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The Laplacian in Spherical Polar Coordinates

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I. SYNOPSIS

In treating the Hydrogen Atom's electron quantum mechanically, we normally convert the Hamiltonian from its Cartesian to its Spherical Polar form, since the problem is variable separable in the latter's coördinate system. This reading treats the brute-force method of effecting the transformation of the kinetic energy operator, normally called the Laplacian, from one to the other coördinate systems.

II. PRELIMINARY DEFINITIONS

We start with the primitive definitions

$$x = r\sin\theta\cos\phi$$

$$y = r\sin\theta\sin\phi$$

and

$$z = r\cos\theta$$

and their inverses

$$r = \sqrt{x^2+y^2+z^2}$$

$$\theta = cos^{-1}\frac{z}{r} = cos^{-1}\frac{z}{\sqrt{x^2+y^2+z^2}}$$

and

$$\phi = tan^{-1}\frac{y}{x}$$

and attempt to write (using the chain rule)

$$\frac{\partial}{\partial x} = \left(\frac{\partial r}{\partial x}\right)_{y,z}\left(\frac{\partial}{\partial r}\right)_{\theta,\phi} + \left(\frac{\partial \theta}{\partial x}\right)_{y,z}\left(\frac{\partial}{\partial \theta}\right)_{r,\phi} + \left(\frac{\partial \phi}{\partial x}\right)_{y,z}\left(\frac{\partial}{\partial \phi}\right)_{r,\theta}$$

and

$$\frac{\partial}{\partial y} = \left(\frac{\partial r}{\partial y}\right)_{x,z}\left(\frac{\partial}{\partial r}\right)_{\theta,\phi} + \left(\frac{\partial \theta}{\partial y}\right)_{x,z}\left(\frac{\partial}{\partial \theta}\right)_{r,\phi} + \left(\frac{\partial \phi}{\partial y}\right)_{x,z}\left(\frac{\partial}{\partial \phi}\right)_{r,\theta}$$

and

$$\frac{\partial}{\partial z} = \left(\frac{\partial r}{\partial z}\right)_{x,y}\left(\frac{\partial}{\partial r}\right)_{\theta,\phi} + \left(\frac{\partial \theta}{\partial z}\right)_{x,y}\left(\frac{\partial}{\partial \theta}\right)_{r,\phi} + \left(\frac{\partial \phi}{\partial z}\right)_{x,y}\left(\frac{\partial}{\partial \phi}\right)_{r,\theta}$$

III. PRELIMINARY PARTIAL DERIVATIVES

The needed (above) partial derivatives are:

$$\left(\frac{\partial r}{\partial x}\right)_{y,z} = \sin\theta\cos\phi \tag{3.1}$$

$$\left(\frac{\partial r}{\partial y}\right)_{x,z} = \sin\theta\sin\phi \tag{3.2}$$

$$\left(\frac{\partial r}{\partial z}\right)_{x,y} = \cos\theta \tag{3.3}$$

and we have as a starting point for doing the θ terms,

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$$d\cos\theta = -\sin\theta d\theta = \frac{dz}{r} + z\cdot d\left(\frac{1}{r}\right) = \frac{dz}{r} - z\cdot\left(\frac{1}{r^2}\right)dr$$
$$= \frac{dz}{r} - \frac{z}{r^2}\frac{1}{r}\left(xdx + ydy + zdz\right) \tag{3.4}$$

so that, for example (when dy = dz = 0) we have

$$-\sin\theta d\theta = -\frac{z}{r^2}\frac{x}{r}dx$$

which is

$$-\sin\theta d\theta = -\frac{r\cos\theta}{r^2}\sin\theta\cos\phi dx = \frac{r^2\left(1-\cos^2\vartheta\right)}{r^3}dz$$

so that

$$\left(\frac{\partial\theta}{\partial x}\right)_{y,z} = \frac{\cos\theta\cos\phi}{r} \tag{3.5}$$

$$\left(\frac{\partial\theta}{\partial y}\right)_{x,z} = \frac{\cos\theta\sin\phi}{r} \tag{3.6}$$

but, for the z-equation, we have

$$-\sin\theta d\theta = \frac{dz}{r} - \frac{z}{r^2}\frac{1}{r}zdz$$

which is

$$-\sin\theta d\theta = \left(\frac{1}{r} - \frac{z^2}{r^3}\right)dz = \frac{r^2 - z^2}{r^3}dz$$

$$-\sin\theta d\theta = \left(\frac{1}{r} - \frac{z^2}{r^3}\right)dz = \frac{r^2\sin^2\theta}{r^3}dz$$

so one has

$$\left(\frac{\partial\theta}{\partial z}\right)_{x,y} = -\frac{\sin\theta}{r} \tag{3.7}$$

Next, we have (as an example)

$$\tan\phi = \frac{\sin\phi}{\cos\phi} = \tan^{-1}\frac{y}{x}$$

and taking the partial derivatives on both sides, we obtain

$$\frac{1}{\cos\vartheta}d(\sin\phi) + \frac{\sin\phi}{-\cos^2\phi}d(\cos\phi)$$

so

$$\left(1 + \frac{\sin^2\phi}{\cos^2\phi}\right)d\phi = \frac{dy}{x} - \frac{y}{x^2}dx$$

or

$$\left(\frac{1}{\cos^2\phi}\right)d\phi = \frac{dy}{x} - \frac{y}{x^2}dx$$

so, after multiplying across by $\cos^2\phi$ leads to (at constant x)

$$\left(\frac{\partial\phi}{\partial y}\right)_{x,z} = \frac{\cos\phi}{r\sin\theta} \tag{3.8}$$

and (at constant y)

$$\left(\frac{\partial\phi}{\partial x}\right)_{y,z} = -\frac{\sin\phi}{r\sin\theta} \tag{3.9}$$

$$\left(\frac{\partial\phi}{\partial z}\right)_{x,y} = 0 \tag{3.10}$$

IV. THE FIRST PARTIAL DERIVATIVE TERMS

Given these results (above) we write

$$\frac{\partial}{\partial z} = \cos\theta\frac{\partial}{\partial r} - \left(\frac{\sin\theta}{r}\right)\frac{\partial}{\partial\theta} \tag{4.1}$$

and

$$\frac{\partial}{\partial y} = \left(\sin\theta\sin\phi\right)\frac{\partial}{\partial r} + \left(\frac{\cos\theta\sin\phi}{r}\right)\frac{\partial}{\partial\theta} + \left(\frac{\cos\phi}{r\sin\theta}\right)\frac{\partial}{\partial\phi} \tag{4.2}$$

and

$$\frac{\partial}{\partial x} = \left(\sin\theta\cos\phi\right)\frac{\partial}{\partial r} + \left(\frac{\cos\theta\cos\phi}{r}\right)\frac{\partial}{\partial\theta} + \left(-\frac{\sin\phi}{r\sin\theta}\right)\frac{\partial}{\partial\phi} \tag{4.3}$$

V. GATHERING TERMS TO FORM THE LAPLACIAN

From Equation 4.1 we form

$$\frac{\partial^2}{\partial z^2} = \cos\theta \frac{\partial \left[\cos\theta \frac{\partial}{\partial r} - \left(\frac{\sin\theta}{r}\right)\frac{\partial}{\partial \theta}\right]}{\partial r} - \left(\frac{\sin\theta}{r}\right)\frac{\partial\left(\cos\theta\frac{\partial}{\partial r} - \left(\frac{\sin\theta}{r}\right)\frac{\partial}{\partial\theta}\right]}{\partial\theta} \tag{5.1}$$

while from Equation 4.2 we obtain

$$\begin{aligned}\frac{\partial^2}{\partial y^2} = (\sin\theta\sin\phi)\,&\frac{\partial\left[\sin\theta\sin\phi\frac{\partial}{\partial r} + \left(\frac{\cos\theta\sin\phi}{r}\right)\frac{\partial}{\partial\theta} + \left(\frac{\cos\phi}{r\sin\theta}\right)\frac{\partial}{\partial\phi}\partial r\right]}{\partial r} \\ + \left(\frac{\cos\theta\sin\phi}{r}\right)&\frac{\partial\left[\sin\theta\sin\phi\frac{\partial}{\partial r} + \left(\frac{\cos\theta\sin\phi}{r}\right)\frac{\partial}{\partial\theta} + \left(\frac{\cos\phi}{r\sin\theta}\right)\frac{\partial}{\partial\phi}\right]}{\partial\theta} \\ + \left(\frac{\cos\phi}{r\sin\theta}\right)&\frac{\partial\left[\sin\theta\sin\phi\frac{\partial}{\partial r} + \left(\frac{\cos\theta\sin\phi}{r}\right)\frac{\partial}{\partial\theta} + \left(\frac{\cos\phi}{r\sin\theta}\right)\frac{\partial}{\partial\phi}\right]}{\partial\phi}\end{aligned} \tag{5.2}$$

and from Equation 4.3 we obtain

$$\begin{aligned}\frac{\partial^2}{\partial x^2} = (\sin\theta\cos\phi)\,&\frac{\partial\left[\sin\theta\cos\phi\frac{\partial}{\partial r} + \left(\frac{\cos\theta\cos\phi}{r}\right)\frac{\partial}{\partial\theta} - \left(\frac{\sin\phi}{r\sin\theta}\right)\frac{\partial}{\partial\phi}\right]}{\partial r} \\ + \left(\frac{\cos\theta\cos\phi}{r}\right)&\frac{\partial\left[\sin\theta\cos\phi\frac{\partial}{\partial r} + \left(\frac{\cos\theta\cos\phi}{r}\right)\frac{\partial}{\partial\theta} - \left(\frac{\sin\phi}{r\sin\theta}\right)\frac{\partial}{\partial\phi}\right]}{\partial\theta} \\ - \left(\frac{\sin\phi}{r\sin\theta}\right)&\frac{\partial\left[\sin\theta\cos\phi\frac{\partial}{\partial r} + \left(\frac{\cos\theta\cos\phi}{r}\right)\frac{\partial}{\partial\theta} - \left(\frac{\sin\phi}{r\sin\theta}\right)\frac{\partial}{\partial\phi}\right]}{\partial\phi}\end{aligned} \tag{5.3}$$

Expanding, we have

$$\begin{aligned}\frac{\partial^2}{\partial z^2} = \cos^2\theta\frac{\partial^2}{\partial r^2} + \frac{\cos\theta\sin\theta}{r^2}\frac{\partial}{\partial\theta} - \frac{\sin\theta\cos\theta}{r}\frac{\partial^2}{\partial r\partial\theta} \\ - \left(\frac{\sin\theta}{r}\right)\left(-\sin\theta\frac{\partial}{\partial r} - \cos\theta\frac{\partial}{\partial\theta}\right) - \frac{\sin\theta\cos\theta}{r^2}\frac{\partial}{\partial\theta} + \left(\frac{\sin\theta}{r}\right)^2\frac{\partial^2}{\partial\theta^2}\end{aligned} \tag{5.4}$$

while for the y-equation we have

$$\frac{\partial^2}{\partial y^2} = \sin^2\theta\sin^2\phi\frac{\partial^2}{\partial r^2} \tag{5.5}$$

$$+\sin\theta\sin\phi\left[+\left(\frac{\cos\theta\sin\phi}{r^2}\right)\frac{\partial}{\partial\theta} + \left(\frac{\cos\theta\sin\phi}{r}\right)\frac{\partial^2}{\partial r\partial\theta}\right] \tag{5.6}$$

$$+\sin\theta\sin\phi\left[\left(-\frac{\cos\phi}{r^2\sin\theta}\right)\frac{\partial}{\partial\phi} + \left(\frac{\cos\phi}{r\sin\theta}\right)\frac{\partial^2}{\partial r\partial\phi}\right] \tag{5.7}$$

$$+\left(\frac{\cos\theta\sin\phi}{r}\right)\left[\cos\theta\sin\phi\frac{\partial}{\partial r} + \sin\theta\sin\phi\frac{\partial^2}{\partial r\partial\theta}\right] \tag{5.8}$$

$$+\left(\frac{\cos\theta\sin\phi}{r}\right)\left[-\left(\frac{\sin\theta\sin\phi}{r}\right)\frac{\partial}{\partial\theta} + \left(\frac{\cos\theta\sin\phi}{r}\right)\frac{\partial^2}{\partial\theta^2}\right] \tag{5.9}$$

$$+\left(\frac{\cos\theta\sin\phi}{r}\right)\left[-\left(\frac{\cos\phi\cos\theta}{r\sin^2\theta}\right)\frac{\partial}{\partial\phi} + \left(\frac{\cos\phi}{r\sin\theta}\right)\frac{\partial^2}{\partial\phi\partial\theta}\right] \tag{5.10}$$

$$+\left(\frac{\cos\phi}{r\sin\theta}\right)\left[\sin\theta\cos\phi\frac{\partial}{\partial r} + \sin\theta\sin\phi\frac{\partial^2}{\partial r\partial\phi}\right] \tag{5.11}$$

$$+\left(\frac{\cos\phi}{r\sin\theta}\right)\left[+\left(\frac{\cos\theta\cos\phi}{r}\right)\frac{\partial}{\partial\theta} + \left(\frac{\cos\theta\sin\phi}{r}\right)\frac{\partial^2}{\partial\theta\partial\phi}\right] \tag{5.12}$$

$$+\left(\frac{\cos\phi}{r\sin\theta}\right)\left[-\left(\frac{\sin\phi\cos\phi}{r\sin\theta}\right)\frac{\partial}{\partial\phi} + \left(\frac{\cos\phi}{r\sin\theta}\right)\frac{\partial^2}{\partial\phi^2}\right] \tag{5.13}$$

and finally

$$\frac{\partial^2}{\partial x^2} = (\sin\theta\cos\phi)\sin\theta\cos\phi\frac{\partial^2}{\partial r^2}$$

$$+(\sin\theta\cos\phi)\left[-\left(\frac{\cos\theta\cos\phi}{r^2}\right)\frac{\partial}{\partial\theta}+\left(\frac{\cos\theta\cos\phi}{r}\right)\frac{\partial^2}{\partial\theta\partial r}\right] \quad (5.14)$$

$$-(\sin\theta\cos\phi)\left[-\left(\frac{\sin\phi}{r^2\sin\theta}\right)\frac{\partial}{\partial\phi}+\left(\frac{\sin\phi}{r\sin\theta}\right)\frac{\partial^2}{\partial\phi\partial r}\right] \quad (5.15)$$

$$+\left(\frac{\cos\theta\cos\phi}{r}\right)\left[\cos\theta\cos\phi\frac{\partial}{\partial r}+\sin\theta\cos\phi\frac{\partial^2}{\partial r\partial\theta}\right] \quad (5.16)$$

$$+\left(\frac{\cos\theta\cos\phi}{r}\right)\left[-\left(\frac{\sin\theta\cos\phi}{r}\right)\frac{\partial}{\partial\theta}+\left(\frac{\cos\theta\cos\phi}{r}\right)\frac{\partial^2}{\partial\theta^2}\right] \quad (5.17)$$

$$+\left(\frac{\cos\theta\cos\phi}{r}\right)\left[+\left(\frac{\sin\phi}{r\sin^2\theta}\right)\frac{\partial}{\partial\phi}-\left(\frac{\sin\phi}{r\sin\theta}\right)\frac{\partial^2}{\partial\phi\partial\theta}\right] \quad (5.18)$$

$$-\left(\frac{\sin\phi}{r\sin\theta}\right)\left[\sin\theta\sin\phi\frac{\partial}{\partial r}+\sin\theta\cos\phi\frac{\partial^2}{\partial r\partial\phi}\right] \quad (5.19)$$

$$-\left(\frac{\sin\phi}{r\sin\theta}\right)\left[-\left(\frac{\cos\theta\sin\phi}{r}\right)\frac{\partial}{\partial\theta}+\left(\frac{\cos\theta\cos\phi}{r}\right)\frac{\partial^2}{\partial\theta\partial\phi}\right] \quad (5.20)$$

$$-\left(\frac{\sin\phi}{r\sin\theta}\right)\left[-\left(\frac{\cos\phi}{r\sin\theta}\right)\frac{\partial}{\partial\phi}-\left(\frac{\sin\phi}{r\sin\theta}\right)\frac{\partial^2}{\partial\phi^2}\right] \quad (5.21)$$

Now, one by one, we expand completely each of these three terms. We have

$$\frac{\partial^2}{\partial z^2} = \cos^2\theta\frac{\partial^2}{\partial r^2} \quad (5.22)$$

$$+\frac{\cos\theta\sin\theta}{r^2}\frac{\partial}{\partial\theta} \quad (5.23)$$

$$-\frac{\sin\theta\cos\theta}{r}\frac{\partial^2}{\partial r\partial\theta} \quad (5.24)$$

$$+\left(\frac{\sin^2\theta}{r}\right)\frac{\partial}{\partial r} \quad (5.25)$$

$$-\left(\frac{\sin\theta\cos\theta}{r}\right)\frac{\partial^2}{\partial r\partial\theta} \quad (5.26)$$

$$+\frac{\sin\theta\cos\theta}{r^2}\frac{\partial}{\partial\theta} \quad (5.27)$$

$$+\left(\frac{\sin^2\theta}{r^2}\right)\frac{\partial^2}{\partial\theta^2} \quad (5.28)$$

and, for the y-equation:

$$\frac{\partial^2}{\partial y^2} = \sin^2\theta\sin^2\phi\frac{\partial^2}{\partial r^2} \quad (5.29)$$

$$(5.6)\to+\left(\frac{\sin\theta\cos\theta\sin^2\phi}{r^2}\right)\frac{\partial}{\partial\theta} \quad (5.30)$$

$$+\left(\frac{\cos\theta\sin\theta\sin^2\phi}{r}\right)\frac{\partial^2}{\partial r\partial\theta} \quad (5.31)$$

$$(5.7)\to-\left(\frac{\sin\phi\cos\phi}{r^2}\right)\frac{\partial}{\partial\phi} \quad (5.32)$$

$$+\left(\frac{\cos\phi\sin\phi}{r}\right)\frac{\partial^2}{\partial r\partial\phi} \quad (5.33)$$

$$(5.8)\to+\left(\frac{\cos^2\theta\sin^2\phi}{r}\right)\frac{\partial}{\partial r} \quad (5.34)$$

$$+\left(\frac{\cos\theta\sin\theta\sin^2\phi}{r}\right)\frac{\partial^2}{\partial r\partial\theta} \quad (5.35)$$

$$-\left(\frac{\sin\theta\cos\theta\sin^2\phi}{r^2}\right)\frac{\partial}{\partial\theta} \quad (5.36)$$

$$(5.9)\to+\left(\frac{\cos^2\theta\sin^2\phi}{r^2}\right)\frac{\partial^2}{\partial\theta^2} \quad (5.37)$$

$$-\left(\frac{\cos^2\theta\cos\phi\sin\phi}{r\sin^2\theta}\right)\frac{\partial}{\partial\phi} \quad (5.38)$$

$$+\left(\frac{\cos\theta\cos\phi\sin\phi}{r^2\sin\theta}\right)\frac{\partial^2}{\partial\phi\partial\theta} \quad (5.39)$$

$$(5.10)\to+\left(\frac{\cos^2\phi}{r}\right)\frac{\partial}{\partial r} \quad (5.40)$$

$$+\left(\frac{\cos\phi\sin\phi}{r}\right)\frac{\partial^2}{\partial r\partial\phi} \quad (5.41)$$

$$+\left(\frac{\cos^2\phi\cos\theta}{r^2\sin\theta}\right)\frac{\partial}{\partial\theta} \quad (5.42)$$

$$(5.12)\to+\left(\frac{\cos\theta\cos\phi\sin\phi}{r^2\sin\theta}\right)\frac{\partial^2}{\partial\theta\partial\phi} \quad (5.43)$$

$$(5.13)\to-\left(\frac{\cos^2\phi\sin\phi}{r\sin^2\theta}\right)\frac{\partial}{\partial\phi} \quad (5.44)$$

$$+\left(\frac{\cos^2\phi}{r^2\sin^2\theta}\right)\frac{\partial^2}{\partial\phi^2} \quad (5.45)$$

and finally, for the x-equation, we have

$$\frac{\partial^2}{\partial x^2} = \sin^2\theta\cos^2\phi\frac{\partial^2}{\partial r^2} \quad (5.46)$$

$$(5.14) \to -\left(\frac{\sin\theta\cos\theta\cos^2\phi}{r^2}\right)\frac{\partial}{\partial\theta} \quad (5.47)$$

$$(5.14) \to +\left(\frac{\sin\theta\cos\theta\cos^2\phi}{r}\right)\frac{\partial^2}{\partial\theta\partial r} \quad (5.48)$$

$$\left(\frac{\cos\phi\sin\phi}{r^2}\right)\frac{\partial}{\partial\phi} \quad (5.49)$$

$$-\left(\frac{\sin\phi\cos\phi}{r}\right)\frac{\partial^2}{\partial\phi\partial r} \quad (5.50)$$

$$(5.15) \to +\left(\frac{\cos^2\theta\cos^2\phi}{r}\right)\frac{\partial}{\partial r} \quad (5.51)$$

$$+\left(\frac{\sin\theta\cos\theta\cos^2\phi}{r}\right)\frac{\partial^2}{\partial r\partial\theta} \quad (5.52)$$

$$(5.15) \to -\left(\frac{\sin\theta\cos\theta\cos^2\phi}{r^2}\right)\frac{\partial}{\partial\theta} \quad (5.53)$$

$$+\left(\frac{\cos^2\theta\cos^2\phi}{r^2}\right)\frac{\partial^2}{\partial\theta^2} \quad (5.54)$$

$$(5.16) \to +\left(\frac{\cos\theta\cos\phi}{r}\right)\left(\frac{\cos\phi\sin\phi}{r\sin\theta}\right)\frac{\partial}{\partial\phi} \quad (5.55)$$

$$-\left(\frac{\sin\phi\cos\phi\cos\theta}{r^2\sin\theta}\right)\frac{\partial^2}{\partial\phi\partial\theta} \quad (5.56)$$

$$(5.17) \to -\left(\frac{\sin^2\phi}{r}\right)\frac{\partial}{\partial r} \quad (5.57)$$

$$-\left(\frac{\sin\phi\cos\phi}{r}\right)\frac{\partial^2}{\partial r\partial\phi} \quad (5.58)$$

$$(5.19) \to +\left(\frac{\cos\theta\sin^2\phi}{r^2\sin\theta}\right)\frac{\partial}{\partial\theta} \quad (5.59)$$

$$-\left(\frac{\cos\theta\sin\phi\cos\phi}{r^2\sin\theta}\right)\frac{\partial^2}{\partial\theta\partial\phi} \quad (5.60)$$

$$(5.20) \to +\left(\frac{\sin\phi\cos\phi}{r\sin^2\theta}\right)\frac{\partial}{\partial\phi} \quad (5.61)$$

$$+\left(\frac{\sin^2\phi}{r^2\sin^2\theta}\right)\frac{\partial^2}{\partial\phi^2} \quad (5.62)$$

Gathering terms as coefficients of partial derivatives, we obtain (from Equations 5.22, 5.29 and 5.46)

$$\frac{\partial^2}{\partial r^2}\left(\cos^2\theta + \sin^2\theta\sin^2\phi + \sin^2\theta\cos^2\phi\right) \to \frac{\partial^2}{\partial r^2}$$

and (from Equations 5.23, 5.26, 5.30, 5.36, 5.42, 5.47, 5.53, and 5.59)

$$\frac{\partial}{\partial\theta}\left(+\frac{\cos\theta\sin\theta}{r^2} + \frac{\sin\theta\cos\theta}{r^2} - \frac{\sin\theta\cos\theta\sin^2\phi}{r^2} - \frac{\sin\theta\cos\theta\sin^2\phi}{r^2} + \frac{\cos^2\phi\cos\theta}{r^2\sin\theta}\right.$$
$$\left. - \frac{\sin\theta\cos\theta\cos^2\phi}{r^2} - \frac{\sin\theta\cos\theta\cos^2\phi}{r^2} + \frac{\cos\theta\sin^2\phi}{r^2\sin\theta}\right)$$
$$\to \frac{\cos\theta}{r^2\sin\theta}\frac{\partial}{\partial\theta} \quad (5.63)$$

while we obtain from Equations 5.28, 5.37, and 5.54:

$$\frac{\partial^2}{\partial\theta^2}\left(\frac{\sin^2\theta}{r^2} + \frac{\cos^2\theta\sin^2\phi}{r^2} + \frac{\cos^2\theta\cos^2\phi}{r^2}\right) \to \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} \quad (5.64)$$

From Equations 5.25, 5.34, 5.40, 5.51, 5.57,

$$\frac{\partial}{\partial r}\left(+\frac{\sin^2\theta}{r} + \frac{\cos^2\theta\sin^2\phi}{r} + \frac{\cos^2\phi}{r} + \frac{\cos^2\theta\cos^2\phi}{r} - \frac{\sin^2\phi}{r}\right) \to \frac{2}{r}\frac{\partial}{\partial r} \quad (5.65)$$

From Equations 5.32, 5.38, 5.44, 5.49, 5.55 and 5.61 we obtain

$$\frac{\partial}{\partial\phi}\left(-\frac{\sin\phi\cos\phi}{r^2} - \frac{\cos^2\theta\cos\phi\sin\phi}{r\sin^2\theta} - \frac{\cos^2\theta\cos\phi\sin\phi}{r\sin^2\theta} + \frac{\cos\phi\sin\phi}{r^2} + \left(\frac{\cos\theta\cos\phi}{r}\right)\right.$$
$$\left. + \left(\frac{\cos\theta\cos^2\phi\sin\phi}{r^2\sin\theta}\right) + \frac{\sin\phi\cos\phi}{r\sin^2\theta}\right) \to 0 \quad (5.66)$$

From Equations 5.45 and 5.62 we obtain

$$\frac{\partial^2}{\partial\phi^2}\left(\frac{\cos^2\phi}{r^2\sin^2\theta} + \frac{\sin^2\phi}{r^2\sin^2\theta}\right) \to \left(\frac{1}{r^2\sin^2\theta}\right)\frac{\partial^2}{\partial\phi^2} \quad (5.67)$$

The mixed derivatives yield, first, from Equations 5.33, 5.41, 5.50, and 5.58 leading to

$$\frac{\partial^2}{\partial r\partial\phi}\left(\frac{\cos\phi\sin\phi}{r}+\frac{\cos\phi\sin\phi}{r}-\frac{\sin\phi\cos\phi}{r}-\frac{\sin\phi\cos\phi}{r}\right)\to 0 \tag{5.68}$$

From Equations 5.24, 5.27, 5.35, 5.31 5.52, 5.48

$$\frac{\partial^2}{\partial r\partial\theta}\left(-\frac{\sin\theta\cos\theta}{r}-\frac{\sin\theta\cos\theta}{r}+\frac{\cos\theta\sin\theta\sin^2\phi}{r}\right.$$
$$\left.+\frac{\sin\theta\cos\theta\cos^2\phi}{r}+\frac{\cos\theta\sin\theta\sin^2\phi}{r}+\frac{\sin\theta\cos\theta\cos^2\phi}{r}\right)\to 0 \tag{5.69}$$

From Equations 5.39 5.43 5.56 5.60

$$\frac{\partial^2}{\partial\phi\partial\theta}\left(\frac{\cos\theta\cos\phi\sin\phi}{r^2\sin\theta}+\frac{\cos\phi\sin\phi}{r^2\sin\theta}\right.$$
$$\left.-\left(\frac{\sin\phi\cos\phi\cos\theta}{r^2\sin\theta}\right)-\left(\frac{\cos\theta\sin\phi\cos\phi}{r^2\sin\theta}\right)\right)\to 0 \tag{5.70}$$

Gathering together the non-vanishing terms, we obtain

$$\frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}+\frac{\cos\theta}{r^2\sin\theta}\frac{\partial}{\partial\theta}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$

which is one of the two “classic” forms for ∇^2. The other is

$$\frac{1}{r^2}\frac{\partial\left(r^2\frac{\partial}{\partial r}\right)}{\partial r}+\frac{1}{r^2\sin^2\theta}\left(\sin\theta\frac{\partial\left(\sin\theta\frac{\partial}{\partial\theta}\right)}{\partial\theta}+\frac{\partial^2}{\partial\phi^2}\right)$$

VI. MAPLE EQUIVALENT

A. Example 1

Here is a set of Maple instructions adjusted from the 2-dimensional code [1] for our 3-dimensional case, which will get you the same result:

```
restart;
f:=g(r,theta,phi);
tx :=
sin(theta)*cos(phi)*diff(f,r)+((cos(theta)*cos(phi))/r)*diff(f,theta)
-(sin(phi)/(r*sin(theta)))*diff(f,phi);
tx2:=expand(
sin(theta)*cos(phi)*diff(tx,r)+((cos(theta)*cos(phi))/r)*diff(tx,theta)
-(sin(phi)/(r*sin(theta)))*diff(tx,phi));
ty :=
sin(theta)*sin(phi)*diff(f,r)+((cos(theta)*sin(phi))/r)*diff(f,theta)
+(cos(phi)/(r*sin(theta)))*diff(f,phi);

ty2:=expand(sin(theta)*sin(phi)*diff(ty,r)+((cos(theta)*sin(phi))/r)
*diff(ty,theta)+(cos(phi)/(r*sin(theta)))*diff(ty,phi));
tz := cos(theta)*diff(f,r)
-(sin(theta)/r)*diff(f,theta);
tz2 := expand(cos(theta)*diff(tz,r)-(sin(theta)/r)*diff(tz,theta));

del := tx2+ty2+tz2:
del := algsubs( cos(theta)^2=1-sin(theta)^2, del ):
del := expand(algsubs( cos(phi)^2=1-sin(phi)^2, del ));
```

B. Example 2

Here is another version of the same thing:

```
> #CARTESIAN TO SPHERICAL POLAR
> restart;
> with(plots):
Warning, the name changecoords has been redefined
> uu:=u(sqrt(x^2+y^2+z^2),arccos(z/sqrt(x^2+y^2+z^2)),arctan(y,x));
```

$$uu := \mathrm{u}(\sqrt{x^2+y^2+z^2},\, \arccos(\frac{z}{\sqrt{x^2+y^2+z^2}}),\, \arctan(y,\, x))$$

```
> ux:=diff(uu,x):
> uy:=diff(uu,y):
> uz:=diff(uu,z):
> uxx:=diff(ux,x):
> uyy:=diff(uy,y):
> uzz:=diff(uz,z):
> Lapu:=simplify(uxx+uyy+uzz):
> assume(r,positive);
> Lapu:=simplify(subs(x=r*sin(theta)*cos(phi),
> y=r*sin(theta)*sin(phi),
> z = r*cos(theta),
> arctan(sin(theta)*sin(phi),sin(theta)*cos(phi))=phi,
> arccos(cos(theta))=theta,
> Lapu),trig):
> Lapu := subs(arctan(sin(theta)*sin(phi),sin(theta)*cos(phi))=phi,
> arccos(cos(theta))=theta,
> Lapu):
> Lapu := algsubs(-1+cos(theta)^2=-sin(theta)^2,Lapu):
> Lapu:=expand(Lapu);
```

$$\mathit{Lapu} := \frac{\mathrm{D}_2(u)(r\tilde{},\, \theta,\, \phi)\sin(\theta)^2\cos(\theta)}{r\tilde{}^2\,(\sin(\theta)^2)^{(3/2)}} + \frac{\mathrm{D}_{2,\,2}(u)(r\tilde{},\, \theta,\, \phi)}{r\tilde{}^2} + \frac{\mathrm{D}_{3,\,3}(u)(r\tilde{},\, \theta,\, \phi)}{r\tilde{}^2\sin(\theta)^2} + \frac{2\,\mathrm{D}_1(u)(r\tilde{},\, \theta,\, \phi)}{r\tilde{}} + \mathrm{D}_{1,\,1}(u)(r\tilde{},\, \theta,\, \phi)$$

It takes some getting used to Maple notation to see that this is the expected result.

VII. COMMENTS

The reader should be aware that the brute force methods used here are primitive in the extreme, and that there are better ways to carry out the transformation from Cartesian to Spherical Polar (and indeed any orthogonal) coördinate system.

[1] Mathias Kawski, http://math.la.asu.edu/∼kawski/MAPLE/MAPLE.html