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Invitation to Magnetism and Magentic Resonance

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I. SYNOPSIS

With NMR so ubiquitous in modern life, it is reasonable to study spins in magnetic fields, hence this first attempt.

II. UNITS

Coulomb's Law

$$F = k\frac{QQ'}{r^2}$$

is an empirical law linking the force (F) between two charges, Q and Q', separated by a distance of r. The constant is required to make the equation true depending on the units involved. In elementary physics, using the mks system, we have

$$F(newtons) = \frac{1}{4\pi\epsilon_0}\frac{Q(Coulomb)Q'(Coulomb)}{r^2(m^2)}$$

where ϵ_0 is called the permittivity constant. Its measured value is about $8.85 \times 10^{-12} coul^2/(nt - m^2)$.

In our atomic/molecular work, it is more convenient to use units which are less common. Specifically, in modified cgs units, we have

$$F(dynes) = \frac{q(statcoulomb)q'(statcoulomb)}{r^2(cm^2)}$$

which defines a statcoulomb. Thus, a force of 1 dyne is exerted by two charges of a statcoulomb each if separated by 1 cm. These are not SI units! But they allow us to do Bohr theory (and other work) without worrying about the permittivity of free space! In these units, the charge on the electron is $4.8 \times 10^{-10} statcoulomb$ which is the same as $1.6 \times 10^{-19} Coulomb$, a convenient reference for conversion between the two systems.

The electric field associated with a charge (Q) at the origin, is

$$E = \frac{1}{4\pi\epsilon_0}\frac{Q(Coulomb)}{r^2(m^2)}$$

where we are using a unit test charge (1 Coulomb) to probe the force at the point (x, y, z).

It is the magnetic field which causes the problems. First, we start with the Lorentz force:

$$\vec{F} = q\vec{v} \otimes \vec{B} + q\vec{E}$$

which includes the electric field just discussed [1]. In the cgs system, B is measured in maxwells/(square cm) i.e, in gauss. Then

$$1gauss = 10^{-4}\frac{weber}{m^2} = 1\frac{maxwell}{cm^2}$$

(See Figure 2).

We start with rationalized units versus unrationalized, see Figure 1.

III. EQUIVALENCE OF ORBITING ELECTRON AND MAGNETIC MOMENT

For a magnetic field (vector) $\vec{B}$ acting on an arm of a current loop, a square current loop and a Bohr orbit are similar (see Figure 3). The force on each single charge (q) traveling in the arm comes from the Lorentz force [2]:

$$\vec{F} = q\vec{v} \otimes \vec{B} \tag{3.1}$$

Since $\vec{v}$ is perpendicular to $\vec{B}$ (in our case), the cross product simplifies. The current $\imath$ is given by:

$$\imath \rightarrow n(\frac{charge}{cm^3}) \times q(\frac{statcoulomb}{charge}) \times v(\frac{cm}{sec}) \times \sigma(cm^2) = \frac{statcoulomb}{sec} \tag{3.2}$$

where σ is the cross sectional area of the wire-loop, and the force on each charge is:

$$\mid \vec{F} \mid = q \mid \vec{v} \mid\mid \vec{B} \mid \sin 90^o = qvB \tag{3.3}$$

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which, in the c.g.s. system is

$$statcoulomb \times \left(\frac{cm}{sec}\right)\left(\frac{dyne\,sec}{statcoulomb\,cm}\right) = dyne \quad (3.4)$$

where we have defined B's c.g.s. units as

$$\frac{dyne\,sec}{statcoulomb\,cm} \quad (3.5)$$

or, in m.k.s. terms, we have for the force (equivalent of Equation 3.4)

$$Coulomb \times \left(\frac{meter}{second}\right) \times \left(\frac{Newton\ second}{Coulomb\ meter}\right) = Newton \quad (3.6)$$

which defines one Tesla [3] as (the equivalent of Equation 3.5)

$$1\ Tesla = \frac{Newton\ second}{Coulomb\ meter} \quad (3.7)$$

or

$$B \equiv \frac{Newton\ second}{Coulomb\ meter} \times 10^5 \frac{dynes}{Newton} \times 10^{-2} \frac{meter}{cm} \quad (3.8)$$

or

$$1\ Tesla = 10^3 \frac{dyne\ sec}{Coulomb\ cm} \quad (3.9)$$

and since 1 C = 2.99790×10^9statcoulomb, we have

$$1\ Tesla = 10^3 \frac{dyne\ sec}{Coulomb\ cm} \times \frac{1 Coulomb}{2.99790 \times 10^9\ statcoulomb} \quad (3.10)$$

which is

$$1\ Tesla = 0.3335683 \times 10^{-6} \frac{dyne\ sec}{statcoulomb\ cm} \quad (3.11)$$

and remember that from Coulomb's law, a statcoulomb is a $dyne^{1/2}cm$. i.e.,

$$1\ Tesla = 0.3335683 \times 10^{-6} \frac{dyne\ sec}{dyne^{1/2}cm^2} = \frac{dyne^{1/2}sec}{cm^2} \quad (3.12)$$

which is the useful c.g.s form of the Tesla

Since the number of charges is $n \times a \times \sigma$, the force on one arm of the loop (of length 'a') is

$$F = (n \times a \times \sigma) \times q \times v \times B \rightarrow \imath a B \quad (3.13)$$

$$\left(\frac{Coulomb}{second}\right) \times meter \times \left(\frac{Newton\ second}{Coulomb\ meter}\right) = Newton \quad (3.14)$$

which is reversed on the other (opposite arm) leg of the loop.

The loop is $a \times b$ in area (and $2a+2b$ in circumference), the moment arm about the pivot point is $\frac{b}{2}\sin\alpha$ if α is the angle between the loop and the field. The moment arm is of length 'b/2', i.e., from the axis to a horizontal arm is b/2 (cm). The torque (τ) then is

$$\tau(orque) = |\vec{r} \otimes \vec{F}| = 2\left(\frac{b}{2}\sin\alpha\right)\imath a B \rightarrow Newton\ meter \quad (3.15)$$

but, since a $\times$b is the area (A) of the loop, we have

$$\tau(orque) = iAB\sin\alpha \rightarrow Newton\ meter \quad (3.16)$$

Commonly, this torque is related to a magnetic moment equivalent, i.e.,

$$\vec{\tau}(orque) = \vec{\mu} \times \vec{B} \rightarrow \left(\frac{meter^2 Coulomb}{second}\right)\left(\frac{Newton\ second}{Coulomb\ meter}\right) \rightarrow Newton\,meter \quad (3.17)$$

in analogy with an electric dipole in an electric field. A current loop is equivalent to a magnetic moment, a tiny bar magnet.

We assume that the above would hold for a Bohr orbit.

IV. BOHR THEORY INTERPRETATION OF THE ORBITING ELECTRON'S MAGNETIC MOMENT

At the atomic level, the (Newton) Meter-Kilogram-Second (m.k.s) Scale is a bit cumbersome, especially when we are dealing with charged particles. It is more convenient to work in the cgs system, with charges in statcoulomb. The major advantage of using statcoulombs is that it translates directly, without permitivities and dielectric constant considerations, into forces when employed in Coulomb's Law. This allows us to write that

$$1\ statcoulomb = 1\ dyne^{1/2} cm \tag{4.1}$$

since

$$F = \frac{q \times q'}{r^2} \to \frac{statcoulomb^2}{cm^2} = dynes \tag{4.2}$$

From Bohr Theory we had

$$r = \frac{n^2 \hbar^2}{Zme^2} \to \frac{erg^2\ sec^2}{(gram\ statcoulomb^2)} = \left(\frac{erg^2\ sec^2}{(gram\ dyne\ cm^2)} \right) = cm \tag{4.3}$$

and [4]

$$mr^2 \dot{\theta} = n\hbar = p_\theta \to\ erg\ sec \tag{4.4}$$

so, solving for $\dot{\theta}$ we have

$$\dot{\theta} = \frac{n\hbar}{mr^2} \to \frac{erg\ sec}{gram\ cm^2} = \frac{dyne\ sec}{g\ cm} = \frac{gm\ (cm/sec^2)\ sec}{g\ cm} = \frac{1}{sec} \tag{4.5}$$

and then the current (= statcoulomb/sec) = charge/transit-time = $-e/\tau$, where [5] τ is the period.

$$\frac{\Delta\theta}{\Delta t} = \dot{\theta} = \frac{n\hbar}{mr^2} = \frac{2\pi}{\tau} \tag{4.6}$$

so

$$\frac{charge}{period} = \frac{e}{\tau} = i = -\frac{en\hbar}{2\pi mr^2} \to \frac{statcoulomb\ erg\ sec}{gram\ cm^2} = \frac{statcoulomb}{sec} \tag{4.7}$$

but, since the area is πr^2, we have

$$iA = -\frac{en\hbar}{2\pi mr^2}\pi r^2 = -\frac{en\hbar}{2m} \to \frac{statcoulomb\, cm^2}{sec} = \frac{statcoulomb\, erg\, sec}{gram} \tag{4.8}$$

which defines the Bohr magnetic moment

$$\mu_B = -\frac{e\hbar}{2m} \to \frac{dyne^{1/2} cm\ erg\ sec}{gram} = \frac{dyne^{1/2} cm^2}{sec} \tag{4.9}$$

i.e.,

$$= \frac{4.8 \times 10^{-10} statcoulomb \times 6.627 \times 10^{-27} erg\ sec}{2 \times 9.1094 \times 10^{-28} grams \times 2 \times \pi} = 0.27781 \times 10^{-9} \frac{dyne^{1/2} cm^2}{sec} \tag{4.10}$$

known as the Bohr magneton. This works out to be (since the charge on the electron is $1.6 \times 10^{-19} Coulomb$ and $4.8 \times 10^{-10} statcoulomb$)

$$0.27788 \times 10^{-9} \frac{statcoulomb\ erg\ second}{Tesla} \left(\frac{1.6 \times 10^{-19} Coulomb}{4.8 \times 10^{-10} statcoulomb} \right) \times \frac{10^3 \frac{grams}{kilograms}}{10^7 \frac{erg}{Joule}} \tag{4.11}$$

$$= 0.0926 \times 10^{-22} \frac{Joule\ Coulomb\ second}{kilogram} \tag{4.12}$$

$$= 0.0926 \times 10^{-22} \frac{Joule\ Coulomb\ second}{kilogram} \frac{\frac{Newton\ second}{Coulomb\ meter}}{Tesla} = \frac{J}{T} \tag{4.13}$$

which compares to the literature values of $9.27402 \times 10^{-24} J/T$. or $9.27402 \times 10^{-21} erg/G$.

V. THE NUCLEAR MAGNETIC MOMENT

The nuclear magnetic moment, obtained by changing the mass from that of the electron to that of the proton, is

$$\mu_N = 9.26 \times 10^{-24} \frac{J}{T} \times \frac{9.1094 \times 10^{-28}}{1.67 \times 10^{-24}} \tag{5.1}$$

which is, in mks,

$$50.5 \times 10^{-28} \frac{J}{T} \tag{5.2}$$

which compares with literature values of $5.0505 \times 10^{-24} erg/G$ i.e., $5.0508 \times 10^{-27} J/T$

HOWEVER, the true (measured) nuclear magnetic moment (for a bare proton) is $\mu_{proton} = 2.79277 \mu_N$.

VI. THE TORQUE

We then have

$$\tau(orque) = \imath AB \sin\alpha = -\frac{en\hbar}{2m} B \sin\alpha \tag{6.1}$$

which is, finally (using Equation 4.9),

$$\tau(orque) = n\mu_B B \sin\alpha = \vec{\mu} \times \vec{B} \ \rightarrow \left(\frac{dyne^{1/2} cm^2}{sec}\right) \times \left(\frac{dyne\ sec}{statcoulomb\ cm}\right) = dyne\ cm \tag{6.2}$$

The magnetic moment, $\vec{\mu}$, is proportional to the angular momentum ($\ell = n\hbar$, Bohr), with a proportionality constant γ, i.e.,

$$\vec{\mu}_B = \frac{e\hbar}{2m_e} = \gamma\hbar \tag{6.3}$$

so as the angular momentum precesses in an external magnetic field, so does the magnetic moment vector, *vide infra*.

VII. ENERGY OF AN ORBITING ELECTRON IN AN ARBITRARY MAGNETIC FIELD

The energy associated with rotating the current loop (Bohr orbit) about the axis perpendicular to the field (orienting the loop relative to the field), is

$$E = \int_{reference\ position}^{\alpha,\ final\ angular\ position} \tau(\alpha) d\alpha \tag{7.1}$$

i.e.,

$$E = \int_{90^o}^{\alpha} \tau(x) dx \tag{7.2}$$

which gives

$$E = -n\mu_B B \cos\alpha \tag{7.3}$$

or

$$E = -n\vec{\mu}_B \cdot \vec{B} \rightarrow dyne\ cm = ergs \tag{7.4}$$

VIII. LARMOUR'S THEOREM

We have

$$m\ddot{\vec{r}} = \vec{F} + e\dot{\vec{r}} \times \vec{B} \tag{8.1}$$

If we specialize (arbitrarily) to make the magnetic field point along the z-axis, then

$$m\ddot{x} = F_x + eB\dot{y} \tag{8.2}$$

$$m\ddot{y} = F_y - eB\dot{x} \tag{8.3}$$

and

$$m\ddot{z} = F_z \tag{8.4}$$

What is $\vec{F}$? It is

$$\vec{F} = \frac{Ze^2\hat{r}}{r^3} = \frac{statcoulomb^2}{cm^2} \rightarrow dynes \qquad (8.5)$$

i.e., the Coulomb force. Z is the atomic number, e is the charge on the electron (in $dyne^{1/2}$cm, i.e., statcoulombs). This means

$$F_x = \frac{Ze^2x}{\sqrt{x^2+y^2+z^2}^3} \qquad (8.6)$$

with similar terms for y and z,

$$F_y = \frac{Ze^2y}{\sqrt{x^2+y^2+z^2}^3} \qquad (8.7)$$

$$F_z = \frac{Ze^2z}{\sqrt{x^2+y^2+z^2}^3} \qquad (8.8)$$

It is common to begin treating Larmour precession using a rotating coördinate system attached to the original one, sharing a common z-axis. Therefore, we need to make a "side trip" to coördinate transformations. The situation at hand concerns two dimensions, x and y. Relative to this 'couple' {x(t),y(t)}, we wish to introduce one which is rotating i.e., {x'(t),y'(t)}, where the primed coördinate system is rotating about the z-axis relative to the original {x(t),y(t)} system. Once per revolution, x' and y' will lie exactly juxtaposed upon x and y.

The transformation equations are:

$$x'(t) = x(t)\cos\omega t - y(t)\sin\omega t \qquad (8.9)$$

$$y'(t) = x(t)\sin\omega t + y(t)\cos\omega t \qquad (8.10)$$

which we re-write in matrix notation

$$\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} \cos\omega t & -\sin\omega t \\ \sin\omega t & \cos\omega t \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \qquad (8.11)$$

(see Figure 4).

This set of equations are used to transform from {x(t),y(t)} $\hookrightarrow$ {x'(t),y'(t)}. The reverse of this can be obtained in two steps. First, multiply the top equation (1) by cosine, and the lower one (2) by sine, and add. We obtain

$$x(t) = x'(t)\cos\omega t + y'(t)\sin\omega t \qquad (8.12)$$

and then multiple the top equation by sine and the lower one by cosine, and subtract, yielding

$$y(t) = -x'(t)\sin\omega t + y'(t)\cos\omega t \qquad (8.13)$$

which, in matrix form is:

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} \cos\omega t & \sin\omega t \\ -\sin\omega t & \cos\omega t \end{pmatrix} \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} \qquad (8.14)$$

You tell me an instantaneous value of x and y, and I will tell you the instantaneous values of x' and y' (Equation 8.11), and *vice versa* using Equation 8.14.

It is interesting to substitute the result of one transformation into the other. This should result in 'no transformation', since we are doing and undoing the transformation. We have:

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} \cos\omega t & \sin\omega t \\ -\sin\omega t & \cos\omega t \end{pmatrix} \begin{pmatrix} \cos\omega t & -\sin\omega t \\ \sin\omega t & \cos\omega t \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

which is

$$\begin{pmatrix} \cos\omega t & \sin\omega t \\ -\sin\omega t & \cos\omega t \end{pmatrix} \begin{pmatrix} \cos\omega t & -\sin\omega t \\ \sin\omega t & \cos\omega t \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

which seems perfectly reasonable!

A. Taking time derivatives of coördinates

It is clear that

$$\dot{x} = \frac{dx(t)}{dt} = \frac{dx'(t)}{dt}\cos\omega t + \frac{dy'(t)}{dt}\sin\omega t - x'(t)\omega\sin\omega t + y'(t)\omega\cos\omega t \qquad (8.15)$$

and

$$\dot{y} = \frac{dy(t)}{dt} = -\frac{dx'(t)}{dt}\sin\omega t + \frac{dy'(t)}{dt}\cos\omega t - x'(t)\omega\cos\omega t - y'(t)\omega\sin\omega t \qquad (8.16)$$

and therefore

$$\ddot{x} = \frac{d^2x(t)}{dt^2} = \ddot{x}'(t)\cos\omega t + \ddot{y}'(t)\sin\omega t + 2\omega\left(-\dot{x}'\sin\omega t + \dot{y}'\cos\omega t\right) - \omega^2\left(x'(t)\cos\omega t + y'(t)\sin\omega t\right)$$

so

$$\ddot{y} = \frac{d^2y(t)}{dt^2} = -\ddot{x}'(t)\sin\omega t + \ddot{y}'(t)\cos\omega t + 2\omega\left(-\dot{x}'\sin\omega t + \dot{y}'\cos\omega t\right) + \omega^2\left(-x'(t)\sin\omega t + y'(t)\cos\omega t\right)$$

and substituting into Equation 8.2

$$\ddot{x} = \ddot{x}'(t)\cos\omega t + \ddot{y}'(t)\sin\omega t + 2\omega\left(-\dot{x}'\sin\omega t + \dot{y}'\cos\omega t\right) - \omega^2\left(x'(t)\cos\omega t + y'(t)\sin\omega t\right)$$

$$= \frac{F_x}{m} + \frac{eB}{m}\left(-\dot{x}'\sin\omega t + \dot{y}'\cos\omega t - \omega(x'(t)\cos\omega t + y'(t)\sin\omega t)\right)$$

and,

$$\ddot{y}' = -\ddot{x}'(t)\sin\omega t + \ddot{y}'(t)\cos\omega t + 2\omega\left(-\dot{x}'\cos\omega t + \dot{y}'\sin\omega t\right) + \omega^2\left(-x'(t)\cos\omega t + y'(t)\sin\omega t\right)$$

so, substituting into Equation 8.3

$$\ddot{y}' = \frac{F_y}{m} + \frac{eB}{m}\left(-\dot{x}'\cos\omega t + \dot{y}'\sin\omega t - x'(t)\omega\sin\omega t + y'(t)\omega\cos\omega t\right)$$

We note that the Coulomb force on the electron is not effected substantially by the transformed coördinates, since x and y are merely "changing places", while r stays constant (z stays constant no matter what) (see Figure 5). First, we assume that terms of the order ω^2 are small enough to neglect. Second, we start our clock at t=0 exactly when the x(t) and x'(t) axis are superimposed. Then, we have:

$$\ddot{x}'(t)\cos\omega t + 2\omega\left(\dot{y}'\cos\omega t\right) - \omega^2\left(x'(t)\cos\omega t\right)$$

$$= \frac{F_x}{m} + \frac{eB}{m}\left(\dot{y}'\cos\omega t + x'(t)\omega\cos\omega t\right) \tag{8.17}$$

and

$$\ddot{y}'(t)\cos\omega t + 2\omega\left(\dot{y}'\cos\omega t\right) - \omega^2\left(y'(t)\cos\omega t\right)$$

$$= \frac{F_y}{m} + \frac{eB}{m}\left(\dot{y}'(t)\cos\omega t + x'(t)\omega\cos\omega t\right) \tag{8.18}$$

(see Figure 6). If we were to choose

$$2\omega = \frac{eB}{m} \rightarrow \frac{statcoulomb\,\frac{dynesec}{statcoulomb\,cm}}{gram} = sec^{-1}$$

(see Equation 3.4) then the velocity terms would disappear! This is the Larmour frequency:

$$\omega = \frac{eB}{2m}$$

Approximately, if we choose to rotate a coördinate system at this particular frequency, then the magnetic moment would appear stationary in this rotating coördinate system.

IX. PRECESSION OF A MAGNETIC MOMENT

Regardless of whether a magnetic moment originates in orbital motion of charges or intrinsically in a nucleus, externally, we see a magnetic dipole in space, interacting with external (to it) magnetic fields. How does it react?

We start with the elementary definition of torque,

$$\vec{L} = \vec{r} \otimes \vec{p}$$

so that

$$\frac{d\vec{L}}{dt} = \frac{\vec{p}}{m} \otimes \vec{p} + \vec{r} \otimes \vec{F} = \vec{r} \otimes \vec{F} = \vec{\tau}orque$$

where we have assumed Newton's second Law in equate the time derivative of the momentum with the force, and that $\vec{p} \otimes \vec{p}$ is zero!

$$\vec{\tau}(orque) = \vec{\mu} \times \vec{B} \tag{9.1}$$

and $\vec{\mu} = \gamma\vec{L}$ Since $\frac{d\vec{\mu}}{dt} = -\gamma\vec{B} \otimes \mu$ (and we assume the $\vec{B}$ points along the z-axis) which is

$$\frac{d\mu_x}{dt}\hat{i} + \frac{d\mu_y}{dt}\hat{j} + \frac{d\mu_z}{dt}\hat{k} = -\gamma \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & B \\ \mu_x & \mu_y & \mu_z \end{vmatrix}$$

we see immediately that the z-component of the magnetic moment never changes! [6] Expanding, we have

$$\frac{d\mu_x}{dt} + \gamma B\mu_y = 0$$

$$\frac{d\mu_y}{dt} + \gamma B\mu_x = 0$$

a solution of which is

$$\vec{\mu} = (Acos\gamma Bt)\,\hat{i} + (Asin\gamma Bt)\,\hat{j} + a\,constant\,\hat{k}$$

which means that the μ vector is precessing about the z-axis, with its x- and y-components interchanging during the precession.

X. AN ALTERNATIVE DERIVATION OF THE PRECESSION

$$\vec{t}orque = \gamma\vec{\mu}\otimes\vec{B}$$

is the torque on a spin (1/2) particle in a static magnetic field $\vec{B}$. Then

$$\frac{d\vec{\mu}}{dt} = \vec{t}orque$$

or

$$\frac{d\vec{\mu}}{dt} = -\gamma\vec{B}\otimes\vec{\mu}$$

If we write

$$\vec{\mu} = -|\vec{\mu}|\left(\hat{i}\sin\theta\cos\phi + \hat{j}\sin\theta\sin\phi + \hat{k}\cos\theta\right)$$

then

$$\vec{B}\cdot\vec{\mu} = -|\vec{\mu}|\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & B \\ \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\phi \end{vmatrix}$$

So

$$\frac{d\mu_x}{dt} = -\gamma B|\vec{\mu}|\sin\theta\sin\phi = \frac{d|\vec{\mu}|\sin\theta\cos\phi}{dt} \quad (10.1)$$

$$\frac{d\mu_y}{dt} = \gamma B|\vec{\mu}|\sin\theta\cos\phi = \frac{d|\vec{\mu}|\sin\theta\sin\phi}{dt} \quad (10.2)$$

$$\frac{d\mu_z}{dt} = 0 = \frac{d|\vec{\mu}|\cos\theta}{dt} \quad (10.3)$$

The last equation may be integrated directly, leaving

$$\cos\theta = constant$$

assuming the $|\vec{\mu}|$ is constant (as it must be).

In order to integrate Equation 10.1 and Equation 10.2 we write

$$\gamma B\sin\theta\sin\phi = \frac{d\sin\theta\cos\phi}{dt} = -\sin\theta\cos\phi\frac{d\phi}{dt} \quad (10.4)$$

$$-\gamma B\sin\theta\cos\phi = \frac{d\sin\theta\sin\phi}{dt} = \sin\theta\sin\phi\frac{d\phi}{dt} \quad (10.5)$$

which means that

$$-\gamma B = \frac{d\phi}{dt}$$

or

$$\phi = -\gamma Bt + C$$

where C is a constant (usually called the phase angle).

XI. NMR

If we assume that the magnetic moment we are dealing with is due to a proton, i.e., a spin one half particle, then we know there are two orientations possible, i.e., two possibilities of the state of the proton, one with the magnetic moment 'up', and the other 'down'. The energy of these two states are

$$E_{down} = \frac{1}{2}\gamma B$$

and

$$E_{up} = -\frac{1}{2}\gamma B$$

with a ΔE

$$\Delta E = \gamma B = g_N\mu_N B$$

(see Figure 7). where $g_N = \mu$ is the "nuclear factor", and B is the magnetic field strength (see Figure 8). Empirically, $g_N = \gamma = 5.5856912$. One has, then,

$$\frac{\Delta E}{h} = \nu = \frac{g_N\mu_N B}{h}$$

which, for a 1 Tesla Field would be

$$\nu = \frac{5.5856912\times 5.0508014\times 10^{-27}J/T\times 1T}{6.627\times 10^{-34}J-sec} = 4.26\times 10^{7}seconc$$

XII. PERTURBING WITH AN EXTERNAL MAGNETIC FIELD

In the Continuous Wave (CW) experiment, we superimpose a rotating magnetic perturbation field on the above system, choosing to apply that field in the x-y plane only, at right angles to the main field which originally set up the energy differences. We make this field $B_o\cos\omega t$ where the strength of the perturbation B_o is small, and the frequency ω is tunable. As we change ω we tune through the frequency ω set up by the main magnetic field, and at this ω_{tuned} value, transitions are observed.

XIII. BLOCH'S EQUATIONS

In order to talk about spin relaxation, one needs to first obtain the differential equations governing the macroscopic spin. Normally, this discussion concerns the macroscopic magnetic moment, rather than individual spins. The macroscopic magnetic moment is normally called $\vec{M}$. with three components, one of which, chosen arbitrarily, coincides with the z-axis and the external applied magnetic field.

If the microscopic spins have aligned themselves properly in the external field, with N_α in the α state and N_β in the β state, then

$$M_z = \gamma_N \hbar (N_\alpha - N_\beta)$$

where γ_N is the magnetogyric ratio appropriate.

We know that at thermal equilibrium

$$\frac{N_\alpha}{N_\beta} = e^{g_N \frac{e\hbar}{2mc} B/kT} = e^{g_N \mu_B B/kT}$$

When the field is turned off, the z-component of magnetic moment will approach its final value exponentially in time, so that the above ratio approaches 1.

$$\frac{1}{\gamma_N \hbar}\frac{dM_z}{dt} = -\frac{N_\alpha - N_\beta}{T_1}$$

where T_1 is defined as the spin-lattice relaxation characteristic time. or

$$\frac{dM_z}{dt} = -\frac{M_z}{T_1}$$

When the magnetic field is different from zero but constant, M_z tends to a constant value, M_0, related to the macroscopic static magnetic susceptibility. Now, one has

$$\frac{dM_z}{dt} = -\frac{M_z - M_0}{T_1}$$

and

$$\frac{M_x}{dt} = -\frac{M_x}{T_2}$$

and

$$\frac{M_y}{dt} = -\frac{M_y}{T_2}$$

where T_2 is the transverse relaxation time.

Since the bulk magnetic moment is the sum of all the individual magnetic moments, and since they are actually precessing about the magnetic field, one has

$$\frac{d\vec{M}}{dt} = \gamma_N \vec{M} \otimes \vec{B}$$

in the absence of relaxation. We know that this latter leads to

$$\frac{M_x}{dt} = -\frac{M_x}{T_2} + \gamma_N M_y B_z$$

and

$$\frac{M_y}{dt} = -\frac{M_y}{T_2} + \gamma_N M_x B_z$$

and finally

$$\frac{dM_z}{dt} = -\frac{M_z - M_0}{T_1}$$

XIV. ALTERNATIVE VIEW OF CURRENT LOOP (ADDRESSING SPIN-SPIN ETC., COUPLING ORIGINS

We can view the current loop as a single turn of a solenoid, i.e., creating a magnetic field which passes through the center of the loop. We have, using the Law of Biot and Savart

$$\delta\vec{B} = i\frac{\vec{d\sigma} \times \vec{r}}{r^3}$$

$$\delta\vec{B} = Ki\frac{d\sigma \times r\cos 90^o}{r^3} = Ki\frac{d\sigma}{r^2}$$

where $\delta\vec{B}$ is perpendicular to the $\vec{r}$ vector, indicated as δB and K is a proportionality constant whose value is exactly $10^{-7}\frac{weber}{ampere\ meter}$ which is often written as

$$\frac{\mu_o}{4\pi} = K$$

where

$$\mu_o = 12.57 \times 10^{-7}\frac{weber}{ampere\ meter}$$

(see Figure 9). on the figure. The component of δB along the x-axis from this piece of current loop is

$$\delta B_x = \frac{\mu_o}{4\pi} i \frac{d\sigma}{r^2}\sin\alpha$$

and if we add up the contributions from all elements of the loop by integrating around it, obtaining $2\pi R$ (the circumference) we obtain

$$B_x = \frac{\mu_o}{4\pi} i \frac{\sin\alpha 2\pi R}{r^2}$$

But $\sin\alpha$ is R/r, so we obtain

$$B_x = \frac{\mu_o}{4\pi} i \frac{\pi R^2}{r^3}$$

and if we now translate the point of interest to the origin, so that $r \rightarrow R$, we obtain

$$B_x = \frac{\mu_o}{4\pi} i \frac{\pi R^2}{R^3} \rightarrow \frac{weber}{ampere\ meter}\frac{ampere}{meter} = \frac{weber}{meter^2} = tesla$$

$$B_x = \frac{\mu_o}{4}\frac{i}{R} \rightarrow tesla = \frac{Newton\ second}{Coulomb\ meter}$$

This result says that there is a magnetic field at the origin, caused by the current loop. It is as if there were no current loop. but instead there was a little bar magnet at the origin (see Figure 10). It is clear that if the current element is an electron (in an atomic orbit) and if that electron carries a spin with it, then that spin would interact with the magnetic field generated by the dipole at the origin (itself the result of the electron orbiting), so that there would be two states, one in which the electron spin was 'up' relative to the magnetic dipole's field, and the other 'down'. This is a quasi-classical model for spin orbit interaction, wrong in several details, but sufficient to illustrate approximately where spin orbit interaction comes from.

We had

$$\mu = -\frac{ne\hbar}{2m}$$

for any n, and we can write this as

$$\mu = \mid \vec{\mu} \mid \Rightarrow \vec{\mu} \equiv \vec{L}\frac{e}{2m} \equiv \gamma\vec{L}$$

where γ is the magnetogyric ratio. We are equating the magnetic moment and the angular momentum through a proportionality constant. This means that for a classical orbit with $\ell \neq 0$ there exists an equivalent magnetic moment, located at the origin (which is where the nucleus lies), i.e., a tiny bar magnet associated with the *orbital angular momentum.*

XV. FIGURES

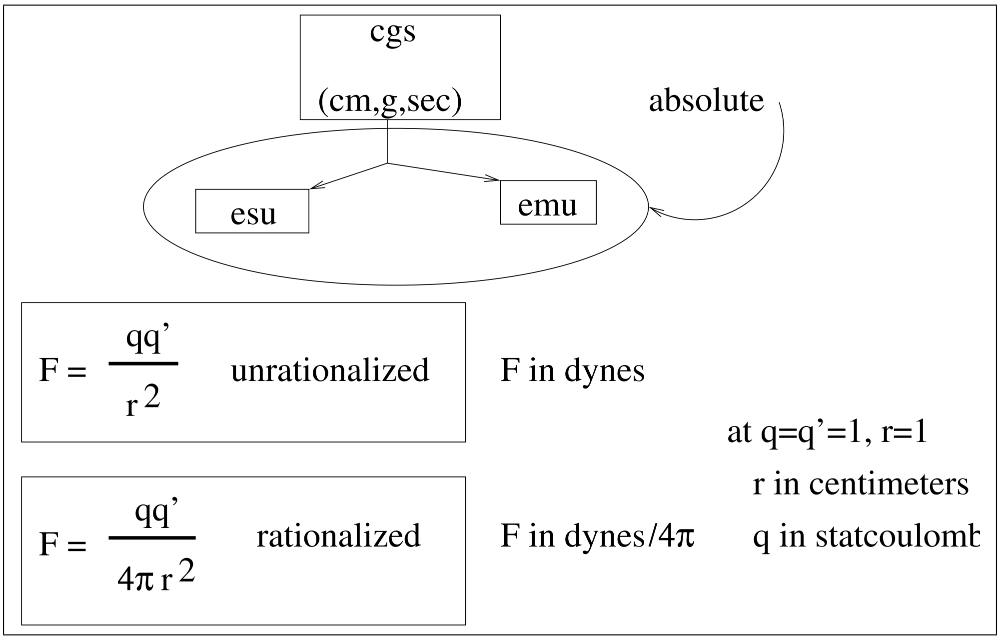

FIG. 1: Unrationalized versus rationalized units

[1] In order to remember directions in the cross product, one need only remember that

$$\hat{\imath} \otimes \hat{\jmath} \rightarrow \hat{k}$$

[2] If there is an electric field, the Lorentz force would be $\vec{F} = q\vec{v} \otimes \vec{B} + q\vec{E}$

[3] $10^{-4} Tesla = 1 Gauss.$

[4] remember, $e = 4.8 \times 10^{-10} statcoulomb$, or $1.6 \times 10^{-19} Coulomb$

[5] we use τ twice here, once for the torque, and once for the period- both usages are common

[6] Reiterating, $\vec{B} = B\hat{k}$ (a convenience).

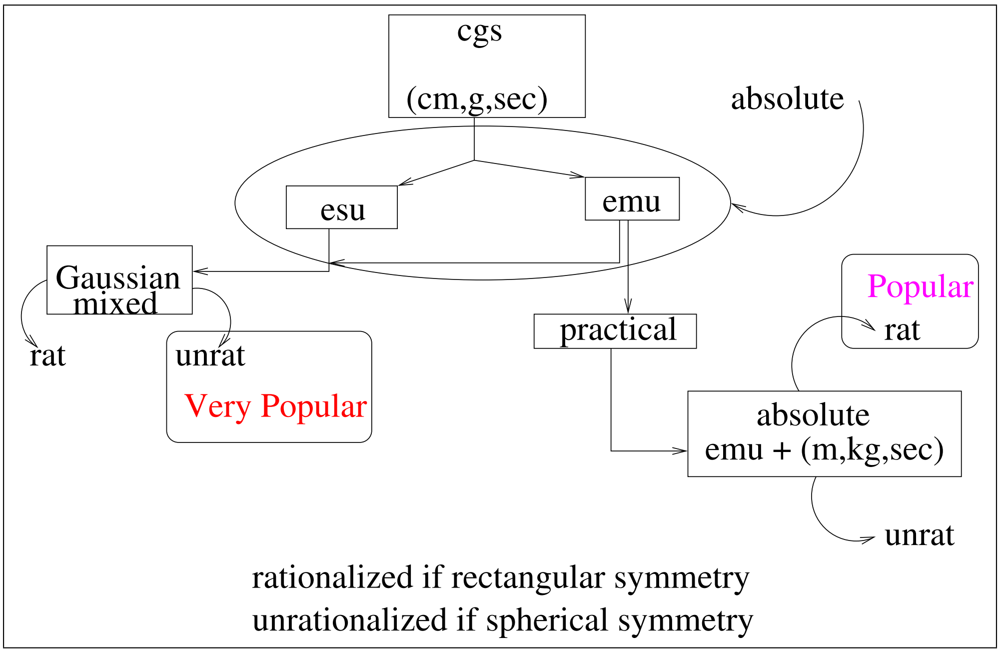

FIG. 2: Units, units, units

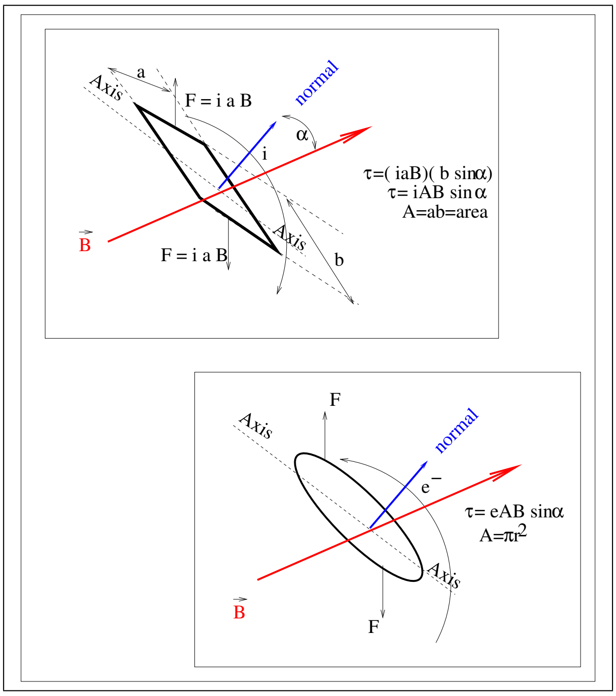

FIG. 3: A Current Loop in a Magnetic Field

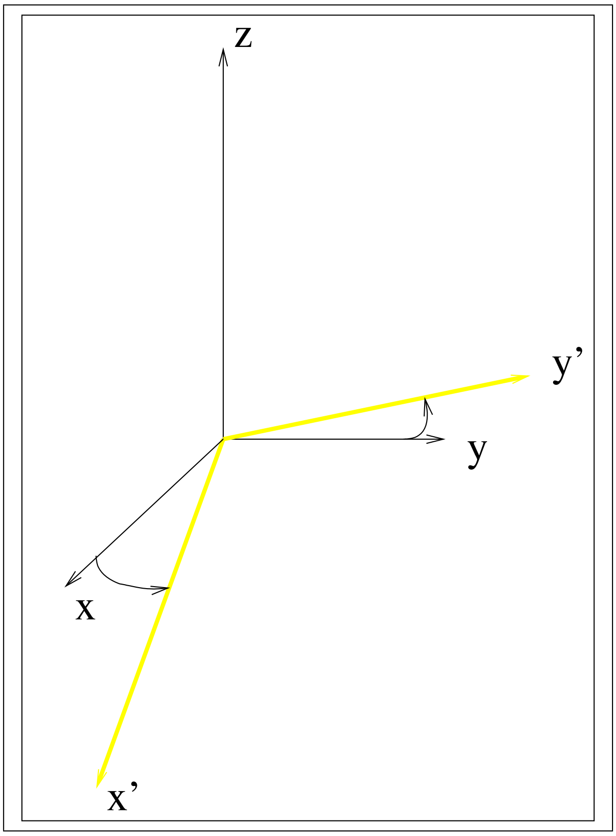

FIG. 4: Relationship between ‘stationary’ and ‘rotating’ coördinates

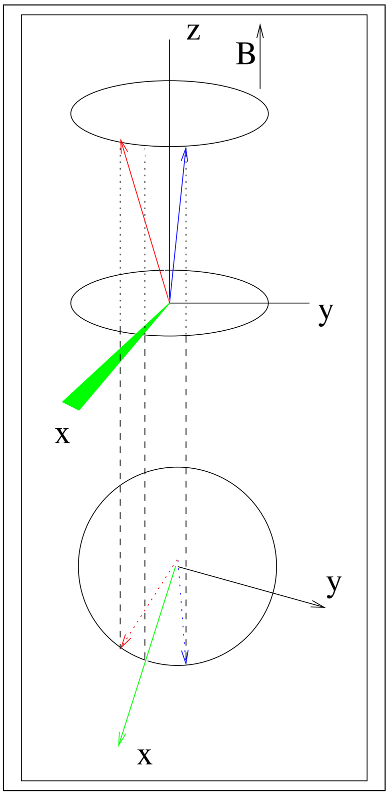

FIG. 5: Precessing Magnetic Moment in Magnetic Field

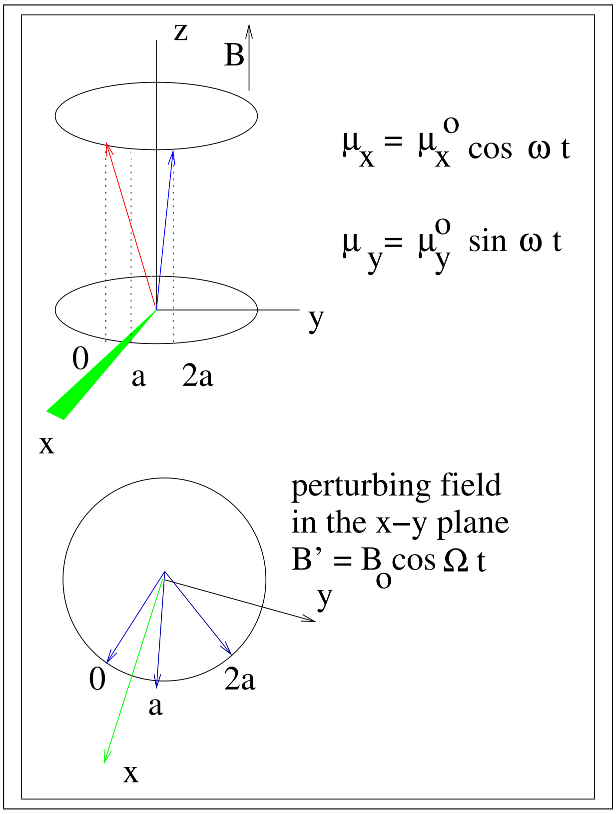

FIG. 6: Relation of Precessing Magnetic Moment to Instrumental Perturbing Magnetic Field in x-y plane

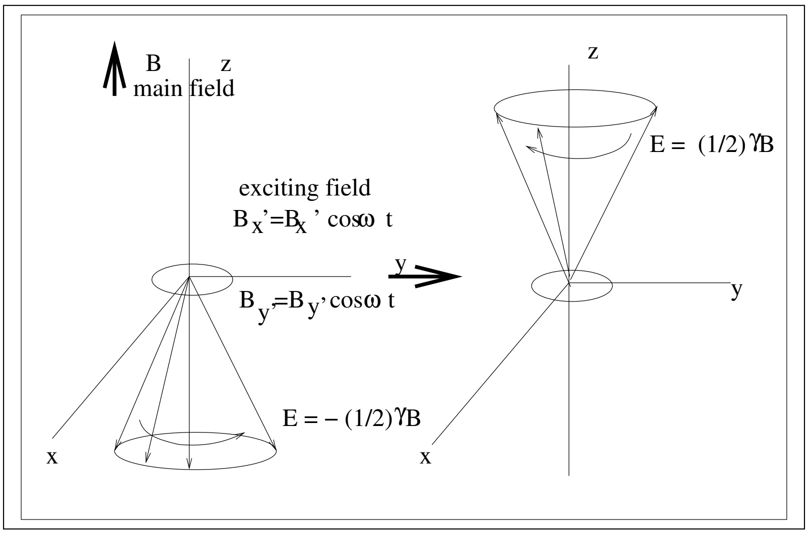

FIG. 7: NMR Transition, from Low to High Energy State

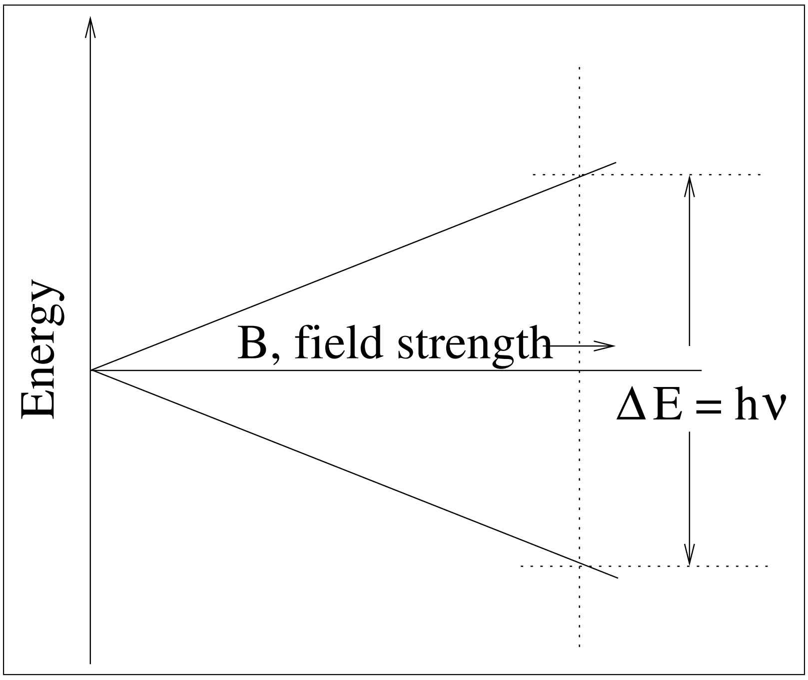

FIG. 8: NMR Transition at a Chosen Field Strength

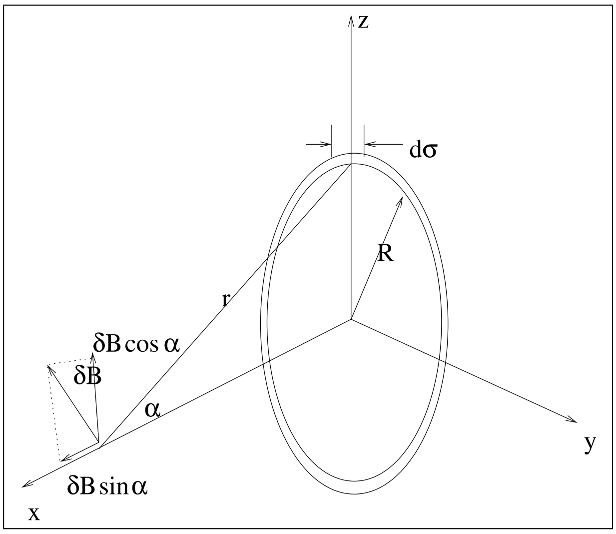

FIG. 9: A Current Loop Creating a Magnetic Field

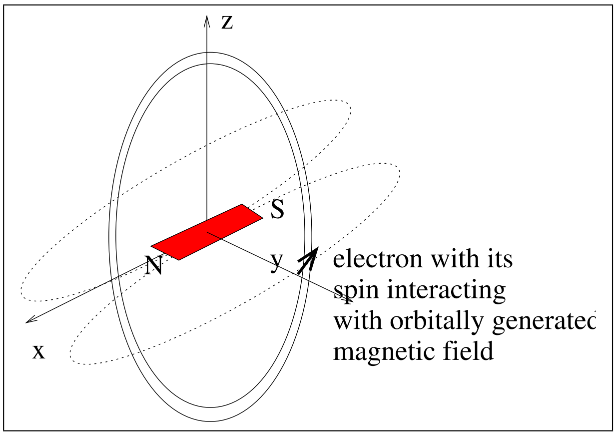

FIG. 10: A Current Loop Creating a Magnetic Field is Equivalent to a Bar Magnet