of the method of proof of Theorem 1 is needed since this time \succ^N *does* have the property in question and restrictions of \succ^N *do not*, rather than the reverse situation as in Example 2, Example 3 and the proof of Theorem 1.

5. Proofs of Propositions 1 and 2.

Proof of Proposition 1.

Suppose k, l and n satisfy the hypotheses of Proposition 1 and $v: X^n \rightarrow \mathfrak{R}^l$ is a Pareto representation for $\succ_{n,k}$. For $i \in \{1, 2, \dots, l\}$ let $v_i: X^n \rightarrow \mathfrak{R}$ be the i th coordinate of v .

Since $y^{j'} \succ_{n,k} x^j$ if $j \notin \{j \oplus 1, j \oplus 2, \dots, j \oplus k\}$, for each j there are $n - k$ values of j' such that $y^{j'} \succ_{n,k} x^j$. Therefore $|\{< j, j' >: y^{j'} \succ_{n,k} x^j\}| = n(n - k)$ and consequently

$$|\{< i, j, j' >: v_i(y^{j'}) > v_i(x^j)\}| \geq n(n - k) \quad (3)$$

Next we find another estimate for the cardinality of the set in (3).

Lemma 1. *If $A \subseteq \{1, 2, \dots, n\}$ and $|A| = r > 0$, then $|\{j': x^j \succ_{n,k} y^{j'} \text{ for some } j \in A\}| \geq \min\{k + r - 1, n\}$.*

Proof. If $|\{j': x^j \succ_{n,k} y^{j'} \text{ for some } j \in A\}| = n$, then the conclusion of Lemma 1 holds. Therefore suppose $|\{j': x^j \succ_{n,k} y^{j'} \text{ for some } j \in A\}| < n$. By the symmetry of $\succ_{n,k}$, suppose without loss of generality that $1 \notin \{j': x^j \succ_{n,k} y^{j'} \text{ for some } j \in A\}$. Write $A = \{j_1, j_2, \dots, j_r\}$. Here are $k + r - 1$ distinct elements of $\{j': x^j \succ_{n,k} y^{j'} \text{ for some } j \in A\}$: $j_1 + 1, j_2 + 1, \dots, j_r + 1, J + 2, J + 3, \dots, J + k$ where $J = \max\{j_1, j_2, \dots, j_r\}$. ■

Now let j_1, j_2, \dots, j_n be a permutation of $1, 2, \dots, n$ such that

$$v_1(x^{j_n}) \geq v_1(x^{j_{n-1}}) \geq \dots \geq v_1(x^{j_1}) \quad (4)$$

Suppose $r \in \{1, 2, \dots, n\}$. By Lemma 1, $|\{j': x^{j_m} \succ_{n,k} y^{j'} \text{ for some } m \leq r\}| \geq \min\{k + r - 1, n\}$. Since v is a Pareto representation for $\succ_{n,k}$, $|\{j': v_1(x^{j_m}) \geq v_1(y^{j'}) \text{ for some } m \leq r\}| \geq \min\{k + r - 1, n\}$. By (4) $|\{j': v_1(x^{j_r}) \geq v_1(y^{j'})\}| \geq \min\{k + r - 1, n\}$, which implies $|\{j': v_1(y^{j'}) > v_1(x^{j_r})\}| \leq \max\{n - k - r + 1, 0\}$. Therefore,

$$\begin{aligned} |\{< j, j' >: v_1(y^{j'}) > v_1(x^j)\}| &= \sum_{r=1}^{n-k} |\{j': v_1(y^{j'}) > v_1(x^{j_r})\}| \\ &\leq (n - k) + (n - k - 1) + \dots + 2 + 1 = \frac{(n-k)(n-k+1)}{2} \end{aligned}$$

$$\text{Then } |\{< i, j, j' >: v_i(y^{j'}) > v_i(x^j)\}| \leq \frac{l(n-k)(n-k+1)}{2} \quad (5)$$

Inequalities (3) and (5) imply $l(n-k)(n-k+1) \geq 2n(n-k)$. Since $n-k > 0$, $l(n-k+1) \geq 2n$ which implies $(l-2)n \geq l(k-1)$ contradicting a hypothesis of Proposition 1. The assumption that $\succ_{n,k}$ is Pareto representable in \mathfrak{R}^l has led to a contradiction. ■

Proof of Proposition 2.

The proof will be seen to follow easily from the following lemma.

Lemma 2. *Suppose k, l , and n satisfy the hypotheses of Proposition 2. For $j_0 \in \{1, 2, \dots, \frac{n}{l} - (l-1)\}$ let $S^{j_0} = \cup_{m=0}^{l-1} \{x^{j_0+mn/l}, x^{j_0+mn/l+1}, \dots, x^{j_0+mn/l+(l-1)}, y^{j_0+mn/l}, y^{j_0+mn/l+1}, \dots, y^{j_0+mn/l+(l-1)}\}$. Then $\succ_{n,k} \upharpoonright_{X^n - S^{j_0}}$ is Pareto representable in \mathfrak{R}^l .*

Proof. By the symmetry of $\succ_{n,k}$ it is enough to prove Lemma 2 for the case $j_0 = \frac{n}{l} - (l-1)$.

In the following definition of $v_1: X^n - S^{\frac{n}{l} - (l-1)} \rightarrow \mathfrak{R}$ we break the string of inequalities into $2l-2$ pieces for easy reference. Define v_1 to satisfy the following inequalities.

$$v_1(x^1) > v_1(x^2) > \dots > v_1(x^{\frac{n}{l}-l}) > \quad (6.1)$$

$$v_1(x^{\frac{n}{l}+1}) > v_1(x^{\frac{n}{l}+2}) > \dots > v_1(x^{\frac{2n}{l}-l}) > \quad (6.2)$$

$$\begin{array}{ccc} \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \end{array}$$

$$v_1(x^{\frac{(l-3)n}{l}+1}) > v_1(x^{\frac{(l-3)n}{l}+2}) > \dots > v_1(x^{\frac{(l-2)n}{l}-l}) > \quad (6.l-2)$$

$$v_1(y^{\frac{(l-2)n}{l}+1}) > v_1(x^{\frac{(l-2)n}{l}+1}) > \dots > v_1(y^{\frac{(l-1)n}{l}-l}) > v_1(x^{\frac{(l-1)n}{l}-l}) > \quad (6.l-1)$$

$$\left. \begin{array}{l} v_1(x^{n-l}) > v_1(x^{n-l-1}) > \dots > v_1(x^{n-2l+1}) > \\ v_1(y^{\frac{(l-2)n}{l}-l}) > v_1(x^{n-2l}) > \dots > v_1(y^{\frac{(l-3)n}{l}+l+1}) > v_1(x^{\frac{(l-1)n}{l}+1}) > \\ v_1(y^{\frac{(l-3)n}{l}+l}) > \dots > v_1(y^{\frac{(l-3)n}{l}+2}) > v_1(y^{\frac{(l-3)n}{l}+1}) > \end{array} \right\} \quad (6.l)$$

$$v_1(y^1) > v_1(y^2) > \dots > v_1(y^{\frac{n}{l}-l}) > \quad (6.l+1)$$

$$\begin{array}{ccc} \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \end{array}$$

$$v_1(y^{\frac{(l-4)n}{l}+1}) > \dots > v_1(y^{\frac{(l-3)n}{l}-l}) > \quad (6.2l-3)$$

$$v_1(y^{\frac{(l-1)n}{l}+1}) > \dots > v_1(y^{n-l}) \quad (6.2l-2)$$

Notice that the first $\frac{n}{l} - l$ x^j 's appear in (6.1), the second $\frac{n}{l} - l$ x^j 's appear in (6.2), and more generally the m^{th} $\frac{n}{l} - l$ x^j 's appear in (6. m) for $1 \leq m \leq l-2$. The $(l-1)^{\text{st}}$ $\frac{n}{l} - l$ x^j 's and the $(l-1)^{\text{st}}$ $\frac{n}{l} - l$ y^j 's appear in (6. $l-1$). The final $\frac{n}{l} - l$ x^j 's and the $(l-2)^{\text{nd}}$ $\frac{n}{l} - l$ y^j 's appear in (6. l). In (6. $l+1$) through (6. $2l-3$) the first through $(l-3)^{\text{rd}}$ $\frac{n}{l} - l$ y^j 's appear and in (6. $2l-2$), the final $\frac{n}{l} - l$ y^j 's appear.

It may help the reader to write out the string of inequalities for the case $l = 3$, $k = 10$, $n = 24$. For this example, since $l - 2 = 1$, (6.1) – (6. $l-2$) will be represented by (6.1), and since $2l - 3 < l + 1$, (6. $l+1$) through (6. $2l-3$) will not be used at all. The string of inequalities will be written using only (6.1), (6. $l-1$), (6. l) and (6. $2l-2$).

Define $v_i: X^n - S^{\frac{n}{l}-(l-1)} \rightarrow \Re$ for $i = 2, 3, \dots, l$ by setting, for $x^j, y^j \in X^n - S^{\frac{n}{l}-(l-1)}$,

$$v_i(x^j) = v_1(x^{j \oplus \frac{(i-1)n}{l}}) \text{ and } v_i(y^j) = v_1(y^{j \oplus \frac{(i-1)n}{l}}) \quad (7)$$

To show that $v = \langle v_1, v_2, \dots, v_l \rangle: X^n - S^{\frac{n}{l}-(l-1)} \rightarrow \Re^l$ is a Pareto representation for $\succ_{n,k} \mid_{X^n - S^{\frac{n}{l}-(l-1)}}$, we first show that $v(x^j) \geq v(x^{j'})$ for all $x^j, x^{j'} \in X^n - S^{\frac{n}{l}-(l-1)}$. By the symmetry introduced in (7) we can assume $1 \leq j \leq \frac{n}{l} - l$. Then x^j appears in (6.1). If $j' \geq j$, then by (6.1) – (6. l), $v_1(x^j) > v_1(x^{j'})$. If $j' < j$, then by (7) and (6. l) $v_l(x^j) = v_1(x^{j + \frac{(l-1)n}{l}}) > v_1(x^{j' + \frac{(l-1)n}{l}}) = v_l(x^{j'})$.

The proof that $v(y^j) \geq v(y^{j'})$ for all $y^j, y^{j'} \in X^n - S^{\frac{n}{l}-(l-1)}$ is similar, but (6. $l+1$) takes the place of (6.1).

To show that $v(x^j) \geq v(y^{j'})$ for all $x^j, y^{j'} \in X^n - S^{\frac{n}{l}-(l-1)}$, first use the symmetry introduced in (7) to assume $1 \leq j \leq \frac{n}{l} - l$. Then x^j appears in (6.1) and clearly $v_1(x^j) > v_1(y^{j'})$.

Finally to show $x^j \succ_{n,k} y^{j'}$ if and only if $v(x^j) > v(y^{j'})$ for all $x^j, y^{j'} \in X^n - S^{\frac{n}{l}-(l-1)}$, suppose $x^j, y^{j'} \in X^n - S^{\frac{n}{l}-(l-1)}$ and by the symmetry introduced in (7) assume $1 \leq j \leq \frac{n}{l} - l$.

Then

$$v(y^{j'}) \geq v(x^j)$$

if and only if $v_i(y^{j'}) > v_i(x^j)$ for some i

if and only if $v_{l-1}(y^{j'}) > v_{l-1}(x^j)$ or $v_l(y^{j'}) > v_l(x^j)$ [since $v_i(x^j) = v_1(x^{j + \frac{(i-1)n}{l}})$],

and for $i \neq l-1, l$, $x^j + \frac{(l-1)n}{l}$ appears in (6.1), (6.2), ..., or $(6.l-2)$

if and only if

$$\begin{aligned} \frac{(l-2)n}{l} + 1 \leq j' + \frac{(l-2)n}{l} \leq j + \frac{(l-2)n}{l} \text{ [in which case} \\ v_1(y^{j'} + \frac{(l-2)n}{l}) > v_1(x^j + \frac{(l-2)n}{l}) \text{ by (6.l-1) so that} \\ v_{l-1}(y^{j'}) > v_{l-1}(x^j)] \text{ or} \end{aligned}$$

$$\begin{aligned} \frac{(l-2)n}{l} + 1 \leq j' \oplus \frac{(l-1)n}{l} \leq \frac{(l-1)n}{l} - l \text{ [in which case} \\ v_1(y^{j'} \oplus \frac{(l-1)n}{l}) > v_1(x^j + \frac{(l-1)n}{l}) \text{ by (6.l-1) and (6.l), so that} \\ v_l(y^{j'}) > v_l(x^j)] \text{ or} \end{aligned}$$

$$\begin{aligned} j + \frac{(l-3)n}{l} + l \leq j' \oplus \frac{(l-1)n}{l} \leq \frac{(l-2)n}{l} - l \text{ [in which case} \\ v_1(y^{j'} + \frac{(l-1)n}{l}) > v_1(x^j + \frac{(l-1)n}{l}) \text{ by (6.l),} \\ \text{so that } v_l(y^{j'}) > v_l(x^j)]. \end{aligned}$$

if and only if

$$1 \leq j' \leq j \text{ or } \frac{(l-1)n}{l} + 1 \leq j' \leq n-l \text{ or } j + \frac{(l-2)n}{l} + l \leq j' \leq \frac{(l-1)n}{l} - l$$

if and only if

$$j' \leq j \text{ or } j' \geq j + \frac{(l-2)n}{l} + l.$$

In summary, for $x^j, y^{j'} \in X^n - S^{\frac{n}{l}-(l-1)}$ and $j \leq \frac{n}{l} - l$,

$$v(x^j) > v(y^{j'}) \text{ if and only if } j < j' < j + \frac{(l-2)n}{l} + l \quad (8)$$

Next, for $x^j, y^{j'} \in X^n - S^{\frac{n}{l}-(l-1)}$ and $j \leq \frac{n}{l} - l$,

$$x^j \succ_{n,k} y^{j'} \text{ if and only if } j < j' < j + k + 1$$

. From the hypotheses, $n = \frac{l(k-1)}{l-2} - l$. Solving for k , $k = \frac{(l-2)n}{l} + l - 1$. Therefore

$$x^j \succ_{n,k} y^{j'} \text{ if and only if } j < j' < j + \frac{(l-2)n}{l} + l \quad (9)$$

Comparing (8) and (9), $x^j \succ_{n,k} y^{j'}$ if and only if $v(x^j) > v(y^{j'})$. This completes the proof that v is a Pareto representation for $\succ_{n,k} \mid_{X^n - S^{\frac{n}{l}-(l-1)}}$. ■

Returning to the proof of Proposition 2, suppose $Y \subseteq X^n$ and $Y \cap S^{j_0} \neq \emptyset$ for all $j_0 \in \{1, 2, \dots, \frac{n}{l} - (l-1)\}$ (see the statement of Lemma 2 for the definition of S^{j_0}). Since an element of X can be an element of at most l S^{j_0} 's, $|Y| \geq \frac{n - (l-1)}{l}$. Therefore if $S \subseteq X^n$ and $|S| < \frac{n - (l-1)}{l}$, then $S \cap S^{j_0} = \emptyset$ for some j_0 . For any such j_0 , $S \subseteq X^n - S^{j_0}$ so that by Lemma 2, $\succ_{n,k}|_S$ is Pareto representable in \mathfrak{R}^l . ■

6. Concluding Remarks and A Voting Paradox.

Suppose a finite set of alternatives is submitted to a committee and subsequently that committee publishes its preferences over that set. Suppose also that neither the size of the committee nor the procedure it uses to generate its preferences is known to you, and that you would like to know whether the committee could possibly be using unanimous voting to generate its preferences. Here we define unanimous voting to mean that each member of the committee has preferences represented by a utility function; that the members vote on each pair of alternatives; and that alternative x is preferred to alternative y if in the x versus y vote all members who do not abstain vote for x , and at least one member does not abstain.

Since a set of preferences could have been generated by unanimous voting by a committee of unknown size if and only if it has a Pareto representation in \mathfrak{R}^l for some l if and only if it is asymmetric and transitive (Dushnik and Miller, 1941) there is a characterization of type 3 that can be used to decide whether the committee's preferences could have been generated by unanimous voting.

However, if you somehow discover the size of the committee, then by Theorem 1, there is no characterization of finite type that can be used to decide whether the committee's preferences could have been generated by unanimous voting.

In short, it is easy to decide whether a committee could have generated its preferences by unanimous voting, unless you know the size of the committee. Then it is hard.

One final remark. Fix $l \geq 2$. Despite Theorem 1, there might exist an algorithm that checks any binary relation on a finite set for Pareto representability in \mathfrak{R}^l , and performs that check in polynomial time. However, by Theorem 1 if such an algorithm exists, it will not be a simple one; that is, it will not issue from a characterization of finite type.

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